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QUANTUM STOCHASTIC DIFFERENTIAL EQUATIONS WITH UNBOUNDED COEFFICIENTS AND DILATIONS OF FELLER'S MINIMAL SOLUTION

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SUMMARY. Quantum stochastic evolutions are constructed for unbounded coefficients and infinitely many noise components. A sufficient condition for the evolution to be conservative is obtained. The theory is then used in dilating Feller's minimal process, associated with an unbounded Markov generator, in boson Fock space. A necessary and sufficient condition for the dilation to be conservative is obtained. It is also shown how to realise the minimal process as a commutative stochastic flow. A notion of quantum exit stop time is introduced.

1. Introduction

The basic tools for bosonic stochastic calculus were developed and a necessary and sufficient condition for existence of a unitary evolution, satisfying a quantum stochastic differential equation (q.s.d.e.) with bounded coefficients was obtained in [17]. In [23] these results are extended to include the cases where infinitely many noise components are present.

This theory has many applications : the dilation of dynamical semigroups [7], the construction of quantum diffusions in the sense of [9] and modelling physical systems [4] etc.

However, in the context of [2-4, 6, 10-13, 18, 22], the coefficients are irregular and therefore there arises the problem of extending these results. In [11], improving the basic inequalities concerning iterative integrals, a sufficient condition on the coefficients is obtained to guarantee the existence of a unitary evolution. In particular it successfully deals with the quantum harmonic oscillator. On the other hand in [10], equicontinuity method has been employed to guarantee the existence of a unique contractive evolution associated with a pure birth (pure death) process and a necessary and sufficient condition for the evolution to be unitary is obtained. Model dependent studies have been carried out and some more results in this direction can be found in [3, 5, 6, 12, 22].

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In this paper we consider a class of q.s.d.e.s with unbounded coefficients and infinite degrees of freedom. In Section 2, mostly, we review the basic results in quantum stochastic calculus with regular coefficients [8, 9, 16, 17, 18, 23, 24, 25]. In particular we recall the 'time reversal property' indicated in [18], which allows us to derive some analytical properties of the evolution from that of the dual process.

In Section 3 we exploit the method outlined in [10] to ensure the existence of a contractive operator valued process satisfying a quantum stochastic differential equation with unbounded coefficients which admits an approximating sequence of regular elements. The spirit is similar to that of semigroup theory developed as in [27]. The approximating sequence of evolutions being non-commutative, it only guarantees a contractive solution as a 'weak operator limit ' of a subsequence of the evolutions. Analyticity of exponential vectors (Wiener chaos expansion) plays an important role in setting up an inductive procedure to get a sufficient condition for the solution to be unique or isometric. Analysing the dual process we also obtain a sufficient condition for the evolution to be co-isometric. It is worth noting that the condition for the evolution to be isometric (co-isometric) is similar to that of Feller's condition for the minimal process, associated with a Kolmogorov's differential equation, to be faithful. To emphasise this point we shall deal with two classes of dilations associated with countable state Markov processes. To this end, in Section 4, we review a construction of Feller's minimal solution and some basic analytic facts from the literature ([14], [19], [20]) on classical theory of Markov processes.

In Section 5, a generalised quantum harmonic oscillator [11], associated with a Markov generator, is constructed. It is a contractive process satisfying a cocycle property in the sense of [18], and the induced family of Evans-Hudson maps dilates Feller's minimal solution in Fock space. Feller's condition is still necessary and sufficient for the dilation to be conservative. For an unbounded generator it is not clear whether this dilation admits a diffusion equation in the sense of [9].

In Section 6, we continue the programme begun in [21]. In a series of papers [24, 23, 10], it has been shown how to realise a classical Markov process with countable state space as a commutative Evans-Hudson flow. But it is restricted only to processes with bounded Markov generators. Here, in Section 6, we consider the general situation and realise Feller's minimal solution as a commutative Evans-Hudson flow. Motivated by the construction of Feller's minimal solution, as outlined in [20], we introduce a special sequence of commutative Evans-Hudson flows which approximates the induced Evans-Hudson flow on a suitable algebra in the strong of operator topology. A necessary and sufficient condition for the flow to be conservative is obtained. A notion of quantum exit stop time is introduced. It is a commutative adapted family of strongly continuous increasing projections on Fock space. Feller's exit stop time is realised as the vacuum expectation of these projections. In view of Feller's boundary theory [14] we expect this stop time to play a crucial role in describing the dilations of other solutions associated with Kolmogorov's first and second differential equations. In the spirit of [19] we hope to deal with the dilation and non-uniqueness problems associated with birth and death processes. Finally imposing a weak hypothesis on the Markov generator, we show that the dilation admits a diffusion equation for suitable elements.

2. Notations and Preliminaries

All the Hilbert spaces that appear here are assumed to be complex and separable with inner product $<.,.>$ linear in the second variable. For any Hilbert space H, we denote by $\Gamma(H)$ the symmetric Fock space over H and $B(H)$ the C^* algebra of all bounded linear operators in H. For any $u \in H$, we denote by $e(u)$ the exponential vector in $\Gamma(H)$ associated with u:

$$e(u) = \bigoplus_{n \geqslant 0} u^{(n)}$$

where

$$u^{(n)} = \begin{cases} 1 & ; n = 0 \\ \dfrac{1}{\sqrt{n!}}\, u^{\otimes^n} & ; n \geqslant 1. \end{cases}$$

The family $\{e(u) : u \in \mathscr{M}\}$ is total for any dense linear manifold $\mathscr{M}$ in H and linearly independent in $\Gamma(H)$. So operators may be defined densely on $\Gamma(H)$ by giving their action on each $e(u)$. Thus, when C is a bounded operator on H and u is an element of H, the second quantized $\Gamma(C)$ of C and the Weyl operator $W(u)$ are determined uniquely by the relations:

$$\Gamma(C)\, e(v) = e(Cv)$$

$$W(u)\, e(v) = \exp\left\{-\frac{1}{2}\, \|u\|^2 - < u, v >\right\}\, e(u+v)$$

for all $v \in H$.

Fix two Hilbert spaces h_0 and k and write Γ_+, $\Gamma_{s,t}$ for $\Gamma(H)$ when $H = L^2(I, k)$ and $I = \mathbf{R}_+, [s, t)$ respectively. Set

$$\tilde{H} = h_0 \otimes \Gamma_+, \tilde{H}_{t]} = h_0 \otimes \Gamma_{0,t}, \tilde{H}_{[t} = \Gamma_{t,\infty}$$

we have the decomposition $\tilde{H} = \tilde{H}_{t]} \otimes \tilde{H}_{[t}$. The Hilbert space $\tilde{H}_{t]}$ will be identified with the subspace $\tilde{H}_{t]} \otimes \Phi_{[t}$ of $\tilde{H}$ where $\Phi_{[t}$ is the vacuum vector in $H_{[t}$. Every operator defined on a tensorial factor of $\tilde{H}$ will be identified with its canonical ampliation to the whole space and denoted by the same symbol.

Fix dense linear manifolds $\mathcal{D}$ in h_0 and $\mathcal{M}$ in $L^2 \mathbf{R}_+, k)$. The algebraic tensor product $\mathcal{D} \underline{\otimes} \varepsilon(\mathcal{M})$ is dense in $\tilde{H}$, where $\varepsilon(\mathcal{M})$ is the linear manifold generated by the vectors $e(u) : u \in \mathcal{M}$.

Definition 2.1 [17]. A family $X = \{X(t) : t \geqslant 0\}$ of operators in $\tilde{H}$ is called an *adapted process* with respect to $(\mathcal{D}, \mathcal{M})$ if

(a) $\mathcal{D}(X(t)) \supseteq \mathcal{D} \underline{\otimes} \varepsilon(\mathcal{M})$

(b) $X(t) fe(u\chi_{[0,t]}) \in \tilde{H}_{t]}$ and $X(t) fe(u) = \{X(t) fe(u\chi_{[0,t)})\}\, e(u\chi_{[t,\infty)})$ for all $t \geqslant 0, f \in \mathcal{D}, u \in \mathcal{M}$.

It is said to be *regular*, if in addition, the map $t \to X(t) fe(u)$ from $\mathcal{R}_+$ into $\tilde{H}$ is continuous for each $f \in \mathcal{D}, u \in \mathcal{M}$. An adapted process is called *bounded, contractive, isometric, co-isometric* or *unitary* according as the operators $X(t)$ are bounded, contractive, isometric, co-isometric or unitary for every $t \geqslant 0$.

For $0 \leqslant s \leqslant t$ denote by $a_{s,t}$ the von-Neumann subalgebra of $a \equiv B(\Gamma_+)$ given by

$$\{W(u) : \text{supp}\ u \subseteq [s, t]\}$$

This is simply $I_{0,s} \otimes B(\Gamma_{s,t}) \otimes I_{t,\infty}$. The family $\{N_{s,t} = a_0 \otimes a_{s,t} ; 0 \leqslant s \leqslant t\}$ forms a filtration of the Von-Neumann algebra $N := a_0 \otimes a$ where $a_0 := B(h_0)$. Vacuum conditional expectations $\{\mathcal{E}_{s,t} : 0 \leqslant s \leqslant t\}$ on each of these subalgebras exist and are characterized by

$$\mathcal{E}_{s,t}\, [B \otimes W(u)] = \langle e(0), W(u\chi_{[s,t]^c}) e(0) \rangle B \otimes W(u\chi_{[s,t]})$$

where $[s, t]^c = \mathcal{R}_+ \setminus [s, t]$. They satisfy the relations :

$$\mathcal{E}_{s,t}\, 0\, \mathcal{E}_{s',t'} = \mathcal{E}_{s,t}$$

where $[s, t] \subseteq [s', t']$. We also write $\mathcal{E}_s$ for $\mathcal{E}_{0,s}$.

Definition 2.2 [25]. A bounded operator valued process $X \equiv (X(t) : t \geqslant 0$ is called a *martingale* if

$$\mathcal{E}_s [X(t)] = X(s)$$

for all $0 \leqslant s \leqslant t$ and a *regular* martingale if there is a Randon measure μ on $\mathcal{R}_+$ for which

$$\|[(X(t)-X(s)]\psi\|^2+\|[X(t)^*-X(s)^*]\psi\|^2 \leqslant \mu([s,t])\|\psi\|^2$$

whenever $0 \leqslant s \leqslant t$ and $\psi \in \Gamma_{0,s} \otimes \Phi_{t}$.

We fix an orthonormal basis $\{e_i : i \in \overline{S}\}$ in k and set $E^i_j = |e_j > < e_i| : i, j \in S$. The basic quantum stochastic processes $\{\Lambda^i_j : i, j \in \overline{S} := S \bigcup \{0\}\}$ are defined by

$$\Lambda^i_j(t) = \begin{cases} \Lambda(\chi_{[0,t]} \otimes E^i_j) & ; i, j \in S \\ a(\chi_{[0,t]} \otimes e_i) & ; i \in S, j = 0 \\ a^\dagger(\chi_{[0,t]} \otimes e_j) & ; i = 0, j \in S \\ tI & ; i = 0 = j. \end{cases}$$

Then quantum Ito's formula can be expressed as :

$$d\Lambda^i_j d\Lambda^k_l = \hat{\delta}^i_l d\Lambda^k_j \qquad \ldots \quad (2.1)$$

for all $i, j, k, l \in S$ where

$$\hat{\delta}^i_l = \begin{cases} 0 & ; l = 0 \text{ or } i = 0 \\ \delta^i_l & ; \text{otherwise.} \end{cases}$$

We denote by $u^j(s) = < e_j, u(s) >, u_j(s) = u^j(s)$ for $j \in S$ and $u_0(s) = u_0(s) = 1$. Choose $\mathcal{M} = \{u \in H : u^j(.) = 0$ for all but finitely many $j \in \overline{S}\}$ and set $N(u) = \{j ; u^j(.) \neq 0\}$. So $\# N(u) < \infty$ for $u \in \mathcal{M}$.

Definition 2.3 ([17], [25]). $L = \{L^i_j(s) : i, j \in \overline{S}\}$ is said to be a $(\mathcal{D}, \mathcal{M})$ adapted *square integrable* family of processes of each L^i_j is $(\mathcal{D}, \mathcal{M})$ adapted and for each $j \in \overline{S}, f \in \mathcal{D}, u \in \mathcal{M}$ and $t \geqslant 0$

$$\sum_{i \in S} \int_0^t \|L^i_j(s) f e(u)\|^2 \, d\nu_u(s) < \infty \qquad \ldots \quad (2.2)$$

where

$$\nu_u(t) = \int_0^t (1+\|u(s)\|^2) ds.$$

We shall denote by $\mathcal{L}(\mathcal{D}, \mathcal{M})$ the class of all such square integrable families. For further details on these definitions and quantum Ito's formula the reader is refered to Hudson-Parthasarathy [17], Evans [8] and Mohari-Sinha [23]. A complete account is available in Parthasarathy [26].

Theorem 2.4 [17, 23]. *Suppose $L \in \mathcal{L}(\mathcal{D}, \mathcal{M})$. Then*

$$X(t) = \sum_{i,j \in \bar{S}} \int_0^t L^i_j(s)\, d\Lambda^j_i(s)$$

exists in the strong sense on $\mathcal{D} \otimes \varepsilon(\mathcal{M})$ and defines a regular adapted process satisfying for $f, g \in \mathcal{D}$ and $u, v \in \mathcal{M}$

$$< fe(u), X(t)ge(v) > = \sum_{i,j \in \bar{S}} \int_0^t ds\, u_i(s) v^j(s) < fe(u), L^i_j(s) ge(v) > \quad \ldots \quad (2.3)$$

$$\|X(t)fe(u)\|^2 \leqslant 2\, exp\,(v_*(t)) \sum_{i \in \bar{S}, j \in N(u)} \int_0^t \|L^i_j(s) fe(u)\|^2\, dv_*(s) \quad \ldots \quad (2.4)$$

If M is another element in $\mathcal{L}(\mathcal{D}, \mathcal{M})$ and

$$Y(t) = \sum_{i,j \in \bar{S}} \int_0^t M^i_j(s)\, d\Lambda^j_i(s)$$

then

$$< Y(t)fe(u), X(t)ge(v) > = \sum_{i,j \in \bar{S}} \int_0^t ds\, u_i(s) v^j(s) \{ < Y(s)fe(u), L^i_j(s)ge(v) >$$

$$+ < M^i_j(s)fe(u), X(s)ge(v) > + \sum_{k \in S} < M^k_i(s)fe(u), L^k_j(s)ge(v) > \} \ldots \quad (2.5)$$

Proof. This is a generalisation of the basic result of Hudson-Parthasarathy [17]. We omit the proof since, in this generality, it is presented in Mohari-Sinha [23] and a complete and self-contained account is included in Parthasarathy [26].

Denote by $\mathcal{K}_B$ the class of elements $Z = (Z^i_j : i, j \in \bar{S})$ where $Z^i_j \in \mathcal{B}(h_0)$ for all $i, j \in \bar{S}$ and there exist non-negative constants (depending on Z) $c_j, j \in \bar{S}$ satisfying

$$\sum_{i \in \bar{S}} \|Z^i_j f\|^2 \leqslant c_j \|f\|^2 \quad \ldots \quad (2.6)$$

for all $f \in h_0$. Also denote by $\mathcal{J}_B$ the class of elements $Z \in \mathcal{K}_B$ satisfying for all $i, j \in \bar{S}$

$$Z^i_j + (Z^j_i)^* + \sum_{k \in S} (Z^k_i)^* Z^k_j = 0 \quad \ldots \quad (2.7)$$

The necessary convergence in (2.7) follows from (2.6) and the following Lemma 2.5.

Lemma 2.5. *Suppose* $\{A_k\}_{k\geq 1}$ *and* $\{B_k\}_{k\geq 1}$ *are two families of bounded operators in* h_0 *such that* $\sum_{k\geq 1} A_k^* A_k$ *and* $\sum_{k\geq 1} B_k^* B_k$ *converge in strong operator topology. Then* $\Sigma\, A_k^* B_k$ *also converges in strong operator topology.*

Proof. For a proof, see Mohari-Sinha [23].

For any $Z \equiv (Z_j^i : j \in \bar{S})$, denote by $\widetilde{Z} \equiv (\widetilde{Z}_j^i : i, j \in \bar{S})$ where

$$\widetilde{Z}_j^i = (Z_i^j)^*.$$

Also set $\widetilde{\mathcal{Y}}_R \equiv \{Z : \widetilde{Z} \in \mathcal{Y}_R\}$ and $\widetilde{\mathcal{J}}_R \equiv \{Z : \widetilde{Z} \in \mathcal{J}_R\}$.

Theorem 2.6 [17, 22, 23]. *Suppose* $Z \in \mathcal{Z}_R$ *and* $Z \in \widetilde{\mathcal{J}}_R$. *Then there exists a unique strongly continuous co-isometric operator valued* $(h_0, \mathcal{M})$ *adapted process* $V = \{V(t) : t \geq 0\}$ *satisfying*

$$dV(t) = \sum_{i,j\in\bar{S}} V(t) Z_j^i d\Lambda_i^j(t)\ ;\ V(0) = I \qquad \ldots \quad (2.8)$$

on $h_0 \otimes e(\mathcal{M})$.

And (a) for all $f \in h_0$, $u \in \mathcal{M}$, $0 \leq s \leq t < T$

$$\|[V(t) - V(s)]fe(u)\|^2 \leq K_T(f, u)[\nu_u(t) - \nu_u(s)] \qquad \ldots \quad (2.9)$$

where

$$K_T(f, u) = 2\ exp(\nu_u(T))\|e(u)\|^2 \sum_{i\in\bar{S}, j\in N(u)} \|Z_j^i f\|^2.$$

(b) *V is also isometric if and only if* $Z \in \mathcal{J}_R$.

Proof. It is essentially a restatement of the basic result in Bosonic calculus developed in Hudson-Parthasarathy [17]. For this generality, see Mohari-Sinha [23] except (2.9) which follows from the basic estimate (2.4) and the fact that $\|V(t)\| \leq 1$ for all $t \geq 0$.

Denote by θ the right shift on $L^2(\mathcal{R}_+, k)$ so that for all $t \geq 0$

$$(\theta_t u)(x) = \begin{cases} u(x-t) & ;\ x \geq t \\ 0 & ;\ 0 \leq x \leq t. \end{cases}$$

For any bounded operator A in $\widetilde{H}$, $\Gamma(\theta_t) A \Gamma(\theta_t^*)$ takes $h_0 \otimes \Phi_{t]} \otimes \widetilde{H}_{[t}$ into itself. Denote by $\overline{\Gamma(\theta_t) A \Gamma(\theta_t^*)}$ the canonical ampliation to the whole space $\widetilde{H}$.

Definition 2.7 [18]. An adapted bounded process $V \equiv \{V(t) : t \geq 0\}$ is said to be a *cocycle* if for all $s, t \geq 0$

$$V(t+s) = V(t)\overline{\Gamma(\theta_t) V(s) \Gamma(\theta_t^*)} \qquad \ldots \quad (2.10)$$

For a cocycle set $P_t := \mathscr{E}_0[V(t)]$ and observe [1, 16] that $P = \{P_t : t \geq 0\}$ is a semigroup. $V = V(t) : t \geq 0\}$ is said to be a *regular cocycle* if P is norm continuous.

We quote the following theorem without proof.

Theorem 2.8 [1, 16]. *Suppose $V = \{V(t) : t \geq 0\}$ is a strongly continuous contractive cocycle. Then there exist two weakly* continuous semigroups $\tau = \{\tau_t : t \geq 0\}$, $\tilde{\tau} = \{\tilde{\tau}_t : t \geq 0\}$ of positivity preserving contractions on $\mathscr{B}(h_0)$ such that*

$$\tau_t(B) = \mathscr{E}_0[V(t)(B \otimes I)V(t)^*]$$

$$\tilde{\tau}_t(B) = \mathscr{E}_0[V(t)^*(B \otimes I)V(t)] \qquad \ldots \quad (2.11)$$

for all $B \in \mathscr{B}(h_0)$.

Denote by R_t the time reversal operator on $\mathscr{L}^2(\mathscr{R}_+, k)$ so that for $t \geq 0$

$$(R_t u)(x) = \begin{cases} u(t-x) & ; \ 0 \leq x \leq t \\ u(x) & ; \ t < x \end{cases}$$

and $\mathscr{U}_t := \Gamma(R_t)$. For any bounded adapted process $V = \{V(t) : t \geq 0\}$ we write $\tilde{V} = \{\tilde{V}(t) : t \geq 0\}\}$ for the dual process ([18]) defined by

$$\tilde{V}(t) = \mathscr{U}_t V(t)^* \mathscr{U}_t^{-1}. \qquad \ldots \quad (2.12)$$

Proposition 2.9 [18]. *$V = \{V(t) : t \geq 0\}$ is a (regular) cocycle if and only if $\tilde{V} = \{\tilde{V}(t) : t \geq 0\}$ is a (regular) cocycle.*

Proof. Since $\tilde{\tilde{V}} = V$, it suffices to show 'only if' part. As in Journé [18] observe that

(a) for $X \in \mathscr{B}(h_0) \otimes \mathscr{B}(\Gamma_{0,s})$ and $Y = \overline{\Gamma(\theta_t) X \Gamma(\theta_t^*)}$ we have $\mathscr{U}_s X \mathscr{U}_s^{-1} = \mathscr{U}_{t+s} Y \mathscr{U}_{t+s}^{-1}$

(b) for $X \in \mathscr{B}(h_0) \otimes \mathscr{B}(\Gamma_{0,t})$ we have

$$\mathscr{U}_{t+s} X \mathscr{U}_{t+s}^{-1} = \overline{\Gamma(\theta_s) \mathscr{U}_t X \mathscr{U}_t^{-1} \Gamma(\theta_s^*)}$$

and

$$\begin{aligned} \tilde{V}(t+s) &= \mathscr{U}_{t+s} \overline{\Gamma(\theta_t) V(s)^* \Gamma(\theta_t^*)} \mathscr{U}_{t+s}^{-1} \mathscr{U}_{t+s} V(t)^* \mathscr{U}_{t+s}^{-1} \\ &= \mathscr{U}_s V(s)^* \mathscr{U}_s^{-1} \overline{\Gamma(\theta_s) \mathscr{U}_t V(t)^* \mathscr{U}_t^{-1} \Gamma(\theta_s^*)} \\ &= \tilde{V}(s) \overline{\Gamma(\theta_s) \tilde{V}(t) \Gamma(\theta_s^*)} \end{aligned}$$

when (a) and (b) have been used to get the second equality. So $\widetilde{V}$ is a cocycle. Now set $\widetilde{P}_t := \mathcal{E}_0[\widetilde{V}(t)]\ t \geqslant 0$ and observe that $\widetilde{P}_t = P_t^*$. Hence this completes the proof.

Theorem 2.10 [16]. *$V \equiv \{V(t) : t \geqslant 0\}$ is a regular unitary cocycle if and only if it satisfies (2.8) for some $Z \in \mathcal{J}_R \cap \widetilde{\mathcal{J}}_R$. The choice of Z is unique.*

Proof. 'If' part is similar to Proposition 3.1 in Hudson-Lindsay [16]. To show the converse we shall adopt the method outlined in Hudson-Lindsay [16]. First observe that $P \equiv (P_t : t \geqslant 0)$ is a norm continuous semigroup with $P_0 = I$, hence it has a bounded generator, say Z_0^0. Define the bounded adapted process

$$X(t) = V(t) - \int_0^t V(s)\, Z_0^0 ds.$$

Now exploiting the cocycle property and boundedness of the generator Z_0^0 as in [16] observe that $X \equiv (X(t) : t \geqslant 0)$ is a regular martingale. So by Theorem 3.8 in Parthasarathy-Sinha [25] we have the representation

$$dV(t) = \sum_{i,j \in \bar{S}} V(t)\, Z_j^i(t) d\Lambda_i^j(t)\ ;\ V(0) = I \qquad \text{... (2.13)}$$

on $h_0 \otimes e(\mathcal{M})$ where $Z_j^i(s)$ are bounded $(h_0, \mathcal{M})$ adapted processes for $i, j \in S$ and for $j \in \bar{S}$, the series

$$\sum_{i \in \bar{S}} (Z_j^i(s))^* Z_j^i(s) \qquad \text{... (2.14)}$$

converges in strong operator topology. Now employ the method used in [16] to conclude that $Z_j^i(s)$ are independent of $s \geqslant 0$, say Z_j^i. So by (2.14) $Z \equiv (Z_j^i : i, j \in \bar{S})$ is an element in $\mathcal{Z}_R$. Quantum Ito's formula (2.1) and $V(t)^* V(t) = 1 (t \geqslant 0)$ implies that $Z \in \mathcal{J}_R$. To show $Z \in \widetilde{I}_R$, consider the dual cocycle $\widetilde{V}$ and employ the above argument to get a representation

$$d\widetilde{V}(t) = \sum_{i,j \in \bar{S}} \widetilde{V}(t) L_j^i d\Lambda_i^j(t)\ ;\ \widetilde{V}(0) = I \qquad \text{... (2.15)}$$

on $h_0 \otimes e(\mathcal{M})$ for some $L \in \mathcal{J}_R$. The proof is complete once we have shown that $L = \widetilde{Z}$. To this end we introduce for any fixed $f, g \in h_0$, $u, v \in \mathcal{M}$

$$\lambda(t) = \langle fe(u), \widetilde{V}(t) ge(v) \rangle - \langle fe(u), ge(v) \rangle \quad (t \geqslant 0)$$

we have from (2.14)

$$\lambda(t) = \sum_{i,j \in \bar{S}} \int_0^t \bar{u}_i(s) v^j(s) \langle fe(u), \widetilde{V}(s) L_j^i ge(v) \rangle\, ds \qquad \text{... (2.16)}$$

and from (2.15)

$$\lambda(t) = \sum_{i,j \in \bar{S}} u_i(t-s)v^j(t-s) < f e(u), \widetilde{Z}^i_j \mathcal{U}_s V(s)^* \mathcal{U}_t^{-1} g e(v) > ds. \quad \ldots \quad (2.17)$$

Note that for u, v continuous at 0, $\lim_{t \downarrow 0} \frac{1}{t} \lambda(t)$ exists and equating the limiting values one gets from (2.16) and (2.17)

$$\sum_{i,j \in \bar{S}} u_i(0)v^j(0) < f, (L^i_j - \widetilde{Z}^i_j)g > = 0. \quad \ldots \quad (2.18)$$

Since (2.18) holds for all $f, g \in h_0$, taking $\{u = 0, v = 0\}$, $\{u = 0, v = \chi_{[0,1]} e_j\}$, $\{u = \chi_{[0,1]} e_i, v = 0\}$ and $\{u = \chi_{[0,1]} e_i, v = \chi_{[0,1]} e_j\}$ in (2.18) we obtain the required result.

Theorem 2.11 (Journé's time reversal principle). *Fix any $Z \in \mathcal{J}_R \cap \widetilde{\mathcal{J}}_R$. $V \equiv \{V(t) : t \geq 0\}$ is the unique unitary solution for* (2.8) *with coefficients Z if and only if $\widetilde{V} \equiv \{\widetilde{V}(t) : t \geq 0\}$ is so for* (2.8) *with coefficients $\widetilde{Z}$.*

Proof. It follows from the last part of the argument employed in Theorem 2.10. □

Let $\mathcal{A}$ be a unitai subalgerba of $\mathcal{B}(h_0)$.

Definition 2.12 [9, 23]. $\mu \equiv \{\mu^i_j : i, j \in \bar{S}\}$ is said to be a family of regular structure maps if it satisfies the following : For $x, y \in \mathcal{A}$

(1) μ^i_j is linear on $\mathcal{A}$;

(2) $\mu^i_j(I) = 0$;

(3) $\mu^i_j(x)^* = \mu^j_i(x^*)$;

(4) for each $j \in \bar{S}$, there exist constant $\alpha_j > 0$, a countable index set S_j; and a family $(D^i_j; i \in S_j) \in \mathcal{B}(h_0)$ such that for all $f \in h_0$

$$\sum_{i \in S} \| \mu^i_j(x) f \|^2 \leq \sum_{i \in S_j} \| x D^i_j f \|^2$$

where

$$\sum_{i \in S_j} \| x D^i_j f \|^2 \leq \alpha_j \| f \|^2 ;$$

(5) $\mu^i_j(xy) = \mu^i_j(x)y + x\mu^i_j(y) + \sum_{k \in S} \mu^i_k(x)\mu^k_j(y)$

where the necessary convergence in (5) follows from (4) and and Lemma 2.5.

We shall quote without proof the following result.

Theorem 2.13 [9, 23, 24]. *Let μ be a family of regular structure maps. Then there exists a unique contractive $(h_0, \mathcal{M})$ adapted family $\{j_t : t \geqslant 0\}$ of identity preserving $*$ homomorphisms from $\mathcal{A}$ into $\mathcal{B}(H)$ satisfying: for $x \in \mathcal{A}, t \geqslant 0$*

$$dj_t(x) = \sum_{i j \in \bar{S}} j_t(\mu_j^i(x)) d\Lambda_i^j(t) \; ; \; j_0(x) = x$$

on $h_0 \underline{\otimes} \epsilon(\mathcal{M})$.

And if $\mathcal{A}$ is commutative then for $s, t \geqslant 0, x, y \in \mathcal{A}$

$$j_t(y) j_s(x) - j(x) j_s(y) = 0.$$

3. A QUANTUM STOCHASTIC DIFFERENTIAL EQUATION (QSDE) WITH UNBOUNDED COEFFICIENTS

In this section we shall consider stochastic evolutions satisfying a q.s.d.e. with unbounded coefficients. To this end we introduce some notations. For a dense linear manifold $\mathcal{D}$ in h_0, we denote by $Z(\mathcal{D})$ the class of densely defined operators $Z \equiv (Z_j^i : i, j \in \bar{S})$ satisfying

$$(a) \quad \mathcal{D} \subseteq \mathcal{D}(Z_j^i) \; ; (i, j \in \bar{S}) \; ; \qquad \ldots \quad (3.1)$$

(b) There exists a sequence $Z(n) \in \mathcal{L}_R \cap \tilde{\mathcal{J}}_R, n \geqslant 1$ so that for all $f \in \mathcal{D}, i, j \in \bar{S}$

$$\underset{n \to \infty}{s\text{-lim}} \; Z_j^i(n) f = Z_j^i f \qquad \ldots \quad (3.2)$$

and for each $j \in \bar{S}$

$$\sup_{n \geqslant 1} \sum_{i \in \bar{S}} \| Z_j^i(n) f \|^2 < \infty. \qquad \ldots \quad (3.3)$$

Lemma 3.1. *Let $Z \equiv (Z_j^i : i, j \in \bar{S})$ be a family of densely defined operators satisfying* (3.1) *and* (3.2) *where $Z(n) \in \mathcal{J}_R$; $n \geqslant 1$. Then* (3.3) *holds.*

Proof. $Z(n) \in \mathcal{J}_R$ implies that for each fixed $j \in \bar{S}$

$$\sum_{i \in \bar{S}} \|Z_j^i(n) f\|^2 = \| Z_j^0(L) f\|^2 - < Z_j^j(n) f, f > - < f, Z_j^j(n) f >$$
$$\leqslant \| Z_j^0(n) f\|^2 + 2\|f\| \| Z_j^j(n) f \|. \qquad \ldots \quad (3.4)$$

Now the required results follows once we apply (3.2) in (3.4). □

Lemma 3.2. *Let $Z \in \mathscr{Z}(\mathscr{D})$ then for each $f \in \mathscr{D}$, $j \in \tilde{S}$ there exists a constant $c_j(f) \geq 0$ such that*

$$\sum_{i \in \tilde{S}} \|Z^i_j f\|^2 \leq c_j(f). \quad \ldots \quad (3.5)$$

Proof. A simple application of Fatou's lemma in (3.3) and (3.2) establish (3.5). □

Fix $Z \in \mathscr{Z}(\mathscr{D})$ and $Z(n) \in \mathscr{Z}_R \cap \mathscr{I}_R$ satisfying (3.2) and (3.3). We denote $V^{(n)} \equiv \{V^{(n)}(t) : t \geq 0\}$ the unique co-isometric operator valued $(h_0, \mathscr{M})$ adapted process satisfying (Theorem 2.6)

$$dV^{(n)}(t) = \sum_{i,j \in \tilde{S}} V^{(n)}(t) Z^i_j(n) d\Lambda^j_i(t)(t) \; ; \; V^{(n)}(0) = I \quad \ldots \quad (3.6)$$

on $h_0 \otimes e(\mathscr{M})$.

Following an idea of Frigerio as outlined in Fagnola [10] and Mohari, Parthasarathy [22] we shall investigate the asymptotic behaviour of $\{V^{(n)}\}$ as $n \to \infty$.

Proposition 3.3. *The sequence $\{V^{(n)}\}$ defined as in (3.6) admits a subsequence $\{V^{(n_k)}\}$ satisfying the following :*

(i) *$w\text{-}\lim_{k \to \infty} V^{(n_k)}(t) = V(t)$ exists for all $t \geq 0$;* ... (3.7)

(ii) *$V \equiv \{V(t) : t \geq 0\}$ is a contractive $(h_0, \mathscr{M})$-adapted process for which*

$$\lim_{k \to \infty} \sup_{t \leq T} | < \psi, [V^{(n_k)}(t) - V(t)] f e(u) > | = 0$$

for $0 \leq T < \infty$, $\psi \in \tilde{H}$, $f \in \mathscr{D}$, $u \in \mathscr{M}$;

(iii) *For each $0 \leq T < \infty$, $f \in \mathscr{D}$, $u \in \mathscr{M}$ there exists a constant $c = c(f, u, T)$ such that*

$$\|[V(t) - V(s)] f e(u)\| \leq c[\nu_u(t) - \nu_u(s)]^{1/2} \; ; \; 0 \leq s \leq t \leq T \; ; \quad \ldots \quad (3.8)$$

(iv) *$V = \{V(t) : t \geq 0\}$ is a strongly continuous $(h_0, \mathscr{M})$ adapted process, $\{V(t)Z^i_j\} \in \mathscr{L}(\mathscr{D}, \mathscr{M})$ and*

$$dV(t) = \sum_{i,j \in \tilde{S}} V(t) Z^i_j d\Lambda^j_i(t) \; ; \; V(0) = I \quad \ldots \quad (3.9)$$

holds on $\mathscr{D} \otimes e(\mathscr{M})$;

(v) *If (3.9) admits a unique contractive solution then V is a cocycle and*

$$w. \lim_{n \to \infty} V^{(n)}(t) = V(t) \; (t \geq 0).$$

Proof. As in [22] consider the sequence $\{\rho_n\}$ of continuous functions on $\mathcal{R}_+$ defined by

$$\rho_n(t) = \langle \psi, V^{(n)}(t)fe(u) \rangle$$

where $\psi \in \widetilde{\mathcal{H}}, f \in \mathcal{D}, u \in \mathcal{M}$ are fixed. By (2.10) and (3.3) we have for $0 \leqslant s \leqslant t < T$

$$\begin{aligned} |\rho_n(t)-\rho_n(s)| &\leqslant \|\psi\| \|[V^{(n)}(t)-V^{(n)}(s)]fe(u)\| \\ &\leqslant \|\psi\| \, c(f, u, T)\, [\nu_u(t)-\nu_u(s)]^{1/2} \end{aligned}$$

where $c(f, u, T)$ is a non-negative constant independent of n. Furthermore $|\rho_n(t)| \leqslant \|\psi\| \|fe(u)\|$ for all $t \geqslant 0$ and $n \geqslant 1$. Hence by Arzela-Ascoli theorem $\{\rho_n\}$ is conditionally compact in the topology of uniform convergence on compacta. Using the seperabality of the spaces involved and usual diagonalisation procedure extract a subsequence $\{V^{(n_k)}\}$ satisfying (i) and (ii). For (iii) observe that for any $\psi \in \mathcal{H}, f \in \mathcal{D}, u \in \mathcal{M}$

$$\begin{aligned} |\langle \psi, [V(t)-V(s)]fe(u) \rangle| &= \lim_{k \to \infty} |\langle \psi, [V^{(n_k)}(t)-V^{(n_k)}(s)]fe(u) \rangle| \\ &\leqslant \|\psi\| c(f, u, T)[\nu_u(t)-\nu_u(s)]^{1/2}. \end{aligned}$$

So taking supremum over all unit vectors ψ we get

$$\|[V(t)-V(s)]fe(u)\| \leqslant c(f, u, T)\,[\nu_u(t)-\nu_u(s)]^{1/2}.$$

$V = \{V(t) : t \geqslant 0\}$ being contractive, strong continuity follows from (3.8) and also $\{V(t)Z_j^i\} \in \mathcal{L}(\mathcal{D}, \mathcal{M})$ is immediate from Lemma 3.2. Now by (2.8) and (3.6) we have for each $f, g \in \mathcal{D}, u, v \in \mathcal{M}$ and $t \geqslant 0$

$$\langle fe(u), V(t)ge(v) \rangle = \lim_{k \to \infty} \langle fe(u), V^{(n_k)}(t)ge(v) \rangle$$

$$= \langle fe(u), ge(v) \rangle + \lim_{k \to \infty} \sum_{i,j \in \bar{S}} \int_0^t ds\, u_i(s)v^j(s) \langle fe(u), V^{(n_k)}(s)Z_j^i(n_k)ge(v) \rangle$$

$$= \langle fe(u), ge(v) \rangle + \sum_{i,j \in \bar{S}} \int_0^t ds\, u_i(s)v^j(s) \langle fe(u), V(s)Z_j^i ge(v) \rangle$$

which implicos (3.9) and proves (iv).

Fix any $s \geqslant 0$ and define as in [16] the contractive adapted process $V_s = (V_s(t)\,;\, t \geqslant 0)$ by

$$V_s(t) = \begin{cases} V(t) & ;\ 0 \leqslant t \leqslant s, \\ V(s)\overline{\Gamma(\theta_s)V(t-s)\Gamma(\theta_s^*)} & ;\ t \geqslant s. \end{cases}$$

The proof of 'first part' is complete once we have shown that V_s is also a solution of (3.9). V being a solution of (3.9), the following holds for $t \geqslant s$:

$$dV_s(t) = V(s)\overline{\Gamma(\theta_s) V(t-s) \{ \sum_{ij \in \bar{S}} Z^i_j d\Lambda^j_i(t-s)\Gamma(\theta^*_s)\}}$$

on $\mathscr{D} \underline{\otimes} \varepsilon(\mathscr{M})$. Also observe that $\overline{\Gamma(\theta_s) d\Lambda^j_i(t-s)\Gamma(\theta^*_s)} = d\Lambda^j_i(t)$ and $\Gamma(\theta^*_s)\Gamma(\theta_s) = I$. So

$$dV_s(t) = V(s)\overline{\Gamma(\theta_s) V(t-s)\Gamma(\theta^*_s)} \left\{ \sum_{ij \in \bar{S}} Z^i_j \overline{\Gamma(\theta_s) d\Lambda^j_i(t-s)\Gamma(\theta^*_s)} \right\}.$$

Hence we obtain the required result. The 'second part' of (v) follows by a standard subsequence argument. □

Lemma 3.4. *Suppose $X = \{X(t) : t \geqslant 0\}$ is a strongly continuous bounded operator value $(h_0, \mathscr{M})$ adapted process satisfying*

$$dX(t) = \sum_{ij \in \bar{S}} X(t)Z^i_j d\Lambda^j_i(t) \; ; \; X(0) = 0 \qquad \ldots \quad (3.10)$$

on $\mathscr{D} \underline{\otimes} \varepsilon(\mathscr{M})$. Then for all m, $n \geqslant 0, f, g \in \mathscr{D}, u, v \in \mathscr{M}$ and $t \geqslant 0$ the following holds:

$$< fu^{(m)}, X(t)gv^{(n)} > = \sum_{i,j \in \bar{S}} \int_0^t ds u_i(s)v^j(s) < fu^{(m_i)}, X(s)gv^{(n_j)} > \qquad \ldots \quad (3.11)$$

where $u^{(-1)} = 0$ and for any $n \geqslant 0$

$$n_i = \begin{cases} n & , \; i = 0 \\ n-1, & i \in S. \end{cases}$$

Proof. X being strongly continuous, for any $T \geqslant 0$, $\sup_{0 \leqslant t \leqslant T} \|X(t)\| < \infty$. Now use the fact that $s \to e(su)$ is real analytic for any fixed $u \in \mathscr{M}$ and dominated convergence theorem to get (3.11) from (3.10). □

Lemma 3.5. *Suppose $T \equiv (T(t) : \geqslant 0)$ is a family of strongly contiuous operators in h_0 such that $\sup_{t \geqslant 0} \|T(t)\| < \infty$ and*

$$dT(t) = T(t)K \, dt, \; T(0) = 0 \qquad \ldots \quad (3.12)$$

holds on $\mathscr{D}$. If K is the generator of a contraction C_0-semigroup with $\mathscr{D}$ as a core then $T(t) = 0$ for all $t \geqslant 0$.

Proof. $\mathscr{D}$ being a core for K, for all $\lambda > 0$ we ge.

$$\overline{(K-\lambda)\mathscr{D}} = h_0 \qquad \ldots \quad (3.13)$$

Define bounded operators R_λ ; $n > 0$ by

$$R_\lambda = \int_0^\infty e^{-\lambda t}\, T(t) dt$$

and from (3.12) observe that

$$\lambda R_\lambda = R_\lambda K \qquad \ldots \quad (3.14)$$

on $\mathcal{D}$. Hence by (3.13) and (3.14) we have $R_\lambda = 0$ for all $\lambda > 0$, so $T(t) = 0$ for all $t \geqslant 0$.

Proposition 3.6. *If Z_0^0 is the generator of a contractive C_0-semigroup with $\mathcal{D}$ as a core then euation* (3.9) *has a unique contractive solution.*

Proof. Let $V' = \{V'(t) : t \geqslant 0\}$ be an another contractive process satisfying (3.9). Using the basic estimate (2.4) and (3.5) observe that V' also satisfies (3.8). Hence V' is strongly continuous. Define $X(t) = V(t) - V'(t)$ $(t \geqslant 0)$. To show that $X(t) \equiv 0 (t \geqslant 0)$ it is enough to show that for any $u, v \in \mathcal{M}$

$$T_{u^{(m)}, v^{(n)}}(t) = 0 \qquad \ldots \quad (3.15)$$

where $T_{u^{(m)}, v^{(n)}}(t) \in \mathcal{B}(h_0)$ is defined by

$$< f, T_{u^{(m)}, v^{(n)}}(t) g > = < f u^{(m)}, X(t) g v^{(n)} >$$

In view of Lemma 3.5, we are to show that $T_{u^{(m)}, v^{(n)}}(t)$ satisfies (3.12), We shall do this by induction on $m, n \geqslant 0$. For $m = 0 = n$ it is immediate from (3.11) $(u = 0 = v)$. Assume that (3.15) holds for all $u, v \in \mathcal{M}$ and $m, n \geqslant 0$ such that $m+n \leqslant k$. Then by induction hypothesis and (3.11) observe that $T_{u^{(m)}, v^{(n)}}(t)$ satisfies (3.12) for all $u, v \in \mathcal{M}$ and $m, n \geqslant 0$ where $m+n = k+1$. Now an application of Lemma 3.5 completes the proof. □

For any $X \in \mathcal{B}(\widetilde{\mathcal{K}})$ we define the bilinear forms $\mathcal{L}_j^i(X)\,(i, j \in \overline{S})$ on $\mathcal{D} \underline{\otimes}\, \varepsilon(\mathcal{M})$

$$< fe(u), \mathcal{L}_j^i(X)\, ge(v) > = < fe(u), X Z_j^i\, ge(v) > + < Z_i^j\, fe(u), X ge(v) >$$
$$+ \sum_{k \in S} < Z_i^k\, fe(u), X Z_j^k\, ge(v) > \qquad \ldots \quad (3.16)$$

where the necessary convergence follows from (3.5) and Cauchy-Schwarz inequality. In order that the solution $V = \{V(t) : t \geqslant 0\}$ of (3.9) be isometric it is necessary that $\mathcal{L}_j^i(I) = 0 (i, j \in \overline{S})$. Here our aim is to get a sufficient condition for $V = \{V(t) : t \geqslant 0\}$ to be isometric. To this end we introduce a few more notations :

$$I = \{Z \in \mathcal{L}(\mathcal{D}) : \mathcal{L}_j^i(I) = 0 \,;\, i, j \in \overline{S}\}$$

and for $\lambda > 0$

$$\beta_\lambda \equiv \{B \geqslant 0 : B \in \mathcal{B}(h_0) \; ; \; \mathcal{L}_0^0(B) = \lambda B\}$$

Lemma 3.7. *If $Z \in \mathcal{J}$ then for all $m, n \geqslant 0, f, g \in \mathcal{S}\; u, v \in \mathcal{M}$ and $t \geqslant 0$*

$$< fu^{(m)}, X(t)gv^{(n)} > = \sum_{i,j \in \bar{S}} \int_0^t ds u_i(s) v^j(s) < fu^{(m_i)}, \mathcal{L}_j^i (X(t))gv^{(n_j)} > \quad \ldots \quad (3.17)$$

where $m_i, n_j (i, j \in \bar{S})$ are as in (3.11) *and*

$$X(t) = I - V(t)^* V(t) \; (t \geqslant 0).$$

Proof. $Z \in \mathcal{J}$ and quantum Ito's formula implies that for all $f, g \in \mathcal{S}$, $u, v \in \mathcal{M}$ and $t \geqslant 0$

$$< fe(u), X(t)ge(v) > = \sum_{i,j \in \bar{S}} \int_0^t ds u_i (s) v^j(s) < e(u), \mathcal{L}_j^i (X(t))ge(v) > \quad \ldots \quad (3.18)$$

We obtain (3.17) from (3.18) and analyticity of the map $s \to e(sv)$ (for any $v \in \mathcal{M}$, where the necessary convergence follows from (3.5). □

Proposition 3.8. *If $Z \in \mathcal{J}$ and $\beta_\lambda = \{0\}$ for some $\lambda > 0$ then the solution $V \equiv \{V(t) : t \geqslant 0\}$ of* (3.9) *is isometric.*

Proof. Note that $0 \leqslant X(t) \leqslant I, X(0) = 0$. Denote non-negative operators $Y_\lambda \in B(\widetilde{H})$ and $B_\lambda^{(n)}(u) \in B(h_0) \; (\lambda > 0, n \geqslant 0, u \in \mathcal{M})$ defined by

$$Y_\lambda = \int_0^\infty e^{-\lambda t} X(t) dt$$

and

$$< f, B_\lambda^{(n)}(u) g > = < fu^{(n)}, Y_\lambda g u^{(n)} >.$$

Observe that for any fixed $n \geqslant 0, u \in \mathcal{M}, B_\lambda^{(n)}(u) = 0$ for some $\lambda > 0$ if and only if $X(t) fu^{(n)} = 0$ for all $f \in h_0$ and $t \geqslant 0$. We shall show by induction on $n \geqslant 0$ that for all $f \in h_0, u \in \mathcal{M}, t \geqslant 0$

$$X(t) fu^{(n)} = 0 \quad \ldots \quad (3.19)$$

Taking $u = 0 = v$ in (3.17) observe that $B_\lambda^{(0)}(0) \in \beta_\lambda$. So (3.19) follows for $n = 0$ by our earlier observation and the assumption that $\beta_\lambda \equiv \{0\}$ for some $\lambda > 0$. Now assuming (3.19) for $n-1 (n \geqslant 1)$ we get for $(i, j) \neq (0, 0)$ and $t \geqslant 0$

$$< fu^{(n_i)}, \mathcal{L}_j^i (X(t)) gu^{(n_j)} > = 0.$$

Hence (3.17) implicees that $B_\lambda^{(n)}(u) \in \beta_\lambda$ for all $u \in \mathcal{M}, \lambda > 0$, so $B_\lambda^{(n)}(u) = 0$ for some $\lambda > 0$, which by the observation made earlier implies (3.19) and completes the proof. □

Now our aim is to exploit the time reversal principle to obtain a sufficient condition for $V = \{V(t) : \geqslant 0\}$ to be co-isometric. To this end we impose some additional conditions on Z.

Assumption 3.9. For the triad $(\mathscr{D}, Z, Z(n) ; n \geqslant 1)$ satisfying (3.1)-(3.2) and $Z(n) \in \mathscr{J}_R \cap \widetilde{\mathscr{J}}_R$ there exists a dense linear manifold $\widetilde{D}$ in h_0 such that $(\widetilde{D}, \widetilde{Z}, \widetilde{Z}(n) ; n \geqslant 1)$ also satisfies (3.1) and (3.2).

If Z satisfies Assumption 3.9, Lemma 3.1 implies that $Z \in \mathscr{Y}(\mathscr{D})$ and $\widetilde{Z} \in \mathscr{Y}(\widetilde{\mathscr{D}})$. For any $X \in \beta(\widetilde{H})$ define the bilinear forms $\widetilde{\mathscr{L}}^i_j(X)\ (i, j \in \overline{S})$ on $\widetilde{D} \underline{\otimes}\, \varepsilon(\mathscr{M})$ as in (3.16) with Z replaced by $\widetilde{Z}$ and set

$$\widetilde{\mathscr{J}} \equiv \{Z : \widetilde{\mathscr{L}}^i_j(I) = 0 \ ; i, j \in \overline{S}\}$$

and for $\lambda > 0$

$$\widetilde{\beta}_\lambda \equiv \{B \geqslant 0 : B \in B(h_0) : \widetilde{\mathscr{L}}^0_0(B) = \lambda B\}$$

Since $Z(n) \in \mathscr{J}_R \cap \widetilde{\mathscr{J}}_R (n \geqslant 1)$, $V^{(n)} \equiv \{V^{(n)}(t) : t \geqslant 0\}$ defined as in (3.6) is a regular unitary cocycle and by Theorem 2.11 the dual unitary cocycle $\widetilde{V}^{(n)} \equiv \{\widetilde{V}^{(n)}(t) : t \geqslant 0\}$ satisfies

$$d\widetilde{V}^{(n)}(t) = \widetilde{V}^{(n)}(t)\widetilde{Z}^i_j(n)d\Lambda^j_i(t) \ ; \ \widetilde{V}^{(n)}(0) = I \qquad \ldots \quad (3.20)$$

on $h_0 \underline{\otimes}\, \varepsilon(\widetilde{\mathscr{M}})$. Also from (3.7) we get

$$w.\lim_{k \to \infty} \widetilde{V}^{(n_k)}(t) = \widetilde{V}(t)\ (t \geqslant 0) \qquad \ldots \quad (3.21)$$

where

$$\widetilde{V}(t) = \mathscr{U}_t V(t)^* \mathscr{U}_t^{-1} (t \geqslant 0).$$

Proposition 3.10. *Let for Z Assumption* 3.9 *be valid. Then*

(i) $\widetilde{V} \equiv \{\widetilde{V}(t) : t \geqslant 0\}$ *is a strongly continuous* $(h_0, \mathscr{M})$ *adapted process,* $\{\widetilde{V}(t)\widetilde{Z}^i_j\} \in \mathscr{L}(\widetilde{\mathscr{D}}, \mathscr{M})$ *and*

$$d\widetilde{V}(t) = \sum_{i, j \in \overline{S}} \widetilde{V}(t)\, \widetilde{Z}^i_j\, d\Lambda^j_i(t) : \widetilde{V}(0) = I$$

holds on $\widetilde{\mathscr{D}} \otimes \varepsilon(\mathscr{M})$.

(ii) $V^* \equiv \{V(t)^* : t \geqslant 0\}$ *is strongly continuous.*

(iii) *If* V *is co-isometric then* $Z \in \widetilde{\mathscr{J}}$.

(iv) *If* $Z \in \widetilde{\mathscr{J}}$ *and* $\widetilde{\beta}_\lambda \equiv \{0\}$ *for some* $\lambda > 0$ *then* V *is co-isometric.*

Proof. $\widetilde{Z} \in \mathscr{Y}(\widetilde{\mathscr{D}})$, so i) is immediate from Proposition 3.3 and (3.21). (ii) follows from (i) because $t \to \mathscr{U}_t$ is continuous in strong operator topology. For (iii) and (iv) observe that V is co-isometric if and only if $\widetilde{V}$ is isometric. Hence the required results follow from Proposition 3.8 and (i). □

4. Classical Markov processes

Here our aim is to recall some basic facts from classical theory of Markov processes. In particular we shall quote without proof the construction of Feller's 'minimal' solution as outlined in Ledermann-Reuter [20] and a necessary and sufficient condition for the minimal solution to be strictly stochastic.

Definition 4.1. A family of matrices $\Omega \equiv \{\Omega(t) = (\Omega_{ij}(t) : i, j \in \mathbb{Z});\ t \geqslant 0$ is said to be *regular Markov* if the following holds :

$$\text{(a)} \quad \Omega_{ij}(t) \geqslant 0 (i \neq j)\ ; \quad \Omega_{ii}(t) = -\sum_{j \neq i} \Omega_{ij}(t)\ ; \qquad \dots (4.1)$$

(b) $t \to \Omega_{ij}(t)$ is continuous for each $i, j \in \mathbb{Z}$.

For $\Omega(t) \equiv \Omega$, $\Omega \equiv (\Omega_{ij}\ ; i, j \in \mathbb{Z})$, independent of t, Ω will be called a *Markov matrix*. For any $n \geqslant 1$ denote the family of finite matrices

$$\Omega^{(n)} := \{\Omega^{(n)}(t) = (\Omega_{ij}(t) : -n \leqslant i, j \leqslant n)\ ; t \geqslant 0\}$$

and $F^{(n)} := \{F^{(n)}(s, t) = (F^{(n)}_{ij}(s, t) : -n \leqslant i, j \leqslant n\ ;\ 0 \leqslant s \leqslant t\}$, the unique solution of

$$\frac{\partial}{\partial t} F^{(n)}(s, t) = F^{(n)}(s, t)\, \Omega^{(n)}(t),\ F^{(n)}(s, s) = I\ ;\ 0 \leqslant s \leqslant t.$$

Lemma 4.2. *For all $n \geqslant 1$, $0 \leqslant s \leqslant t < \infty$, $-n \leqslant i, j \leqslant n$ the following holds :*

$$\text{(i)} \quad F^{(n)}_{ik}(s, s) = \delta_{ik} \qquad \dots (4.2)$$

$$\text{(ii)} \quad \frac{\partial}{\partial t} F^{(n)}_{ik}(s, t) = \sum_{-n \leqslant j \leqslant n} F^{(n)}_{ij}(s, t)\, \Omega_{jk}(t) \qquad \dots (4.3)$$

$$\text{(iii)} \quad \frac{\partial}{\partial s} F^{(n)}_{ik}(s, t) = -\sum_{-n \leqslant j \leqslant n} \Omega_{ij}(s) F^{(n)}_{jk}(s, t) \qquad \dots (4.4)$$

$$\text{(iv)} \quad F^{(n)}_{ik}(s, t) = \sum_{-n \leqslant j \leqslant n} F^{(n)}_{ij}(s, r) F^{(n)}_{jk}(r, t)\ ;\ (s \leqslant r \leqslant t) \qquad \dots (4.5)$$

$$\text{(v)} \quad F^{(n)}_{ik}(s, t) \geqslant 0, \quad \sum_{-n \leqslant j \leqslant n} F^{(n)}_{ij}(s, t) \leqslant 1 \qquad \dots (4.6)$$

$$\text{(vi)} \quad F^{(n+1)}_{ik}(s, t) \geqslant F^{(n)}_{ik}(s, t) \qquad \dots (4.7)$$

(vii) *If $\Omega(t) \equiv \Omega$, set $F^{(n)}_{ik}(t) = F^{(n)}_{ik}(0, t)$, then $F^{(n)}_{ik}(s, t) = F^{(n)}_{ik}(t - s)$.* ... (4.8)

So as $n \to \infty$, $F^{(n)}_{ik}(s, t)$ tends to a limit say $F^{\Omega}_{ik}(s, t)$. From Lemma 4.2 we have the following theorem.

Theorem 4.3. *For any fixed* $s \geqslant 0$, $F_{ik}(s, t)$ *is absolutely continuous in* t, *for any fixed* $t \geqslant 0$, $F_{ik}(s, t)$ *is continuously differentiable in* s. *For all* $0 \leqslant s \leqslant t < \infty$ *and* $i, k \in \mathscr{Z}$ *the following holds* :

(i) $F_{ik}(s, s) = \delta_{ik}$ ... (4.9)

(ii) $\dfrac{\partial}{\partial t} F_{ik}(s, t) = \sum_j F_{ij}(s, t)\, \Omega_{jk}(t)$ ... (4.10)

for almost all $t \geqslant s$ (*s held fixed*)

(iii) $\dfrac{\partial}{\partial s} F_{ik}(s, t) = -\sum_j \Omega_{ij}(s)\, F_{jk}(s, t)$ ... (4.11)

(iv) $F_{ik}(s, t) = \sum_j F_{ij}(s, r)\, F_{jk}(r, t)$ ... (4.12)

(v) $F_{ik}(s, t) \geqslant 0, \sum_j F_{ij}(s, t) \leqslant 1.$ ... (4.13)

(vi) *If* $\Omega(t) \equiv \Omega$ *as in Lemma* 4.2 (vii) *set* $F_{ik}(t) = F_{ik}(0, t)$, *then*

$$F_{ik}(s, t) = F_{ik}(t-s) \quad \text{... (4.14)}$$

and (4.10) *is valid for all* $t \geqslant s$.

Theorem 4.4. *If a family of matrices* $P(s, t) \equiv \{P_{ik}(s, t) : i, k \in \mathbf{Z} : 0 \leqslant s \leqslant t < \infty\}$ *satisfying*

$$P_{ik}(s, s) = \delta_{ik}, P_{ik}(s, t) \geqslant 0$$

and either (4.10) *or* (4.11) *then*

$$P_{ik}(s, t) \geqslant f_{ik}(s, t) \quad \text{... (4.15)}$$

for all $0 \leqslant s \leqslant t < \infty$.

Proof. For a complete account of these results see Ledermann-Reuter [20].

Consider the situation when $\Omega(t) \equiv \Omega$ and set $F_{ik}(t) : t \geqslant 0$ as in Theorem 4.3 (vi). It is clear from (4.13) that for all $t \geqslant 0$

$$\sum_k F_{ik}(t) \leqslant 1 \quad \text{... (4.16)}$$

The following theorem indicates a necessary and sufficient condition for equality in (4.16).

Theorem 4.5. *For all* $i \in \mathbf{Z}$ *and* $t \geqslant 0$ *equality holds in* (4.16) *if and only if* $B_\lambda \equiv \{0\}$ *for some* $\lambda > 0$ *where*

$$B_\lambda \equiv \{x \geqslant 0, x \in l_\infty(\mathbf{Z}). \quad \Omega x = \lambda x\}$$

Proof. See Feller [14]. □

For a more explicit description of Feller's condition for birth and death processes, the reader is refered to Karlin-McGregor [19].

5. A class of non-commutative Markov processes

In this section we shall deal with a class of quantum stochastic evolutions initiated by Fagnola [11]. Some restuts in this direction will be found in [6]. We extend the results obtained in [11] and improve some unsatisfactory parts in [6].

Fix a Markov matrix $\Omega = (\Omega_{ij}\,;\, i\, j \in \mathbf{Z})$ and choose complex numbers m_{ij} $(i, j \in \mathbf{Z})$ such that

$$\Omega_{ij} = \begin{cases} |m_{ij}|^2 & ; i \neq j \\ -|m_{ii}|^2 & ; i = j \end{cases} \qquad \ldots \quad (5.1)$$

and $S \subseteq \mathbf{Z}\setminus\{0\}$ so that for all $k \in \mathbf{Z}, i \notin \overline{S}$

$$m_{k,k+i} = 0$$

So for each $i \in \mathbf{Z}, -\Omega_{ii} = \sum_{j \in S} \Omega_{ij}$ holds. Also fix an orthonormal basis $\{f_k : k \in \mathbf{Z}\}$ for h_0 and denote by $\mathcal{D}$ the linear manifold generated by the basis vectors. Define unitary operators $S_i, (i \in S)$ and projections $\phi_k(k \in \mathbf{Z})$, $\Pi_n(n \geqslant 1)$ in h_0 by

$$\begin{aligned} S_i f_k &= f_{k+i}, \\ \phi_k &= |f_k><f_k|, \qquad \ldots \quad (5.2) \\ \Pi_n &= \sum_{|k| \leqslant n} \phi_k \end{aligned}$$

and denote by $\mathcal{A}$ the von-Neumann algebra generated by $\{\phi_k\,;\, k \in \mathbf{Z}\}$. Also consider the normal operators Z_i $(i \in S)$ satisfying

$$Z_i f_k = m_{k,k+i} f_k.$$

Observe that for each $f \in \mathcal{D}$ there exists a constant $c(f) \geqslant 0$ such that

$$\sum_{i \in S}{}^{*} \|Z_i f\|^2 \leqslant c(f) \qquad \ldots \quad (5.3)$$

Now consider operators $Z = (Z^i_j\,;\, i, j \in \overline{S})$ defined by

$$Z^i_j = \begin{cases} 0 & ; i, j \in S \\ -S_i Z_i & ; i \in S, j = 0 \\ Z^*_j S^*_j & ; i = 0, j \in S \\ -\frac{1}{2} \sum_{k \in S} Z^*_k Z_k & ; i = 0 = j \end{cases} \qquad \ldots \quad (5.4)$$

Taking $Z(n)\,(n \geqslant 1)$ as in (5.4) with $Z_i(i \in S)$ replaced by $Z_i^{(n)} := Z_i \Pi_n$ a routine verification shows that Z satisfies Assumption 3.9, $\mathcal{D}$ is a core for Z_0^0 which is the generator of a contractive C_0-semigroup. Also $Z \in \mathcal{J} \cap \tilde{\mathcal{J}}$.

Theorem 5.1. *Suppose operators $Z \equiv (Z_j^i : i, j \in \bar{S})$ are as in (5.4). Then*

(i) *there exists a unique strongly continuous $(h_0, \mathcal{M})$ adapted contractive evolution $V \equiv \{V(t) : t \geqslant 0\}$ satisfying*

$$dV(t) = \sum_{i,j \in \bar{S}} V(t) Z_j^i \, d\Lambda_i^j(t) \,; \quad V(0) = I$$

on $\mathcal{D} \underline{\otimes} \mathcal{E}(\mathcal{M})$

(ii) *V is a cocycle and for all $i, j \in \mathcal{Z}$, $t \geqslant 0$ the following holds :*

(a) $< f_i, \tau_t(\phi_j) f_i > = F_{ij}(t)$

(b) $< f_i, \tilde{\tau}_t(\phi_j) f_i > = F_{ij}(t)$

where $\tau \equiv (\tau_{t \geqslant 0} : t \geqslant 0)$ and $\tilde{\tau} \equiv (\tilde{\tau}_t : t \geqslant 0)$ are as in (2.11) and $F(t) \equiv (F_{ij}(t) : i, j \in \mathbf{Z})$ is the minimal solution for the Markov matrix Ω.

(iii) *The following statements are equivalent :*

(a) *$V \equiv \{V(t) : t \geqslant 0\}$ is isometric.*

(b) *$V \equiv \{V(t) : t \geqslant 0\}$ is co-sometric.*

(c) *$B_\lambda = 0$ for some $\lambda > 0$.*

where $B_\lambda(\lambda > 0)$ are defined as in Theorem 4.5.

Proof. (i) is immediate from Proposition 3.3 and Proposition 3.6. For (ii) set matrices $P^{(m,n)}(t) \equiv \{P_{ij}^{(m,n)}(t) : -n \leqslant i, j \leqslant n\}$; $m \geqslant n$ defined by

$$P_{ij}^{(m,n)}(t) = < f_i\, e(0),\ V^{(m)}(t)^* \, \phi_j V^{(n)}(t) f_i\, e(0) >$$

We shall show that for each $n \geqslant 1$ and $m \geqslant n$

$$P^{(m,n)}(t) = F^{(n)}(t) \quad \ldots \quad (5.5)$$

where $F^{(n)}(t)(t \geqslant 0)$ is described in Lemma 4.2 (vii). To show this first observe that (5.5) is true for $t = 0$. Quantum Ito's formula (2.1) implies that

$$\frac{d}{dt} P^{(m,n)}(t) = \Omega^{(n)} P^{(m,n)}(t) \quad \ldots \quad (5.6)$$

where $\Omega^{(n)} \equiv (\Omega_{ij} : -n \leqslant i, j \leqslant n)$.

But (5.6) admits a unique solution, so (5.5) is immediate. Now using the fact $w\text{-}\lim_{n\to\infty} V^{(n)}(t) = V(t)(t \geqslant 0)$ we have for all $t \geqslant 0, i\, j \in \mathbf{Z}$

$$\lim_{n\to\infty} F_{ij}^{(n)}(t) = \lim_{n\to\infty} \lim_{m\to\infty} P_{ij}^{(n,m)}(t) = < f_i, \tilde{\tau}_t(\phi_j) f_i >$$

Hence (b) in (ii) follows from Theorem 4.3 (vi). (a) in (ii) follows by an identical method and we omit the details. For (iii) we shall show that (a) $\Longleftrightarrow$ (c), a similar method will yield (b) $\Longleftrightarrow$ (c). For (a) $\Longrightarrow$ (c), observe that $V \equiv \{V(t) : t \geqslant 0\}$ being an isometric process we have from (ii), for each $j \in \mathbf{Z}, \sum_i F_{ij}(t) = 1(t \geqslant 0)$. Hence by Theorem 4.5 we get $B_\lambda \equiv \{0\}$ for some $\lambda > 0$. To show the converse recall the sufficient condition for $V \equiv \{V(t) \geqslant 0\}$ to be isometric, described in Proposition 3.8. Let $B \in \beta_\lambda$ for some $\lambda > 0$. Denote $x \equiv (x(k) : k \in \mathbf{Z})$ defined by

$$x(k) = < f_k, Bf_k >$$

A simple computation shows that $x \in B_\lambda$. Hence by our hypothesis $x = 0, B$ being a non-negative element we have $B = 0$. Hence $\beta_\lambda \equiv \{0\}$ for some $\lambda > 0$. This completes the proof. □

It is known [6] that $\{\alpha_t(\phi) := V(t)\phi V(t)^* \,;\, t \geqslant 0 \,;\, \phi \in \mathcal{A}\}$ is a non-commutative family of bounded operators. By Theorem 5.1, α_t is an identity preserving homomorphism if and only if $B_\lambda = 0$ for some $\lambda > 0$. For an unbounded Markov generaror it is not clear whether it satisfies a diffusion equation in the sense of [8].

6. A class of commutative quantum Markov processes

Here we shall continue the programme initiated by Meyer [21], studied subsequently in a series of articles Parthasarathy-Sinha [24], Mohari-Sinha [23], Fagnola [10].

As in Section 5, Ω is a Markov matrix and operators $Z_i, S_i (i \in S)$ and $\Pi_n(n \geqslant 1)$ are as in (5.1)—(5.3). Now consider operators $Z = (Z_j^i : i, j \in \bar{S})$ defined by

$$E_j^i \equiv \begin{cases} (S_i^* - I)\delta_{ij} & ;\ i, j \in S, \\ -Z_i & ;\ i \in S, j = 0, \\ Z_j^* \cdot S_j^* & ;\ i = 0, j \in S, \\ -\frac{1}{2} \sum_{k \in S} Z_k^* Z_k & ;\ i = 0 = j. \end{cases} \qquad \dots (6.1)$$

Taking $Z(n)(n \geqslant 1)$ as in (6.1) with $Z_i(i \in S)$ replaced by $Z_i^{(n)} = Z_i\Pi_n$ a routine verification shows that Z satisfies Assumption 3.9 and $Z \in \mathcal{J} \bigcap \widetilde{\mathcal{J}}$. Moreover $\mathcal{D}$ is a core for Z_0^0 which is the generator of a contractive C_0-semigroup. Exploiting the results proved in Section 3 and Section 4 we have the following theorem.

Theorem 6.1. *Suppose the operators* $Z = (Z_j^i\,; i, j \in \bar{S})$ *are as in* (6.1). *Then*

(i) *There exists a unique strongly continuous* $(h_0\ \mathcal{M})$ *adapted isometric evolution* $V \equiv \{V(t)\} : t \geqslant 0\}$ *satisfying*

$$dV(t) = \sum_{i,j \in \bar{S}} V(t)\, Z_j^i d\Lambda_i^j(t)\,;\ V(0) = I \qquad \ldots \quad (6.2)$$

on $\mathcal{D} \underline{\otimes}\, e(\mathcal{M})$.

(ii) V *is a cocycle and for all* $i, j \in \mathbf{Z}, t \geqslant 0$

$$< f_i, \tau_t(\phi_j) f_i > = F_{ij}(t)$$

where $\tau \equiv (\tau_t : t \geqslant 0)$ *is as in* (2.12) *and* $F(t) \equiv (F_{ij}(t) : i, j \in \mathrm{Z})$ *is the minimal solution for the Markov matrix* Ω.

(iii) $V \equiv \{V(t) : t \geqslant 0\}$ *is coisometric if and only if* $B_\lambda = \{0\}$ *for some* $\lambda > 0$.

Proof. (i) is immediate from Proposition 3.3 and Proposition 3.6 except that V is isometric which follows once we verify the sufficient condition indicated in Proposition 3.8. To this end let $B \in \beta_\lambda$ and set $x(k) : \equiv < f_k, Bf_k > (k \in \mathcal{Z})$. B being an element in β_λ we have from (3.16)

$$\lambda x(k) = -\frac{1}{2}\ |m_{kk}|^2 x(k) - \frac{1}{2}\ |m_{kk}|^2\ x(k) + \sum_{j \in S}\ |m_{k,k+j}|^2 x(k)$$

$$= (\sum_{i \in \bar{S}} \Omega_{ki}) x(k) = 0$$

Hence $\beta_\lambda = \{0\}$ for all $\lambda > 0$. This completes the proof of (i).

(ii) follows by a similar method employed for the proof of (b) in Theorem 5.1 (ii). Now for the 'only if' part in (iii) use (ii) and Theorem 4.5. For the converse recall the sufficient condition indicated in Proposition 3.10 for $V \equiv \{V(t) : t \geqslant 0\}$ to be co-isometric and observe that it is the same as that for $V = \{V(t) : t \geqslant 0\}$ in Theorem 5.1 to be co-isometric. So $B_\lambda = \{0\}$ for some $\lambda > 0$ implies $\beta_\lambda \equiv \{0\}$ for some $\lambda > 0$. Hence this completes the proof of (iii). □

Consider the family of maps $\alpha = \{\alpha_t\ ;\ t \geqslant 0\}$ defined by

$$\alpha_t(\phi) = V(t)\phi V(t)^*(\phi \in \mathcal{A}) \tag{6.3}$$

It is shown in [23, 24] if the Markov generator is a bounded operator i.e. $\sup |\Omega_{ii}| < \infty$ then $\alpha \equiv \{\alpha_t\ ;\ t \geqslant 0\}$ is the unique family of strongly continuous identity preserving* homomorphisms satisfying :

$$d\alpha_t(\phi) = \sum_{ij\in\bar{S}} \alpha_t(\theta^i_j(\phi))d\Lambda^j_i(t)\ ;\ \alpha_0(\phi) = \phi \qquad \dots \quad (6.4)$$

on $h_0 \underline{\otimes} e(\mathcal{M})$, where $\theta \equiv (\theta^i_j : i, j \in \bar{S})$ is a regular family of structure maps on $\mathcal{A}$ given by

$$\theta^i_j(\phi_k) = \begin{cases} (\phi_{k-i} - \phi_k)\delta_{ij} & ;\ i, j \in S \\ m_{k-i,k}\, \phi_{k-i} - m_{k,\,k+i}\phi_k & ,\ i \in S, j = 0 \\ \bar{m}_{k-j,k}\, \phi_{k-j} - \bar{m}_{k,k+j}\phi_k & ,\ i = 0, j \in S \\ \sum_{r\in S} |m_{k-r,k}|^2\phi_{k-r} - |m_{kk}|^2\phi_k, & ,\ i = 0 = j \end{cases} \qquad \dots \quad (6.5)$$

Furthermore $\{\alpha_t(\phi) : t \geqslant 0, \phi \in \mathcal{A}\}$ is a commutative family of bounded operators.

Here our aim is to drop the boundedness assumption of Ω and investigate the family $\alpha \equiv \{\alpha_t : t \geqslant 0\}$ in detail.

By Theorem 6.1 observe that $\alpha \equiv \{\alpha_t : t \geqslant 0\}$ is a family of strongly continuous* homomorphisms and it preserves identity if and only if $B_\lambda \equiv \{0\}$ for some $\lambda > 0$.

In [22] the asymptotic behaviour of the induced Evans-Hudson flows $j^{(n)}_t(\phi) := V^{(n)}(t)\phi V^{(n)}(t)^*(t \geqslant 0, \phi \in \mathcal{A})$ as $n \to \infty$ has been investigated but it is not clear whether it approximates the process $\alpha \equiv \{\alpha_t : t \geqslant 0\}$ in a reasonable topology. Here we shall modify the approximating sequence to ensure it and conclude some properties of $\alpha \equiv \{\alpha_t : t \geqslant 0\}$. In particular, we shall show the commutativity of the process and prove that the differential equation (6.4) is satisfied in weak sense. Finally with an additional hypothesis on Ω, we shall show it satisfies (6.4) in strong sense. To this end we introduce some notations.

Define bounded operators $S^{(n)}_i(i \in S)$, $Z(n) \equiv (Z^i_j(n) : i, j \in \bar{S})$ $(n \geqslant 1)$ by

$$S^{(n)}_i = \begin{cases} S_i\Pi_{[-n} + I - \Pi_{[-n} & ,\ i > 0 \\ S_i\Pi_{n]} + I - \Pi_{n]} & ,\ i < 0 \end{cases}$$

where

$$\Pi_{[-n} = \sum_{k \geq -n} \phi_k, \quad \Pi_{n]} = \sum_{k \leq n} \phi_k$$

and

$$Z^i_j(n) = \begin{cases} ((S^{(n)}_i)^* - I)\delta_{ij} & , i, j \in S, \\ -Z^{(n)}_i & , i \in S, j = 0, \\ (Z^{(n)}_j)^* (S^{(n)}_j)^* & , i = 0, j \in S, \\ -\frac{1}{2} \sum_{k \in S} (Z^{(n)}_k)^* Z^{(n)}_k & , i = 0 = j. \end{cases} \quad \ldots \quad (6.6)$$

A simple computation shows that for each $n \geq 1$, $Z(n) \in \mathcal{A}_R \cap \bar{\mathcal{J}}_R$ and satisfies (3.1)–(3.2) where Z is defined as in (6.1). Also for all $i, j \in \bar{S}$

$$Z^i_j(n) + Z^j_i(n)^* + \sum_{k \in S} Z^k_i(n)^* Z^k_j(n) = \begin{cases} (S^{(n)}_i (S^{(n)}_i)^* - I)\delta_{ij}, & i, j \in S, \\ 0 & , \text{otherwise}. \end{cases}$$

So for each $j \in \bar{S}$ and $f \in \mathfrak{h}$ we have

$$\sum_{i \in \bar{S}} \|Z^i_j(n) f\|^2 \leq \|f\|^2 + \|Z^0_j(n) f\|^2 + 2\|f\| \, \|Z^j_j(n) f\|$$

and (4.2) implies (4.4). Denote $C^{(n)} \equiv \{C^{(n)}(t) : t \geq 0\}$ the unique co-isometric solution of (3.6) where $Z(n)$ $(n \geq 1)$ are as in (6.6). So by Proposition 3.3 and Theorem 6.1 we have

$$s\text{-}\lim_{n \to \infty} C^{(n)}(t) = V(t) \; (t \geq 0) \quad \ldots \quad (6.7)$$

Now consider the maps $\alpha^{(m,n)} = (\alpha^{(m,n)}_t : t \geq 0)$ $m, n \geq 1)$ defined by

$$\alpha^{(m,n)}_t (\phi) = C^{(m)}(t) \phi C^{(n)}(t)^*, \; \phi \in \mathcal{A}.$$

We also write $\alpha^{(n)}$ for $\alpha^{(n,n)}$ $(n \geq 1)$.

A simple application of quantum Ito's formula (2.1) shows that

$$d\alpha^{(m,n)}_t(\phi) = \sum_{i, j \in \bar{S}} \alpha^{(m,n)}_t({}^{(m,n)}\mu^i_j(\phi)) d\Lambda^j_i(t) \; ; \; \alpha^{(m,n)}_0(\phi) = \phi \quad \ldots \quad (6.9)$$

where

$${}^{(m,n)}\mu^i_j(\phi) = \begin{cases} (\sigma_i(\phi) - \phi)\delta_{ij} & , i, j \in S, \\ \sigma_i(\phi) Z^{(n)}_i - Z^{(m)}_i \phi & , i \in S, j = 0, \\ (Z^{(m)}_i)^* \sigma_j(\phi) - \phi (Z^{(n)}_j)^* & , i = 0, j \in S, \\ \sum_{k \in S} \{(Z^{(m)}_k)^* \sigma_k(\phi) Z^{(n)}_k - \frac{1}{2} (Z^{(m)}_k)^* Z^{(m)}_k \phi \\ \quad - \frac{1}{2} \phi (Z^{(n)}_k)^* Z^{(n)}_k\} & , i = 0 = j \end{cases} \quad \ldots \quad (6.10)$$

and

$$\sigma_k(\phi) = (S^{(m)}_k)^* \phi S^{(n)}_k, \; k \in S.$$

We also write ${}^{(n)}\mu$ for ${}^{(n,n)}\mu$ for each $n \geqslant 1$. For $n \geqslant 1$ denote ${}^{(n)}\theta \equiv \{{}^{(n)}\theta^i_j : i, j \in \bar{S}\}$ the regular structure maps defined by (6.10) where $m = n$ and $\sigma_k(\phi) = S_k^*\phi S_k, (k \in S)$. Some algebraic relations among these maps are listed in the following lemma.

Lemma 6.2. *Fix any $n \geqslant 1$. The following holds for all $i, j \in \bar{S}$:*

(a) *for $\phi \in \mathcal{A}$*

$$ {}^{(n)}\mu^i_j(\phi) = \begin{cases} \Pi_{[-n}\, {}^{(n)}\theta^i_j(\phi), & i, j \geqslant 0, \\ \Pi_{n]}\, {}^{(n)}\theta^i_j(\phi), & \textit{otherwise}; \end{cases} $$

(b) *for $|k| \leqslant n \leqslant m$*

$$ {}^{(n)}\mu^i_j(\phi_k) = {}^{(m,n)}\mu^i_j(\phi_k). $$

Proof. Note that for all $i \in S, n \geqslant 1$

(i) $S_i^{(n)} Z_i^{(n)} = S_i\, Z_i^{(n)}$;

(ii) for $k \in \mathbb{Z}$

$$ (S_i^{(n)})^* \phi_k S_i^{(n)} - \phi_k = \begin{cases} \Pi_{[-n}\, \phi_{k-i} - \phi_k), & i > 0, \\ \Pi_{n]}\, (\phi_{k-i} - \phi_k), & i < 0; \end{cases} $$

(iii) for $|k| \leqslant n \leqslant m$

$$ (S_i^{(m)})^* \phi_k S_i^{(n)} = (S_i^{(n)})^* \phi_k S_i^{(n)}. $$

With these observations a routine co-mputation implies (a) and (b).□

Let $\mathcal{A}_0$ be the linear manifold generated by $\{\phi_k : k \in \mathbb{Z}\}$. So $\mathcal{A}_0$ is weakly dense in $\mathcal{A}$.

Proposition 6.3. *For any $n \geqslant 1$,*

(a) *$\alpha^{(n)} = \{\alpha_t^{(n)} : t \geqslant 0\}$ is a family of $*$ homomorphisms. The family $\{\alpha_t^{(n)}(\phi) : t \geqslant 0, \phi \in \mathcal{A}\}$ is commutative.*

(b) *for $|k| \leqslant n \leqslant m$; $t \geqslant 0$*

$$ \alpha_t^{(n)}(\phi_k) = \alpha_t^{(m,n)}(\phi_k) \qquad \dots \quad (6.11) $$

(c) *for $\phi \in \mathcal{A}_0, t \geqslant 0$*

$$ s\text{-}\lim_{n \to \infty} \alpha_t^{(n)}(\phi) = \alpha_t(\phi) \qquad \dots \quad (6.12) $$

(d) *the family of operators $\{\alpha_t(\phi) ; t \geqslant 0, \phi \in \mathcal{A}\}$ is commutative.*

Proof. Since ${}^{(n)}\theta$ is a family of regular structure maps Lemma 6.1 (a) implies that ${}^{(n)}\mu$ is also a family of regular structure maps on $\mathcal{A}$. Hence (a) follows from Theorem 2.13.

For any fixed $f, g \in h_0$, $u, v \in \mathcal{M}$, $n \geqslant 1$ denote $x^{(m)}(t) \equiv \{x_k^{(m)}(t) : |k| \leqslant n\}$; $t \geqslant 0$, $m \geqslant n$ defined by

$$x_k^{(m)}(t) = \langle fe(u), \alpha_t^{(m,n)}(\phi_k)\, ge(v) \rangle$$

From (6.9) we get for $m \geqslant n$

$$\frac{d}{dt}\, x^{(m)}(t) = x^{(m)}(t)\Omega^{(n)}(t)(t \geqslant 0) \qquad \dots \quad (6.13)$$

where $\Omega^{(n)}(t) = \{\Omega_{ij}(t) : -n \leqslant i, j \leqslant n\}$; $t \geqslant 0$ defined by

$$\Omega_{ij}(t) = \begin{cases} (u_{i-j}(t)+m_{ij})\,(v^{i-j}(t)+\overline{m}_{ij}) & ;\ i \neq j \\ -\sum\limits_{r \neq i} \Omega_{ir}(t) & ;\ i = j \end{cases}$$

Also observe that $x^{(m)}(0)$ is independent of $m \geqslant n$. Since (6.13) admits a unique solution we have for all $m \geqslant n \geqslant |k|$, $f, g \in h_0$, $u, v \in \mathcal{M}$ and $t \geqslant 0$

$$\langle fe(u), \alpha_t^{(n)}(\phi_k) ge(v) \rangle = \langle fe(u), \alpha_t^{(m,n)}(\phi_k) ge(v) \rangle .$$

Now a standard argument implies (b). For (c) it is enough to show (6.12) for $\phi = \phi_k$, $k \in \mathscr{Z}$. From (6.7) and (6.11) we have for each $n \geqslant |k|$

$$\alpha_t^{(n)}(\phi_k) = w\text{-}\lim_{m \to \infty} \alpha_t^{(m,n)}(\phi_k) = V(t)\phi_k C^{(n)}(t)^*(t \geqslant 0). \qquad \dots \quad (6.14)$$

Hence we get applying (6.7) once more in (6.14)

$$w\text{-}\lim_{n \to \infty} \alpha_t^{(n)}(\phi_k) = \alpha_t(\phi_k) \quad (t \geqslant 0).$$

Since $\alpha_t^{(n)} : n \geqslant 1$ and $\alpha_t(t \geqslant 0)$ are * homomorphisms, (6.12) follows. This completes the proof of (c). For (d) use (a) and (c) to show that $\{\alpha_t(\phi) : t \geqslant 0, \phi \in \mathcal{A}_0\}$ is a commutative family. Since $\mathcal{A}_0$ is strongly dense in $\mathcal{A}$, (d) follows by a standard approximation argument. □

We shall show that $\alpha = \{\alpha_t : t \geqslant 0\}$ is indeed, a quantum analogue of Feller's minimal solution. To this end we introduce a few notations. For any fixed $u \in \mathcal{M}_c$ consider the family of matrices $P(s, t) \equiv \{P_{ij}(s, t) : -\infty < i, j < \infty\}$, $P^{(n)}(s, t) \equiv \{P_{ij}^{(n)}(s, t) : -n \leqslant i, j \leqslant n\}$; $0 \leqslant s \leqslant t$ and $n \geqslant 1$, $\Omega^{(n)}(t) = \{\Omega_{ij}(t) : n \leqslant i, j \leqslant n\}$; $t \geqslant 0$ where

$$P_{ij}^{(n)}(s\, t) = \langle f_i e(u), C^{(n)}(s)^* C^{(n)}(t)\phi_j C^{(n)}(t)^* C^{(n)}(s) f_i e(u) \rangle \; \| e(u)\|^{-2}$$

$$P_{ij}(s, t) = \langle f_i e(u), V(s)^* V(t)\phi_j V(t)^* V(s) f_i e(u) \rangle \; \|e(u)\|^{-2}$$

$$\Omega_{ij}(t) = \begin{cases} |m_{ij} + u_{j-i}(t)|^2, & i \neq j \\ -\sum\limits_{k \neq i} \Omega_{ik}(t) ; & i = j \end{cases}$$

and

$$\mathcal{M}_c \equiv \{u \in \mathcal{M} ; u \text{ is continuous}\}$$

Proposition 6.4. *For any fixed $u \in \mathcal{M}_c$ the following holds :*

(a) $\lim_{n \to \infty} P^{(n)}_{ij}(s, t) = P_{ij}(s, t) \ (0 \leqslant s \leqslant t < \infty)$

(b) $\{P(s, t) ; 0 \leqslant s \leqslant t\}$ *is the minimal solution satisfying* (4.9)–(4.13) *and* (4.15) *where* $\Omega(t) \equiv \{\Omega_{ij}(t) : -\infty \leqslant i, j \leqslant \infty\}, t \geqslant 0$.

(c) *If* $u = 0$

$$P(0, t) = F(t)$$

where $F \equiv (F(t) : t \geqslant 0)$ *is the minimal solution for the Markov matrix* $\Omega \equiv (\Omega_{ij} : -\infty < i, j < \infty)$.

(d) $\sigma \equiv (\sigma(t) := I - V(t)V(t)^* ; t \geqslant 0)$ *is a strongly continuous increasing projection valued commutative adapted process.*

Proof. (a) follows from (6.7) and (6.12). For (b) using quantum Ito's formula (2.3) we have for $0 \leqslant s \leqslant t < \infty$ and $n \geqslant 1$

$$\frac{\partial}{\partial t} P^{(n)}(s, t) = P^{(n)}(s, t)\Omega^{(n)}(t) \qquad \ldots \quad (6.15)$$

Since (6.15) admits a unique solution, we have for any $i, j \in \mathbb{Z}$ $n \geqslant \max(|i|, |j|)$

$$P^{(n)}_{ij}(s, t) = \sum_{|k| \leqslant n} P^{(n)}_{ik}(s, s)F^{(n)}_{kj}(s ; t) \qquad \ldots \quad (6.16)$$

where $F^{(n)}(s, t)$ is the unique solution of (4.3). Now taking limit as $n \to \infty$ in (6.16) we get for all $i, j \in \mathbb{Z}$

$$P_{ij}(s, t) = F_{ij}(s, t)$$

where (4.7) and (4.13) have been used to employ dominated convergence theorem. Hence (b) follows by Theorem 4.3. (c) follows from (vi) of Theorem 4.3.

Fix any $u, v \in \mathcal{M}$ and $i \neq j$. Since (6.13) admits a unique solution, in particular we have for $|k| \leqslant n$

$$< f_i e(u), \alpha^{(n)}_t(\phi_k) f_j e(v) > = 0 \ (t \geqslant 0).$$

Taking limit as $n \to \infty$ in the above expression, we get for all $k \in \mathbb{Z}$

$$< f_i e(u), \alpha_t(\phi_k) f_j e(v) > = 0 \quad (t \geqslant 0).$$

Hence for all $u, v \in \mathcal{M}$ and $i \neq j$

$$< f_i e(u), \alpha_t(I) f_j e(v) > = 0 \quad (t \geqslant 0). \qquad \ldots \quad (6.17)$$

So for an element $\psi = \sum_i c_i f_i e(u_i)$, where finitely many c_i are non-zero, $u_i \in \mathcal{M}$ we have from (6.17)

$$\|V(t)^* \psi\|^2 = \sum_i |c_i|^2 \|V(t)^* f_i e(u_i)\|^2 \quad (t \geqslant 0) \qquad \ldots \quad (6.18)$$

For any fixed $u \in \mathcal{M}_0$, $i \in \mathbf{Z}$, $0 \leqslant s \leqslant t$ using (4.12) and (4.13) we get

$$\|V(t)^* f_i e(u)\|^2 = \sum_j F_{ij}(t) = \sum_k F_{ik}(s) \sum_j F_{kj}(t-s)$$

$$\leqslant \sum_k F_{ik}(s) = \|V(s)^* f_i e(u)\| \qquad \ldots \quad (6.19)$$

Now exploiting the fact that $\mathcal{A} \underline{\otimes} e(\mathcal{M})$ is dense in $h_0 \otimes \Gamma_+$, (6.18) and (6.19) implies that σ is an increasing process. By Proposition 3.10 (ii) V^* is strongly continuous, hence Theorem 6.1 implies that σ is a strongly continuous projection valued process. Commutativity follows from Proposition 6.3(d). This completes the proof. □

For the rest of this section we shall impose the following hypothesis on the Markov matrix Ω :

$[\mathcal{N}]$ for each $j \in \mathbf{Z}$, $\sup_i \Omega_{ij} < \infty$.

Observe that for Ω satisfying $(\mathcal{N})$, $\theta = \{\theta^i_j : i, j \in \overline{S}\}$ described as in (6.5) indeed maps $\mathcal{A}_0$ into $\mathcal{A}$. Furthermore we have the following lemma.

Lemma 6.5. *Let $(\mathcal{N})$ be valid. Then for $\phi \in \mathcal{A}_0$ the following holds :*

(a)
$$\sum \theta^i_j(\phi)^* \theta^i_j(\phi) \qquad \ldots \quad (6.20)$$

is convergent in strong operator topology for $j \in \overline{S}$.

(b)
$$w\text{-}\lim_{n \to \infty} \alpha_t^{(n)}({}^{(n)}\mu^i_j(\phi)) = \alpha_t(\theta^i_j(\phi)) \qquad \ldots \quad (6.21)$$

for $t \geqslant 0$, $i, j \in \overline{S}$.

Proof. In view of Lemma 2.5 to show (a) it is enough to verify (6.20) for $\phi = \phi_k$; $k \in \mathbf{Z}$. For $j \in S$ (6.20) is always valid since only finitely many terms are non-zero. For $j = 0$, $i \in S$ we have

$$\theta^i_0(\phi_k)^* \theta^i_0(\phi_k) = \Omega_{k-i,\,k} \phi_{k-i} + \Omega_{k,\,k+i} \phi_k$$

so for each $f \in h_0$

$$\sum_{i \in \overline{S}} \|\theta^i_0(\phi_k) f\|^2 \leqslant 2(|\Omega_{kk}| + \sup_{i : i \neq k} \Omega_{ik}) \|f\|^2$$

Hence this completes the proof of (a). For (b) note that it suffices to verify (6.21) for $\phi = \phi_k$; $k \in \mathbf{Z}$. For $(i, j) \neq (0, 0)$, ${}^{(n)}\mu^i_j(\phi_k)$ being equal to

$\theta_j^i(\phi_k)$ for sufficiently large n, (6.21) follows from (6.12). Proof of (b) will be complete once we verify (6.21) for $i = 0 = j$. To show this observe the following :

(i) ${}^{(n)}\mu_0^0(\phi_k)$ being an element in the linear span of $\{\phi_r : |r| \leqslant n\}$, (6.14) implies that

$$\alpha_t^{(n)}({}^{(n)}\mu_0^0(\phi_k)) = V(t)\,({}^{(n)}\mu_0^0(\phi_k))C^{(n)}(t)^*(t \geqslant 0)$$

(ii) ${}^{(n)}\mu_0^0(\phi_k) : n \geqslant |k|$ is a sequence of self-adjoint operators and

$$s\text{-}\lim{}^{(n)}\ \mu_0^0(\phi_k) = \theta_0^0(\phi_k).$$

A standard argument coupled with these observations and (6.7) lead us to the required result. This completes the proof. □

Theorem 6.6. *Consider the family of maps* $\alpha = (\alpha_t : t \geqslant 0)$ *defined as in* (6.3). *Then the following holds* :

(a) $\alpha_t : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\Gamma_+)$; $t \geqslant 0$ *is a family of strongly continuous* homomorphisms and* $\{\alpha_t(\phi) : t \geqslant 0, \phi \in \mathcal{A}\}$ *is a commutative family of bounded operators* ;

(b) α *is identity preserving if and only if* $B_\lambda = \{0\}$ *for some* $\lambda > 0$.

(c) *If* $\mathcal{H}$ *holds then for all* $\phi \in \mathcal{A}_0$

$$\alpha_0(\phi) = \phi,\ d\alpha_t(\phi) = \sum_{ij \in \bar{S}} \alpha_t(\theta_j^i(\phi))d\Lambda_j^i(t)\ \ (t \geqslant 0) \qquad \ldots \quad (6.22)$$

holds on $h_0 \underline{\otimes}\, e(\mathcal{M})$.

(d) *For any* $f \in h_0, u \in \mathcal{M}, t \geqslant 0, \phi \geqslant 0$ *and a positivity preserving bounded process* $j = \{j_t(\phi) : t \geqslant 0, \phi \in \mathcal{A}\}$ *satisfying* (6.22) *the following inequality holds* :

$$< fe(u), j_t(\phi)fe(u) > \geqslant < fe(u), \alpha_t(\phi)\, fe(u) > .$$

Proof. By Proposition 3.10 (ii) observe that V^* is strongly continuous, hence Theorem 6.1 implies the first part of (a). For the rest of (a) appeal to Proposition 6.3 (d). (b) follows from Theorem 6.1 (iii).

First observe that for all $f, g \in h_0$, $u, v \in \mathcal{M}$, $\phi \in \mathcal{A}_0$ and $t \geqslant 0$

$$< fe(u), \alpha_t(\phi)ge(v) > = \lim_{n \to \infty}\ < fe(u), \alpha_t^{(n)}(\phi)ge(v)) >$$

$$= < fe(u), \phi ge(v)> + \sum_{i,j \in \bar{S}}\ \lim_{n \to \infty} \int_0^t ds\bar{u}_i(s)v^j(s)< fe(u), \alpha_s^{(n)}({}^{(n)}\mu_j^i(\phi))ge(v)>$$

$$= < fe\,(u), \phi ge(v) > + \sum_{i,j \in \bar{S}} \int_0^t\ ds\bar{u}_i(s)v^j(s) < fe(u), \alpha_s(\theta_j^i(\phi))ge(v) >$$

where (6.12), $u, v \in \mathcal{M}$ and (6.12) have been used in the first, second and last equality respectively. Now for (c) it is enough to show for each $\phi \in \mathcal{A}_0$, $\{\alpha_t(\theta_j^i(\phi))\} \in \mathcal{L}(h_0, \mathcal{M})$. Adaptedness of the processes is clear from Theorem 6.1 and for each $\phi \in \mathcal{A}_0, j \in \overline{S}$, α_t being a homomorphism we get from (6.20)

$$\sum_{i \in \overline{S}} \alpha_t(\theta_j^i(\phi))^* \alpha_t(\theta_j^i(\phi)) = \alpha_t(\sum_{i \in \overline{S}} \theta_j^i(\phi)^* \theta_j^i(\phi)). \qquad \dots \quad (6.23)$$

where the series converge in strong operator topology. α_t being a contractive map for each $t \geqslant 0$, we get the required result from (6.23). This completes the proof of (c).

For (d) we need to show for each $f \in h_0$, $u \in \mathcal{M}$ and $|k| \leqslant n$

$$y_k(t) \geqslant x_k^{(n)}(t) \quad (t \geqslant 0)$$

where

$$y_k(t) = < fe(u), j_t(\phi_k) fe(u) >$$

and

$$x_k^{(n)}(t) = < fe(u), \alpha_t^{(n)}(\phi_k) fe(u) >.$$

Fix any $n \geqslant 1$ observe by our assumption on $j = \{j_t : t \geqslant 0\}$

$$\frac{d}{dt} \, y^{(n)}(t) = y^{(n)}(t) \Omega^{(n)}(t) + z^{(n)}(t) (t \geqslant 0) \qquad \dots \quad (6.24)$$

where $y^{(n)}(t) = \{y_k(t) : -n \leqslant k \leqslant n\}$ and $z^{(n)}(t) = \{z_k^{(n)}(t) : -n \leqslant k \leqslant n\}$ is given by

$$z_k^{(n)}(t) = \sum_{|j| > n} y_j(t) \, \Omega_{jk}(t) \quad (t \geqslant 0)$$

and $z^{(n)}(t) \geqslant 0$. Also note that $x^{(n)}(t) = \{x_k^{(n)}(t) : -n \leqslant k \leqslant n\}$ is the unique solution of (6.24) where $z^{(n)}(t) \equiv 0$. With these observations we get the required inequality by integrating the differential equation. This completes the proof. □

In analogy with the classical Feller minimal process, we expect an operator inequality in Theorem 6.6 (d). However, with an additional assumption on j_t, namely for all $i \neq j$ and $u, v \in \mathcal{M}$, $< f_i e(u), j_t(\phi) f_j e(v) > = 0$ for all $t \geqslant 0$, we have

$$j_t(\phi) \geqslant \alpha_t(\phi)$$

whenever $\phi \geqslant 0$. It remains an open question whether Feller's condition is also sufficient for the existence of a unique positivity preserving contractive flow satisfying (6.22).

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