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ON SOME PROPERTIES OF THE GEOMETRIC DISTRIBUTION

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SUMMARY. In this note, a necessary and sufficient condition for the n-divisibility of a random variable (r.v.) with support contained in $\{0, 1, 2, \ldots\}$ is given. Next a characterization of the geometric distribution is obtained. Also bounds for $P(X = k+1)$ are given when the distribution of an infinitely divisible r.v. X is known to coincide with the geometric distribution at the points $0, 1, \ldots, k$.

1. A necessary and sufficient condition

Let X be a r.v. taking values in $\{0, 1, 2, \ldots\}$ and let $g(t)$ be its probability generating function (p.g.f.). Then we have the following result.

Theorem 1 : *In order that a r.v. X taking values in $\{0, 1, 2, \ldots\}$ with the probability distribution $\{P_k\}$, $P_0 > 0$ is n-divisible, it is necessary and sufficient that there exists a sequence $\{\pi_k\}$ of non-negative numbers satisfying*

$$\pi_0 = P_0^{1/n}, \quad \pi_1 = n^{-1} P_1 P_0^{(1-n)/n},$$

$$nP_0\pi_k = P_k\pi_0 - \sum_{r=1}^{k-1} n\{1-r(1+n)/kn\}P_r\pi_{k-r}, \quad k \geq 2. \qquad \ldots \quad (1)$$

Proof : Clearly X is n-divisible if and only if $[g(t)]^{1/n} = h_n(t)$ is a p.g.f. Taking logarithmic derivatives we have $h_n g' = ngh_n'$. Writing $h_n(t) = \Sigma t^k \pi_k(n)$ and equating the coefficients we get the required result.

Remark 1 : It is easily seen that $n\,\pi_k/\pi_0$ tends to a constant r_k as $n \to \infty$ resulting in the condition that if $\{P_n\}$ is infinitely divisible (i.d.), then there exists a sequence of non-negative constants $\{r_n\}$ satisfying

$$(n+1)P_{n+1} = \sum_{k=0}^{n} r_k P_{n-k}, \qquad n = 0, 1, 2, \ldots \qquad \ldots \quad (2)$$

Katti (1967) and Steutel (1970) proved that condition (2) is also sufficient for $\{P_n\}$ to be i.d.

2. A CHARACTERIZATION RESULT AND BOUNDS FOR A CERTAIN PROBABILITY

A r.v. taking values in $\{0, 1, 2, \ldots\}$ is said to have a compound geometric distri bution if its p.g.f. $P(t)$ is of the form

$$P(t) = P_0(1-tq(t))^{-1}, \qquad \ldots \quad (3)$$

where $P_0 \in (0, 1)$ and $q(t)(1-P_0)^{-1}$ is a p.g.f. It is well known that $P(t)$ is i.d.

Theorem 2 : *If $\{P_k\}$ is compound geometric, and if $P_1 = P_0(1-P_0)$ then $\{P_k\}$ is geometric.*

Proof : Since $\{P_k\}$ (cf (3)) is compound geometric we have the relation

$$P_{k+1} = \sum_{j=0}^{k} q_j P_{k-j}, \quad k = 0, 1, 2, \ldots$$

where q_j are non-negative constants. Summing the equalities we get $1-P_0 = \sum_{k=0}^{\infty} q_k$. Since $q_0 = P_1/P_0 = 1-P_0$ it follows that $q_k = 0$ for $k \neq 0$. Then $\{P_k\}$ is geometric.

It would be interesting to know whether the assumption that $\{P_k\}$ is compound geometric can be weakened to the assumption that $\{P_k\}$ is i.d. in Theorem 2. The fact is that even if $\{P_n\}$ is i.d. with $P_j = P_0(1-P_0)^j$, $j = 1, \ldots, k$ the distribution $\{P_n\}$ may not be geometric.

To see this consider the p.g.f.

$$q(t) = \tfrac{1}{2} \exp\left\{\frac{t}{2}+\alpha t^{k+1}\right\}, \quad \alpha = \log 2-\tfrac{1}{2},$$

which proves the point.

We shall now obtain some useful bounds for the probability P_{k+1}.

Theorem 3 : *Let a r.v. X be i.d. with support $\{0, 1, 2, \ldots\}$ and suppose that*

$$P(X = r) = pq^r, \qquad r = 0, 1, \ldots, k$$

with $p+q = 1$, $0 < p < 1$. Then

$$\frac{k}{k+1}\, pq^{k+1} \leqslant P(X = k+1) < q^{k+1}.$$

Proof: The right side inequality is obvious. To prove the other, we have from (2)

$$(k+1)P_{k+1} = \sum_{j=0}^{k} P_j r_{k-j} \geqslant \sum_{j=1}^{k} P_j r_{k-j}$$

$$= \sum_{1}^{k} p q^j q^{k-j+1} = kp q^{k+1}$$

because $r_j = q^{j+1}$ for $j = 0, 1, \ldots, k-1$ by the assumption.

Remark 2: Similar procedure can be used to obtain lower bounds for $P(X = m)$ with $m > k$.

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REFERENCES

KATTI, S. K. (1977): Infinite divisibility of integer valued random variables. *Ann. Math. Statist.*, 38, 1306-1308.

STEUTEL, F. W. (1970): Preservation of infinite divisibility under mixing and related topics. *Math. Centre Tracts.*, 33.

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