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ON THE ADMISSIBILITY OF POLYNOMIAL ESTIMATORS IN THE ONE-PARAMETER EXPONENTIAL FAMILY

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SUMMARY. Karlin (1958), Ping (1964), Gupta (1966), Zidek (1970), and Ghosh and Meeden (1977) dealt with the problem of finding sufficient conditions for admissibility of linear estimators in the one-parameter exponential family. In this paper it has been shown that following Karlin's argument one can also find sufficient conditions for admissibility of polynomial estimators for estimating linear combinations of the raw moments.

1. Introduction

Consider a random variable X whose distribution admits a density function $p(x, w) = \beta(w)e^{xw}$ with respect to some σ-finite measure μ on the real line. Here 'w' represents a typical point in the parameter space $\Omega = \{w : \int e^{xw}\, d\mu(x) < \infty\}$. It is well known that Ω is an interval $(\underline{w}, \bar{w})$, which may be finite or infinite.

Karlin (1958) investigated the conditions for admissibility of estimators of the form aX for estimating the mean $E_w(X) = -\beta'(w)/\beta(w)$, under squared-error loss. Sufficient conditions for admissibility of $aX+b$ for the same problem were later obtained by Ping (1964) using the Rao-Cramér inequality and by Gupta (1966) following Karlin's argument. Zidek (1970) gave sufficient conditions for the admissibility of X when the parametric function is any arbitrary piece-wise continuous function $\gamma(w)$. Recently, Ghosh and Meeden (1977) used Karlin's arguments to derive sufficient conditions for the admissibility of $aX+b$, for estimating the same parametric functions as Zidek's.

The motivation for the present investigation stems from the following question : what can be said about the admissibility or otherwise (under squared-error loss) of estimates which are not necessarily linear ? This question is in general very hard to be dealt with for an arbitrary $\gamma(w)$. In what follows, we have practically restricted ourselves to parametric functions of the form $\gamma_m(w) = C_m E_w(X^m)+\ldots+C_1 E_w(X)$ and estimators of the form $\delta_m(X) = a_m X^m+\ldots+a_1 X+a_0$, and have established, following Karlin's technique, sufficient conditions for the admissibility of $\delta_m(X)$ for estimating $\gamma_m(w)$. It is demonstrated that the joint tail—behaviour of a certain prior and of a linear combination of the raw moments of X determines the admissibility of $\delta_m(X)$. The technique adopted is general, as we shall observe later, but the actual algebra gets involved and clumsy so much so that we have actually stated in the main theorem sufficient conditions for $m = 2$, although exactly similar conditions have been obtained also for $m = 3$ and 4.

2. Heuristics : how does the prior look like ?

Let

$$h(w) = E_w(X) = -\beta'(w) \mid \beta(w),$$

then it turns out that

$$E_w(X^2) = h^2(w) + h'(w),$$

$$E_w(X^3) = h^3(w) + 3h(w)\, h'(w) + h''(w).$$

Let

$$\gamma(w) = c_0 h'(w) + c_1 h(w) + c_2 h^2(w)$$

be the parametric function to be estimated and

$$\delta(X) = a_0 + a_1 X + a_2 X^2$$

its estimate. Note that $\gamma(w)$ is slightly more general than $\gamma_2(w)$. We assume that the prior is absolutely continuous with respect to the Lebesgue measure on Ω and denote the Radon-Nikodym derivative by $\pi(w)$.

If $\delta(X)$ were to be generalized Bayes with respect to $\pi(w)$ for estimating $\gamma(w)$, then one has

$$\delta(x) = \int \gamma(w) e^{xw} \beta(w)\, \pi(w) dw / \int e^{xw} \beta(w)\, \pi(w) dw,$$

for every x, where the integrals are over $\Omega = (\underline{w}, \bar{w})$.

Let

$$\rho_0(w) = \pi(w) \beta(w), \text{ for } w \in \Omega.$$

Hence,

$$\delta(x) \int e^{xw} \rho_0(w)\, dw = \int \gamma(w) e^{xw}\, \rho_0(w)\, dw$$

$$\Longrightarrow (a_2 x^2 + a_1 x + a_0) \int e^{xw} \rho_0(w) dw = \int \gamma(w) e^{xw}\, \rho_0(w)\, dw. \qquad \ldots \quad (2.1)$$

Now for $\underline{w} \leqslant a < b \leqslant \bar{w}$,

$$\int_a^b \exp(xw)\, \rho_0'(w)\, dw = [e^{xb}\rho_0(b) - e^{xa}\rho_0(a)]$$

$$-x \int_a^b e^{xw}\, \rho_0(w)\, dw \qquad \dots \quad (2.2)$$

$$\int_a^b e^{xw}\rho_0''(w)\, dw = [e^{xb}\rho_0'(b) - e^{xa}\, \rho_0'(a)]$$

$$-x[e^{xb}\, \rho_0(b) - e^{xa}\, \rho_0(a)]$$

$$+x^2 \int_a^b e^{xw}\, \rho_0(w)\, dw. \qquad \dots \quad (2.3)$$

Hence, if the quantities $[e^{x\bar{w}}\, \rho_0(\bar{w}) - e^{x\underline{w}}\, \rho_0(\underline{w})]$ and $[e^{x\bar{w}}\rho_0'(\bar{w}) - e^{x\underline{w}}\, \rho_0'(\underline{w})]$ are zero, then from (2.1) one gets

$$a_2 \int e^{xw}\, \rho_0''(w)\, dw - a_1 \int e^{xw}\, \rho_0'(w)\, dw + a_0 \int e^{xw}\, \rho_0(w)\, dw$$

$$= \int \gamma(w) e^{xw} \rho_0(w)\, dw. \qquad \dots \quad (2.4)$$

Hence, by the uniqueness property of the Laplace transforms, we have

$$a_2\rho_0''(w) - a_1\rho_0'(w) = (\gamma(w) - a_0)\, \rho_0(w),\ \forall w \in (\underline{w}, \bar{w}). \qquad \dots \quad (2.5)$$

It is clear from (2.5) that a solution to it for an arbitrary $\gamma(w)$ is difficult to find out, unless we confine ourselves to linear estimators, as in Ghosh and Meeden (1977). In view of the specific form of $\gamma(w)$ under consideration, we suggest the following form of $\rho_0(w)$, namely,

$$\rho_0(w) = \exp\{d_1 w + d_2 \int h(w)\, dw\}$$

where $h(w)$ is any differentiable function and d_1, $d_2 \neq 0$ are constants to be suitably chosen. Then one has

$$\rho_0'(w) = \{d_1 + d_2 h(w)\}\, \rho_0(w),$$

$$\rho_0''(w) = [\{d_1 + d_2 h(w)\}^2 + d_2 h'(w)]\, \rho_0(w). \qquad \dots \quad (2.6)$$

Hence, from (2.5) and (2.6), we get

$$a_2(d_1^2 + 2d_1 d_2 h + d_2^2\, h^2) + a_2 d_2 h' - a_1(d_1 + d_2 h) + a_0 = c_2 h^2 + c_1 h + c_0 h'. \qquad \dots \quad (2.7)$$

Clearly, the following are sufficient for (2.7) to hold for every w :

$$a_2d_2^2-c_2 = 0, \quad 2a_2d_1d_2-a_1d_2-c_1 = 0,$$
$$a_2d_1^2-a_1d_1+a_0 = 0, \qquad a_2d_2-c_0 = 0. \qquad \ldots \quad (2.8)$$

Henceforth, we shall refer to the above system of equations (2.8) as the 'Consistency conditions'.

3. The main result

Theorem : *Let X have a density given by $p(x, w) = \beta(w)e^{xw}$, with respect to some σ-finite measure μ on the real line. Let the natural parameter space be denoted by*

$$\Omega = \{w : \int e^{xw}d\mu(x) < \infty\} = (\underline{w}, \bar{w}).$$

Consider any differentiable function $h(w)$ on Ω. Let

$$\gamma(w) = c_2h^2(w)+c_1h(w)+c_0h'(w).$$

Let

$$\rho_0(w) = exp\{d_1w+d_2 \int h(w)\, dw\},$$

where $\int h(w)\, dw$ is a primitive of $h(w)$, and $d_1, d_2 \neq 0$ are real numbers. Let

$$\pi(w) = \beta^{-1}(w)\rho_0(w), \text{ for } w \in \Omega.$$

Let

$$\delta(X) = a_2X^2+a_1X+a_0,$$

where a_0, a_1, a_2 are real numbers.

Suppose a_0, a_1, a_2, d_1, d_2 subject the consistency conditions (2.8) are such that

$$\text{(i)} \quad \int_b^{\bar{w}} \pi^{-1}(w)f^{-1}(w)\, dw = \infty, \qquad \text{for } \underline{w} < b < \bar{w},$$

$$\text{(ii)} \quad \int_{\underline{w}}^{c} \pi^{-1}(w)\, f^{-1}(w)\, dw = \infty, \qquad \text{for } \underline{w} < c < \bar{w},$$

where $f(w)$ is as in (3.7). Then $\delta(X)$ is an admissible estimator of $\gamma(w)$ under squared-error loss function.

Proof: Suppose $\delta(X)$ is not admissible. Then there exists another estimator $\delta'(x)$ such that

$$E_w(\delta'(X)-\gamma(w))^2 \leqslant E_w(\delta(X)-\gamma(w))^2 \;\forall\; w,$$

with strict inequality for at least one 'w'. In terms of integrals (3.1) can be written as

$$\int (\delta'(x)-\gamma(w))^2 e^{xw}\beta(w)\, d\mu(x) \leqslant \int (\delta(x)-\gamma(w))^2 e^{xw}\beta(x) d\mu(x),\ \forall\ w \in \Omega \quad \ldots \quad (3.1)$$

$$\Longleftrightarrow (\delta'(x)-\delta(x))^2 e^{xw}\beta(w) d\mu(x) \leqslant 2 \int (\delta(x)-\delta'(x))(\delta(x)-(\gamma w)) e^{xw}\beta(w) d\mu(x),$$
$$\forall\ w \in \Omega. \quad \ldots \quad (3.2)$$

Hence, for $(a, b) \subset (\underline{w}, \overline{w})$,

$$\int_a^b \int [\ (\delta'(x)-\delta(x))^2 e^{xw}\beta(w) d\mu(x)]\pi(w) dw$$

$$\leqslant 2 \int_a^b [\ \int (\delta(x)-\delta'(x))(\delta(x)-\gamma(w))\, e^{xw}\beta(w)\, d\mu(x)]\pi(w)\, dw$$

$$= 2 \int [\delta(x)-\delta'(x)] \left\{ \int_a^b (\delta(x)-\gamma(w))\, e^{xw}\rho_0(w)\, dw \right\} d\mu(x), \quad \ldots \quad (3.3)$$

on changing the order of integration.

The inner integral is

$$\int_a^b (\delta(x)-\gamma(w))\, e^{xw}\rho_0(w) dw = (a_2x^2+a_1x+a_0) \int_a^b e^{xw}\rho_0(w) dw$$

$$-\int_a^b (c_2h^2(w)+c_1h(w)+c_0h'(w)) e^{xw}\rho_0(w) dw. \quad \ldots \quad (3.4)$$

Now using (2.2), (2.3), (2.6) and the consistency conditions (2.8), straightforward computations yield

$$\int_a^b (\delta(x)-\gamma(w)) \exp(xw)\rho_0(w) dw$$
$$= \exp(xb)\rho_0(b)[\alpha_1 x+\alpha_2+\alpha_3 h(b)]$$
$$-\exp(xa)\rho_0(a)[\alpha_1 x+\alpha_2+\alpha_3 h(a)] \quad \ldots \quad (3.5)$$

where $\alpha_1 = a_2,\ \alpha_2 = a_1-a_2d_1,\ \alpha_3 = -a_2d_2$.

Setting

$$T(w) = \int (\delta(x)-\delta'(x))^2 \exp(xw)\beta(w)\, d\mu(x),$$

we have from (3.3), after some standard manipulations (Karlin, 1958)

$$\int_a^b T(w)\pi(w)dw \leqslant K[\sqrt{T(b)\pi(b)}\sqrt{\pi(b)f(b)}+\sqrt{T(a)\pi(a)}\ \sqrt{\pi(a)f(a)} \qquad \ldots \quad (3.6)$$

where K is a constant and

$$f(w) = E_w(\alpha_1 X+\alpha_2+\alpha_3 h(w))^2. \qquad \ldots \quad (3.7)$$

Proceeding now as in Karlin (1958), the proof of the theorem is completed.

Remark 1 : To justify our claim given in the introduction that the technique of proof works for any general $m \geqslant 1$, it is enough to note that it is indeed possible to carry out the same matching procedure as in (2.8) for every $m \geqslant 1$. The reason for this is that, for an arbitrary $m \geqslant 1$, the fundamental equation analogous to (2.5) would involve in the LHS quantities $\rho_0^{(j)}/\rho_0$, for $j = 1, 2, \ldots, m$. A careful analsyis now shows that $\rho_0^{(j)}/\rho_0$ can be expressed as the j-th raw moment of a certain distribution whose first j cumulants are $d_1+d_2k_1, d_2k_2, \ldots ,d_2k_j$ respectively, where k_i's are the cumulants of X. This justifies our claim since $\gamma(w)$ is again a linear combination of the raw moments of X. However, although this matching procedure can be theoretically carried out, by using the well-known relations between moments and cumulants, determining the consistency conditions for an arbitrary m an analogy with (2.8) is algebriacally difficult. In particular, if $m = 3$,

$$\gamma_3(w) = c_0h'(w)+c_1h(w)+c_2h^2(w)+c_3h^3(w)+c_4h(w)h'(w)+c_5h''(w),$$

and

$$\delta_3(X) = a_3X^3+a_2X^2+a_1X+a_0,$$

the consistency conditions reduce to

$$\begin{aligned}
&a_3d_2^3+c_3 = 0,\\
&-3a_3d_1d_2^2+a_2d_2^2-c_2 = 0,\\
&-3a_3d_1^2d_2+2a_2d_1d_2-a_1d_2-c_1 = 0,\\
&-a_3d_1^3+a_2d_1^2-a_1d_1+a_0 = 0,\\
&-3a_3d_1d_2+a_2d_2-c_0 = 0,\\
&3a_3d_2^2+c_4 = 0,\\
&a_3d_2+c_5 = 0.
\end{aligned} \qquad \ldots \quad (3.8)$$

An application of the case $m = 3$ is provided later by using the above equations (vide Example 1(b)).

4. Further remarks and examples

Remark 2 : If h', h, h^2 have an explicit linear relationship, the consistency conditions (2.8) can be reformed from (2.7) by blocking together their coefficients and matching these modified coefficients with the corresponding coefficients in the LHS. In the process, the number of independent equations reduces and extensive control is achieved over coefficients in the estimate, as shown in the examples.

Remark 3 : When $h(w) = w$, $\gamma(w)$ in general (for an arbitrary m) is a polynomial in the natural parameter w. Therefore, according to the theorem, certain polynomial estimators in X are admissible for such functions $\gamma(w)$. However, in all the standard one-parameter exponential set-ups, $\pi(w)$ in this case becomes a proper prior if the "divergence conditions" of the theorem have to hold, thus making the estimators trivially admissible. Specialized to the normal distribution in which case $\gamma(w)$ becomes the expectation of a polynomial in X, our theorem yields only proper Bayes admissible estimators for $m > 1$.

Remark 4 : Our theorem does not provide any quadratic admissible estimate for a completely arbitrary $\gamma(w)$. In this sense our theorem is not strictly a generalization of the result due to Ghosh and Meeden (1977), which however follows trivially from our theorem by putting $a_2 = 0$ in $\delta(X)$, and $c_2 = c_0 = 0$, $c_1 = 1$ in $\gamma(w)$. Note that we need not demand the differentiability of $h(w)$. Since $f(w)$ turns out to be a constant, our conditions are exactly the same as theirs.

Example 4 provides an application of our theorem when $h(w)$ is not the mean.

Remark 5 : It is also possible to prove the theorem by generalizing the Cramér-Rao type inequalities (Blyth, 1974) for Bhattacharya bounds, as done in Roy and Mallik (1966).

Remark 6 : In contrast to the traditional way of forming admissible estimators for a given parametric function, the sufficient conditions of our theorem can also be used to generate a class of parametric functions for which a given polynomial in X is admissible. Each such $\gamma(w)$ is obtained by choosing the c_i's which satisfy the consistency conditions (2.8) and the divergence conditions of the theorem.

Example 1 : Let $X \sim$ Poisson (e^{w}), $-\infty < w < \infty$.

(a) Consider $\gamma(w) = b_2 e^{2w} + b_1 e^{w}$, b_1, b_2 real numbers. If we propose a quadratic estimator, the consistency conditions reduce to

$$\begin{aligned} & a_2 d_2^2 - b_2 = 0 \\ & 2a_2 d_1 d_2 - a_1 d_2 + a_2 d_2 - b_1 = 0 \qquad \ldots \text{(A)} \\ & a_2 d_1^2 - a_1 d_1 + a_0 = 0. \end{aligned}$$

Here,

$$\rho_0(w) = e^{d_1 w} \cdot e^{d_2 e^{w}} \Longrightarrow \pi(w) = e^{d_1 w + (1+d_2)e^{w}}.$$

Also, $f(w)$ is the mean of a quadratic in X; say,

$$f(w) = ae^{2w} + be^{w} + c,$$

where a, b, c are some constants.

Now,

$$\pi^{-1}(w) f^{-1}(w) = \frac{e^{-d_1 w - (1+d_2)e^{w}}}{ae^{2w} + be^{w} + c} \to \frac{1}{c} \text{ as } w \to -\infty$$

$$\to \infty \text{ as } w \to \infty,$$

if $d_1 = 0$, $1 + d_2 < 0$.

It is interesting to note $f(w)$ does not play any role in the divergence of $\int \pi^{-1}(w) f^{-1}(w) dw$. The same phenomenon occurs for any $f(w)$ which is a polynomial in e^{w}. Moreover, if $d_1 = 0$, then $\pi(w) \to 1$ as $w \to \infty$ so that π is improper.

Since $d_1 = 0$, (A) boils down to

$$\begin{aligned} & a_2 d_2^2 - b_2 = 0 \\ & a_2 d_2 - a_1 d_2 - b_1 = 0 \\ & a_0 = 0 \end{aligned}$$

$$\Longrightarrow a_2 = \frac{b_2}{d_2^2}; \quad \text{also } a_1 = a_2 - \frac{b_1}{d_2} = \frac{b_2}{d_2^2} - \frac{b_1}{d_2}.$$

Hence,

$$\delta(X) = \frac{b_2}{d_2^2}(X^2 + X) - \frac{b_1}{d_2} X$$

is an admissible estimator of

$$\gamma(w) = b_2 e^{2w} + b_1 e^{w}, \text{ for any } d_2 < -1.$$

In particular, if $\gamma(w) = e^{2w} = E_w\{X(X-1)\}$, then $a(X^2+X)$ is admissible for any $0 < a < 1$. In contrast to this, Ghosh and Meeden suggested the linear estimator X as an admissible estimator of e^{2w}.

(b) Let

$$\gamma(w) = E_w\{X(X-1)(X-2)\} = e^{3w}.$$

If we let

$$h(w) = E_w(X) = e^w,$$

then

$$h'(w) = h''(w) = h(w), \; \forall w \in (-\infty, \infty).$$

Identifying $\gamma(w)$ as $\gamma_3(w)$ (vide Remark 1) gives

$$c_2+c_4 = 0$$

$$c_1+c_5+c_0 = 0$$

$$c_3 = 1.$$

The consistency conditions (3.8) (vide Remark 1) reduce to

$$a_0 = 0$$

$$a_3 d_2^3 = 1$$

$$3a_3 d_2^2 - a_2 d_2^2 = 0$$

$$a_1 d_2 + a_3 d_2 - a_2 d_2 = 0,$$

when we invoke the divergence condition $d_1 = 0$. Thus

$$a_2 = 3a_3, \; a_1 = 2a_3, \; a_3 = -\frac{1}{d_2^3} < 1 \text{ as } d_2 < -1.$$

Hence, $\delta(X) = a(X^3+3X^2+2X)$ is admissible for $\gamma(w) = e^{3w}$, for any $0 < a < 1$. Again, the estimator suggested by Ghosh and Meeden in this case was X.

Example 2 : Let

$$X \sim \text{Bin}\left(n, \frac{e^w}{1+e^w}\right), \; -\infty < w < \infty.$$

Let

$$h(w) = E_w(X) = \frac{ne^w}{1+e^w}.$$

Then

$$h'(w) = \text{var}_w(X) = h(w) - \frac{h^2(w)}{n}, \quad -\infty < w < \infty \qquad \ldots \text{(A)}$$

Let

$$\gamma(w) = \text{var}_w(X) = h'(w) = -\frac{h^2(w)}{n+\delta} + \frac{nh(w)}{n+\delta} + \frac{\delta h'(w)}{n+\delta},$$

because of (A), where $0 < \delta < 1$.

In this case

$$\pi(w) = e^{d_1 w}(1+e^w)^{n(1+d_2)}$$

$f(w)$, as in Example 1, is the mean of a certain quadratic in X say,

$$f(w) = a\left(\frac{e^w}{1+e^w}\right)^2 + b\left(\frac{e^w}{1+e^w}\right) + c$$

for some constants a, b, c.

Hence,

$$\pi^{-1}(w)f^{-1}(w) = \frac{e^{-d_1 w}(1+e^w)^{-n(1+d_2)}}{a\left(\frac{e^w}{1+e^w}\right)^2 + b\left(\frac{e^w}{1+e^w}\right) + c} \begin{array}{l} \to \frac{1}{c} \text{ as } w \to -\infty \\ \to \infty \text{ as } w \to \infty, \end{array}$$

if we let

$$d_1 = 0, \; 1+d_2 < 0.$$

Also, $\pi(w) \to 1$ as $w \to -\infty$, implying π is improper. The consistency conditions are

$$a_2 d_2^2 = -\frac{1}{n+\delta}, \quad a_2 d_2 = \frac{\delta}{n+\delta},$$

$$a_1 d_2 = -\frac{n}{n+\delta}, \quad a_0 = 0.$$

Hence,

$$d_2 = -\frac{1}{\delta} < -1 \text{ (as } 0 < \delta < 1).$$

Also,

$$a_2 = -\frac{\delta^2}{n+\delta}, \text{ and, } a_1 = \frac{-n/(n+\delta)}{-1/\delta} = \frac{n\delta}{n+\delta}.$$

Thus

$$\delta(X) = -\frac{\delta^2}{n+\delta} X^2 + \frac{n\delta}{n+\delta} X$$

$$= \frac{\delta}{n+\delta}(nX - \delta X^2)$$

is admissible for $\text{var}_w(X)$ for every $0 < \delta < 1$.

Incidentally, no linear estimator of $\text{var}_w(X)$ satisfies the sufficient condition of Ghosh and Meeden (1977).

Example 3 : Let X have the negative binomial distribution given by

$$P(X = x) = \binom{r+x-1}{x} p^r q^x, \; x = 0, 1, 2, \ldots, r > 1 \text{ known}, 0 < p < 1.$$

In other words,

$$X \sim NB(e^w), \; -\infty < w < 0.$$

Then

$$h(w) = E_w(X) = \frac{re^w}{1-e^w},$$

$$h'(w) = \text{var}_w(X) = \frac{re^w}{(1-e^w)^2}.$$

It turns out that

$$h'(w) = h(w) + \frac{h^2(w)}{r}.$$

Let

$$\gamma(w) = \text{var}_w(X) = h'(w).$$

Hence,

$$\gamma(w) = c_2 h^2(w) + c_1 h(w) + c_0 h'(w)$$

$$= \left(c_2 - \frac{c_1}{r}\right) h^2(w) + (c_1 + c_0) h'(w),$$

where

$$c_2 - \frac{c_1}{r} = 0, \; c_1 + c_0 = 1. \qquad \dots \text{(A)}$$

Here,

$$\pi(w) = e^{d_1 w}.(1-e^w)^{-r(1+d_2)}.$$

Hence,

$$\pi^{-1}(w) f^{-1}(w) = \frac{(1-e^w)^{r(1+d_2)} e^{-d_1 w}.}{a \frac{re^w}{1-e^w} + b \frac{re^w}{(1-e^w)^2} + c \frac{re^{2w}}{(1-e^w)^2} + d}.$$

It turns out from (3.4) and (3.6) that

$$b = a_2^2 \neq 0, \; c = a_2^2(1-d_2^2) \neq 0.$$

If we let $d_1 = 0$, and $r(1+d_2)+2 < 0$, then,

$$\pi^{-1}(w) f^{-1}(w) \to \frac{1}{d} \text{ as } w \to -\infty$$

$$\to \infty \text{ as } w \to 0.$$

Also, $\pi(w) \to 1$ as $w \to \infty$, whence π is improper.

By (A), the consistency conditions reduce to

$$a_2 d_2^2 + \frac{a_1 d_2}{r} = 0, \; a_2 d_2 - a_1 d_2 = 1, \; a_0 = 0.$$

Hence,

$$a_2 = -\frac{a_1}{rd_2} \Longrightarrow -a_1 \left(\frac{1}{rd_2} + 1 \right)$$

$$= \frac{1}{d_2} \Longrightarrow a_1 = -\frac{r}{1+rd_2} > 0.$$

Since $d_2 < -1 - \frac{2}{r}$, it turns out that $a_1 < \frac{r}{r+1}$.

Also,

$$a_2 = a_1 + \frac{1}{d_2} = \frac{1}{d_2(1+rd_2)} < \frac{r}{(r+1)(r+2)}.$$

Hence, $\delta(X) = a_2X^2+a_1X$ is admissible for $\text{var}_w(X)$ for every

$$0 < a_2 < \frac{r}{(r+1)(r+2)}, \quad 0 < a_1 < \frac{r}{r+1}.$$

Example 4 : In the preceding examples, d_1 was taken as zero and $h(w)$ was always taken as $E_w(X)$. Here is an example where neither d_1 is 0 nor $h(w)$ is the mean of the distribution.

Let

$$X \sim \text{Bin}\left(n, \frac{e^w}{1+e^w}\right), \quad -\infty < w < \infty.$$

Let $\gamma(w) = e^{2w}$ (which is the square of the ratio $\frac{p}{q}$ in the usual Binomial (n, p) set-up). Thus with

$$h(w) = e^w, \ \gamma(w) = c_2e^{2w}+c_1e^w+c_0e^w,$$

where $c_2 = 1$, $c_1+c_0 = 0$.

We let

$$d_1 = -1 \text{ and } d_2 < 0.$$

Then,

$$\pi(w) = \beta^{-1}(w)\rho_0(w) = \frac{(1+e^w)^n}{e^w} \cdot e^{d_2e^w} \to \infty \text{ as } w \to -\infty,$$

implying π is improper.

Also,

$$f(w) = E_w(a_2X-a_2d_2e^w)^2 \text{ (if we let } a_1-a_2d_1 = a_1+a_2 = 0)$$

$$= a_2^2E_w(X-d_2e^w)^2 = a_2^2\left[\frac{ne^w}{(1+e^w)^2}+\frac{n^2e^{2w}}{(1+e^w)^2}-2nd_2\frac{e^{2w}}{1+e^w}+d_2^2e^{2w}\right]$$

$$= \frac{a_2^2}{(1+e^w)^2}\,[d_2^2e^{4w}-(2nd_2-2d_2^2)e^{3w}+(n-d_2)^2e^{2w}+ne^w]. \qquad \ldots \text{ (A)}$$

Hence,

$$\pi^{-1}(w)f^{-1}(w) = \frac{1}{a_2^2}\cdot\frac{e^w.e^{-d_2e^w}}{[d_2^2e^{4w}-2d_2(n-d_2)e^{3w}+(n-d_2)^2e^{2w}+ne^w]}$$

$$\to \infty \text{ as } w \to \infty$$

$$\to \frac{1}{na_2^2} \text{ as } w \to -\infty.$$

Thus the sufficient conditions of our theorem hold. The coefficients a_2, a_1, a_0 are obtained by using the consistency conditions and (A) as

$$a_2 = \frac{1}{d_2^2}, \quad a_1 = -\frac{1}{d_2^2}, \quad a_0 = 0.$$

Hence, $\delta(X) = a(X^2 - X)$ is admissible for $\gamma(w) = e^{2w}$ for every $a > 0$.

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