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Bartlett-Type Modification for Rao's Efficient Score Statistic

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This paper suggests simple Bartlett-type modifications for a wide class of test statistics that includes in particular the efficient score and the likelihood ratio statistics.

1. INTRODUCTION

Ever since the early work due to Bartlett [5, 6], corrections leading to a better approximation of the null distribution of the likelihood ratio (LR) statistic by the chi-square distribution received considerable attention in the literature (see, e.g., Lawley [18], Barndorff-Nielsen and Cox [2, 3], Cox [16], Cordeiro and Paula [15], Bickel and Ghosh [9] and the references therein). Recently, C. R. Rao, in a private communication, and also Cox [16] posed the problem of developing similar corrections for other popularly used statistics like Rao's efficient score statistic (Rao [25, p. 417]). It is attempted here to settle this problem to some extent.

In order to motivate the ideas, we begin with the one-parameter case and develop simple Bartlett-type modifications for a large family of statistics

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which includes, in particular, the LR, Rao's and Wald's statistics. It is noted that the usual Bartlett's correction for the LR statistic follows as a special case of our results. This is done in Section 2. In Section 3, we extend the ideas to the multiparameter case and consider the modifications for Rao's statistic. It is noted that the suggested modifications do not alter the powers of the corresponding tests, at least up to the second order, in the sense of Chandra and Joshi [12] and Mukerjee [24]. Some possible extensions have been briefly indicated in the concluding remarks. It may be mentioned that the technique of proof employed is essentially of a standard type known from the field of Edgeworth expansions and Cornish–Fisher expansions. The technique is applied to the signed square root of test statistics of the chi-squared type. This kind of approach has been applied to the log-likelihood ratio statistic by McCullagh [20, Sect. 7.4.5] and by Barndorff-Nielsen [1]—see also Chandra and Ghosh [11], Chandra and Joshi [12], and Bickel and Ghosh [9].

2. The One-Parameter Case

For a sequence $\{X_n\}$, $n \geqslant 1$, of i.i.d., possibly vector-valued, random variables with a common density $f(x, \theta)$ $\theta \in \Theta$, an open subset of $\mathscr{R}^1$, consider the problem of testing $H_0: \theta = \theta_0$ against the alternative $\theta \neq \theta_0$. Without loss of generality, by a reparametrization if necessary, let $\mathscr{I} = 1$, where $\mathscr{I}$ is the per observation information at θ_0. Consider a family $\mathscr{F}$ of test statistics λ_n such that for every $\lambda_n \in \mathscr{F}$, a set A_n with $P_{\theta_0}(A_n) = 1 + o(n^{-1})$ can be obtained with the property that on A_n

$$\lambda_n = (W_n)^2 + o(n^{-1}), \tag{2.1}$$

where

$$\begin{aligned} W_n &= H_1 + n^{-1/2}(v_1 H_1 H_2 + v_2 H_1^2) \\ &\quad + n^{-1}(y_1 H_1 H_2^2 + y_2 H_1^2 H_2 + y_3 H_1^3 + y_4 H_1^2 H_3), \\ H_i &= n^{-1/2} \left\{ \sum_{j=1}^{n} (d^i \log f(X_j, \theta_0)/d\theta^i) - n l_i \right\}, \\ l_i &= E_{\theta_0}\{d^i \log f(X, \theta_0)/d\theta^i\}, \end{aligned} \tag{2.2}$$

$i = 1, 2, 3$, and $v_1, v_2, y_1, y_2, y_3, y_4$ are real numbers which do not involve n. As noted in Chandra and Mukerjee [13], the family $\mathscr{F}$ is very rich and includes, in particular, the LR, Rao's and Wald's test statistics to be denoted here by λ_{1n}, λ_{2n}, λ_{3n}, respectively—for suitable choices of $v_1, v_2, y_1, y_2, y_3, y_4$. The forms of the expressions (2.1) and (2.2) for the LR statistic λ_{1n} were given by different authors (Lawley [18], McCullagh and Cox [21, Sect. 2.2]).

In the one-parameter case, Bartlett's correction for the LR statistic is essentially based on the observation that

$$E_{\theta_0}(\lambda_{1n}) = 1 + a/n + o(n^{-1})$$

and that

$$P_{\theta_0}[\lambda_{1n}/(1 + a/n) \leqslant x] = \int_0^x g_1(z)\, dz + o(n^{-1}), \qquad \forall x \geqslant 0, \tag{2.3}$$

where a is a constant free from n and $g_u(\cdot)$ is the density of the chi-square distribution with u degrees of freedom. The structure of the Bartlett correction, as in (2.3), is easily explained by McCullagh and Cox [21]—see expression (11) in their paper. A detailed expression for the constant a has been given at the end of this section.

That the above simple technique will not be applicable to the tests in the family $\mathscr{F}$, in general, follows if one simply considers Rao's statistic and notes that $E_{\theta_0}(\lambda_{2n}) \equiv 1$, so that no appropriate divisor as in the left-hand side of (2.3) is available. This difficulty can be overcome by considering the square root version of the statistics as in (2.1) and (2.2). Denoting the W_n corresponding to λ_{1n}, λ_{2n}, λ_{3n} by W_{1n}, W_{2n}, W_{3n} respectively, note that on A_n,

$$\lambda_{1n}/(1 + a/n) = (W_{1n} - \tfrac{1}{2}n^{-1}aH_1)^2 + o(n^{-1}). \tag{2.4}$$

The relation (2.4) suggests that for any $\lambda_n \in \mathscr{F}$ one can consider a modified version λ_n^*, where

$$\lambda_n^* = (W_n^*)^2, \tag{2.5a}$$

$$W_n^* = W_n + n^{-1/2}(bH_1^2) + n^{-1}(cH_1 + sH_1^3), \tag{2.5b}$$

the constants b, c, s, free from n, being so determined that the relation

$$P_{\theta_0}(\lambda_n^* \leqslant x) = \int_0^x g_1(z)\, dz + o(n^{-1}), \qquad \forall x \geqslant 0, \tag{2.6}$$

holds. We emphasize that the modifications of the form (2.5a), (2.5b) are quite simple and that the random terms in the modifications involve only the first derivative of the log-likelihood which, anyway, one has to compute for almost any inference problem. The constant coefficients b, c, s may, of course, depend on expectations involving the higher order derivatives, but, in a given context, these coefficients can be computed once and for all (see Theorem 2.1 below). As will be shown now, these modifications are applicable to the entire family of test statistics under consideration. Let $h^{(i)} = d^i \log f(X, \theta_0)/d\theta^i$ $(1 \leqslant i \leqslant 4)$, $L_{ijuw} = E_{\theta_0}\{(h^{(1)})^i (h^{(2)})^j (h^{(3)})^u (h^{(4)})^w\}$, $L_{iju} = L_{iju0}$, $L_{ij} = L_{ij0}$, $L_i = L_{i0}$.

THEOREM 2.1. *For every $\lambda_n \in \mathscr{F}$, there exists a unique choice of the constant coefficients b, c, s in* (2.5b) *such that* (2.6) *holds. This unique choice is given by*

$$b = -\tfrac{1}{6}L_3 - (v_1 L_{11} + v_2),$$

$$c = \tfrac{1}{8}(L_4 - 3) - \tfrac{5}{24}L_3^2 + \tfrac{1}{2}v_1(L_{21} + 1 - L_{11}L_3) + (v_1^2 - y_1)(L_{02} - 1 - L_{11}^2),$$

$$s = -\tfrac{1}{24}\{L_4 - 3 - \tfrac{8}{3}L_3^2 + 12v_1(L_{21} + 1 - L_{11}L_3) + 12v_1^2(L_{02} - 1 - L_{11}^2) + 24(y_1 L_{11}^2 + y_2 L_{11} + y_3 + y_4 L_{101})\}.$$

Proof. Using the findings in Chandra and Samanta [14] (we use the corrected version of a printing mistake there), the approximate cumulants of W_n^* as defined in (2.5b), under θ_0, are given by

$$k_{1n} = n^{-1/2}\rho_1 + o(n^{-1}), \qquad k_{2n} = 1 + n^{-1}\rho_2 + o(n^{-1}),$$
$$k_{3n} = n^{-1/2}\rho_3 + o(n^{-1}), \qquad k_{4n} = n^{-1}\rho_4 + o(n^{-1}),$$
$$k_{rn} = o(n^{-1}) \qquad (r \geqslant 5),$$

where the ρ_i's, which are free from n, are

$$\rho_1 = v_1 L_{11} + (v_2 + b), \tag{2.7a}$$

$$\rho_2 = 2\{v_1(L_{21} + 1) + (v_2 + b)L_3 + y_1(L_{02} - 1 + 2L_{11}^2) + 3y_2 L_{11} + 3(y_3 + s) + 3y_4 L_{101} + c\} + v_1^2(L_{02} - 1 + L_{11}^2) + 2(v_2 + b)^2 + 4v_1(v_2 + b)L_{11}, \tag{2.7b}$$

$$\rho_3 = L_3 + 6v_1 L_{11} + 6(v_2 + b), \tag{2.7c}$$

$$\rho_4 = L_4 - 3 + 12v_1(L_{21} + 1 + L_{11}L_3) + 24(v_2 + b) \times L_3 + 12v_1^2(L_{02} - 1 + 3L_{11}^2) + 48(v_2 + b)^2 + 96v_1(v_2 + b)L_{11} + 24(y_1 L_{11}^2 + y_2 L_{11} + y_3 + s + y_4 L_{101}). \tag{2.7d}$$

Hence, the approximate characteristic function of W_n^*, under θ_0, is given by (here $\xi = \sqrt{-1}\, t$)

$$\phi_n(\xi) = \exp(\tfrac{1}{2}\xi^2)[1 + n^{-1/2}(\rho_1 \xi + \tfrac{1}{6}\rho_3 \xi^3) + n^{-1}\{\tfrac{1}{2}(\rho_2 + \rho_1^2)\xi^2 + (\tfrac{1}{6}\rho_1\rho_3 + \tfrac{1}{24}\rho_4)\xi^4 + \tfrac{1}{72}\rho_3^2\xi^6\}] + o(n^{-1}). \tag{2.8}$$

Recalling the symmetry of the normal distribution and making use of an Edgeworth expansion (see Bhattacharya and Ghosh [8], Bhattacharya [7]) for the distribution of W_n^* under θ_0, it is clear that (2.6) will hold provided b, c, s are so chosen that

$$\rho_2+\rho_1^2=0, \qquad \tfrac{1}{6}\rho_1\rho_3+\tfrac{1}{24}\rho_4=0, \qquad \rho_3^2=0. \tag{2.9}$$

By (2.7a)–(2.7d), it can be seen that the unique solutions for b, c, s satisfying (2.9) are as in the statement of the theorem. ∎

In particular, for Rao's efficient score statistic, $v_1=v_2=y_1=y_2=y_3=y_4=0$, so that the solutions for b, c, s are simple and given by

$$b^{\mathrm{Rao}}=-\tfrac{1}{6}L_3, \qquad c^{\mathrm{Rao}}=\tfrac{1}{8}(L_4-3)-\tfrac{5}{24}L_3^2, \qquad s^{\mathrm{Rao}}=\tfrac{1}{24}(L_4-3-\tfrac{8}{3}L_3^2). \tag{2.10}$$

In particular, if $L_3=0$, as happens in many situations of practical interest (see Example 2.1 below), then $b^{\mathrm{Rao}}=0$, $c^{\mathrm{Rao}}=\frac{1}{8}(L_4-3)$, $s^{\mathrm{Rao}}=-\frac{1}{24}(L_4-3)$, so that by (2.5a) (2.5b),

$$\begin{aligned}\lambda_{2n}^*&=\{1+\tfrac{1}{4}n^{-1}(L_4-3)\}\,\lambda_{2n}-\tfrac{1}{12}n^{-1}(L_4-3)\,\lambda_{2n}^2\\&=\lambda_{2n}/\{1-n^{-1}a_0(1-\lambda_{2n}/3)\}+o(n^{-1}),\end{aligned}\tag{2.11}$$

where $a_0=(L_4-3)/4$, and this resembles the Bartlett correction for the LR statistic. It is also interesting to derive the usual Bartlett correction for the LR statistic λ_{1n} from Theorem 2.1. Since for λ_{1n}, $v_1=\frac{1}{2}$, $v_2=\frac{1}{6}L_{001}$, $y_1=\frac{3}{8}$, $y_2=\frac{5}{12}L_{001}$, $y_3=\frac{1}{24}L_{0001}+\frac{1}{9}L_{001}^2$, $y_4=\frac{1}{6}$ (see Chandra and Joshi [12]), the solutions for b, c, s are given by

$$b^{\mathrm{LR}}=s^{\mathrm{LR}}=0, \qquad c^{\mathrm{LR}}=\tfrac{1}{8}L_4-\tfrac{5}{24}L_3^2+\tfrac{1}{4}(L_{21}-L_{11}L_3)-\tfrac{1}{8}(L_{02}-L_{11}^2),$$

using simple regularity conditions (see Chandra and Mukerjee [13]). Since $b^{\mathrm{LR}}=s^{\mathrm{LR}}=0$, the above agrees with (2.4) and hence with (2.3). Also, from (2.4), (2.5b), the constant a in (2.3) eqals $-2c^{\mathrm{LR}}$; i.e.,

$$a=\tfrac{1}{4}(L_{02}-L_{11}^2-L_4)+\tfrac{5}{12}L_3^2-\tfrac{1}{2}(L_{21}-L_{11}L_3).$$

EXAMPLE 2.1. Let $X_1, X_2, \ldots,$ be i.i.d. 2×1 vector random variables each distributed as bivariate normal with zero means, unit variances, and an unknown correlation coefficient θ ($|\theta|<1$). Consider $H_0\colon \theta=0$ against the alternative $\theta\neq 0$. It can be seen that here $\mathscr{I}=1$, $L_3=0$, $L_4=9$, so that by (2.11),

$$\lambda_{2n}^*=(1+\tfrac{3}{2}n^{-1})\,\lambda_{2n}-\tfrac{1}{2}n^{-1}\lambda_{2n}^2=\lambda_{2n}/\{1-\tfrac{1}{2}n^{-1}(3-\lambda_{2n})\}+o(n^{-1}).$$

3. The Multiparameter Case: Rao's Test

The ideas of Section 2 can be extended to the multiparameter case with reference to a general class of tests along the line of (2.1), (2.2). However, in this section we present results pertaining only to Rao's test in order to simplify notations and to save space. It may also be emphasized that in this article we are primarily concerned with Rao's test in consideration of the recent studies on its optimality properties (Chandra and Joshi [12], Mukerjee [22, 24]).

Consider the setup of Section 2 with the exception that $\theta = (\theta_1, ..., \theta_p)'$ is now $p(\geqslant 2)$-dimensional. We are interested in testing $H_0: \theta = \theta_0$ against $\theta \neq \theta_0$. Also, without loss of generality, if necessary by a reparametrization, let the per observation information matrix at θ_0 be I, the $p \times p$ identity matrix. Then Rao's test statistic is given by $\lambda_{2n} = H_1' H_1$, where H_1 is a $p \times 1$ vector with its ith element given by $H_{1i} = n^{-1/2} \sum_{j=1}^{n} \partial \log f(X_j, \theta_0)/\partial\theta_i$, $1 \leqslant i \leqslant p$. Generalizing the ideas of Section 2, we consider a modified version of λ_{2n} as

$$\lambda_{2n}^* = (H_1^*)' (H_1^*), \tag{3.1a}$$

where

$$H_1^* = H_1 + n^{-1/2} B(H_1 \otimes H_1) + n^{-1}\{CH_1 + S(H_1 \otimes H_1 \otimes H_1)\}, \tag{3.1b}$$

the elements of the matrices B, C, S, which are of orders $p \times p^2$, $p \times p$, $p \times p^3$ respectively, being constants, free from n, to be so chosen that the relation

$$P_{\theta_0}(\lambda_{2n}^* \leqslant x) = \int_0^x g_p(z)\, dz + o(n^{-1}), \qquad \forall x \geqslant 0, \tag{3.2}$$

holds. Here $\otimes$ stands for Kronecker product. Note that the random terms in the modification in (3.1a), (3.1b), like those in (2.5a), (2.5b), involve only the first partial derivatives of the log-likelihood.

The following notations will be helpful in the derivation. For $1 \leqslant i, j, u, w \leqslant p$, let

$$G_{iju}^{(1)} = E_{\theta_0}\{(\partial \log f(X, \theta_0)/\partial\theta_i)(\partial \log f(X, \theta_0)/\partial\theta_j) \times (\partial \log f(X, \theta_0)/\partial\theta_u)\},$$

$$G_{ijuw}^{(2)} = E_{\theta_0}\{(\partial \log f(X, \theta_0)/\partial\theta_i)(\partial \log f(X, \theta_0)/\partial\theta_j) \times (\partial \log f(X, \theta_0)/\partial\theta_u)(\partial \log f(X, \theta_0)/\partial\theta_w)\}.$$

Note that $G_{iju}^{(1)}$, $G_{ijuw}^{(2)}$ are invariant under permutation of the subscripts. Also, for $1 \leqslant i \leqslant p$, let the elements in the ith rows of B and S be b_{iju} and

s_{ijuw}, arranged in lexicographic orders of j, u and j, u, w, respectively $(1 \leqslant j, u, w \leqslant p)$. Let c_{ij} be the (i, j)th element of C and δ_{ij} stand for Kronecker delta $(1 \leqslant i, j \leqslant p)$.

A tedious algebra, the details of which are omitted here to save space, shows that the approximate cumulants of $H_1^* = (H_{11}^*, ..., H_{1p}^*)'$, say, under θ_0, are

$$k_{1n}(H_{1i}^*) = n^{-1/2}\rho_i^{(1)} + o(n^{-1}),$$
$$k_{2n}(H_{1i}^*, H_{1j}^*) = \delta_{ij} + n^{-1}\rho_{ij}^{(2)} + o(n^{-1}),$$
$$k_{3n}(H_{1i}^*, H_{1j}^*, H_{1u}^*) = n^{-1/2}\rho_{iju}^{(3)} + o(n^{-1}),$$
$$k_{4n}(H_{1i}^*, H_{1j}^*, H_{1u}^*, H_{1w}^*) = n^{-1}\rho_{ijuw}^{(4)} + o(n^{-1}), \qquad 1 \leqslant i, j, u, w \leqslant p,$$

where for $1 \leqslant i, j, u, w \leqslant p$,

$$\rho_i^{(1)} = \sum_{q=1}^{p} b_{iqq}, \tag{3.3a}$$

$$\rho_{ij}^{(2)} = \sum \sum_{q,r=1}^{p} (b_{iqr}G_{jqr}^{(1)} + b_{jqr}G_{iqr}^{(1)}) + \sum \sum_{q,r=1}^{p} (b_{iqr}b_{jrq} + b_{iqr}b_{jqr})$$
$$+ \sum_{q=1}^{p} (s_{ijqq} + s_{iqjq} + s_{iqqj} + s_{jiqq} + s_{jqiq} + s_{jqqi}) + c_{ij} + c_{ji}, \tag{3.3b}$$

$$\rho_{iju}^{(3)} = G_{iju}^{(1)} + (b_{iju} + b_{iuj} + b_{jiu} + b_{jui} + b_{uij} + b_{uji}), \tag{3.3c}$$

$$\rho_{ijuw}^{(4)} = G_{ijuw}^{(2)} - (\delta_{ij}\delta_{uw} + \delta_{iu}\delta_{jw} + \delta_{iw}\delta_{ju})$$
$$+ \sum' s_{i_1i_2i_3i_4} + \tfrac{1}{2}\sum' \sum_{q=1}^{p} (b_{i_1qi_2}b_{i_3qi_4} + b_{i_1i_2q}b_{i_3i_4q}$$
$$+ b_{i_1qi_2}b_{i_3i_4q} + b_{i_1i_2q}b_{i_3qi_4}) + \tfrac{1}{2}\sum' \sum_{q=1}^{p} (b_{i_1qi_2} + b_{i_1i_2q})\, G_{qi_3i_4}^{(1)}, \tag{3.3d}$$

where $\sum'$ denotes sum over the 24 possible permutations (i_1, i_2, i_3, i_4) of (i, j, u, w). All higher order cumulants of H_1^* are of order $o(n^{-1})$.

Hence, as in the one-parameter case, if one considers a multivariate Edgeworth expansion for H_1^*, under θ_0, and uses the symmetry of the multivariate normal distribution, then it can be seen that (3.2) holds provided the elements of B, C, S are so chosen that, analogously to (2.9), the following hold:

$$\rho_{iuw}^{(3)}\rho_{jqr}^{(3)} = 0, \qquad \tfrac{1}{24}\rho_{ijuw}^{(4)} + \tfrac{1}{6}\rho_i^{(1)}\rho_{juw}^{(3)} = 0,$$
$$\rho_{ij}^{(2)} + \rho_i^{(1)}\rho_j^{(1)} = 0, \qquad \forall 1 \leqslant i, j, u, w, q, r \leqslant p.$$

From (3.3a)–(3.3d), it can be seen that the above hold provided

$$
\begin{aligned}
b_{iju} &= -\tfrac{1}{6} G^{(1)}_{iju}, \\
s_{ijuw} &= -\tfrac{1}{24}\Big\{ G^{(2)}_{ijuw} - (\delta_{ij}\delta_{uw} + \delta_{iu}\delta_{jw} + \delta_{iw}\delta_{ju}) \\
&\qquad - \tfrac{8}{9} \sum_{q=1}^{p} (G^{(1)}_{ijq} G^{(1)}_{uwq} + G^{(1)}_{iuq} G^{(1)}_{jwq} + G^{(1)}_{iwq} G^{(1)}_{juq}) \Big\}, \\
c_{ij} &= \tfrac{1}{8} \left\{ \sum_{q=1}^{p} G^{(2)}_{ijqq} - (p+2)\,\delta_{ij} \right\} \\
&\qquad - \sum \sum_{q,r=1}^{p} \Big\{ \tfrac{1}{9} G^{(1)}_{ijq} G^{(1)}_{qrr} + \tfrac{1}{72} G^{(1)}_{iqq} G^{(1)}_{jrr} \\
&\qquad + \tfrac{1}{12} G^{(1)}_{iqr} G^{(1)}_{jqr} \Big\}, \qquad 1 \leqslant i, j, u, w \leqslant p,
\end{aligned}
$$

which extend (2.10) to the multiparameter case.

Remark 1. The results in Chandra and Samanta [14] imply that the modifications suggested in (2.5a), (2.5b) in the one-parameter case do not alter the power, up to the third order, in the sense of Chandra and Joshi [12]. In the multiparameter case, it follows from Mukerjee [24] that the modifications suggested in (3.1a), (3.1b) always keep "average" power, up to the second order, unaltered; in fact, average power remains unaltered up to the third order if $G^{(1)}_{iju} = 0$ for each i, j, u (this ensures $B = 0$ in (3.1b)), a condition which holds in many situations—for example, in testing for the vector of location parameters in a multivariate Cauchy distribution.

Remark 2. For the modified versions of the tests as considered in this paper, the remainder terms in (2.6), (3.2) are actually of order $O(n^{-2})$—see, e.g., Chandra and Ghosh [11] and Chandra [10]. A similar phenomenon in connexion with the Bartlett correction for the LR statistic has been observed by Barndorff-Nielsen and Hall [4].

Remark 3. In this paper, we have considered simple null hypotheses. In the presence of nuisance parameter(s) the position is as follows: Let θ be the parameter of interest and m be the nuisance parameter. If θ be one-dimensional then combining the methods in this paper with those in Mukerjee [23] it should be possible to derive appropriate modifications for Rao's statistic. The problem, however, becomes much more complex for multidimensional θ—in particular, if θ and m are both multidimensional then in general one cannot employ parametric orthogonality (Cox and Reid [17]) and tensor methods (McCullagh [19, 20]) should be useful. These aspects deserve further attention.

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