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A PROPERTY OF MAXIMUM LIKELIHOOD ESTIMATOR

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SUMMARY. Roughly speaking our object in this note is to prove that under standard regularity conditions, with probability tending to one, the maximum ikelihood estimate lies in $100(1-\alpha)\%$ confidence set $(0 < \alpha < 1)$ determined by the family of locally most powerful unbiased tests of $H_0(\theta = \theta_0)$ vs $H_1(\theta \neq \theta_0)$; a sort of converse is also proved.

1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of i.i.d r.v.'s with a common d.f. $F_\theta(x), \theta \varepsilon \Theta$; Θ is an open subset of R. Let $f(x, \theta)$ be the density of $F_\theta(x)$ w.r.t. some dominating measure μ.

We assume $f(x, \theta)$ satisfies the regularity assumptions I to VI of the next section.

Roughly speaking our object in this note is to prove that under these conditions, with probability tending to one, the maximum likelihood estimate (m.l.e) lies in the $100(1-\alpha)\%$ confidence set $(0 < \alpha < 1)$ determined by the family of locally most powerful unbiased tests (LMPU tests) of $H_0(\theta = \theta_0)$ vs. $H_1(\theta \neq \theta_0)$; a sort of converse is also proved. A more precise statement is presented later.

We now proceed to a precise formulation of our result.

Our assumptions guarantee (see Lehmann, 1959, p. 83) the existence of a LMPU test of $H_0(\theta = \theta_0)$ vs. $H_1(\theta \neq \theta_0)$ with critical function

$$\phi_{\theta_0} = \begin{cases} 1 & \text{if } W_{n\theta_0} + Z^2_{n\theta_0} > K_{1n\theta_0} + K_{2n\theta_0} Z_{n\theta_0} \\ 0 & \text{if } \quad ,, \quad < \quad ,, \\ \text{arbitrary} & \text{if } \quad ,, \quad = \quad ,, \end{cases}$$

where

$$Z_{n\theta_0} = n^{-\frac{1}{2}} I^{-\frac{1}{2}}(\theta_0) \sum_{1}^{n} \frac{d}{d\theta} \log f(x_i, \theta_0),$$

$$W_{n\theta_0} = n^{-1} I^{-1}(\theta_0) \sum_{1}^{n} \frac{d^2}{d\theta^2} \log f(x_i, \theta_0),$$

and $K_{1n\theta_0}$ and $K_{2n\theta_0}$ are such that

$$E_{\theta_0}(\phi_{\theta_0}) = \alpha \text{ and } E_{\theta_0}(\phi_{\theta_0} Z_{n\theta_0}) = 0.$$

Let V_n be the randomized confidence set arising from this family of tests i.e. it consists of all θ accepted by the test ϕ_θ. The set V_n will depend on the randomising device in addition to $X_1, X_2, \ldots, X_n$ but will contain

$$\omega_n = \{\theta : W_{n\theta} + Z_{n\theta}^2 < K_{1n\theta} + K_{2n\theta} Z_{n\theta}\}.$$

Similarly

$$V_n \subset \{\theta : W_{n\theta} + Z_{n\theta}^2 \leqslant K_{1n\theta} + K_{2n\theta} Z_{n\theta}\} = \omega_n' \text{ (say)}.$$

Now we state our result. Let $\hat{\theta}_n$ denote the maximum likelihood estimate.

Theorem : *Under assumptions* I *to* VI

(a) *For every* $\theta_0 \in \Theta$ *and for every* $0 < \alpha < 1$, $P_{\theta_0}\{\hat{\theta}_n \in V_n\} \to 1$ *as* $n \to \infty$.

(b) *Let* T_n *be any other estimate of* θ; *then for* $\theta_0 \in \Theta$, $P_{\theta_0}\{T_n \in V_n\} \to 1$ *for every* $0 < \alpha < 1$ *if* $\sqrt{n}(\hat{\theta}_n - T_n) \xrightarrow{P_{\theta_0}} 0$.

(c) *Let* T_n *be any consistent estimate of* θ *such that for* $\theta_0 \in \Theta$, $P_{\theta_0}\{T_n \in V_n\} \to 1$ *for every* $0 < \alpha < 1$. *Then* $\sqrt{n}(\theta_n - T_n) \xrightarrow{P_{\theta_0}} 0$.

Remark : If instead of the randomized confidence set V_n one of the nonrandomized confidence sets ω_n or ω_n' be used, the resultant size of the test will be $\alpha_n(\theta)$ which will eventually be α as $n \to \infty$ for every $\theta \in \Theta$ (vide proof of Lemma 3). The theorem remains true if V_n is replaced by ω_n or ω_n' throughout. This is so because the proof of (a) and (b) uses $\{\hat{\theta}_n \in \omega_n\}$ and the proof of (c) uses $\{\hat{\theta}_n \in \omega_n'\}$.

The proof of the theorem is deferred to Section 3. In Section 2 the assumptions are stated and some auxiliary results are proved.

2. Assumptions and lemmas

Assumption I : For each x, $f(x, \theta)$ is twice continuously differentiable in $\theta \in \Theta$.

Assumption II : Let

$$I(\theta) = E_\theta \left[-\frac{d^2}{d\theta^2} \log f(x, \theta) \right];$$

then $0 < I(\theta) < \infty$ for $\theta \in \Theta$ and $I(\theta)$ is continuous in $\theta \in \Theta$.

Assumption III : For every $\theta_0 \in \Theta$, $\exists$ a neighbourhood (nhbd) C_{θ_0} of θ_0 such that

$$\sup_{\theta \in C_{\theta_0}} E_\theta \left| \frac{d}{d\theta} \log f(X, \theta) \right|^3 < \infty.$$

Assumption IV : For every $\theta_0 \in \Theta$, $\exists$ a nhbd C_{θ_0} of θ_0 such that

$$\left| \frac{d^2}{d\theta^2} \log f(x, \theta) \right| \leqslant H(x), \ \forall \theta \in C_{\theta_0}$$

$$\left| \frac{d^2}{d\theta^2} \log f(x, \theta) - \frac{d^2}{d\theta^2} \log f(x, \theta') \right| \leqslant |\theta - \theta'| \, A(x)$$

for $\forall \theta, \theta' \in C_{\theta_0}$; and for some $\delta > 0$

$$\sup_{\theta \in C_{\theta_0}} E_\theta H^{2+\delta}(X) < \infty, \quad \sup_{\theta \in C_{\theta_0}} E_\theta A(X) < \infty.$$

Assumption V : If ϕ_n is any test function based on n observations then $E_\theta \phi_n$ is twice continuously differentiable in $\theta \in \Theta$; moreover

$$\frac{d}{d\theta} E_\theta \phi_n(X_1 \ldots X_n) = \int \phi_n(x_1 \ldots x_n) \frac{d}{d\theta} \prod_{i=1}^{n} f(x_i, \theta) d\mu(x_1 \ldots x_n)$$

$$\frac{d^2}{d\theta^2} E_\theta \phi_n(X_1 \ldots X_n) = \int \phi_n(x_1 \ldots x_n) \frac{d^2}{d\theta^2} \prod_{i=1}^{n} f(x_i \theta) d\mu(x_1 \ldots x_n)$$

for every $\theta \in \Theta$ and $n \geqslant 1$.

Assumption VI : The maximum likelihood estimate (mle) $\hat{\theta}_n$ of θ exists and for every $\theta_0 \epsilon \Theta$ and $\epsilon > 0$.

$$P_{\theta_0}\left\{|\hat{\theta}_n - \theta_0| < \epsilon, \frac{d}{d\theta} \log \prod_{i=1}^{n} f(x_i, \hat{\theta}_n) = 0\right\} \to 1 \text{ as } n \to \infty.$$

Remark : VI holds if conditions of Wald (1949) or Bahadur (1971, p. 34) hold.

We quote a lemma of Ghosh, Sinha and Wieand (1980) to be used later.

Lemma 1 : *Let C be a compact interval and let $U(x, t)$ be a real valued function measurable in x for each $t \epsilon C$ and continuous in t for each x. Let $X_1, X_2, \ldots$ be a sequence of i.i.d r.v s having a common d.f. F_θ, $\theta \epsilon \Theta$ and $H(x)$ and $A(x)$ be measurable functions such that $|U(x, t)| \leqslant H(x)$ for $t \epsilon C$ $|U(x, t) - U(x, t')| \leqslant |t - t'| A(x)$ for $t, t' \epsilon C$, and for some $\delta > 0$*

$$\sup_{\theta \epsilon \Theta} E_\theta H^{2+\delta}(X) < \infty, \quad \sup_{\theta \epsilon \Theta} E_\theta A(X) < \infty.$$

Then for any $\epsilon > 0 \; \exists \; n_0$ and $K (0 < K < \infty)$ such that

$$P_\theta\left\{\sup_{t \epsilon C} \left| n^{-1} \sum_{1}^{n} U(X_i, t) - E_\theta U(X_1, t)\right| < \epsilon\right\} \geqslant 1 - K n^{-\delta_2}, \; \forall \theta \epsilon \Theta$$

and $\forall \; n \geqslant n_0$ and some $\delta_2 > 0$.

Note : The assumptions II and IV enable us to apply lemma 1 to $W_{n\theta}$.

We also quote a version of Theorem 3 of Michel (1976) which will be needed in the sequel.

Lemma 2 : *Let $X_1, X_2, \ldots,$ be a sequence of i.i.d r.v's having a common d.f. F_θ, $\theta \epsilon \Theta$ such that $E_\theta(X_1) = 0$, $E_\theta(X_1^2) = 1$. If for some $\delta > 0$*

$$\sup_{\theta \epsilon \Theta} E_\theta |X_1|^{2+\delta} < \infty$$

then there exists a constant f such that for $n \geqslant 1$, $\forall \; \theta \epsilon \Theta$ and for all $t \epsilon R$,

$$|F_{n\theta}(t) - \Phi(t)| \leqslant f n^{-\delta^*}[1 + |t|^{2+\delta}]^{-1}$$

where $F_{n\theta}$ *is the d.f. of* $n^{-\frac{1}{2}}\sum_{1}^{n} X_i$ *under* F_θ. $\Phi(t) = \int_{-\infty}^{t} \frac{exp(-x^2/2)}{\sqrt{2\pi}}\,dx$ *and* $\delta^* = \frac{1}{2}\,min\,(\delta, 1)$.

Note: The assumptions II and III enable us to apply Lemma 2 to the d.f. of $Z_{n\theta}$.

Let

$$X^{(n)} = (X_1, \ldots, X_n),$$

$$R_{n\theta} = \{X^{(n)} : Z^2_{n\theta} + W_{n\theta} > K_{1n\theta} + K_{2n\theta} Z_{n\theta}\},$$

$$R'_{n\theta} = \{X^{(n)} : Z^2_{n\theta} + W_{n\theta} = K_{1n\theta} + K_{2n\theta} Z_{n\theta}\},$$

$$\widetilde{R}_{n\theta} = \{X^{(n)} : Z^2_{n\theta} \geqslant K_{1n\theta} + K_{2n\theta} Z_{n\theta} + 1\}$$

and $A \Delta B = (A^C \cap B) \cup (A \cap B^C)$ for any two sets A and B, C being the usual notation for complement.

We fix a $\theta_0 \epsilon \Theta$ and a bounded open set Ω containing θ_0, $\Omega \subset \Theta$ such that assumptions III and IV hold on the closure of Ω.

Lemma 3: *Uniformly in* $\theta \epsilon \Omega$

$$P_\theta\{\widetilde{R}_{n\theta}\} \to \alpha \qquad \ldots \quad (2.1)$$

and

$$E_\theta[I_{\widetilde{R}_{n\theta}} Z_{n\theta}] \to 0. \qquad \ldots \quad (2.2)$$

Proof: Note

$$|\, E_\theta[I_{\widetilde{R}_{n\theta}} Z_{n\theta}] - E_\theta[\phi_\theta Z_{n\theta}] \,| \leqslant P_\theta^{\frac{1}{2}}(R_{n\theta} \Delta \widetilde{R}_{n\theta}) + P_\theta^{\frac{1}{2}}(R'_{n\theta})$$

and

$$|\, P_\theta(\widetilde{R}_{n\theta}) - E_\theta \phi_\theta \,| \leqslant P_\theta(R_{n\theta} \Delta \widetilde{R}_{n\theta}) + P_\theta(R'_{n\theta}).$$

Hence (2.1) and (2.2) are proved if we prove uniformly in

$$\theta \epsilon \Omega,\ P_\theta(R_{n\theta} \Delta \widetilde{R}_{n\theta}) \to 0 \text{ and } P_\theta(R'_{n\theta}) \to 0.$$

In view of assumptions II and IV it is clear that for every $\epsilon > 0$

$$P_\theta(|\, W_{n\theta} + 1 \,| \leqslant \epsilon) \to 1 \text{ uniformly in } \theta \epsilon \Omega. \qquad \ldots \quad (2.3)$$

Note that,

$$A_{n,\epsilon,\theta} \equiv \left\{X^{(n)} : \frac{K^2_{2n\theta}}{4}+K_{1n\theta}+1-\epsilon \leqslant \left(Z_{n\theta}-\frac{K_{2n\theta}}{2}\right)^2 \leqslant \frac{K^2_{2n\theta}}{4}+K_{1n\theta}+1+\epsilon\right\}$$

$$\supset \{R_{n\theta}\Delta\widetilde{R}_{n\theta}\} \cap \{|W_{n\theta}+1| \leqslant \epsilon\}. \qquad \ldots \quad (2.4)$$

Also,

$$A_{n,\epsilon,\theta} \supset (|W_{n\theta}+1| \leqslant \epsilon) \cap R'_{n\theta}.$$

On the other hand

$$A_{n,\epsilon,\theta} \subset \left\{X^{(n)} : \left(Z_{n\theta}-\frac{K_{2n\theta}}{2}\right)^2 \leqslant 2\epsilon\right\} \text{ if } \frac{K^2_{2n\theta}}{4}+K_{1n\theta}+1 \leqslant \epsilon \quad \ldots \quad (2.5)$$

and

$$A_{n,\epsilon,\theta} = \left\{X^{(n)} : (x-\epsilon)^{\frac{1}{2}}+\frac{K_{2n\theta}}{2} \leqslant Z_{n\theta} \leqslant \frac{K_{2n\theta}}{2}+(x+\epsilon)^{\frac{1}{2}}\right\}$$

$$\cup \left\{X^{(n)} : \frac{K_{2n\theta}}{2}-(x+\epsilon)^{\frac{1}{2}} \leqslant Z_{n\theta} \leqslant \frac{K_{2n\theta}}{2}-(x-\epsilon)^{\frac{1}{2}}\right\}$$

if $x = \frac{K^2_{2n\theta}}{4}+K_{1n\theta}+1 \geqslant \epsilon$.

Now use the Berry-Essen theorem for $Z_{n\theta}$ along with assumption III.

Since $\alpha < 1$, from Lemma 2 and (2.1) it is clear that $\exists\, n_0$ such that

$$\frac{K^2_{2n\theta}}{4}+K_{1n\theta}+1 \geqslant 0 \; \forall n \geqslant n_0, \; \forall \theta \in \Omega.$$

Let

$$C_{1n\theta} = \frac{K_{2n\theta}}{2}+\left(\frac{1}{4}K^2_{2n\theta}+K_{1n\theta}+1\right)^{\frac{1}{2}}$$

and

$$C_{2n\theta} = \frac{1}{2}\, K_{2n\theta}-\left(\frac{1}{4}\, K^2_{2n\theta}+K_{1n\theta}+1\right)^{\frac{1}{2}}$$

for $n \geqslant n_0$.

Lemma 4 : *Let Z be normal with zero mean and unit variance. Then uniformly in $\theta \epsilon \Omega$*

$$E[I_{(C_{2n\theta} < Z < C_{1n\theta})}] \rightarrow 1-\alpha \qquad \dots \quad (2.6)$$

and

$$E[I_{(C_{2n\theta} < Z < C_{1n\theta})}Z] \rightarrow 0. \qquad \dots \quad (2.7)$$

Proof : Note for $n \geqslant n_0$, $\widetilde{R}_{n\theta} = (Z_{n\theta} \leqslant C_{2n\theta}) \cup (Z_{n\theta} \geqslant C_{1n\theta})$ and hence $P_\theta(C_{2n\theta} \leqslant Z_{n\theta} \leqslant C_{1n\theta}) \rightarrow 1-\alpha$ uniformly in $\theta \epsilon \Omega$ by (2.1). The Berry-Essen theorem along with assumption III completes the proof of (2.6).

Let $F_{n\theta}$ be the d.f. of $Z_{n\theta}$. For a d.f. $F(z)$ we have

$$\int_{C_{2n\theta}}^{C_{1n\theta}} z\, dF(z) = C_{1n\theta}F(C_{1n\theta})-C_{2n\theta}F(C_{2n\theta})- \int_{C_{2n\theta}}^{C_{1n\theta}} F(z)dz.$$

Hence we have, with $\Phi(z) = P(Z \leqslant z)$,

$$\left| \int_{C_{2n\theta}}^{C_{1n\theta}} z\, d\, F_{n\theta}(z)- \int_{C_{2n\theta}}^{C_{1n\theta}} zd\Phi(z) \right|$$

$$\leqslant |C_{1n\theta}|\,|F_{n\theta}(C_{1n\theta})-\Phi(C_{1n\theta})|$$

$$+|C_{2n\theta}|\,|F_{n\theta}(C_{2n\theta})-\Phi(C_{2n\theta})|$$

$$+ \int_{C_{2n\theta}}^{C_{1n\theta}} |F_{n\theta}(z)-\Phi(z)|\,dz. \qquad \dots \quad (2.8)$$

Lemma 2 applied to $F_{n\theta}$ implies,

$$\text{R.H.S. of (2.8)} \leqslant bn^{-1}\{|C_{1n\theta}|(1+|C_{1n\theta}|^3)^{-1}+|C_{2n\theta}|(1+|C_{2n\theta}|^3)^{-1}$$

$$+ \int_{C_{2n\theta}}^{C_{1n\theta}} (1+|t|^3)^{-1}dt\} \text{ for some } b > 0.$$

Lemma 5 : $K_{2n\theta} \rightarrow 0$ *and* $K_{1n\theta} \rightarrow \xi^2_{\alpha/2}-1$ *both uniformly in* $\theta \epsilon \Omega$, *where* $\Phi(\xi_\alpha) = 1-\alpha$.

Proof : Note that $0 < \alpha < 1$ and (2.6) imply existence of n_0, $0 < M < \infty$ and $\delta > 0$ such that

$$C_{1n\theta}-C_{2n\theta} > \delta, \ \forall\, n \geqslant n_0, \ \forall\, \theta \epsilon \Omega \qquad \dots \quad (2.9)$$

$$\min(|C_{1n\theta}|,\ |C_{2n\theta}|) < M, \ \forall\, n \geqslant n_0, \ \forall\, \theta \epsilon \Omega. \qquad \dots \quad (2.10)$$

Hence $\exists\ 0 < M' < \infty$ such that

$$\max\left(e^{-C^2_{1n\theta}/2},\ e^{-C^2_{2n\theta}/2}\right) > M',\ \forall\ n \geqslant n_0,\ \forall\ \theta \in \Omega. \qquad \ldots \quad (2.11)$$

Using (2.7) we get

$$\left| \int_{C_{2n\theta}}^{C_{1n\theta}} z.e^{-\frac{z^2}{2}}\, dz \right| = e^{-C^2_{1n\theta}/2} \left| 1-e^{\frac{1}{2}(C_{1n\theta}+C_{2n\theta})(C_{1n\theta}-C_{2n\theta})} \right|$$

$$= e^{-C^2_{2n\theta}/2}\ \left| 1-e^{-\frac{1}{2}(C_{1n\theta}+C_{2n\theta})(C_{1n\theta}-C_{2n\theta})} \right|$$

$$\rightarrow 0 \text{ uniformly in } \theta \in \Omega. \qquad \ldots \quad (2.12)$$

(2.12) along with (2.11) implies

$$K_{2n\theta} = (C_{1n\theta}+C_{2n\theta}) \rightarrow 0 \text{ uniformly in } \theta \in \Omega. \qquad \ldots \quad (2.13)$$

To prove the second part, note that if for every $n \exists M > n$ and $\theta' \in \Omega$ such that $C_{2n\theta'}$ and $C_{1n\theta'}$ are on the same side of zero then we get a contradiction to (2.7) using (2.9) and (2.10). Hence $\exists\ n_0$ such that

$$C_{2n\theta} \leqslant 0 \leqslant C_{1n\theta},\ \forall\ n \geqslant n_0,\ \forall\ \theta \in \Omega.$$

Now

$$2 \int_0^{C_{1n\theta}} d\Phi(z) = \left(\int_0^{C_{1n\theta}} d\Phi(z) + \int_0^{-C_{2n\theta}} d\Phi(z) \right)$$

$$+ \left(\int_0^{C_{1n\theta}} d\Phi(z) - \int_0^{-C_{2n\theta}} d\Phi(z) \right).$$

$$\rightarrow 1-\alpha \text{ uniformly in } \theta \in \Omega$$

because of (2.6) and (2.13); hence in view of (2.13) we have

$$K_{1n\theta} \rightarrow \frac{\zeta_\alpha^2}{2} - 1 \text{ uniformly in } \theta \in \Omega.$$

3. Proof of the theorem

Choose $\delta > 0$ such that $\{|\theta_0-\theta| < \delta\} \subset \Omega$. By assumption VI for any $\eta > 0 \ \exists\ n_0$ such that

$$P_{\theta_0}\{|\hat\theta_n-\theta_0| < \delta,\ Z_{n,\hat\theta_n} = 0\} \geqslant 1-\eta,\ n \geqslant n_0. \qquad \ldots \quad (3.1)$$

Choose $\varepsilon > 0$ such that $\xi^2_{\alpha/2} > 2\varepsilon$ and use Lemma 5 to get n_0 such that

$$K_{1n\theta} > \xi^2_{\alpha/2} - 1 - \varepsilon > -1 + \varepsilon,\ n \geqslant n_0,\ \forall \theta \in \Omega. \qquad \dots \quad (3.2)$$

Using Lemma 1 we get n_0 such that

$$P_{\theta_0}\{ \sup_{\theta \in \Omega} W_{n\theta} \leqslant -1+\varepsilon\} \geqslant 1-\eta,\ \forall n \geqslant n_0. \qquad \dots \quad (3.3)$$

Note that

$$\theta_n \in \omega_n \text{ iff } W_{n\hat{\theta}_n} < K_{1n\hat{\theta}_n}. \qquad \dots \quad (3.4)$$

Combining (3.1), (3.2), (3.3) and (3.4) we get part (a) of the theorem.

Under hypothesis of part (b) or (c) we have

$$P_{\theta_0}\{T_n \in \Omega\} \to 1. \qquad \dots \quad (3.5)$$

This along with Lemmas 2 and 3 implies

$$K_{2nT_n} \overset{P_{\theta_0}}{\to} 0,\ K_{1nT_n} \overset{P_{\theta_0}}{\to} \xi^2_{\frac{1}{2}\alpha} - 1 \text{ and } W_{nT_n} \overset{P_{\theta_0}}{\to} 1, \qquad \dots \quad (3.6)$$

so that

$$K_{1nT_n} - W_{nT_n} + \frac{1}{4} K^2_{2nT_n} \overset{P_{\theta_0}}{\to} \xi^2_{\frac{1}{2}\alpha} > 0,\ 0 < \alpha < 1. \quad \dots \quad (3.7)$$

Now expanding Z_{nT_n} around θ_n and noting $Z_{n\hat{\theta}_n} = 0$, we have

$$\{T_n \in \omega_n\} = \{W_{nT_n} + Z^2_{nT_n} < K_{1nT_n} + K_{2nT_n} Z_{nT_n}\}$$

$$= \Big[\Big\{ \sqrt{n}(T_n - \theta_n) \cdot \frac{1}{n} \cdot \Sigma \frac{d^2}{d\theta^2} \log f(x_i, \theta_n^*) \Big\}^2 I^{-1}(T_n)$$

$$- K_{2nT_n} \Big\{ \sqrt{n}(T_n - \theta_n) \cdot \frac{1}{n} \cdot \Sigma \frac{d^2}{d\theta^2} \log f(x_i, \theta_n^*) \Big\} I^{-\frac{1}{2}}(T_n)$$

$$+ W_{nT_n} - K_{1nT_n} < 0 \Big] \text{ where } \theta_n^* \text{ is between } \theta_n \text{ and } T_n$$

$$= \Big\{ [\sqrt{n}(T_n - \theta_n) - \tfrac{1}{2} K_{2nT_n} W^{-1}_{n\hat{\theta}_n^*} I^{-1}(\theta_n^*) I^{\frac{1}{2}}(T_n)]^2$$

$$< W^{-2}_{n\hat{\theta}_n^*} I^{-2}(\theta_n^*) I(T_n) \Big[K_{1nT_n} - W_{nT_n} + \frac{K^2_{2nT_n}}{4} \Big] \Big\}.$$

Observe that by (3.5) and Lemma 1,

$$W_{n\hat{\theta}_n^*} \cdot I^{-\frac{1}{2}}(\theta_n^*)I(T_n) \qquad \dots \quad (3.8)$$

is bounded away from zero and infinity with probability tending to one.

Let

$$\Sigma_n = \Big\{ W^{-1}_{n\hat{\theta}_n^*}\, I^{-1}(\theta_n^*)I^{\frac{1}{2}}(T_n) \Big[\frac{K_{2nT_n}}{2} - \Big(K_{1nT_n} - W_{nT_n} + \frac{K^2_{2nT_n}}{4}\Big)^{\frac{1}{2}} \Big]$$

$$< \sqrt{n}(T_n - \theta_n) < W^{-1}_{n\hat{\theta}_n^*}\, I^{-1}(\theta_n^*)I^{\frac{1}{2}}(T_n)\Big[\frac{K_{2nT_n}}{2} + \Big(K_{1nT_n} - W_{nT_n}$$

$$+ \frac{K^2_{2nT_n}}{4}\Big)^{\frac{1}{2}} \Big]\Big\} \text{ and } K_{1nT_n} - W_{nT_n} + \frac{K^2_{2nT_n}}{4} > 0. \qquad \dots \quad (3.9)$$

Let Σ_n' denote the set with strict inequalities replaced by " $\leqslant$ " in (3.9).

If $P_{\theta_0}\{T_n \,\epsilon\, \omega_n'\} \to 1$ for each $0 < \alpha < 1$ then, it is clear from (3.7) that $P_{\theta_0}\{\Sigma_n'\} \to 1$ for each $0 < \alpha < 1$. This in view of (3.6), (3.7), (3.8) and the fact $\xi_{\alpha/2} \to 0$ as $\alpha \to 1$ gives us part (c) of the theorem.

Finally, if $\sqrt{n}(\theta_n - T_n) \overset{P_{\theta_0}}{\to} 0$, clearly $P_{\theta_0}(\Sigma_n) \to 1$ for every $0 < \alpha < 1$ and hence $P_{\theta_0}(T_n \,\epsilon\, \omega_n) \to 1$ for every $0 < \alpha < 1$.

Since consistency of T_n was used only to derive (3.5), we have the following

Corollary : *Suppose* $P_{\theta_0}\{T_n \,\epsilon\, C_{\theta_0}\} \to 1$ *and* $P_{\theta_0}\{T_n \,\epsilon\, V_n)\} \to 1$, *for all* $\theta_0 \,\epsilon\, \Theta$. *Then* $\sqrt{n}(T_n - \theta_n) \overset{P_{\theta_0}}{\to} 0$, $\forall \theta_0 \,\epsilon\, \Theta$.

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