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SOME OPTIMALITY RESULTS ON EFFICIENCY-BALANCED DESIGNS

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SUMMARY. Some optimality results have been derived on efficiency-balanced designs, with special emphasis on binary designs. Several specific applications have also been indicated.

1. Introduction

Although the history of efficiency-balanced (EB) designs can be traced back to the work by Jones (1959), the study of such designs underwent a vigorous development over the last fifteen years. Significant results on characterization problems relating to EB designs were obtained by Williams (1975), Puri and Nigam (1975a), Kageyama (1980) and Dey, Singh and Saha (1981) and useful construction procedures were proposed by Puri and Nigam (1975b, 1977), Dey and Singh (1980), Kageyama (1981), Kageyama and Mukerjee (1986) and Ghosh and Karmakar (1988), among others. The simplicity in the analysis of EB designs has been noted by various authors but, with the availability of computers, this property alone does not seem to be very attractive and further statistical justification, through optimality considerations, is called for. It, however, appears that not much work has been reported on the optimality aspects of EB designs, especially in the non-equireplicate case, except for some results recently obtained by Das (1989). This point has been noted, among others, by A. Dey in a private communication. The present paper attempts to fill in this gap to some extent.

It is well-known (see e.g., Williams (1975), Dey, Singh and Saha (1981)) that an equireplicate EB design is variance-balanced and that for such designs, particularly when they are binary, optimality results are available in the

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literature (cf. Kiefer (1975)). In this paper, therefore, non-equireplicate EB designs have been considered. In pratical situations, when the use of a non-equireplicate design is contemplated, it often happens that there are restrictions on the availability of the treatments and, as such, a particular replication pattern has to be followed. In such situations, in addition to the number of treatments, the number of blocks and the block sizes being fixed, the replication numbers may also be fixed a priori from a practical standpoint. Hence while studying the optimality aspects of a non-equireplicate EB design d^*, it is often natural to restrict attention to the class $S(d^*)$ of designs having the same number of treatments, the same number of blocks, the same block sizes and the same replication numbers as d^*. At this point, we deviate from the traditional optimal design theory where optimality is explored within the broader class $S_0(d^*)$ of designs with the same number of treatments, the same number of blocks and the same block sizes as d^*. This change, indeed, makes our results theoretically restrictive. But still, the findings are practically useful when, as indicated above, the replication numbers have to be fixed a priori from extraneous considerations.

2. Main results

Consider an EB design d^* involving v treatments and b blocks such that the i-th treatment is replicated r_i times in d^* and the j-th block in d^* has size k_j $(1 \leqslant i \leqslant v\,;\ 1 \leqslant j \leqslant b)$. Let $\boldsymbol{r} = (r_1, \ldots, r_v)'$, $R = \text{diag}(r_1, \ldots, r_v)$, and $n = \sum_{i=1}^{v} r_i$. The optimality of d^* within the class $S(d^*)$, as defined above, will be investigated. We assume the usual fixed effects additive model with independence and homoscedasticity of errors.

Let $\boldsymbol{t}$ be the $v \times 1$ vector of (fixed) treatment effects. Then a complete set of R^{-1}-normalized treatment contrasts is given by $P\boldsymbol{t}$, where the $(v-1)\times v$ matrix P satisfies

$$PR^{-1}P' = I,\ P\mathbf{1} = \mathbf{0}, \qquad \ldots \quad (2.1)$$

with $\mathbf{1}$ as the $v\times 1$ vector with all elements unity. For any design $d \in S(d^*)$, let C_d denote the coefficient matrix of the reduced normal equations for $\boldsymbol{t}$ (cf. Raghavarao (1971)). It is not hard to see that, under the design d, the information matrix for $P\boldsymbol{t}$ is proportional to $\mathcal{I}(d) = PR^{-1}C_dR^{-1}P'$, and that by (2.1),

$$tr(\mathcal{I}(d)) = v - \sum_{i=1}^{v} \sum_{j=1}^{b} n_{dij}^2/(r_ik_j) \qquad \ldots \quad (2.2)$$

where $N_d = ((n_{dij}))$ is the incidence matrix of d. For the EB design d^*, C_{d^*} is proportional to $(R - n^{-1} \mathbf{r} \mathbf{r}')$, so that by (2.1), $\mathscr{I}(d^*)$ is proportional to the identity matrix. Hence by (2.2), along the line of Kiefer (1975) (see also Sinha and Mukerjee (1982)), the following result is evident.

Lemma 2.1 : *Let d^* be an EB design and suppose*

$$\sum_{i=1}^{v} \sum_{j=1}^{b} n_{dij}^2/(r_i k_j) \geqslant \sum_{i=1}^{v} \sum_{j=1}^{b} n_{d^*ij}^2/(r_i k_j), \quad \ldots \quad (2.3)$$

for every $d \in S(d^*)$. Then d^* is universally optimal in $S(d^*)$ for the estimation of a full set of R^{-1}-normalized treatment contrasts.

Since universal optimality implies D-optimality and since D-optimality with respect to one complete set of treatment contrasts implies that with respect to every complete set of treatment contrasts, from Lemma 2.1 one obtains the following result which is perhaps more important in terms of applications.

Lemma 2.2 : *Let d^* be an EB design and suppose the condition* (2.3) *holds for every $d \in S(d^*)$. Then d^* is D-optimal in $S(d^*)$ for the estimation of every complete set of treatment contrasts.*

There is yet another implication of Lemma 2.1. For any design d in $S(d^*)$, let ϕ_d denote the minimum efficiency with respect to a treatment contrast in d, where efficiency is relative to the corresponding (unblocked) completely randomized design with the same replication numbers. A typical treatment contrast is of the from $\boldsymbol{\xi}' P \mathbf{t}$, where P is as defined above and $\boldsymbol{\xi}$ is a $(v-1) \times 1$ non-null vector. Hence by (2.1), for a connected design $d \in S(d^*)$,

$$\phi_d = \inf_{\xi \neq 0} \{\boldsymbol{\xi}' P R^{-1} P' \boldsymbol{\xi} / \boldsymbol{\xi}' P C_d^- P' \boldsymbol{\xi}\} = \{\lambda_{\max}(P C_d^- P')\}^{-1} = \lambda_{\min}(\mathscr{I}(d)), \quad \ldots \quad (2.4)$$

where for any square matrix A, $\lambda_{\max}(A)$ ($\lambda_{\min}(A)$) denotes its maximum (minimum) eigenvalue and A^- stands for any generalized inverse of A. Trivially, for a disconnected design $d (\in S(d^*))$, $\phi_d = 0$. Since universal optimality implies E-optimality, the following result is evident from Lemma 2.1.

Lemma 2.3 : *Let d^* be an EB design and suppose the condition* (2.3) *holds for every $d \in S(d^*)$. Then $\phi_{d^*} \geqslant \phi_d$ for every $d \in S(d^*)$.*

Note that the optimality criterion in Lemma 2.3 is analogous to the E-optimality criterion. In order to indicate specific applications of the above lemmas, we now consider EB designs with exactly two distinct replication numbers and exactly two distinct block sizes—most of the EB designs reported in the literature are of this type. With such an EB design d^*, one can find non-empty sets G_1, G_2, T_1, T_2 such that (i) G_1, G_2 provide a disjoint partition of $\{1, 2, \ldots, v\}$; T_1, T_2 provide a disjoint partition of $\{1, 2, \ldots, b\}$, (ii) $r_i = r'$ for $i \in G_1$; $r_i = r''$ for $i \in G_2$; $r' \neq r''$, (iii) $k_j = k'$ for $j \in T_1$; $k_j = k''$ for $j \in T_2$; $k' \neq k''$. Without loss of generality, let

$$r' < r'', \; k' < k''. \qquad \ldots \quad (2.5)$$

For every $d \in S(d^*)$, let $h_d^{(s, s')} = \sum_{i \in G_s} \sum_{j \in T_{s'}} n_{dij}$; $s, \; s' = 1, 2$. As every $d (\in S(d^*))$ has the same replication numbers and the same block sizes as d^* writing $h_d^{(1,1)} = f_d$ for the sake of notational simplicity, it follows from (ii) and (iii) above that for $d \in S(d^*)$,

$$h_d^{(1,2)} = wr' - f_d, \; h_d^{(2,1)} = uk' - f_d, \; h_d^{(2,2)} = (b-u)\, k'' - wr' + f_d, \qquad \ldots \quad (2.6)$$

where w and u are the cardinalities of G_1 and T_1 respectively ($0 < w < v$, $0 < u < b$). Hence for every $d \in S(d^*)$,

$$\sum_{i=1}^{v} \sum_{j=1}^{b} n_{dij}^2/(r_i k_j) \geqslant \sum_{i=1}^{v} \sum_{j=1}^{b} n_{dij}/(r_i k_j)$$

$$= (r'k')^{-1} f_d + (r'k'')^{-1}(wr' - f_d) + (r''k')^{-1}(uk' - f_d)$$

$$+ (r''k'')^{-1}\{(b-u)k'' - wr' + f_d\}$$

$$= \Delta + \left(\frac{1}{r'} - \frac{1}{r''}\right)\left(\frac{1}{k'} - \frac{1}{k''}\right) f_d, \qquad \ldots \quad (2.7)$$

by (2.6), where Δ is the same for all designs in $S(d^*)$. Note that equality holds in (2.7) provided d is binary and that by (2.5), the coefficient of f_d in the right-hand member of (2.7) is positive. Furthermore, by (2.6), $f_d \geqslant \max\{0, \; wr' - (b-u)k''\} = f_0$, say, for each $d \in S(d^*)$. Hence by (2.3), (2.7), the following result holds.

Theorem 2.1 : *Let d^* be a binary EB design such that $f_{d^*} = f_0$. Then d^* is optimal in $S(d^*)$ in the senses considered in Lemmas* 2.1—2.3.

3. Applications

The results in the last section, particulraly Theorem 2.1, are helpful in exploring the optimality of a large variety of EB designs available in the literature. In order to save space, we cannot deal with each and every available EB design. However, it appears that the .examples presented below have a fairly wide coverage

Example 3.1. Let N_1, N_2 be incidence matrices of balanced incomplete block (BIB) designs with parameters $\bar{v}, b_1, r_1, k_1, \lambda_1$ and $\bar{v}, b_2, r_2, k_2, \lambda_2$, respectively. Suppose there exist positive integers x, y such that

$$\frac{x}{y} = \frac{\lambda_2\bar{v}(k_1+1)-k_1k_2r_2}{k_2(r_1-\lambda_1)} = \frac{r_2\{\bar{v}(k_2-k_1-1)+k_1k_2\}}{k_2r_1(\bar{v}-k_1)} \quad \ldots \quad (3.1)$$

Then, as shown in Corollary 2.4 in Kageyama (1981) (see also Theorem 5 in Puri and Nigam (1977)), the design d^* with incidence matrix

$$N = \begin{pmatrix} \underbrace{\begin{matrix} \mathbf{1}'_{b_1} \ldots \mathbf{1}'_{b_1} \\ N_1 \ldots N_1 \end{matrix}}_{x \text{ times}} & \underbrace{\begin{matrix} 0' \ldots 0' \\ N_2 \ldots N_2 \end{matrix}}_{y \text{ times}} \end{pmatrix}$$

is EB with parameters $v = \bar{v}+1$, $b = xb_1+yb_2$, $r_i = xb_1$ or xr_1+yr_2, $k_j = k_1+1$ or k_2 (here for a positive integer a, $\mathbf{1}_a$ is the $a\times 1$ vector with all elements unity). By (3.1),

$$xb_1-(xr_1+yr_2) = yr_2\bar{v}(k_2-k_1-1)/(k_1k_2),$$

so that xb_1 is greater than, less than or equal to xr_1+yr_2 according as k_2 is greater than, less than or equal to k_1+1 respectively. If $k_2 = k_1+1$, then d^* becomes a variance-balanced design. Consider therefore the case $k_2 \neq k_1+1$. If $k_2 < k_1+1$, then using the notations of the last section, it is easy to see that $w = 1$, $u = yb_2$, $r' = xb_1$, $k^* = k_1+1$, so that $f_0 = 0$. Also, $f_{d^*} = 0$. Furthermore, d^* is binary. Hence by Theorem 2.1, d^* is optimal in $S(d^*)$ in the senses considered in Lemmas 2.1—2.3. On the other hand, if $k_2 > k_1+1$, then $w = \bar{v}$, $u = xb_1$, $r' = xr_1+yr_2$, $k^* = k_2$, $f_0 = x\bar{v}r_1 = f_{d^*}$, and as before the optimality of d^* follows.

Example 3.2. Let N_1 be the incidence matrix of a BIB design with parameters $v_1, b_1, r_1, k_1, \lambda_1$ and suppose there exist positive integers x, y such that

$$x/y = 2\lambda_1v_1k_1^{-1}-r_1. \quad \ldots \quad (3.2)$$

Then, as shown in Corollary 2.5 in Kageyama (1981), the design d^* with incidence matrix

$$N = \begin{pmatrix} \underbrace{\begin{matrix} \mathbf{1}'_{v_1} \ldots \mathbf{1}'_{v_1} \\ I_{v_1} \ldots I_{v_1} \end{matrix}}_{x \text{ times}} & \underbrace{\begin{matrix} 0' \ldots 0' \\ N_1 \ldots N_1 \end{matrix}}_{y \text{ times}} \end{pmatrix}$$

is EB with paramters $v = v_1+1$, $b = xv_1+yb_1$, $r_i = xv_1$ or yr_1+x, $k_j = 2$ or k_1 (here for a positive integer a, I_a is the $a\times a$ identity matrix). Consider the case $k_1 > 2$ (if $k_1 = 2$ then it can be seen that d^* reduces to a BIB design for which optimality results are well-known). Then, from (3.2), it follows that $xv_1 > yr_1+x$. Hence $w = v_1$, $u = xv_1$, $r' = yr_1+x$, $k'' = k_1$, so that $f_0 = xv_1$. Also $f_{d^*} = xv_1$ and d^* is binary. Hence by Theorem 2.1, d^* is optimal in $S(d^*)$ in the senses considered in Lemmas 2.1—2.3.

Example 3.3. Let N_1 be as in the last example and let there exist positive integers x, y such that

$$x/y = \lambda_1 v_1^2 k_1^{-1} - r_1(v_1-1). \qquad \ldots \quad (3.3)$$

For positive integers a, a', let $J_{a,a'} = \mathbf{1}_a \mathbf{1}'_{a'}$, $J_a = J_{a,a}$. Then as shown in Corollary 2.8 in Kageyama (1981), the design d^* with incidence matrix

$$N = \left(\underbrace{\begin{matrix} \mathbf{1}'_{v_1} & \ldots & \mathbf{1}'_{v_1} \\ J_{v_1}-I_{v_1} & \ldots & J_{v_1}-I_{v_1} \end{matrix}}_{x \text{ times}} \quad \underbrace{\begin{matrix} \mathbf{0}' \ldots \mathbf{0}' \\ N_1, \ldots, N_1 \end{matrix}}_{y \text{ times}} \right)$$

is EB with parameters $v = v_1+1$, $b = xv_1+yb_1$, $r_i = xv_1$ or $x(v_1-1)+yr_1$, $k_j = k_1$ or v_1. Clearly, $k_1 < v_1$, and by (3.3), $xv_1 < x(v_1-1)+yr_1$. Hence $w = 1$, $u = yb_1$, $r' = xv_1$, $k'' = v_1$, $f_0 = 0$. Also $f_{d^*} = 0$ and d^* is binary. Hence as in the last example, the optimality of d^* in $S(d^*)$ follows.

Example 3.4. Let N_1 be in Example 3.2. Then, as shown in Kageyama and Mukerjee (1986) (see also Das and Ghosh (1985)), the design d^* with incidence matrix

$$N = \begin{pmatrix} N_1 & J_{v_1, r_1-\lambda_1} \\ \mathbf{1}'_{b_1} & \mathbf{0}' \end{pmatrix}$$

is EB with parameters $v = v_1+1$, $b = b_1+r_1-\lambda_1$, $r_i = 2r_1-\lambda_1$ or b_1, $k_j = k_1+1$ or v_1. Considering the non-trivial case $k_1+1 < v_1$, one obtains $w = v_1$, $u = b_1$, $r' = 2r_1-\lambda_1$, $k'' = v_1$, $f_0 = v_1 r_1$. Also, $f_{d^*} = v_1 r_1$ and d^* is binary and as before the optimality of d^* in $S(d^*)$ follows.

Example 3.5. Let N_1 be as before. Assume that $r_1 = 2\lambda_1$. Then, as shown in Corollary 2.6 in Kageyama (1981) (see also Das and Ghosh (1985)), for an arbitrary positive integer y, the design d^* with incidence matrix

$$N = \begin{pmatrix} N_1 & J_{v_1,y} \\ \mathbf{0}' & \mathbf{1}'_y \end{pmatrix}$$

is EB with parameters $v = v_1+1$, $b = b_1+y$, $r_i = r_1+y$ or y, $k_j = k_1$ or v_1+1. Clearly, $w = 1$, $u = b_1$, $r' = y$, $k'' = v_1+1$, $f_0 = 0$. Since $f_{d^*} = 0$ and d^* is binary, the optimality of d^* in $S(d^*)$ again follows by Theorem 2.1.

Example 3.6. Let N_1 be as in Example 3.2 and assume that $y = r_1(v_1+1)/(v_1-2k_1-1)$ is a positive integer. Then, following Kageyama and Mukerjee (1986), the design d^* with incidence matrix

$$N = \begin{pmatrix} N_1 & J_{v_1,y} \\ 1'_{b_1} & 1'_y \end{pmatrix}$$

is EB with parameters $v = v_1+1$, $b = b_1+y$, $r_i = r_1+y$ or b_1+y, $k_j = k_1+1$ or v_1+1. Clearly, $w = v_1$, $u = b_1$, $r' = r_1+y$, $k'' = v_1+1$, $f_0 = \max(0, v_1r_1-y)$. But $f_{d^*} = v_1r_1$. Hecne $f_{d^*} > f_0$, and the sufficient condition for optimality given by Theorem 2.1 fails. However, one can directly check the condition (2.3). After some tedious algebra, it can be seen that (2.3) holds (and hence d^* is optimal in $S(d^*)$ in the senses considered in Lemmas 2.1—2.3) if, in particular,

$$v_1^2-3v_1(3k_1+1)+(2k_1^2-k_1+2) \leqslant 0. \quad \ldots \quad (3.4)$$

The condition (3.4) is satisfied for practically useful values of v_1 (i.e., when v_1 is not too large). The derivation of (3.4) is omitted here but may be obtained from the authors.

It appears that the above examples provide a reasonable coverage of the binary EB designs available in the literature—in particular, they demonstrate that many of the binary EB designs given by Puri and Nigam (1977), Kageyama (1981), Das and Ghosh (1985) and Kageyama and Mukerjee (1986) are optimal in the senses considered in Lemmas 2.1—2.3. It may be noted that the EB designs due to Dey and Singh (1980) and Ghosh and Karmakar (1988) are either non-binary or, when binary, become equireplicate and hence variance-balanced (cf. Kageyama (1982)); therefore, these designs have not been considered here.

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