Sankhyā : The Indian Journal of Statistics
1980, Volume 42, Series A, Pts., 1 & 2 pp. 116–119.

A NOTE ON THE REPRESENTATION OF QUANTILES FOR MIXING PROCESSES

By KESAR SINGH
Indian Statistical Institute

SUMMARY. The results on Bahadur-Kiefer representation of quantiles for ϕ-mixing random variables proved in Babu and Singh (1978) are shown to hold under a weaker mixing condition. A representation is established for strong mixing processes under a polynomial decay condition on mixing coefficients.

1. Introduction

For a stationary process $\{X_i\}$ having marginals as $U[0, 1]$, let $F_n(t)$ and $F_n^{-1}(t)$ (the right continuous versions) denote the empirical and quantile processes. Define

$$R_n(t) = |F_n^{-1}(t)-t+F_n(t)-t|$$

and

$$R_n = \sup\{R_n(t);\quad 0 \leqslant t \leqslant 1\}.$$

Recently, Babu and Singh (1978) (from now on referred to as BS) proved that if $\{X_i\}$ is a ϕ-mixing process with $\Sigma\phi^{1/2}(i) < \infty$, then

$$R_n(t) = O(n^{-3/4}(\log\log n)^{3/4}) \text{ a.s. } \forall t \in [0, 1] \qquad \ldots \quad (1.1)$$

and

$$R_n = O(n^{-3/4}(\log n)^{1/2} (\log\log n)^{1/4}) \text{ a.s.} \qquad \ldots \quad (1.2)$$

BS obtained slightly weaker results for strong mixing processes with $\alpha(n)$ decaying at an exponential rate. In the next section of this note, we conclude that (1.1) and (1.2) remain valid for ϕ-mixing processes under a weaker condition than $\Sigma\phi^{1/2}_{(i)} < \infty$, which is $\Sigma\phi(i) < \infty$. The proofs given in BS for the strong mixing case do not work for polynomially decaying $\alpha(i)$. In the next section, it is shown that $R_n(t) = O(n^{-1/2-\gamma})$ a.s. for a $\gamma > 0$ provided $\alpha(n) = O(n^{-4-\delta})$ for some $\delta > 0$. In particular, the result covers linear processes and Gaussian processes having strong mixing properties with $\alpha(n) = O(n^{-4-\delta})$ (see Goredtsky, 1977; Kolmogorov and Rozanov, 1960, for the strong mixing properties of the processes mentioned).

2. The main results

Let d_i denote absolute constants, throughout. Define, for $0 \leqslant \alpha, \beta \leqslant 1$,

$$z_i(\alpha, \beta) = I(\min(\alpha, \beta) \leqslant X_i \leqslant \max(\alpha, \beta)) - |\alpha - \beta|$$

where I denotes the indicator function. Further, let

$$S_{i,j}(\alpha, \beta) = \sum_{l=i}^{j} z_l(\alpha, \beta)$$

where i, j are positive integers, $i \leqslant j$.

In BS, (1.1) and (1.2) are derived from the probability bound given by Lemma 2.1 of that paper without making use of the dependence structure any more. We present below two lemmas for ϕ-mixing processes with $\Sigma\phi(i) < \infty$, which play the role of Lemma 2.1 of BS in the proofs of (1.1) and (1.2) given in BS; and thus the results (1.1) and (1.2) remain valid under this weaker condition.

Lemma 1 : *Let $\{X_i\}$ be a ϕ-mixing process with $\Sigma\phi(i) < \infty$. Then, there exists a $\rho > 0$ such that, whenever $0 \leqslant \alpha, \alpha+\beta \leqslant 1$, $|\beta| \leqslant b > 0$, $1 \leqslant u \leqslant m$, $H \geqslant 0$ and $0 < D^2 \leqslant bm^{22/15}$, we have*

$$P(|S_{H+1, H+u}(\alpha, \alpha+\beta)| \geqslant 2\rho D) \leqslant d_1 m^{-\theta} + d_2 \exp(-8D^2 m^{-1} b^{-1/2}).$$

Proof: The proof is similar to that of Lemma 2.1 of BS. The main thing that one has to observe is that the moment inequality used there remains valid even under this mixing condition. The block length is taken as $[u^{1/2}]$ and the block sums are truncated at $Dm^{-1}b^{-1/2}$.

Lemma 2 : *Under the conditions of Lemma 1, there exists a $\theta > 0$ such that whenever $0 \leqslant \alpha$, $\alpha+\beta \leqslant 1$, $|\beta| \leqslant s > 0$, $H \geqslant 0$, $1 \leqslant u \leqslant M$ and $sM^{30/66} \leqslant Q^2 \leqslant s^{2/3} M^{14/11}$, we have*

$$P(|S_{H+1, H+u}(\alpha, \alpha+\beta)| \geqslant 2\theta Q) \leqslant d_3 M^{-4} + d_4 \exp(-8Q^2 M^{-1} s^{-1}).$$

Proof: The proof of this lemma also goes parallel to that of Lemma 2.1 of BS. This time the block length is taken $[u^{31/44}]$. Block sums are truncated at $QM^{-1}s^{-2}$ and *the effect of truncation is estimated using the above Lemma 1 and not any moment inequality.*

Next, we prove a representation theorem for quantiles based on strong-mixing random variables with $\alpha(n)$ having a polynomial decay rate.

Theorem : *If $\{X_i\}$ is a strong mixing process with marginals as $U[0, 1]$ and $\alpha(n) = O(n^{-4-\delta})$ for some $\delta > 0$, then $R_n(t) = O(n^{-1/2-\gamma})$ a.s. for some $\gamma > 0$ and $t \in [0, 1]$, fixed.*

Proof: Let $a(n) = n^{-19/40} \log n$; so that $a(n)$ is decreasing after certain n onwards. At first, we show that

$$|F_n^{-1}(t)-t| = O(a_n) \text{ a.s.} \qquad \ldots \quad (2.1)$$

Let us take

$$n_r = r^5, \; C_r = \{n : n_r \leqslant n < n_{r+1}\}.$$

Hence

$$n \in C_r \text{ implies } n_{r+1}-n_r = O(n^{4/5}).$$

Also $n \in C_r$ implies that after certain n onwards,

$$\{F_n^{-1}(t)-t > a_n\} \subseteq \{F_n(t+a_n) \leqslant t\} \subseteq \{F_n(t+a(n_{r+1})) \leqslant t\} \subseteq \{S_{1,n_r}(0, t+a(n_{r+1}))$$

$$\leqslant -2^{-1}n_r a(n_{r+1})\} \bigcup \{S_{n_r+1,n}(0, t+a(n_{r+1})) \leqslant -2^{-1}n_r a(n_{r+1})\} = G_r + H_n \text{ (say)}.$$

Using Markov's inequality and Lemma 3.2 of BS, it is seen that $\Sigma P(G_r) < \infty$ and $\Sigma P(H_n) < \infty$. These estimates along with the above set inequalities imply that $\{F_n^{-1}(t)-t > a_n\}$ happens only finitely often, a.s. By similar steps, we conclude that $\{F_n^{-1}(t)-t < -a_n\}$ happens only finitely often a.s. and hence (2.1) holds.

We now complete the proof of the theorem with $\gamma = 1/80$. In view of the fact that

$$|F_n F_n^{-1}(t)-t| \leqslant |F_n F_n^{-1}(t)-F_n(F_n^{-1}(t)-0)|,$$

it follows easily that

$$R_n(t) \leqslant \sup\{3n^{-1}|S_{1,n}(t,s)|;\; |s-t| \leqslant 2a_n\}, \qquad \ldots \quad (2.2)$$

Let us define

$$m_r = [r^{80/22}], \; T_r = \{n : m_r \leqslant n \leqslant m_{r+1}\},$$

$$w_r = [2a(m_r)m_r^{41/80}]+1$$

and

$$W_r = \{\pm m_r^{-41/80}, \pm 2m_r^{-41/80}, \ldots \pm w_r m_r^{-41/80}\}.$$

If $n \in T_r$, the r.h.s. of (2.2) cannot exceed $3m_r^{-1}L_n + m_r^{-41/80}$ where

$$L_n = \max \{ | S_{1,n}(t, t+b) | ; \; b \in W_r \}.$$

Clearly, it suffices to show that $3m_r^{-1}L_n = O(n_r^{-41/80})$ a.s. To this end, we note that if $n \in T_r$ and $A_n = \{L_n > m_r^{39/80}\}$, then

$$\begin{aligned} A_n \subseteq & \{\max_{b \in W_r} \; | S_{1,m_r}(t, t+b) | > 2^{-1} m_r^{39/80}\} \\ \cup & \{\max_{b \in W_r} \; | S_{m_r+1,n}(t, t+b) | > 2^{-1} m_r^{39/80}\} \\ = & \; G_r' \cup H_n' \quad \text{(say)}. \end{aligned}$$

By Bonferroni inequality, Markov's inequality, Lemma 3.2 of BS and the fact that $n \in m_r$ implies $m_{r+1} - m_r = O(n^{57/80})$, it follows that $\Sigma P(H_n') < \infty$. Also it follows using Chebychev's inequality and Bonferroni inequality that

$$P(G_r') \leqslant w_r \, m_r^{-78/80} \max_{b \in W_r} E | S_{1,m_r}(t, t+b) |^2. \qquad \ldots \quad (2.3)$$

Further, using Davydov's inequality (see Deo, 1973) it follows that $b \in W_r$ implies

$$E | S_{1,m_r}(t, t+b) |^2 \leqslant 2 \, m_r(a(m_r) + d_\delta(a(m_r))^{3/4}). \qquad \ldots \quad (2.4)$$

Combining (2.3) and (2.4), we have $\Sigma P(G_r') < \infty$. The theorem follows from these estimates.

With the techniques of this note, the author has also been able to show that $R_n = O(n^{-1/2-\gamma})$ a.s. for a $\gamma > 0$ in the case of strong mixing random variables with $\alpha(n)$ having a suitable polynomial decay rate. The details are too long to be presented here.

References

Babu, G. J. and Singh, K. (1978): On deviation between empirical and quantile processes for mixing random variables. *Jour. Multi. Anal.*, 8, 532-549.

Deo, C. M. (1973): A note on empirical processes for strong mixing sequences. *Ann. Prob.*, 1, 870-875.

Gorodetsky, V. V. (1977): On the strong mixing properties for linear processes. *Teoria Verojatnostei i ee prim.* 22, 421-423.

Kolmogorov, A. N. and Rozanov, Yu. A. (1960): On strong-mixing conditions for stationary Gaussian processes. *Theo. Prob. Appl.*, 5, 204-208.

Paper received: September, 1978.
Revised: September, 1979.