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PATHWISE SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS

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SUMMARY. A formula for evaluating pathwise, the solution to a stochastic differential equation with continuous semimartingale differentials is obtained under the usual lipschitz condition on the coefficients. As an application of this result, a formula for the natural increasing process associated with a continuous local martingale is obtained.

1. Introduction

In this paper we consider Stochastic Differential Equations (SDE) with semimartingale differentials. These have been considered earlier in the literature by Protter (1977) (for continuous semimartingale integrators), Doleans-Dade (1976), Doleans-Dade and Meyer (1975, 1976), Emery (1978) and Bichteler (1979). We consider the continuous case. Our methods are different from the methods used earlier and are simpler. We use a suitable "time change" to get estimates for the Stochastic integral. Protter also applies a time change (See Portter, 1977, p. 248), but he uses it to get an inequality for the norm he considers on the space of continuous semimartingales. We employ a slight modification of the usual iteration procedure. In the usual iteration procedure, the n-th iterate is a stochastic integral of a functional of the $(n-1)$-th iterate. In our method, the n-th iterate is an approximation to the stochastic integral involved, the approximation being pathwise. Our estimates show that these "approximate iterates" converge almost surely to a solution of the original equation, so that we have a formula to evaluate the solution pathwise. It should be noted that Bichteler (1979) also considers pathwise solutions, but he needs to use repeated limits to evaluate the solution. More recently, Bichteler (1980) has obtained a formula similar to ours using different methods. As a consequence of our main result we get "pathwise integration formula" for stochastic integral. This, with Ito's formula, gives a nice formula for the associated natural increasing process of a continuous local martingale. For a similar formula see Kunita-Watanabe (1967, Theorem 1.3). Simple proof of a result on convergence of solutions of SDE as the coefficients converge can be obtained by our method. But for a time change, the details are same as in Friedman (1975) for the Brownian motion case and hence omitted.

Professor P. A. Meyer has drawn our attention to Kazamaki (1974) where equations with semimartingale differentials were first considered. Our method of time change is same as in Kazamaki where existence and uniqueness were treated in a special case.

Professor H. Kunita has drawn our attention to the works of Wong and Zakai (1965); Stroock and Varadhan (1972). These authors consider SDE with Brownian motion differentials. They approximate the solution to the SDE by those to ODE. The approximations given by Stroock and Varadhan converge in the sense of distribution, those of Wong and Zakai converge in the quadratic mean.

2. Notations and preliminaries

$(\Omega, \mathcal{B})$ is a fixed measurable space. A filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ is an increasing right continuous family of sub σ-fields of $\mathcal{B}$. P will always denote a probability measure on $(\Omega, \mathcal{B})$ and $\mathcal{F}^P$ will denote the filtration obtained by augmenting each $\mathcal{F}_t$ by P-null sets from the P-completion of $\mathcal{B}$. For a $\mathcal{F}$ and P as above, let

$$\mathcal{C}(\mathcal{F}) = \{X : X \text{ is a continuous, } \mathcal{F} \text{ adapted process}\}$$

$$\mathcal{D}(\mathcal{F}) = \{X : X \text{ is a left continuous, } \mathcal{F} \text{ adapted process having right limits.}\}$$

$$\mathcal{A}(\mathcal{F}) = \{X \in \mathcal{C}(\mathcal{F}) : X(0) = 0 \text{ and } t \to X(t, w) \text{ has bounded variation on bounded intervals for all } w \in \Omega.\}$$

$$\mathcal{L}(\mathcal{F}, P) = \{X \in \mathcal{C}(\mathcal{F}) : X(0) = 0 \text{ and } (X(t), \mathcal{F}_t) \text{ is a } P\text{-local Martingale.}\}$$

$$\mathcal{S}(\mathcal{F}, P) = \{X \in \mathcal{C}(\mathcal{F}) : X = M+B,\ M \in \mathcal{L}(\mathcal{F}, P) \text{ and } B \in \mathcal{A}(\mathcal{F}).\}$$

$$\mathcal{T}(\mathcal{F}) = T : \{T \text{ is a } \mathcal{F} \text{ stop time.}\}$$

A vector (or matrix) valued process is said to belong to a class of processes, say $\mathcal{U}$ if each of its components belongs to $\mathcal{U}$.

If $X \in \mathcal{D}(\mathcal{F})$ and $T \in \mathcal{T}(\mathcal{F})$, following Protter (1977) we define the stopped process X^T by $X^T(t, w) = X(t \wedge T(w), w)\ 1_{(T>0)}(w)$.

A process H is said to be locally bounded if

$$\exists T_n \in \mathcal{T}(\mathcal{F});\ T_n \uparrow \infty,\ s.t.\ |Y^{T_n}| \leqslant n.$$

Let $E = E_d = C([0, \infty), \mathcal{R}^d)$ be the metric space of continuous functions on $[0, \infty)$ into $\mathcal{R}^d$ equipped with the topology of uniform convergence on compact subsets of $[0, \infty)$. For $\rho \in E$, $t \geqslant 0$, let $|\rho|_t^* = \sup_{0 \leqslant s \leqslant t} |\rho(s)|$. ($|.|$ denotes the usual norm on $\mathcal{R}^d$. Similarly $\|.\|$ denotes the norm, root of sum of squares of the entries, on matrices.)

For $M \in L(\mathcal{F}^P, P)$, let $< M, M >^P \in \mathcal{C}(\mathcal{F})$ be its associated natural increasing process. For $A \in \mathcal{A}(\mathcal{F})$, let $|A| \in C(\mathcal{F})$ be the process

$$|A|(t, w) = \int_0^t |dA(s, w)|.$$

Definition : $\sigma = (\sigma_t)_{t \geq 0} \subseteq \mathcal{T}(\mathcal{F})$ is called a "strict $\mathcal{F}$ time change" if $\sigma_0(w) = 0$, $\lim_{t \to \infty} \sigma_t(w) = \infty$, and $t \to \sigma_t(w)$ is a strictly increasing continuous function for all $w \in \Omega$.

For a "strict $\mathcal{F}$ time change" σ and $X \in \mathcal{D}(\mathcal{F})$, let $\sigma\mathcal{F}$ denote the filtration $(\mathcal{F}_{\sigma_t})_{t \geq 0}$ and σX denote the process $(\sigma X)(t, w) = X(\sigma_t(w), w)$. Let π_σ denote the map $X \to \sigma X$. Let $\lambda_t(w) = u$ if $\sigma_u(w) = t$, $t \in [0, \infty)$, $w \in \Omega$.

We now list some properties of a strict time change.

Lemma 1 : (a) *If* $T \in \mathcal{T}(\mathcal{F})$ *then* $\lambda_T \in \mathcal{T}(\sigma\mathcal{F})$ *and* $\mathcal{F}_T = (\sigma\mathcal{F})_{\lambda_T}$.

(b) λ *is a "strict* $\sigma\mathcal{F}$ *time change" and* $\lambda(\sigma\mathcal{F}) = \mathcal{F}$.

(c) *If* $X \in \mathcal{D}(\mathcal{F})$ *and* $T \in \mathcal{T}(\mathcal{F})$ *then* $\sigma(X^T) = (\sigma X)^{\lambda_T}$.

(d) π_σ *is a bijection between* ($\mathcal{U}(\mathcal{F})$ *and* $\mathcal{U}(\sigma\mathcal{F})$), *where* $\mathcal{U}$ *is any of the class of processes defined at the beginning of this section.*

(e) *Let* $S \in \mathcal{S}(\mathcal{F}, P)$ *and* $f \in \mathcal{D}(\mathcal{F})$ *and* X *be defined by* $X(t) = \int_0^t f dS$. *Then* $(\sigma X)(s) = \int_0^s (\sigma f) d(\sigma S)$.

Proof : Observe that $\bigvee_{t \geq 0} \mathcal{F}_t = \bigvee_{s \geq 0} \mathcal{F}_{\sigma_s}$, so (a) follows from Lemma 10.5 in Jacod (1979, p. 312). (b), (c), (d) are easy to verify. For a proof of (e) see Lemma 10.8 in Jacod (1979, p. 318).

The following lemma is the key step in all our convergence arguments in the later sections.

Lemma 2 : (a) *Let* $S \in \mathcal{S}(\mathcal{F}, P)$. *($R^l$ valued),* $S = M + A$, $M \in \mathcal{L}(\mathcal{F}^P, P)$ *and* $A \in \mathcal{A}(\mathcal{F}^P)$, *satisfy* (*).

For all $i \leq l$ *and* $w \in \Omega$, $< M_i, M_i >^P (t, w)$ *and* $|A_i|(t, w)$ *are absolutely* (*) *continuous functions of* t *with derivatives bounded by one.*

For $f \in D(\mathcal{F})$, ($\mathcal{R}^d \otimes \mathcal{R}^l$ *valued), we have*

$$E|\int_0^{\cdot} f dS|_t^{*2} \leq 8l(1+t) \int_0^t E\|f\|^2(u, w) du.$$

(b) *Let $S \in \mathcal{S}(\mathcal{F}, P)$. There exists a strict $\mathcal{F}^P$ time schange σ such that σS satisfies* (*).

Proof: (a) To prove (a), suffices to show (i), (ii) hold.

$$\text{(i)} \quad E|\int_0 f dM|_t^{*2} \leqslant 4l \int_0^t E\|f\|^2(u, w)du$$

and

$$\text{(ii)} \quad E|\int_0 f dA|_t^{*2} \leqslant tl \int_0^t E\|f\|^2(u, w)du.$$

Observe that

$$E|\int_0 f dM|_t^{*2} = E \sup_{s \leqslant t} \sum_{i=1}^{d} \Big(\sum_{j=1}^{l} \int_0^s f_{ij} dM_j \Big)^2$$

$$\leqslant l \sum_{i=1}^{d} \sum_{j=1}^{l} E \sup_{s \leqslant t} \Big(\int_0^s f_{ij} dM_j \Big)^2 \text{ by Cauchy-Schwarz inequality.}$$

$$\leqslant 4l \sum_{i=1}^{d} \sum_{j=1}^{l} E \Big(\int_0^t f_{ij} dM_j \Big)^2, \text{ by Doob's maximal inequality.}$$

$$= 4l \sum_{i=1}^{d} \sum_{j=1}^{l} E \int_0^t |f_{ij}|^2 d < M_j, M_j >^P \text{ by } L^2\text{-isometry}$$

for Stochastic integral

$$\leqslant 4l \sum_{i=1}^{d} \sum_{j=1}^{l} E \int_0^t |f_{ij}|^2(u)du \text{ by condition } (*).$$

$$= 4l \int_0^t E\|f\|^2(u, w)du \text{ by Fubini.}$$

(ii) follows similarly from Cauchy-Schwarz inequality, Holder's inequality, condition (*) and Fubini.

For (b), define

$$\lambda_t(w) = \sum_{i=1}^{l} (< M_i, M_i >^P(t, w) + |A_i|(t, w)) + t.$$

(where $S = M + A$ as in (a)). Let $\sigma_t(w) = u$, if $\lambda_u(w) = t$. Then it is easy to verify that σ is a strict $\mathcal{F}^P$ time change and σS satisfies the condition (*).

3. Main result

Let $b : \Omega\times[0,\infty)\times E \to \mathcal{R}^d \otimes \mathcal{R}^l$ be such that (C1) holds.

$$\text{(C1)} \begin{cases} \text{(i) } \forall\, \rho \in E,\ b(.,.,\rho) \in \mathcal{B}(\mathcal{F}). \\ \text{(ii) } \forall\, t_0 \geqslant 0.\ b : \Omega\times[0,t_0)\times E \text{ is } \mathcal{F}_{t_0} \otimes \mathcal{B}_{[0,t_0]} \otimes \mathcal{N}_{t_0} \text{ measurable.} \end{cases}$$

($\mathcal{N}_{t_0}$ is the smallest σ-field on E with respect to which the family of maps $\{\rho \to \rho(s) : s \leqslant t_0\}$ is measurable.

Observe that if b satisfies (C1), and $T \in \mathcal{J}()\mathcal{F}$. then b^T also satisfies (C1) $(b(w,t,\rho) = \underset{(T>0)}{1(w)}\ b^T(w, t \wedge T(w), \rho)$.

For b satisfying (C1), $\phi \in \mathcal{C}(\mathcal{F})$, S and P such that $S \in \mathcal{S}(\mathcal{F}, P)$, we consider the SDE

$$X(t) = \phi(t) + \int_0^t b(., u, X)dS(u). \qquad \ldots \quad \text{(I)}$$

If X satisfies (I) we say X is a solution of (I) for (ϕ, b, S) (or ϕ, b, S, P) if we want to stress that the integral in (I) is on $(\Omega, \mathcal{B}, P)$.

Lemma 3 : (a) *If X is a solution of* (I) *for* (ϕ, b, S) *and* $T \in \mathcal{J}(\mathcal{F})$, *then* X^T *is a solution of* (I) *for* (ϕ^T, b^T, S^T).

(b) *If* $T_n \uparrow \infty$, $T_n \in \mathcal{J}(\mathcal{F})$. $X \in \mathcal{C}(\mathcal{F})$ *are such that* X^{T_n} *is a solution to* (I) *for* $(\phi^{T_n}, b^{T_n}, S^{T_n})$ *for every* n *then* X *is a solution to* (I) *for* (ϕ, b, S).

Proof : We only have to observe that (ii) in (C1) implies that if $|\rho_1-\rho_2|_t^* = 0$, then $b(w,t,\rho_1) = b(w,t,\rho_2)\ \forall\ w \in \Omega$. Hence

$$b^T(w, t, X(w)) = b^T(w, t, X^T(w))\ \forall X \in \mathcal{C}(\mathcal{F}),\ T \in \mathcal{J}(\mathcal{F}).$$

(a) and (b) now follow from properties of stochastic integral.

Now given b such that (C1) holds; ϕ, $S \in \mathcal{C}(\mathcal{F})$, $\mathcal{R}^l$ valued, such that $S(0) = 0$; we define a process X "pathwise" (i.e., $X(t, w)$ is defined only in terms of $\phi(u, w)$, $S(u, w)$, $b(w, u, \rho)$ for $\rho \in E$ and $0 \leqslant u \leqslant t$) and prove (under some conditions on b) that X is the unique solution of (I) for (ϕ, b, S, P), where P is any measure s.t. $S \in \mathcal{S}(\mathcal{F}, P)$. To this end, let

$$a(w, t, \rho) = b(w, t+, \rho),$$
$$X_0(t, w) = \phi(t, w).$$

Define $\{T_i^n\}_{i \geq 0}$, X_n, $n \geq 1$ inductively by

$$T_o^n(w) = 0$$

$$T_{i+1}^n(w) = \inf\{t \geq T_i^n(w) : \|a(w, t, X_{n-1}(w)) - a(w, T_i^n(w), X_{n-1}(w))\| \geq 2^{-n}\}$$

$$X_n(t, w) = \phi(t, w) + \sum_{i=0}^{\infty} a(w, T_i^n(w), X_{n-1}(w))[S(t \wedge T_{i+1}^n(w), w) - S(t \wedge T_i^n(w), w)].$$

(observe that each sum is a finite sum. so $X_n \in \mathcal{C}(\mathcal{F})$ and that for each n; $\lim_i T_i^n = \infty$).

Let

$$\Omega_0 = \{w : X_n(., w) \text{ converges in } E\}$$

and

$$X(t, w) = \begin{cases} \lim X_n(t, w) & \text{if } w \in \Omega_0 \\ 0 & \text{if } w \in N = \Omega_0^c, \end{cases}$$

Theorem 1 : *Let ϕ, b, S, X as above, assume that b satisfies* (C2) :

$$\text{(C2)} \quad \begin{cases} \exists \text{ a locally bounded process } K \text{ s.t.} \\ \|b(w, t, \rho_1) - b(w, t, \rho_2)\| \leq K(t, w) |\rho_1 - \rho_2|_t^* \ \forall\, w, t, \rho_1, \rho_2. \end{cases}$$

Then for every P such that $S \in S(\mathcal{F}^P, P)$, X is a solution of (I) for (ϕ, b, S, P), and $P(N) = 0$.

Proof : Fix a P s.t. $S \in S(\mathcal{F}^P, P)$.

First, assume that ϕ and b satisfy (C3)

$$\text{(C3)} \quad \begin{cases} \exists \text{ a constant } K < \infty, \text{ such that} \\ |\phi(t, w)| \leq K, \|b(w, t, 0)\| \leq K, \\ \|b(w, t, \rho_1) - b(w, t, \rho_2)\| \leq K |\rho_1 - \rho_2|_t^* \text{ for all } w, t, \rho_1, \rho_2. \end{cases}$$

Then we have

$$\|b(w, t, \rho)\| \leq K(1 + |\rho|_t^*).$$

For $n \geq 1$ let

$$f_n(t, w) = b(w, t, X_{n-1}(w))$$

and

$$g_n(t, w) = \sum_{i=0}^{\infty} a(w, T_i^n(w), X_{n-1}(w)) 1^{(t)}_{[(T_i^n(w), T_{i+1}^n(w)]}.$$

Then,

$$\|f_n - g_n\| \leqslant 2^{-n}.$$

and

$$X_n(t) = \phi(t) + \int_0^t g_n dS.$$

Let σ be a strict $\mathscr{F}P$ time change such that $R = \sigma S$ satisfies the condition (*) (Lemma 2). Observe that

$$\begin{aligned} \|\sigma f_n(u) - \sigma f_{n+1}(u)\| &= \|f_n(\sigma_u) - f_{n+1}(\sigma_u)\| \\ &\leqslant K|X_n - X_{n+1}|^*_{\sigma_u} \\ &= K|\sigma X_n - \sigma X_{n+1}|^*_u \end{aligned}$$

and

$$(\sigma X_n)(t) = (\sigma\phi)(t) + \int_0^t (\sigma g_n) dR.$$

Hence

$$E|\sigma X_1 - \sigma X_0|_t^{*2} \leqslant 8(1+t)l[K(1+K)]^2$$

and for $n \geqslant 1$

$$E|\sigma X_{n+1} - \sigma X_n|_t^{*2} \leqslant 8(1+t)l \int_0^t E|\sigma g_{n+1}(u) - \sigma g_n(u)|^2 du$$

$$\leqslant 8(1+t)l(t2^{-2n} + t2^{-2n-1} + \int_0^t E|\sigma f_{n+1}(u) - \sigma f_n(u)|^2 du)$$

$$\leqslant 8(1+t)l(2t2^{-2n} + K\int_0^t E|\sigma X_n - \sigma X_{n-1}|_u^* du).$$

Fix $0 < t_0 < \infty$. Then for some $K_1 < \infty$, we have

$$\alpha_n(t) = E|\sigma X_{n+1} - \sigma X_n|_t^{*2}$$

satisfies

$$\alpha_0(t) \leqslant K_1$$

and

$$\alpha_n(t) \leqslant K_1(2^{-2n} + \int_0^t \alpha_{n-1}(u)du) \quad 0 \leqslant t \leqslant t_0,\ n \geqslant 1.$$

So, by induction it follows that

$$\alpha_n(t) \leqslant 2K_1 2^{-2n} e^{8K_1 t}.$$

This inequality with Borel-Cantelli lemma implies that $X_n(., w)$ converges in E a.s. P, so that $P(N) = 0$,

and $$E|\sigma X_n - \sigma X|_t^{*2} \leqslant \left\{ \sum_{k=n}^{\infty} \sqrt{\alpha_k(t)} \right\}^2 = \beta_n(t), \quad 0 \leqslant t \leqslant t_0$$

where $\beta_n(t) \to 0$ as $n \to \infty$.

Thus

$$\int_0^t \|\sigma g_n(u) - \sigma f(u)\|^2 du \to 0$$

where

$$f(t, w) = b(w, t, X(w)).$$

This implies

$$\int_0^t g_n(u)dS(u) \to \int_0^t f(u)dS(u),$$

which shows that X is solution of (I) for (ϕ, b, S, P).

Uniqueness : Let X' be any solution of (I). Then,

$$E|\sigma X - \sigma X'|_t^{*2} \leqslant 8(1+t)tK \int_0^t E|\sigma X - \sigma X'|_u^{*2} du,$$

and

$$\sigma X(0) = \sigma X'(0) = \phi(0).$$

Hence,

$$E|\sigma X - \sigma X'|_t^{*2} \equiv 0 \text{ for all } t \geqslant 0.$$

Thus $X = X'$ a.s. P.

Now to relax conditions (C3) on ϕ, b, get $U_m \uparrow \infty$, $U_m \in \mathcal{T}(\mathcal{F})$ s.t. $\forall\, m \geqslant 1$ ϕ^{U_m}, $b^{U_m}(., ., 0)$, K^{U_m} are bounded. This can be done because $\phi \in \mathcal{C}(\mathcal{F})$; $b(., ., 0) \in \mathcal{B}(\mathcal{F})$ and K is locally bounded.

Let us observe that X_n^{Um} can be obtained from $(\phi^{Um}, b^{Um}, S^{Um})$ by the same formula by which X_n was defined in terms of (ϕ, b, S). Thus by earlier part of the proof; X_n^{Um} converges to a solution of (I) for $(\phi^{Um}, b^{Um}, S^{Um}, P)$. Thus $P(N) = 0$; and $X_n^{Um} \to X^{Um}$ a.s. P, for all $m \geqslant 1$.

Thus X^{Um} is a solution of (I) for $(\phi^{Um}, b^{Um}, S^{Um}, P)$.

Lemma 3 now implies that X is a solution of (I) for (ϕ, b, S, P). Uniqueness follows from Lemma 3 and uniqueness in bounded case.

Remarks : (1) Let $d = l' = 1$, $h \in \mathcal{B}(\mathcal{F})$. By taking $\phi = 0$ and $b(t, w, \rho) = h(t, w)$ in Theorem 1, we get a formula for evaluating the stochastic integral $\int h dS$ pathwise, when $S \in \mathcal{S}(\mathcal{F}, P)$. The same formula is valid when S is a right-continuous semimartingale, See Bichteler (1979, p. 65).

(2) If the process K in condition (C2) is not locally bounded, but if $\exists T_n \in \mathcal{T}(\mathcal{F})$; $T_n \uparrow T$. such that K^{T_n} is bounded, let X_n be as in earlier construction. Let $\Omega_0 = \{w : X_n(t, w)$ converges in u.c.c. topology, $t \in [0, T(w))\}$.

and

$$X(t, w) = \begin{cases} \lim_n X_n(t, w) & \text{if } t \in [0, T(w)) \text{ and } w \in \Omega_0. \\ 0 & \text{otherwise.} \end{cases}$$

Then X is a continuous process on $[[0, T))$ and the same proof will now imply that X is a solution of (I) for (ϕ, b, S) on the interval $[[0, T))$. (This is same as saying that for any $\mathcal{F}$ stop time T' such that $T' < T$ on "$T > 0$", $X^{T'}$ is a solution of (I) for $(\phi^{T'}, b^{T'}, S^{T'})$. Thus the main theorem of Protter (1977) on existence and uniqueness follows as a consequence of Theorem 1.

(3) If we consider the equation $Y(t) = \phi(t) + \int_0 FYdS$, as in Emery (1978, p. 248), our method gives existence and uniqueness of its solution when $S \in \mathcal{S}(\mathcal{F}^P, P)$. Further our solution does not depend on P.

As a consequence of pathwise integration formula, we have

Theorem 2 : *Let $(\mathcal{G}_t)$ be an increasing family of sub σ-fields of $\mathcal{B}$, $F_t = \mathcal{G}_{t+}$. Let $Y \in \mathcal{C}(\mathcal{G})$ be such that $Y(0) = 0$. Let $K_n(t, w) = j$ if there exists t_i s.t. $0 = t_0 < t_1 < \ldots < t_j \leqslant t < t_{j+1}$; $|Y(t_i) - Y(t_{i+1})| = 2^{-n}$ $0 < i < j$; $|Y(t_i) - Y(s)| < 2^{-n}$ if $s \in [t_i, t_{i+1})$, $0 \leqslant i \leqslant j$.*

Let

$$X'(t, w) = \overline{\lim_n} \frac{K_n(t, w)}{2^{2n}}.$$

Let

$$U(w) = \inf\{t > 0 : X'(t-, w) \neq X'(t^+, w)\}$$

Let

$$X(t, w) = X''(t \wedge U(w), w)$$

where

$$X''(t, w) = X'(t_-, w).$$

Then X is a continuous, $\mathcal{G}_t$ adapted increasing process. Further, for all P s.t. $Y \in \mathcal{L}(\mathcal{F}^P, P)$; X is a version of $< Y, Y >^P$.

Proof: Fix P s.t. $Y \in \mathcal{L}(\mathcal{F}, P)$. Let $\{T_i^n\}_{i \geqslant 0}, n \geqslant 1$ be defined by

$$T_0^n = 0$$

$$T_{i+1}^n(w) = \inf\{t \geqslant T_n^i(w) : | Y(t, w) - Y(T_n^i w), w) | \geqslant 2^{-n}\}.$$

Observe that continuity of paths implies $| Y(T_i^n) - Y(T_{i+1}^n) | = 2^{-n}$.

Let

$$X_n(t, w) = Y^2(t, w) - 2 \sum_{i=0}^{\infty} Y(T_i^n(w), w)[Y(t \wedge T_{i+1}^n(w), w) - Y(t \wedge T_i^n(w), w)].$$

Then by Remark 1 and Ito's formula,

$$X_n(t) \rightarrow Y^2(t) - 2 \int_0^t Y(s) dY(s) = < Y, Y >^P (t), \text{ a.s. } P.$$

Writing

$$Y^2(t) = \sum_{i=0}^{\infty} (Y^2(t \wedge T_{i+1}^n) - Y^2(t \wedge T_i^n)),$$

we get

$$X_n(t) = \sum_{i=0}^{\infty} (Y(t \wedge T_{i+1}^n) - Y(t \wedge T_i^n))^2.$$

Observe that $K_n(t, w) = i$ if $T_i^n \leqslant t < T_{i+1}^n$.

Thus,

$$\frac{K_n(t, w)}{2^{2n}} \leqslant X_n(t, w) < \frac{K_n(t, w) + 1}{2^{2n}},$$

and hence

$$X'(t, w) = \lim_n X_n(t, w).$$

But X_n converges to $< Y, Y >^P$ in E a.s. P. Thus outside a P null set, $U(w) = \infty$, $X(w) = X'(w) = < Y, Y >^P (w)$.

Further, each X_n is $\mathcal{F}_t$ adapted and hence so is X'. Alxo, X' is an increasing process. Thus U is a $\mathcal{F}_t$ stop time and X is itself $\mathcal{F}_t$ adapted. It is clear that X is a continuous process. Now $\mathcal{F}_t = \mathcal{G}_{t+}$ implies X is $\mathcal{G}_t$ adapted. This completes the proof.

4. Convergence theorem

Fix a P on $(\Omega, \mathcal{B})$.

Following Protter we say $X_m \in \mathcal{C}(\mathcal{F})$ converges locally in maximal quadratic mean to $X \in \mathcal{C}(\mathcal{F})$ ($X_n \xrightarrow{LMQM} X$), if $\exists\ T_i \in \mathcal{T}(\mathcal{F})$; $T_i \uparrow \infty$ such that $\lim_{m \to \infty} E|X_m^{T_i} - X^{T_i}|_\infty^{*2} = 0$ for all $i \geqslant 1$.

Let us observe that for a strict $\mathcal{F}$ time change σ, $X_m \xrightarrow{LMQM} X$ in $\mathcal{C}(\mathcal{F})$ iff $\sigma X_m \xrightarrow{LMQM} \sigma X$ in $\mathcal{C}(\sigma\mathcal{F})$, so that by first using a time change and then proceeding as in Friedman (1975, p. 118), we can easily prove

Theorem 3 : *Let $\{b_m\}_{1 \leqslant m \leqslant \infty}$ satisfy* (C1), (C2) *for one locally bounded process K. Further assume*

$$\|b_m(w, t, 0)\| \leqslant K(t, w), \forall\, m : \ 1 \leqslant m \leqslant \infty$$

and for all $N \geqslant 1$, for all bounded stop times T,

$$\sup_{|\rho| \leqslant N} \|b_m(w, T, \rho) - b_\infty (w, T, \rho)\| \to 0 \ in\ Probability\ as\ m \to \infty.$$

Let $\{\phi_m\}_{1 \leqslant m \leqslant \infty} \subseteq \mathcal{C}(\mathcal{F})$ be such that $\phi_m \xrightarrow{LMQM} \phi$.

Let X_m be the solution to (I) for (ϕ_m, b_m, S); $1 \leqslant m \leqslant \infty$. then $X_m \xrightarrow{LMQM} X\infty$.

Remark : This implies Theorem 6.4 of Protter (1977, p. 258). But by our methods we could not get convergence of solutions when the semimartingales converge (Theorem 6.1 of Protter, 1977, p. 256)). The difficulty is that given a sequence $\{S_k\}$ of semimartingales, it may not always be possible to get a strict $\mathcal{F}$ time change σ, such that σS_k satisfies (*) for all k.

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REFERENCES

BICHTELER, KLAUS (1979): Stochastic integration and L^p theory of semimartingales. *Technical Report*, Department of Mathematics, The University of Texas at Austin.

BICHTELER, KLAUS (1980): The stochastic integral as a vector measure. *Lecture Notes in Mathematics*, **794**, Springer Verlag.

DOLEANS-DADE, C and MAYER, P. A. (1975-76): Equations Differentielles Stochastiques. *Lecture notes in Mathematics*, **581**. Springer-Verlag.

DOLEANS-DADE, C. (1976): On the existence and unicity of solutions of a stochastic differential equation. *Z. Wahrsheinlichkeitstheorie Verw. Geb.*, **36**, 93-101.

EMERY, MICHEL (1978): Stabilite des solutions des equations differentielles stochastiques application aux integrales multiplicatives stochastiques. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, **41**, 241-262.

FRIEDMAN, AVNER (1975): *Stochastic Differential Equations and Applications*, **1**. Academic Press.

JACOD, J. (1979): Calcul stochastique et problems de martingles. *Lecture notes in Mathematics*, **714**. Springer-Verlag.

KAZAMAKI, N. (1974): On a stochastic integral equation with respect to a weak martingele. *Tohoku Math. Journ.*, **26**, 53-63.

KUNITA, HIROSHI and WATANABE, SHINZO (1967): On Square Integrable Martingales. *Nagoya Mathematical Journal*, **30**, 209-245.

PROTTER, PHILIP E. (1977): On the existence, uniqueness, convergence and explosion of solution of systems of stochastic integral equations. *Ann. Prob.*, **5**, 243-261.

STROOCK, D. W. and VARADHAN S. R. S. (1972): On the support of diffusion processes with applications to the strong maximum principle. *Proc. VIth Berkeley symp.*, III. 333-359.

WONG, E. and ZAKAI, M. (1965): On the relationship between ordinary and stochastic differential equations. *Internat. J. Engin. Sci.*, **3**, 213-229.

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