On Homogeneous Contractions And Unitary Representations of SU(1,1)

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1 Introduction

Let Möb($\mathbb{D}$) be the group of biholomorphic automorphisms of the unit disk, and T be a contraction on a Hilbert space $\mathcal{H}$. Each $\varphi_{2\theta,a}$ in Möb($\mathbb{D}$) has the form

$$\varphi_{2\theta,a}(z) = e^{2i\theta}(z-a)(1-\bar{a}z)^{-1}, |a| \leq 1 \text{ and } \theta \in [0,\pi).$$

Define $\varphi_{2\theta,a}(T)$, by the usual functional calculus. We call an operator T *homogeneous*, if T is unitarily equivalent to $\varphi_{2\theta,a}(T)$ for all $\varphi_{2\theta,a}$ in Möb($\mathbb{D}$). In this paper, we obtain a family of homogeneous operators using the Sz.-Nagy-Foias model for contractions, and we study a corresponding class of projective representations of Möb($\mathbb{D}$).

In a recent paper [8], D.R. Wilkins has studied operators in $B_1(\mathbb{D})$, which are homogeneous under the action of certain Fuchsian groups. Homogeneous tuples of bounded operators on a Hilbert space are discussed in [5].

Let us fix some notation. Let

$$SU(1,1) = \left\{ \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} : |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

The group SU(1,1) acts on the unit disk by

$$\tilde{\varphi}_g(z) = (\alpha z + \beta)(\bar{\beta} z + \bar{\alpha})^{-1}, \text{ for } g = \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} \text{ in } SU(1,1).$$

Note that as a topological group $SU(1,1)$ is homeomorphic (in fact, diffeomorphic) to the product space $\mathbb{T} \times \mathbb{D}$; where $\mathbb{T}$ is the unit circle. For g in $SU(1,1)$, if we set $\theta = \,arg\, \alpha \pmod{2\pi}$ and $a = -\frac{\beta}{\alpha}$, then the map $g \to (e^{i\theta}, a)$ is a diffeomorphism, and the inverse of this map is obtained by setting $\alpha = e^{i\theta}(1 - |a|^2)^{-1/2}$ and $\beta = -a\, e^{i\theta}(1 - |a|^2)^{-1/2}$. The map $\tilde{\varphi}_g$ can now be rewritten as (we will drop the tilde)

$$\varphi_g(z) = e^{2i\theta}(z - a)(1 - \bar{a}z)^{-1}.$$

Thus, if g in $SU(1,1)$ is identified with $(e^{i\theta}, a)$, where $0 \le \theta < 2\pi$, and $|a| < 1$, then the map $q : SU(1,1) \to \text{Möb}(\mathbb{D})$, defined by

$$q(g) = q(e^{i\theta}, a) = \varphi_g = \varphi_{2\theta, a}, \; \theta \in [0, 2\pi) \tag{1.1}$$

exhibits $SU(1,1)$ as a two fold cover of Möb($\mathbb{D}$). The covering map is just q.

We define a function on $SU(1,1) \times \mathbb{D}$ as follows

$$j(g, z) = \varphi_g'(z)^{1/2} = (\bar{\beta} z + \bar{\alpha})^{-1} = e^{i\theta} \frac{(1 - |a|^2)^{1/2}}{1 - \bar{a}z}. \tag{1.2}$$

Note that j satisfies the relations

$$j(g_1 g_2, z) = j(g_1, \varphi_{g_2}(z)) j(g_2, z),$$

$$j(e, z) = 1.$$

Recall that a *projective representation* is a mapping $U : g \to U_g$ of the group G into the unitary group $\mathcal{U}(\mathcal{H})$ on some Hilbert space such that

1. $U_e = 1$, where e is the identity of G,
2. $U_g U_h = c(g, h) U_{g \circ h}$, where $c(g, h)$ is in $\mathbb{T}$,
3. $g \to \langle U_g \zeta, \eta \rangle$, is a Borel function for each $\zeta, \eta \in \mathcal{H}$.

The function c is the multiplier associated with U and is uniquely determined by U. It has the following properties

$c(g, e) = 1 = c(e, g)$, where e is the identity of the group G, $g \in G$.

$c(k, gh)c(g, h) = c(k, g)c(kg, h),\ g, h,$ and k in G.

The set of all multipliers M on the group G is itself a group, called the multiplier group. If there is a continuous function $f : G \to \mathbb{T}$ such that

$$c(g, h) = f(g)f(h)f(gh)^{-1},$$

then the multiplier c is said to be trivial. Note that in this case, if we set

$$V_g = f(g)^{-1}U_g,$$

then $g \to V_g$ is a *linear representation* of the group G, that is a strongly continuous homomorphism ([7], Lemma 8.28, p.34).

It was pointed out in [4], that if a homogeneous operator is irreducible then it gives rise to a projective representation of Möb($\mathbb{D}$). Since the map $g \to \varphi_g$ is a continuous homomorphism of groups, we may lift any projective representation to the group SU(1,1). However, it turns out that the projective representations of Möb($\mathbb{D}$) we obtain from our examples of homogeneous operators are in fact linear representations when lifted to $SU(1, 1)$. In the following section, we discuss the characteristic function for a contraction, and obtain some simple properties of a homogeneous contraction. In particular, we show that a contraction with constant characteristic function must be homogeneous. Next, we point out that the study of homogeneous operators is related to that of *systems of imprimitivity*, introduced by Mackey (cf. [7], p.58). We then obtain explicitly the projective representation associated with the class of homogeneous contractions which have constant characteristic function and show that the projective representations of Möb($\mathbb{D}$), obtained in this manner, lift to linear representations of $SU(1, 1)$.

2 The Characteristic Operator Function for a Contraction

Sz.-Nagy-Foias model theory for contractions associates to each contraction an operator valued holomorphic function $\Theta_T(z)$ on the unit disk.

Let us fix the following notation.

$$D_T \;=\; \sqrt{I - T^*T}$$

$$
\begin{aligned}
D_{T^*} &= \sqrt{I - TT^*} \\
\mathcal{D}_T &= ran\, D_T \\
\mathcal{D}_{T^*} &= ran\, D_{T^*} \\
\Theta_T(z) &= -T + zD_{T^*}(I - zT^*)^{-1}D_T \in \mathcal{L}(\mathcal{D}_T, \mathcal{D}_{T^*}) \\
\Delta_T &= \sqrt{I - \Theta_T\Theta_{T^*}} \\
H &= H^2_{\mathcal{D}_{T^*}} \oplus \Delta_T L^2_{\mathcal{D}_T} \\
\mathcal{M} &= \{(\Theta_T f, \Delta_T f) : f \in H^2_{\mathcal{D}_T}\} \\
\mathcal{M}^\perp &= H \ominus \mathcal{M}.
\end{aligned}
$$

By Sz.-Nagy-Foias theory, T is unitarily equivalent to the operator

$$\mathcal{T} : (f, g) \longrightarrow (zf, e^{it}g)$$

on H, compressed to $\mathcal{M}^\perp$. The compression of $\mathcal{T}$ will again be denoted T. It is the basic theorem of Sz.-Nagy and Foias that *two completely non unitary contractions operators T_1 and T_2 are unitarily equivalent if and only if their characteristic functions coincide*, that is, there exist (constant) unitary operators U and V such that $U\Theta_{T_1}(z)V = \Theta_{T_2}(z)$, for all z in the unit disk (cf. [6], Proposition 3.3, p.256). The dimensions of $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$ are called the defect indices of T.

THEOREM 2.1 *Let T be a completely nonunitary contraction with at least one of the defect indices equal to 1. The operator T is homogeneous if and only if the characteristic operator function for T is a constant.*

Proof: If $\Theta_T(z)$ denotes the characteristic operator function for T, then the characteristic operator function $\Theta_{\varphi_g(T)}$ coincides with that of $\Theta_T(z)$, that is

$$U_g\Theta_{\varphi_g(T)}(z)V_g^* = \Theta_T(\varphi_g^{-1}(z)), \tag{2.1}$$

(cf. [6], p. 240). If T is unitarily equivalent to $\varphi_g(T)$ for all g in G then

$$U'_g\Theta_{\varphi_g(T)}(z)V_g'^* = \Theta_T(z).$$

It follows that

$$U_g^*U'_g\Theta_T(z)V_g'^*V_g = \Theta_T(\varphi_g^{-1}(z)).$$

Since φ_g acts transitively on the unit disk, setting $z = 0$ and $\omega = \varphi_g^{-1}(0)$, we obtain

$$U_g^*U'_g\Theta_T(0)V_g'^*V_g = \Theta_T(\omega).$$

We note that $\|\Theta_T(\omega)\|$ is in fact equal to $\|\Theta_T(0)\|$, and if one of the defect indices is 1, then the characteristic function $\Theta_T(\omega)$ is either a $\mathcal{D}_T$ or a $\mathcal{D}_{T^*}$ valued holomorphic function on the unit disk. In any case, the unit ball of the range is strictly convex, and by the strong form of the maximum principle for vector valued analytic functions (cf. [1], Corollary III.1.5, p.270), it follows that $\Theta_T(z)$ is a constant.

The converse statement is trivial. Certainly if the characteristic function $\Theta_T(z)$ is constant, then using 2.1, we find that

$$U_g \Theta_{\varphi_g(T)}(z) V_g^* = \Theta_T(\varphi_g^{-1}(z)) = \Theta_T(z),$$

that is, the characteristic functions Θ_T and $\Theta_{\varphi_g(T)}$ coincide. In other words, T is homogeneous and the proof is complete.

Unfortunately, there exist completely non unitary contractions with non constant characteristic functions, which are homogeneous. In fact, all the homogeneous operators in $B_1(\mathbb{D})$ discussed in [4], except the unilateral shift, are contractions of class $C_{\cdot 0}$, and their characteristic functions are inner. If the characteristic function of any of these operators were to be a constant then $T\,|_{\mathcal{D}_T}$ would have to be an isometry. However, this is not the case for any of the homogeneous operators in $B_1(\mathbb{D})$.

Corollary 2.1 *The unitary dilation U of a homogeneous operator T is itself homogeneous and is therefore a bilateral shift of uniform multiplicity.*

Proof: Since T is unitarily equivalent to $\varphi_g(T)$, it follows that the unitary dilation U is also unitarily equivalent to $\varphi_g(U)$. However, φ_g acts transitively on the unit circle, and if μ is the spectral measure for U then $\mu \circ \varphi_g$ must be equivalent to the measure μ for all g, that is, the measure μ is a quasi invariant (cf. [7], p.14) measure on the unit circle, the measure class of such a measure μ is the same as that of the Lebesgue measure on $\mathbb{T}$. If T is homogeneous, then

$$\|\Theta_T(\omega)\| = \|\Theta_T(0)\| \leq 1,$$

and consequently, $\Delta_T(\omega)$ is invertible for all ω. This implies that the multiplicity is constant and the proof is complete.

Let $\mathcal{L}_{inv}(\mathcal{H})$ denote the set of invertible operators on $\mathcal{H}$ and let $L : G \to \mathcal{L}_{inv}(\mathcal{H})$ be a uniformly bounded homomorphism. The map L is said to be *unitarizable*, if there exists a invertible operator $\mathcal{L}$ such that $\mathcal{L} L_g \mathcal{L}^{-1}$ is unitary for all g in G. There are known examples (cf. [3], Theorem 5) of uniformly bounded homomorphisms $L : SU(1,1) \to \mathcal{L}_{inv}(\mathcal{H})$, which are not unitarizable.

PROPOSITION 2.1 *An irreducible contraction S is similar to a homogeneous operator T if and only if $L_g^{-1} S L_g = \varphi_g(S)$ for all g in G, and the map $L : g \to L_g$ is an uniformly bounded map into $\mathcal{L}_{inv}(\mathcal{H})$, which is also unitarizable.*

Proof: Suppose $\mathcal{L} T \mathcal{L}^{-1} = S$. Let $U : g \to U_g$ be the projective representation associated with the homogeneous operator $T = \mathcal{L}^{-1} S \mathcal{L}$. The map $L : g \to \mathcal{L}^{-1} U_g \mathcal{L}$ is a uniformly bounded representation of G, which is evidently unitarizable, and $L_g^{-1} S L_g = \varphi_g(S)$.

On the other hand, if S is any operator such that $L_g^{-1} S L_g = \varphi_g(S)$ and the map $L : g \to L_g$ is uniformly bounded, then to say $g \to L_g$ is unitarizable means that for some invertible operator $\mathcal{L}$, the operator $\mathcal{L} L_g \mathcal{L}^{-1}$ is unitary and we have

$$\mathcal{L} L_g \mathcal{L}^{-1} (\mathcal{L} S \mathcal{L}^{-1}) \mathcal{L} L_g^{-1} \mathcal{L}^{-1} = \mathcal{L}(\varphi_g(S)) \mathcal{L}^{-1} = \varphi_g(\mathcal{L} S \mathcal{L}^{-1}).$$

Thus, the operator $T = \mathcal{L} S \mathcal{L}^{-1}$ is homogeneous and is similar to S. The proof is now complete.

If T and $\varphi_g(T)$ are similar for all g, we say that the operator T is *weakly homogeneous*. How are the homogeneous operators related to weakly homogeneous operators? If, for example, we can find an operator T, which is weakly homogeneous but not similar to any homogeneous operator, with the added property that the map $L : g \to L_g$ implementing the similarity is both uniformly bounded and is a homomorphism, then in view of the proposition, we would have obtained a representation of SU(1,1), which is not unitarizable.

3 Systems of Imprimitivity

Let G be a locally compact, second countable, continuous group and X be a locally compact metrizable space. If G acts continuously and transitively on X, then X is a transitive, G-space. Let ϕ be a $*$-homomorphism of $C(X)$ into $\mathcal{L}(\mathcal{H})$ and $U : g \to U_g$ be a projective unitary representation of G on $\mathcal{H}$. Then (U, ϕ, X) is a *system of imprimitivity* based on X, for the group G if we also have

$$U_g \phi(f) U_g^* = \phi(f \circ g^{-1}) \text{ for all } g \text{ in } G. \tag{3.1}$$

If X is compact then classification of such systems of imprimitivity is obtained through classification of $*$-homomorphisms of the C*-algebra $C(X)$. Mackey shows that, if $X = G/H$ for some closed subgroup H of G, then there is a one-one correspondence between systems of imprimitivity based on X and representations of G induced from the subgroup H. A good reference for all this material is ([2],[7]).

Let $U : G \to \mathcal{U}(\mathcal{H})$ be a projective representation of a locally compact group G, and X be a transitive G-space. Let $\mathcal{A}$ be a function algebra, that is, a subalgebra (not necessarily closed with respect to $*$) of the C*-algebra of continuous functions $C(X)$, and $\phi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ be a contractive homomorphism. Define a *system of imprimitivity for the group G over a function algebra $\mathcal{A}$*, to be a triple (U, ϕ, X) satisfying 3.1. Typically, if $G = \text{Möb}(\mathrm{D})$, then there is a subgroup H such that $G/H = \mathbb{D}$, and the algebra $\mathcal{A}$ is the disk algebra $\mathcal{A}(\mathbb{D})$; in this case we identify $\mathcal{A}(\mathbb{D})$ as a subalgebra of the C*-algebra $C(\mathbb{T})$.

Note that if T is homogeneous, then we obtain a projective unitary representation $U : g \to U_g$ of G such that

$$U_g T U_g^* = g \cdot T,$$

here we have set $g \cdot T = \varphi_g(T)$. If ϕ is the contractive homomorphism of the disk algebra $\mathcal{A}(\mathbb{D})$ defined via $p \to p(T)$then we see that

$$U_g \phi(p) U_g^* = U_g p(T) U_g^* = p(U_g T U_g^*) = p \circ \varphi_g(T), \tag{3.2}$$

where we are thinking of $g = h^{-1}$, so that the map $h \to U_g$ is a projective representation. The relation 3.2 is the imprimitivity relation on the disk algebra. On the other hand, given a system of imprimitivity for G over the disk algebra, we obtain a homogeneous operator T by simply setting $T = \phi(z)$. Thus, *there is a natural one to one correspondence between homogeneous contractions and systems of imprimitivity over the disk algebra.*

THEOREM 3.1 *Let $(U, \phi, \mathbb{T})$ be a system of imprimitivity over $C(\mathbb{T})$. If $\mathcal{H}$ is a semi invariant subspace for $\phi(id\,|\mathbb{T})$ and each U_g leaves $\mathcal{H}$ invariant, then the operator $T = P_{\mathcal{H}} \phi(id\,|\mathbb{T})$ is homogeneous with $U_g T U_g^* = \varphi_g(T)$. Conversely, given an irreducible homogeneous operator T (or, equivalently, a system of imprimitivity over $\mathcal{A}(\mathbb{D})$), let $g \to V_g$ be the associated projective representation of G on $\mathcal{H}$ satisfying $V_g T V_g^* = \varphi_g(T)$. Let W_T be the minimal unitary dilation for T on $\mathcal{K}$ containing $\mathcal{H}$ as a semi invariant subspace. Then there exists a projective representation $U : g \to U_g$ of G on $\mathcal{K}$, which leaves $\mathcal{H}$ invariant $U_g W_T U_g^* = \varphi_g(W_T)$ and $U_g\,|_{\mathcal{H}} = V_g$.*

Proof: One half of this theorem is easy to prove. We need only observe that if $\mathcal{H}$ is invariant for U_g, then the projection $P_{\mathcal{H}}$ commutes with U_g and U_g^*. Thus,

$$P_{\mathcal{H}} \phi(f \circ \varphi_g) P_{\mathcal{H}} = P_{\mathcal{H}} U_g \phi(f) U_g^* P_{\mathcal{H}} = U_g P_{\mathcal{H}} \phi(f) P_{\mathcal{H}} U_g^*.$$

For the converse, we take W_T to be the matrix

$$\begin{bmatrix} \ddots & & & & & \\ & I & & & & \\ & & D_T & -T^* & & \\ & & \boxed{T} & D_{T^*} & & \\ & & & & I & \\ & & & & & \ddots \end{bmatrix},$$

where the box as usual denotes the (0,0) entry. If we restrict W_T to the subspace

$$\mathcal{K}_T = \left\{ (h_n) \in \oplus_{n=-\infty}^{\infty} \mathcal{H} : h_n \in \begin{array}{ll} \mathcal{D}_T & \text{for } n < -1, \\ \mathcal{H} & \text{for } n = 0 \text{ and} \\ \mathcal{D}_{T^*} & \text{for } n > 1 \end{array} \right\},$$

then W_T is a minimal unitary dilation of T. However since T is an irreducible homogeneous operator on $\mathcal{H}$, there is a projective representation $g \to V_g$ of G such that $V_g T V_g^* = \varphi_g(T)$. Let U_g be the diagonal operator acting on $\oplus_{-\infty}^{\infty} \mathcal{H}$, with each diagonal entry equal to V_g. Note that $\varphi_g(W_T)$ (cf. [6], Proposition 4.3, p.14) is a minimal unitary dilation for the operator $\varphi_g(T)$. Since the unitary operator V_g intertwines T and $\varphi_g(T)$, it is clear that U_g will map $\mathcal{K}$ onto $\mathcal{K}_{\varphi_g(T)}$. However, $\mathcal{K}_T$ is equal to $\mathcal{K}_{\varphi_g(T)}$. Therefore, U_g is a unitary operator on $\mathcal{K}_T$ which leaves the subspace $\mathcal{H}$ invariant. It is also clear that U_g intertwines W_T and $\varphi_g(W_T)$. Since V_g is a projective representation of the group G and U_g is defined to be a block diagonal matrix with each diagonal block equal to V_g, it follows that U_g is itself a projective representation of the group G. This completes the proof of the theorem.

The second half of the theorem says that *every system of imprimitivity over the disk algebra $\mathcal{A}(\mathbb{D})$ lifts to a system of imprimitivity over the C^*-algebra of continuous functions $C(\mathbb{T})$.*

4 Contractions with Constant Characteristic Function and Unitary Representations of $SU(1,1)$

THEOREM 4.1 *Let T be a completely nonunitary contraction with constant characteristic function*

$$\Theta_T(z) = C \in \mathcal{L}(\mathcal{D}_T, \mathcal{D}_{T^*}),$$

where C is independent of z, and $\|C\| < 1$. Then for any linear fractional transformation φ mapping $\mathbb{D}$ onto $\mathbb{D}$, $\varphi(T)$ is unitarily equivalent to T:

$$\varphi(T) = U_\varphi T U_\varphi^*. \tag{4.1}$$

Furthermore, the unitary operators U_φ can be chosen so that $\varphi \to U_\varphi$ is continuous in the strong operator topology and so that

$$U_\psi U_\varphi = c(\psi, \varphi) U_{\varphi \circ \psi}$$

where $c(\psi, \varphi)$ is a complex constant of modulus 1.

Proof: By Sz.-Nagy-Foias theory, T is unitarily equivalent to the operator

$$\mathcal{T} : (f, g) \longrightarrow (zf, e^{it} g)$$

on H, compressed to $\mathcal{M}^\perp$, in the notation of section 2. The compression of $\mathcal{T}$ will again be denoted T

$$T : (f, g) \longrightarrow P_{\mathcal{M}^\perp}(zf, e^{it} g),$$

since $\mathcal{M}$ is invariant under $\mathcal{T}$, the operator T is a (power) compression. Thus,

$$\varphi(T)(f, g) = P_{\mathcal{M}^\perp}(\varphi(z) f, \varphi(e^{it}) g) \tag{4.2}$$

holds for φ analytic in $|z| \leq 1$. In particular, 4.2 holds for a linear fractional transformation φ as in the statement of the theorem.

The following is a characterization of the space $\mathcal{M}^\perp$:

$$\mathcal{M}^\perp = \{(f, -C^*(I - CC^*)^{-1/2} f + e^{-it} h) : f \in H^2_{\mathcal{D}_{T^*}}, h(e^{-it}) \in H^2_{\mathcal{D}_T}\} \tag{4.3}$$

Indeed, since $C^*(I - CC^*)^{-1/2} = \Delta^{-1} C^*$, we have, for $g \in H^2_{\mathcal{D}_T}$

$$\begin{aligned} &< (f, -C^*(I - CC^*)^{-1/2} f), (Cg, \Delta g) > \\ &\qquad =< f, Cg > - < C^* f, g >= 0 \end{aligned}$$

and $< (0, e^{-it} h), (Cg, \Delta g) >=< e^{-it} h, \Delta g >= 0$, since $\Delta g \in H^2_{\mathcal{D}_T}$ and $e^{-it} h \perp H^2_{\mathcal{D}_T}$. This proves $\supseteq$ in 4.3.

To prove $\subseteq$ in 4.3, suppose $(g_1, g_2) \in \mathcal{H}$ is orthogonal to the right side of 4.3. Since $(g_1, g_2) \perp (0, e^{-it} h)$, we have $g_2 \in H^2_{\mathcal{D}_{T^*}}$. Now for $f \in H^2_{\mathcal{D}_{T^*}}$,

$$(g_1, g_2) \perp (f, -C^*(I - CC^*)^{-1/2} f).$$

So

$$< g_1, f >=< g_2, C^*(I-CC^*)^{-1/2}f >,$$

or

$$g_1 - (I-CC^*)^{-1/2}Cg_2 \perp H^2_{\mathcal{D}_{T^*}}.$$

It follows that

$$g_1 = (I-CC^*)^{-1/2}Cg_2 = C\Delta^{-1}g_2$$

and therefore

$$(g_1, g_2) = (Ch, \Delta h) \in \mathcal{M} \text{ (where } h = \Delta^{-1}g_2 \in H^2_{\mathcal{D}_T}).$$

Now we prove that

$$P_{\mathcal{M}^\perp}(0, h_0) = (-C\Delta h_0, C^*Ch_0) \tag{4.4}$$

for $h_0 \in \mathcal{D}_T$ (i.e. h_0 a constant function in $L^2_{\mathcal{D}_T}$). First,

$$\begin{aligned}
&(-C\Delta h_0, C^*Ch_0)\\
&= (-(I-CC^*)^{1/2}Ch_0, C^*(I-CC^*)^{-1/2}(I-CC^*)^{1/2}Ch_0)\\
&= -((I-CC^*)^{1/2}Ch_0, -C^*(I-CC^*)^{-1/2}(I-CC^*)^{1/2}Ch_0) \in \mathcal{M}^\perp.
\end{aligned}$$

Secondly,

$$\begin{aligned}
(0, h_0) - (-C\Delta h_0, C^*Ch_0) &= (0, h_0) + (C\Delta h_0, -C^*Ch_0)\\
&= (C\Delta h_0, \Delta^2 h_0) \in \mathcal{M}.
\end{aligned}$$

This proves 4.4.

Now, we can characterize the action of T on $\mathcal{M}^\perp$ by

$$\begin{aligned}
&T(f, -C^*(I-CC^*)^{-1/2}f + e^{-it}h)\\
&= P_{\mathcal{M}^\perp}(zf, -C^*(I-CC^*)^{-1/2}e^{it}f + h)\\
&= (zf, -C^*(I-CC^*)^{-1/2}e^{it}f + e^{-it}(e^{it}(h - \hat{h}(0)))) + P_{\mathcal{M}^\perp}(0, \hat{h}(0))\\
&= (zf, -C^*(I-CC^*)^{-1/2}e^{it}f + e^{-it}(e^{it}(h - \hat{h}(0)))) + (-C\Delta\hat{h}(0), C^*C\hat{h}(0))\\
&= (zf - C\Delta\hat{h}(0), -C^*(I-CC^*)^{-1/2}e^{it}f + h - \Delta^2\hat{h}(0)).
\end{aligned}$$

Now, we will write φ for $\varphi_{2\theta,a}$, which has the form

$$\varphi(z) = e^{2i\theta}(z-a)(1-\bar{a}z)^{-1} \in \text{Möb}(\mathbb{D}).$$

We define elements of $\mathcal{M}^{\perp}$ by

$$\begin{aligned}\Phi(f,n) &= \varphi(e^{it})^{n-1}(1-\bar{a}e^{it})^{-1}(f,-C^*(I-CC^*)^{-1/2}f),\, f\in\mathcal{D}_{T^*}\\ \Phi(f,-n) &= \overline{\varphi(e^{it})}^{\,n}(1-\bar{a}e^{-it})^{-1}(0,f),\ f\in\mathcal{D}_T.\end{aligned}$$

for n = 1,2,..., it is clear that, for a given φ and for $n=\pm1,\pm2,\ldots,\{\Phi(f,n)\}$ form a basis for $\mathcal{M}^{\perp}$. Furthermore,

$$<\Phi(f,n),\Phi(g,m)>=0 \text{ if } n\neq m.$$

Also, if $n>0$

$$\begin{aligned}&<\Phi(f,n),\Phi(g,n)>\\ &= \ <(1-\bar{a}e^{it})^{-1}f,(1-\bar{a}e^{it})^{-1}g>\\ &\quad +<(1-\bar{a}e^{it})^{-1}C^*(I-CC^*)^{-1/2}f,(1-\bar{a}e^{it})^{-1}C^*(I-CC^*)^{-1/2}g>\\ &= (1-|a|^2)^{-1}[<f,g>+<(I-CC^*)^{-1/2}CC^*(I-CC^*)^{-1/2}f,g>]\\ &= (1-|a|^2)^{-1}<[I+CC^*(I-CC^*)^{-1}]f,g>\\ &= (1-|a|^2)^{-1}<(I-CC^*)^{-1}f,g>.\end{aligned}$$

and if $n<0$,

$$\begin{aligned}<\Phi(f,n),\Phi(g,n)> &= \ <(0,(1-\bar{a}e^{it})^{-1}f),(0,(1-\bar{a}e^{it})^{-1}g)>\\ &= (1-|a|^2)^{-1}<f,g>.\end{aligned}$$

For $\varphi(e^{it})=e^{it}$, we denote $\Phi(f,n)$ by $I(f,n)$ (I for identity function),

Define the operator $U_\varphi:\mathcal{M}^{\perp}\to\mathcal{M}^{\perp}$ by

$$U_\varphi I(f,n)=(I-|a|^2)^{1/2}\Phi(f,n)$$

for $n\neq0$ and $f\in\mathcal{D}_T$ if $n<0, f\in\mathcal{D}_{T^*}$ if $n>0$. Note that U_φ is unitary and satisfies

$$U_\varphi(f(z),g(e^{it}))=(1-|a|^2)^{1/2}(1-\bar{a}e^{it})^{-1}(f\circ\varphi,g\circ\varphi),$$

for $(f,g)\in\mathcal{M}^{\perp}$.

We compute, for $n>0$ and $f\in\mathcal{D}_{T^*}$,

$$\begin{aligned}U_\varphi TI(f,n) &= U_\varphi T(z^{n-1}f,-C^*(I-CC^*)^{-1/2}e^{i(n-1)t}f)\\ &= U_\varphi(z^nf,-C^*(I-CC^*)^{-1/2}e^{int}f)\\ &= U_\varphi I(f,n+1)\\ &= (1-|a|^2)^{1/2}\Phi(f,n+1).\end{aligned}$$

If $n > 1$ and $f \in \mathcal{D}_T$,

$$\begin{aligned} U_\varphi T I(f, -n) &= U_\varphi T(0, e^{-int} f) = U_\varphi(0, e^{-i(n-1)t} f) \\ &= U_\varphi I(f, -n+1) = (1 - |a|^2)^{1/2} \Phi(f, -n+1) \end{aligned}$$

and, if $f \in \mathcal{D}_T$,

$$\begin{aligned} U_\varphi T I(f, -1) &= U_\varphi T(0, e^{-it} f) = U_\varphi(-C\Delta f, C^* C f) \\ &= U_\varphi I(-(I - CC^*)^{1/2} C f, 1) \\ &= (1 - |a|^2)^{1/2} \Phi(-(I - CC^*)^{1/2} C f, 1). \end{aligned}$$

To complete the proof of 4.1, we apply the relation 4.2, to get, for $n > 0$,

$$\varphi(T)\Phi(f, n) = \Phi(f, n+1),$$

for $n > 1$,

$$\varphi(T)\Phi(f, -n) = \Phi(f, -n+1)$$

and, for $n = -1$,

$$\begin{aligned} \varphi(T)\Phi(f, -1) &= P_{\mathcal{M}^\perp}(1 - \bar{a}e^{it})^{-1}(0, f) \\ &= (1 - \bar{a}e^{it})^{-1}(-C\Delta f, C^* C f) \\ &= \Phi(-(I - CC^*)^{1/2} C f, 1). \end{aligned}$$

(The next to last equality is verified by checking that the right side lies in $\mathcal{M}^\perp$ and the difference of the left and right sides lies in $\mathcal{M}$.)

Thus, for all $n > 0$,

$$U_\varphi T I(f, n) = (1 - |a|^2)^{1/2} \varphi(T)\Phi(f, n) = \varphi(T) U_\varphi I(f, n)$$

so that 4.2 holds.

To prove $\varphi \to U_\varphi$ is continuous from the uniform topology to the strong topology, suppose $\varphi_k(z)$ converges uniformly to $\varphi(z)$ (in $|z| \le 1$). We need to show

$$U_{\varphi_k} f \to U_\varphi f \text{ for } f \in \mathcal{M}^\perp. \tag{4.5}$$

Write

$$f = \sum_{n \neq 0} I(f_n, n),$$

where

$$\sum_{-\infty}^{-1} \|(I - CC^*)^{1/2} f_n\|^2 + \sum_{1}^{\infty} \|f_n\|^2 < \infty.$$

Given $\epsilon > 0$, choose N so that

$$\sum_{N \leq |n|} \|I(f_n, n)\|^2 < \epsilon^2/8.$$

For each n, it is clear that

$$(1 - |a_k|^2)^{1/2} \Phi_k(f_n, n) \to (1 - |a|^2)^{1/2} \Phi(f_n, n)$$

in $\mathcal{M}^\perp$, where a_k is the zero of φ_k and a is the zero of φ. Therefore, there is a positive integer K such that

$$\|(1 - |a_k|^2)^{1/2} \Phi_k(f_n, n) - (1 - |a|^2)^{1/2} \Phi(f_n, n)\| < \epsilon/(2N)$$

for $0 < |n| < N$ and $k > K$. Therefore, if $k > K$,

$$\begin{aligned}
&\|U_{\varphi_k} f - U_\varphi f\| \\
&= \|(1 - |a_k|^2)^{1/2} \sum_{n \neq 0} \Phi_k(f_n, n) - (1 - |a|^2)^{1/2} \sum_{n \neq 0} \Phi(f_n, n)\| \\
&\leq \sum_{0 < |n| < N} \|(1 - |a_k|^2)^{1/2} \Phi_k(f_n, n) - (1 - |a|^2)^{1/2} \Phi(f_n, n)\| \\
&\qquad + 2[\sum_{N \leq |n|} \|I(f_n, n)\|^2]^{1/2} < \epsilon,
\end{aligned}$$

which proves 4.5.

To prove the last assertion of the theorem, let

$$\varphi(z) = e^{2i\theta}(z - a)(1 - \bar{a}z)^{-1}, \psi(z) = e^{2i\eta}(z - b)(1 - \bar{b}z)^{-1},$$

where, $|a|, |b| < 1,\ \ \theta, \eta \in [0, \pi)$. Then

$$\varphi \circ \psi(z) = e^{2i(\theta+\eta)}(1 + \bar{b}ae^{-2i\eta})(1 + b\bar{a}e^{2i\eta})^{-1}(z - d)(1 - \bar{d}z)^{-1},$$

where, $d = (e^{2i\eta}b + a)(e^{2i\eta} + \bar{b}a)^{-1}$. We have

$$1 - |d|^2 = (1 - |a|^2)(1 - |b|^2)|e^{2i\eta} + \bar{b}a|^{-2}$$

and so

$$\begin{aligned}
&U_\psi U_\varphi(f(z), g(e^{it})) \\
&\quad = (1-|a|^2)^{1/2}(1-|b|^2)^{1/2}(1-\bar{a}\psi)^{-1}(1-\bar{b}e^{it})^{-1}.(f\circ\varphi\circ\psi, g\circ\varphi\circ\psi) \\
&\quad = (1-|a|^2)^{1/2}(1-|b|^2)^{1/2}(1+\bar{a}be^{2i\eta})^{-1}(1-\bar{d}e^{it})^{-1}.(f\circ\varphi\circ\psi, g\circ\varphi\circ\psi) \\
&\quad = |e^{2i\eta}+\bar{b}a|(1+\bar{a}be^{2i\eta})^{-1}U_{\varphi\circ\psi}.
\end{aligned}$$

This completes the proof of the theorem.

For the Möbius transformation $\varphi = \varphi_{2\theta,a}$ of the theorem, let

$$f(\varphi) = e^{i\theta}.$$

Then we have

$$U_\varphi^* U_\psi^* U_{\varphi\circ\psi} = f(\varphi)f(\psi)/f(\varphi\circ\psi).$$

Indeed, if we write $\psi(z) = \psi_{2\eta,b}(z) = e^{2i\eta}(z-b)(1-\bar{b}z)^{-1}$ and φ is as above, then

$$\varphi\circ\psi(z) = e^{2i(\theta+\eta)}(1+\bar{b}ae^{-2i\eta})(1+b\bar{a}e^{2i\eta})^{-1}(z-d)(1-\bar{d}z)^{-1},$$

and so $f(\varphi\circ\psi) = e^{i(\theta+\eta)}[(1+\bar{b}ae^{-2i\eta})(1+b\bar{a}e^{2i\eta})^{-1}]^{1/2}$, and

$$\begin{aligned}
f(\varphi)f(\psi)/f(\varphi\circ\psi) &= e^{i\theta}e^{i\eta}e^{-i(\theta+\eta)}[(1+b\bar{a}e^{2i\eta})(1+\bar{b}ae^{-2i\eta})^{-1}]^{1/2} \\
&= [(1+b\bar{a}e^{2i\eta})^2|1+b\bar{a}e^{2i\eta}|^{-2}]^{1/2} \\
&= (1+b\bar{a}e^{2i\eta})|1+b\bar{a}e^{2i\eta}|^{-1} \\
&= U_\varphi^* U_\psi^* U_{\varphi\circ\psi}.
\end{aligned}$$

by the last step in the proof of the theorem. The function f is not continuous on the group $\text{Möb}(\mathbb{D})$ and we cannot infer that that the map $\varphi \to f(\varphi)^{-1}U_\varphi$ is a linear representation.

However, the map $V : SU(1,1) \to \mathcal{U}(\mathcal{M}^\perp)$ defined by

$$V(g) = V(e^{i\theta}, a) = e^{i\theta}U\circ q(e^{i\theta}, a) = e^{i\theta}U_{\varphi_{2\theta,a}},$$

where q is the quotient map (see 1.1); is a linear (anti)representation of $SU(1,1)$. Note that

$$V(g) = j(g,\cdot)R_g, \text{ where } R_g f = f\circ\varphi_g,$$

see 1.2.

How does the representation V decompose in terms of the known irreducible representations of $SU(1,1)$? When both the defect indices of the operator T are one, we can show that the associated representation V is unitarily equivalent to the direct sum of two copies of the discrete series representation of $SU(1,1)$ corresponding to the Hardy space.

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References

[1] T. Franzoni and E. Vesentini, *Holomorphic maps and invariant distances*, North-Holland Mathematics Studies # 40, 1980.

[2] A.A. Kirillov, *Elements of the theory of representations*, Springer Verlag, 1976.

[3] R.A. Kunze and E.M. Stein, *Uniformly bounded representations and harmonic analysis of the real* 2×2 *unimodular group*, Amer. J. Math., 82 (1960), 1-62.

[4] G. Misra, *Curvature and discrete Series representation of* $SL_2(R)$, Integral Equations Operator Theory, 9 (1986), 452-459.

[5] G. Misra and N.S.N. Sastry, *Homogeneous tuples of operators, and holomorphic discrete series representation of some classical groups*, J. Operator Theory, 24 (1990), 23-32.

[6] B. Sz.-Nagy and C. Foias, *Harmonic analysis of Hilbert space operators*, North-Holland, 1970.

[7] V.S. Varadarajan, *Geometry of quantum theory*, vol II, Van Nostrand Reinhold Company, 1970

[8] D.R. Wilkins, *Operators, Fuchsian groups and automorphic vector bundles*, Preprint.