A Note on the Quantitative Measure of Image Enhancement Through Fuzziness

SANKAR K. PAL

***Abstract*—The "index of fuzziness" and "entropy" of an image reflect a kind of quantitative measure of its enhancement quality. Their values are found to decrease with enhancement of an image when different sets of S-type membership functions with appropriate crossover points were considered for extracting the fuzzy property plane from the spatial domain of the image.**

***Index Terms*—Entropy, fuzzy set, image processing, index of fuzziness, property plane.**

I. Introduction

The present correspondence illustrates an application of the theory of fuzzy sets in image processing problems. The problem is to provide a quantitative measure of enhancement quality of an image through the evaluation of its amount of fuzziness. These are explained by the terms "index of fuzziness" and "entropy" [1], [2] of a fuzzy set. Index of fuzziness reflects the ambiguity present in an image by measuring the distance between its fuzzy property plane and nearest ordinary plane. The term "entropy," on the other hand, uses Shannon's function in the property plane but its meaning is quite different from the one of classical entropy because no probabilistic concept is needed to define it. These two terms which give an idea of "indefiniteness" or fuzziness of a set may be regarded as the measure of an average intrinsic information which is received when one has to make a decision (as in pattern analysis) in order to classify the ensembles of patterns described by a fuzzy set. These quantities are found to decrease with enhancement of image.

The fuzzy property plane has been extracted from the spatial domain using S-type membership function [3], [4] along with two fuzzifiers. The role of the fuzzifiers is to introduce different amount of ambiguity in a property plane by changing the crossover point and slope of the transformation function. The effectiveness of the algorithm with different values of these fuzzifiers is demonstrated on a set of enhanced images.

II. Fuzzy Set and Image Definition

A fuzzy set (A) with its finite number of supports $x_1, x_2, \cdots, x_n$ in the universe of discourse U is defined as

$$A = \{(\mu_A(x_i), x_i)\} \tag{1a}$$

or, in union form

$$A = \bigcup_i \mu_i/x_i, \quad i = 1, 2, \cdots, n \tag{1b}$$

where the membership function $\mu_A(x_i)$ having positive value in the interval $[0, 1]$ denotes the degree to which an event x_i may be a member of A. If $\mu_A(x_i) = 0.5$, x_i is said to be the crossover point in A.

Since a gray tone image possesses some ambiguity within the pixels due to the possible multivalued levels of brightness, it is justified to apply the concept and logic of fuzzy set rather than ordinary set theory to an image processing problem. Keeping this in mind, an image X of $M \times N$ dimension and L levels can be considered as an array of fuzzy singletons, each with a value of membership function denoting the degree of having brightness relative to some brightness level l, $l = 0, 1, \cdots, L - 1$. In the notion of fuzzy set, we may therefore write

$$X = \bigcup_m \bigcup_n p_{mn}/x_{mn} \quad m = 1, 2, \cdots, M; n = 1, 2, \cdots, N \tag{2}$$

where p_{mn}/x_{mn} $(0 \leqslant p_{mn} \leqslant 1)$ represents the grade of possessing some property p_{mn} by the (m, n)th pixel x_{mn}. This fuzzy property p_{mn} may be defined in a number of ways with respect to any brightness level depending on the problems to hand. This is explained in Section IV.

III. Evaluation of Fuzziness of an Image

A. Index of Fuzziness

The index of fuzziness of a set A having n supporting points is defined as [1]

$$\gamma(A) = \frac{2}{n^k} d(A, \boldsymbol{A}) \tag{3}$$

where $d(A, \boldsymbol{A})$ denotes the distance between fuzzy set A and its nearest ordinary set $\boldsymbol{A}$. The set $\boldsymbol{A}$ is such that $\mu_{\boldsymbol{A}}(x_i) = 0$ if $\mu_A(x_i) \leqslant 0.5$ and 1 for $\mu_A(x_i) > 0.5$. The number 2 and the positive constant k appear in order to make $\gamma(A)$ lie between 0 and 1. The value of k depends on the type of distance function used. For example, $k = 1$ for a generalized Hamming distance whereas $k = 0.5$ for an Euclidean distance. The corresponding indexes of fuzziness are called the "linear index of fuzziness" $\gamma_l(A)$ and the "quadratic index of fuzziness" $\gamma_q(A)$. Considering "d" to be a generalized Hamming distance we have

$$d(A, \boldsymbol{A}) = \sum_i |\mu_A(x_i) - \mu_{\boldsymbol{A}}(x_i)| = \sum_i \mu_{A \cap \bar{A}}(x_i) \tag{4}$$

and

$$\gamma_l(A) = \frac{2}{n} \sum_i \mu_{A \cap \bar{A}}(x_i), \quad i = 1, 2, \cdots, n \tag{5}$$

where $A \cap \bar{A}$ is the intersection between fuzzy set A and its complement $\bar{A}$. $\mu_{A \cap \bar{A}}(x_i)$ denotes the grade of membership of x_i to such a fuzzy set $A \cap \bar{A}$ and is defined as

$$\mu_{A \cap \bar{A}}(x_i) = \min\{\mu_A(x_i), \mu_{\bar{A}}(x_i)\}, \quad \text{for all } i \tag{6a}$$

$$= \min\{\mu_A(x_i), (1 - \mu_A(x_i))\}, \quad \text{for all } i. \tag{6b}$$

From (5) it is seen that

$$(\gamma_l)_{\min} = 0 \quad \text{for } \mu_i = 0 \text{ or } 1 \tag{7a}$$

and

The author is with the Electronics and Communication Sciences Unit, Indian Statistical Institute, Calcutta, India, on leave at the Department of Electrical Engineering, Imperial College of Science and Technology, London, England.

$$(\gamma_1)_{\max} = 1 \quad \text{for } \mu_1 = \mu_2 = \cdots = \mu_n = 0.5. \tag{7b}$$

Furthermore, it follows that

$$\gamma_1(A) \geqslant \gamma_1(A^*) \tag{7c}$$

and

$$\gamma_1(A) = \gamma_1(\bar{A}) \tag{7d}$$

where A^* is a "sharpened" version of A such that $\mu_A^*(x_i) \geqslant \mu_A(x_i)$ for $\mu_A(x_i) \geqslant 0.5$ and $\mu_A^*(x_i) \leqslant \mu_A(x_i)$ for $\mu_A(x_i) \leqslant 0.5$.

Extending (5) in two-dimensional image plane we may write

$$\gamma_1(X) = \frac{2}{MN} \sum_m \sum_n \mu_{X \cap \bar{X}}(x_{mn})$$
$$m = 1, 2, \cdots, M; \quad n = 1, 2, \cdots, N. \tag{8}$$

Equation (8) defines the amount of fuzziness present in the property plane of an image X. μ corresponds to p_{mn}. $X \cap \bar{X}$ is the intersection between fuzzy image planes $X = \{p_{mn}/x_{mn}\}$ and $\bar{X} = \{(1 - p_{mn})/x_{mn}\}$, the complement of X. $\mu_{X \cap \bar{X}}(x_{mn})$ denotes the degree of membership of (m, n)th pixel x_{mn} to such a fuzzy property plane $X \cap \bar{X}$ so that

$$\mu_{X \cap \bar{X}}(x_{mn}) = p_{mn} \cap \bar{p}_{mn} = \min\{p_{mn}, (1 - p_{mn})\},$$
$$\text{for all } (m, n). \tag{9}$$

Similarly, for an Euclidean distance we have

$$\gamma_q(X) = \frac{2}{\sqrt{MN}} \left[\sum_m \sum_n (\mu_X(x_{mn}) - \mu_{\underline{X}}(x_{mn}))^2 \right]^{0.5}$$
$$m = 1, 2, \cdots, M; n = 1, 2, \cdots, N \tag{10}$$

where $\underline{X}$ is the nearest ordinary image plane of fuzzy plane X.

B. Fuzziness through Entropy

The entropy of a fuzzy set A having n supporting points is defined as [1]

$$H(\varphi_1, \varphi_2, \cdots, \varphi_n) = -\frac{1}{\ln(n)} \sum_i \varphi_A(x_i) \cdot \ln(\varphi_A(x_i)) \tag{11}$$

where

$$\varphi_A(x_i) = \frac{\mu_A(x_i)}{\sum_i \mu_A(x_i)}, \quad i = 1, 2, \cdots, n. \tag{12}$$

The entropy is then seen to lie between 0 and 1 in a way

$$H_{\min} = 0 \quad \text{for } \varphi_j = 1, \quad j \in \{1, 2, \cdots, n\}$$
$$\varphi_i = 0, \quad i \neq j \tag{13a}$$

$$H_{\max} = 1 \quad \text{for } \varphi_1 = \varphi_2 = \cdots = \varphi_n = 1/n. \tag{13b}$$

It is to be mentioned here that this method (unlike the previous one) does not depend on the absolute values of μ but their relative values. In other words, a fuzzy set with a single nonzero μ-value would have zero entropy and a set having a constant μ-value for all the elements would have $H = 1$. Therefore, an image X with $\mu(x_{mn}) = 1$ or 0, $x_{mn} \in X$ (i.e., fully bright or dark) according to this definition would be possessing maximum entropy, but this is intuitively unappealing.

De Luca and Termini [2] defined entropy of a fuzzy set A in analogy with information theoretic entropy, although quite different conceptually, as

$$H(A) = \frac{1}{n \ln 2} \sum_i Sn(\mu_A(x_i)), \quad i = 1, 2, \cdots, n \tag{14}$$

with the Shannon's function

$$Sn(\mu_A(x_i)) = -\mu_A(x_i) \ln \mu_A(x_i)$$
$$- (1 - \mu_A(x_i)) \ln (1 - \mu_A(x_i)). \tag{15}$$

Therefore, like the index of fuzziness, this entropy, (14) is also dependent on the absolute values of μ and satisfies the properties

$$H_{\min} = 0 \quad \text{for } \mu_i = 0 \text{ or } 1 \tag{16a}$$

$$H_{\max} = 1 \quad \text{for } \mu_1 = \mu_2 = \cdots = \mu_n = 0.5 \tag{16b}$$

$$H(A) \geqslant H(A^*) \tag{16c}$$

and

$$H(A) = H(\bar{A}). \tag{16d}$$

In fact, these conditions (7) or (16) may be regarded as the criteria to be satisfied by a function in order to measure fuzziness in a set.

With this notion, we define the entropy of an $M \times N$ dimensional image plane X as

$$H(X) = \frac{1}{MN \ln 2} \sum_m \sum_n Sn(\mu_X(x_{mn})) \tag{17}$$

with

$$Sn(\mu_X(x_{mn})) = -\mu_X(x_{mn}) \ln \mu_X(x_{mn}) - (1 - \mu_X(x_{mn}))$$
$$\cdot \ln(1 - \mu_X(x_{mn}))$$
$$m = 1, 2, \cdots, M; n = 1, 2, \cdots, N. \tag{18}$$

C. Interpretation of $\gamma(X)$ and $H(X)$ for Image Enhancement

In the previous section, we have described $\gamma(X)$ and $H(X)$ for providing a measure of the fuzziness present in two-dimensional image plane X. $\gamma(X)$ measures the distance between fuzzy property plane of X and its nearest ordinary plane. $H(X)$, on the other hand, is based on the well-known property of Shannon's function $Sn(\mu)$ (15)—monotonically increasing in the interval [0, 0.5] and monotonically decreasing in [0.5, 1] with a maximum (= ln 2) at $\mu = 0.5$—in the fuzzy property plane of X.

For gray tone image processing problem, an image pattern X looks ambiguous to a people or device which knows only black and white gray levels. The nature of this ambiguity (fuzziness) in X therefore arises from the "incertitude" present when one has to decide whether the (m, n)th pixel intensity x_{mn} has to be considered white or black. We may measure this incertitude or uncertainty by $\mu_{X \cap \bar{X}}(x_{mn})$ or $Sn(\mu_X(x_{mn}))$ which is 0 if $\mu_X(x_{mn}) = 0$ or 1 and is maximum for $\mu_X(x_{mn}) = 0.5$; the average (normalized) amount of incertitude is measured by the terms $\gamma(X)$ or $H(X)$.

Now through processing, if we can remove partially the uncertainty on the gray levels of X, we say that we have obtained an average amount of information given by $\delta\gamma = \gamma(X) - \gamma(X')$ or $\delta H = H(X) - H(X')$ where X' is the processed (sharpened) version of X. The criteria $\gamma(X') \leqslant \gamma(X)$ and $H(X') \leqslant H(X)$ in order to have positive $\delta\gamma$ and δH-values are followed from (7c) and (16c), respectively. If the uncertainty is completely removed, then $\gamma(X') = H(X') = 0$. In other words, $\gamma(X)$ and $H(X)$ can be regarded as the measure of average amount of information (about the gray levels of pixels) which has been lost for transforming the classical pattern (two tone) into a fuzzy pattern X.

IV. Property Plane and Fuzzifiers

The operations described in Section III are restricted in fuzzy property plane. To enter this domain from the spatial image plane, we need a membership function which will transform each x_{mn} in the spatial domain to its corresponding p_{mn}-value in the property domain. This function may be either S-type or π-type or their complements depending on the problem in hand. The S-function defines the compatibility function corresponding to fuzzy plane "x_{mn} is $x_{\max}$" whereas the π-function corresponds to a plane "x_{mn} is l," $0 < l < x_{\max}$. The corresponding fuzzy p_{mn}-values denote the degree of possessing maximum brightness level $x_{\max}$ and some other level l by the (m, n)th pixel x_{mn}.

Now in our problem of measuring fuzziness of an image we are interested in a monotonic increasing/decreasing function as represented by $S/(1 - S)$-function which will result in an one-to-one mapping of the elements in the x-plane ranging from 0 to $x_{\max}$ to the p-plane in the interval $[0, 1]/[1, 0]$. To represent such an S-function, we define a simple expression

$$p_{mn} = G_S(x_{mn}) = \left[1 + \frac{(x_{\max} - x_{mn})}{F_d}\right]^{-F_e} \tag{19}$$

where F_e and F_d are the exponential and denominational fuzzifiers, respectively. These two positive constants have the effect of altering the ambiguity (fuzziness) in the fuzzy property plane by changing the crossover point and slope of the S-function. Their effect on the γ and H-values has been studied in the next section. The function G_S is symmetric in the interval $[0, x_{\max}]$ if it leads to the crossover point at $x_{\max}/2$. Otherwise, it is said to be nonsymmetric.

If for example, we use $|(x_{\max}/2) - x_{mn}|$ instead of $(x_{\max} - x_{mn})$, (19) would represent a π-function (G_π) symmetric over $x_{\max}/2$. Since such a function would result in the same p-value for any two pixel intensities located symmetrically on opposite sides of $x_{\max}/2$, the conditions (7a), (7b), and (16a), (16b) (except for $\mu_i = 0$) would not convey the appropriate interpretation of fuzziness of an image. For example, an image X with $x_{mn} = x_{\max}/2$, $x_{mn} \in X$ would have $p_{mn} = G_\pi(x_{mn}) = 1$ and hence $\gamma(X) = H(X) = 0$. Similarly, the image X with $x_{kl} = x_{\max}/4$ and $x_{mn} = 3x_{\max}/4$, $(k, l) \neq (m, n)$, x_{kl}, $x_{mn} \in X$ would have (for symmetrical G_π-function) $p_{mn} = 0.5$ and hence $\gamma(X) = H(X) = 1$. Both of the cases are not intuitively appealing. Only the case when $x_{mn} = 0$ or $x_{\max}$ for which $p_{mn} = 0$ and $\gamma(X) = H(X) = 0$, conveys an appropriate information regarding ambiguity in X.

Again, it is to be mentioned here that the above G_S-function results in an α-level property plane where α is the value of p_{mn} for $x_{mn} = 0$. Since this violates the condition (7a) or (16a) for $x_{mn} = 0$, the algorithm includes provision for constraining all the zero x_{mn}-values to zero p_{mn}-value. However, the results without using this constraint are also reported for comparison.

V. Example and Discussion

Fig. 1 shows a 96 × 99, 32-level image of handwritten script ("Shu"). Fig. 2(a), (b), and (c) are its different enhanced versions as obtained by histogram equalization technique, contrast intensification technique, and contrast intensification along with smoothing, respectively [5]. Table I illustrates the values of $\gamma_1(X)$, $\gamma_q(X)$, and $H(X)$ for different slopes of the symmetrical G_S-function (crossover point at 15.5). Results for different nonsymmetrical G_S-functions (for $F_e = 2$) are explained in Table II. Here we have considered the values of F_d to be 70, 60, 50, 45, 35, 30, 25, and 15 so that the crossover point of the nonsymmetrical G_S-function can lie between the gray levels 2 and 3, 6 and 7, 10 and 11, 12 and 13, 14 and 15, 16 and 17, 18 and 19, 20 and 21, and 24 and 25, respectively.

From Table I it is seen that the quadratic distance when used in (3) results in higher effective values of γ as compared to those of linear distance. The absolute γ-values and H-values for a fixed corssover point are decreased as the curve tends to be steeper (with increase in the values of F_e and F_d) resulting

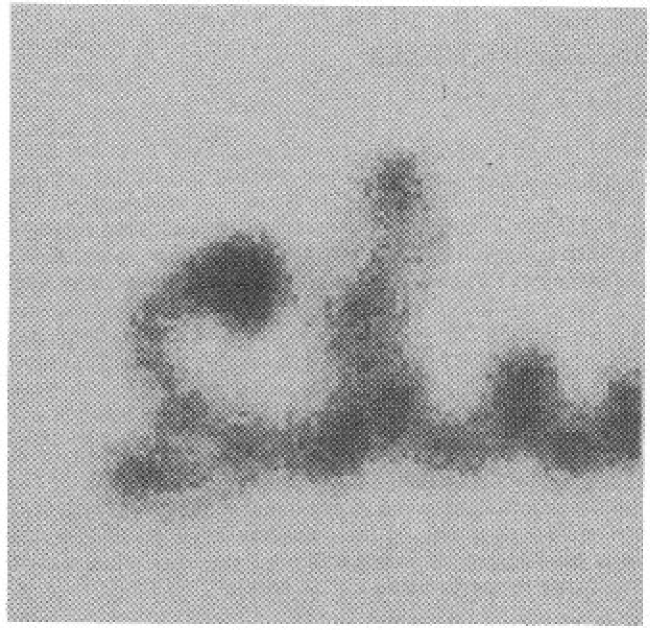

Fig. 1. Original image.

in a decrease in ambiguity in p-plane. These values are minimum for Fig. 2(c) and maximum for Fig. 1. This relative order (as in Table I) for different enhanced images is seen to be maintained as long as the crossover point is restricted in the left half [0, 15.5] of the gray scale. As we keep the crossover point moving from 15.5 towards $x_{\max} = 31$, the amount of fuzziness in the equalized image [Fig. 2(a)] tends to be greater (after the crossover points 16.5, 16.5, and 18.6 are reached for γ_l, γ_q, and H, respectively) than that of input image (Fig. 1). It is revealed under investigation that Fig. 2(a), since it possesses an almost uniform histogram, contains as compared to Fig. 1, a large number of levels around the crossover points (as selected in the right half of gray scale) and it is these levels which cause an increase in $(p_{mn} \cap \bar{p}_{mn})$-value of $\gamma(X)$ and $Sn(p_{mn})$-value for $H(X)$.

As mentioned in the previous section, the above results were obtained using the constraint $\alpha = 0$ in (19). For comparison of these results, the parameters γ and H were also computed 1) without using this constraint and 2) using an ideal S-function which is defined as [3]

$$p_{mn} = 2(x_{mn}/x_{\max})^2 \qquad \text{for } x_{mn} \leqslant \beta \tag{20a}$$

$$= 1 - 2((x_{mn} - x_{\max})/x_{\max})^2 \qquad \text{for } \beta \leqslant x_{mn} \leqslant x_{\max} \tag{20b}$$

with β (crossover point) $= x_{\max}/2$ and $0 \leqslant p_{mn} \leqslant 1$. The results are shown only for γ_l (Table I) as a typical case of illustration. The use of (19) alone (i.e., with $\alpha \neq 0$) results in an increase in the absolute values of fuzziness (especially for Fig. 2(b) and (c) having a large number of zero gray levels) but does not change the relative order of fuzziness for these images, whereas (20) does change. It is also to be noted that (20) is symmetric across β and there is no control over the crossover point in order to make it nonsymmetric.

In a part of the experiment, the entropy under Kaufmann's definition (11) was also computed for these images. Since these images are neither fully bright nor fully dark (for which it leads to an unappealing concept of $H = 1$), the order of their H-values is not changed from that obtained with (14) (Table I). Experiments were also conducted using G_π-functions, but the results as explained in Section IV, did not reflect the appropriate measure of enhancement-quality for different values of the fuzzifiers.

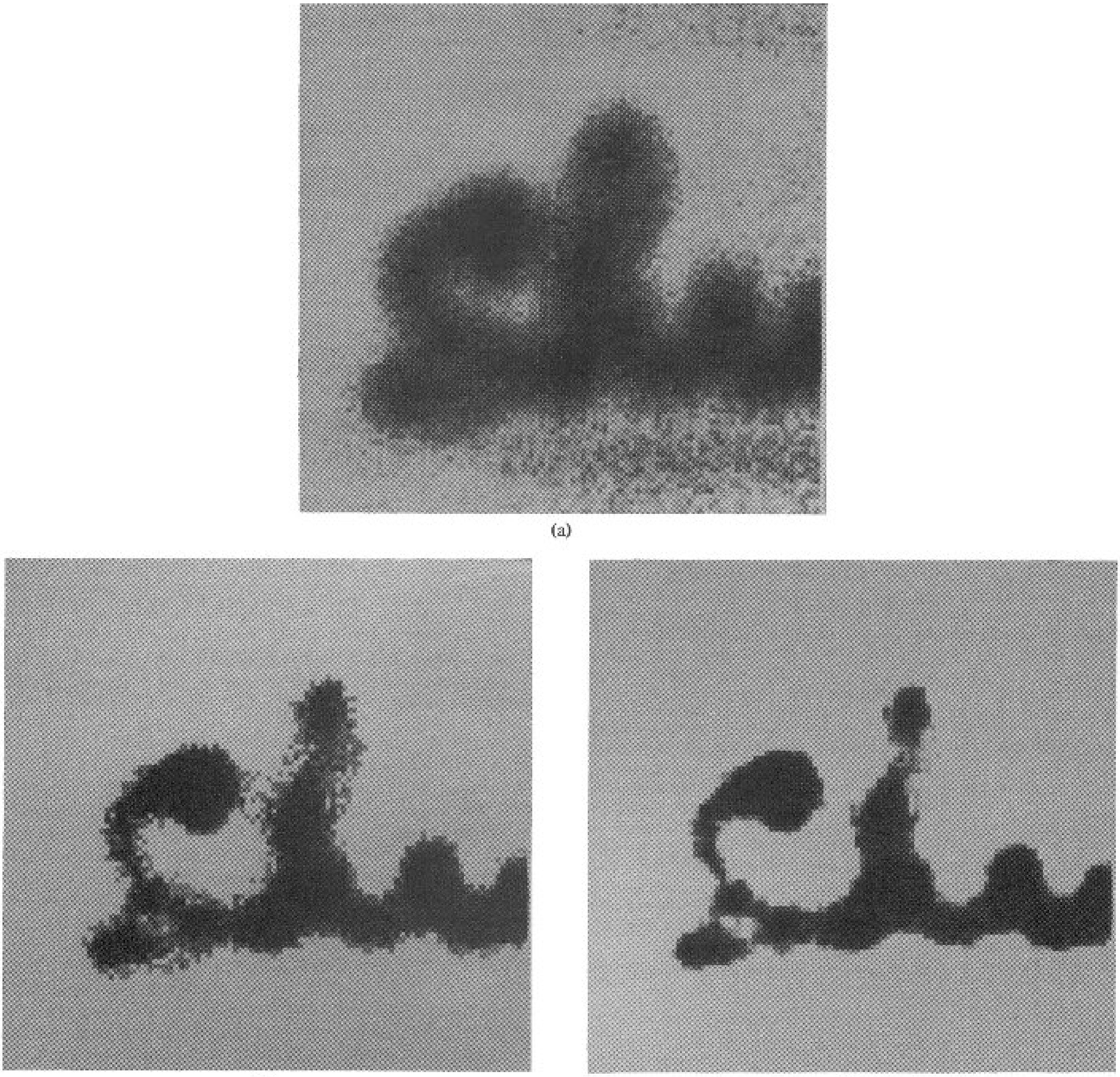

Fig. 2. Different enhanced images.

TABLE I
"INDEXES OF FUZZINESS" AND "ENTROPY" OF IMAGES FOR DIFFERENT SYMMETRICAL S-FUNCTIONS (CROSSOVER POINT AT $x_{max}/2 = 15.5$)

IMAGE	$\gamma_l(X)$					$\gamma_q(X)$			$H(X)$			
	(1)	(2)	(2)*	(3)	ISF	(1)	(2)	(3)	(1)	(2)	(2)**	(3)
Fig. 1	0.800	0.760	0.760	0.744	0.286	0.804	0.767	0.752	0.966	0.951	0.999	0.944
Fig. 2a	0.711	0.681	0.684	0.670	0.367	0.753	0.727	0.717	0.880	0.863	0.994	0.857
Fig. 2b	0.196	0.189	0.641	0.186	0.103	0.403	0.389	0.384	0.233	0.230	0.842	0.229
Fig. 2c	0.096	0.091	0.578	0.089	0.043	0.261	0.250	0.246	0.134	0.129	0.812	0.128

(1): $F_e = 1$ and $F_d = 15.5$; (2): $F_e = 2$ and $F_d = 37.42$; (3): $F_e = 3$ and $F_d = 59.63$

* without the constraint $\alpha = 0$ in equation (19)

ISF using Ideal S-Function (equation 20)

** using equation (11), Entropy under Kaufmann's definition

TABLE II
"INDEXES OF FUZZINESS" AND "ENTROPY" OF IMAGES FOR DIFFERENT NONSYMMETRICAL S-FUNCTIONS (CORRESPONDING CROSSOVER POINTS, CP FOR $F_e = 2$ ARE ALSO MENTIONED)

IMAGE	$F_d = 70$ CP = 2.01	$F_d = 60$ CP = 6.15	$F_d = 50$ CP = 10.28	$F_d = 45$ CP = 12.36	$F_d = 40$ CP = 14.43	$F_d = 35$ CP = 16.5	$F_d = 30$ CP = 18.57	$F_d = 25$ CP = 20.64	$F_d = 15$ CP = 24.79
	0.869	0.917	0.882	0.845	0.794	0.725	0.643	0.547	0.319
Fig. 1	0.876	0.921	0.884	0.849	0.799	0.733	0.652	0.557	0.330
	0.979	0.989	0.987	0.979	0.965	0.937	0.896	0.836	0.624
	0.569	0.619	0.658	0.674	0.678	0.679	0.666	0.637	0.508
Fig. 2a	0.636	0.683	0.715	0.727	0.725	0.723	0.707	0.677	0.558
	0.780	0.810	0.838	0.850	0.860	0.865	0.864	0.852	0.763
	0.166	0.180	0.188	0.191	0.189	0.187	0.179	0.166	0.120
Fig. 2b	0.352	0.380	0.392	0.395	0.391	0.385	0.371	0.346	0.264
	0.215	0.222	0.228	0.230	0.231	0.230	0.227	0.220	0.186
	0.068	0.075	0.082	0.086	0.089	0.092	0.095	0.098	0.097
Fig. 2c	0.202	0.219	0.235	0.242	0.247	0.251	0.256	0.258	0.253
	0.107	0.113	0.120	0.124	0.127	0.131	0.134	0.137	0.138

Upper Score: $\gamma_1(X)$; Middle Score: $\gamma_q(X)$ and Lower Score: $H(X)$ of equation (14)

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Efficient Spiral Search in Bounded Spaces

RICHARD W. HALL

Abstract—This correspondence defines approaches for the efficient generation of a spiral-like search pattern within bounded rectangularly tessellated regions. The defined spiral-like search pattern grows outward from a given source in a two-dimensional space, thus tending to minimize search time in many sequential tracking tasks. Efficient spiral generation is achieved by minimizing the number of operations required for interaction with boundaries. Algorithms are developed for both rectangular search regions and for arbitrary convex search regions.

Index Terms—Convex and rectangular search regions, image tracking component, iterative growth techniques, search pattern generation, spiral search pattern.

The author is with the Department of Electrical Engineering, University of Pittsburgh, Pittsburgh, PA 15261.

I. INTRODUCTION

An important facet of many image processing applications is the search in a given image space for a particular object or objects followed by some form of subsequent analysis (e.g., see [1]-[3]). Although many current applications utilize global processing with exhaustive search strategies, the ordering of a given search can reduce image processing time (e.g., see [4]). In particular, we have been developing an image processing approach to the eye tracking problem [5] which has generated a particular interest in efficient search strategies for the detection of small ($\simeq$4-5 pixels) moving objects in a binary image space. In one approach we are considering the use of a spiral-like search pattern which proceeds outward from the previous known position of the object being tracked. This allows us to economize on the size of the search space when objects to be tracked are not numerous and have relatively small velocities. Spiral-like search patterns found early use in analog character recognition approaches [6] and in light pen tracking for random access CRT display technologies [7].

The definition of a spiral-like search pattern is quite straightforward except when one must interact efficiently with boundaries as in our application. Consequently, characterizations have been developed for spiral-like search which are efficient in use of computation time by minimizing interactions with arbitrary boundaries in both rectangular and convex search regions. These characterizations are the subject of this note.

II. THE SPIRAL SEARCH CHARACTERIZATION IN RECTANGULAR SEARCH REGIONS

A useful class of spiral-like search over a rectangularly tessellated space is illustrated and defined in Fig. 1. Prior to reaching any boundaries, we may characterize the clockwise