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FREIDLIN-WENTZELL TYPE ESTIMATES FOR ABSTRACT WIENER SPACES

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SUMMARY. The paper extends certain estimates for probabilities of Gaussian processes to abstract Wiener spaces. Various applications are studied.

1. Introduction

It is the aim of this paper to extend to abstract Wiener spaces certain estimates for probabilities of Gaussian stochastic processes obtained by Freidlin (1972). These results of Freidlin are, in the context of Gaussian processes, analogous to results obtained by him and Wentzell concerning small random perturbations of dynamical systems (Wentzell and Freidlin 1970).

Theorems 1 and 2 are the basic results of the paper. The notion of an "action functional", introduced in Freidlin (1972) has a natural meaning as the square of the norm of the reproducing kernel Hilbert space of the abstract Wiener space. The remainder of the paper is devoted to the following applications. In Section 4 the problem of high level occupation times of continuous Gaussian processes is considered and a different proof as well as an extension of Marlow's result is obtained (Marlow, 1973). An application to tail probabilities of continuous functionals of Gaussian processes is made in Section 5. The results obtained bear comparison with those of Marlow (1970). An analogue of the Laplace asymptotic formula for integrals on abstract Wiener spaces is derived in Section 6. This formula generalizes the work of Schilder (1966) and Pincus (1968).

2. Freidlin-Wentzell type estimates for abstract Wiener spaces

Let (i, H, B) be an abstract Wiener space, i.e., let H be a separable Hilbert space, $\|\cdot\|$ a measurable norm on H, B the Banach space obtained by completing H with respect to $\|\cdot\|$, and i the injective map of H into B. The dual B^* of B is densely contained in the dual H^* of H, which will be identified with H. Let $< \cdot, \cdot >$ denote the natural pairing between B^* and B, and let $\|\cdot\|_H$

and $(\cdot,\cdot)_H$ denote respectively the norm and the inner product of H. Let $\mathcal{B}$ be the σ-field of Borel subsets of B and let μ be the σ-additive extension to $(B, \mathcal{B})$ of the canonical normal distribution on H, i.e., the Gaussian measure on $(B, \mathcal{B})$ with zero mean and covariance kernel

$$R(y_1, y_2) = \int_B < y_1, x > < y_2, x > \mu(dx) = (y_1, y_2)_H, \quad y_1, y_2 \in B^*.$$

Let K^* be the closed unit ball of B^* and let

$$\sigma^2 = \sup_{y \in K^*} \int_B | < y, x > |^2 \mu(dx) = \sup_{y \in K^*} R(y, y) = \sup_{y \in K^*} \| y \|_H^2.$$

Note that $\sigma = \|i\|$, the norm of the injective map i. Since B is separable, there is a sequence $\{y_n\}$ in K^* such that $\mu\{x | \|x\| = \sup_n | < x, y_n > | < \infty\} = 1$ and $\sigma^2 = \sup_n \int_B | < x, y_n > |^2 \mu(dx)$. Hence, by a result of Fernique (1971, Theorem 8),

$$\int_B \exp[\alpha\|x\|^2] \mu(dx) < \infty \quad \text{for all} \quad \alpha < 1/(2\sigma^2).$$

Let $\{\phi_j\}$ be a sequence in B^*, which forms a complete orthonormal system in H. Then $< \phi_j, x >$ are independent standard Gaussian random variables and the sequence $\left\{x_n = \sum_{j=1}^{n} < \phi_j, x > \phi_j\right\}$ converges to x in $\|\cdot\|$ -norm μ-almost surely as $n \to \infty$. Put

$$\sigma_n^2 = \sup_{y \in K^*} \int_B | < y, x - x_n > |^2 \mu(dx) = \sup_{y \in K^*} \left\{R(y, y) - \sum_{j=1}^{n} < y, \phi_j > ^2\right\}.$$

K^* is w^*-compact, $R(y, y)$ and $\sum_{j=1}^{n} < y, \phi_j >^2$ are continuous in w^*-topology, and $\sum_{j=1}^{n} <y, \phi_j>^2 \uparrow R(y, y)$. Hence, by Dini's theorem, $\sigma_n^2 \to 0$ as $n \to \infty$, and it follows from Fernique's theorem quoted above that, for any given $u, v > 0$, there is an integer N such that

$$\mu\{x | \|x - x_n\| > u\} \leqslant \text{const.} \cdot \exp[-v] \quad \text{for all } n \geqslant N. \qquad \dots \quad (1)$$

We shall now prove the basic estimates which extend the results of Freidlin (1972) to abstract Wiener spaces (see also Wentzell, 1972).

Theorem 1 : *Let $\phi \in H$. Then, for any positive numbers δ and h, there is a positive number $\varepsilon_0 = \varepsilon_0(\delta, h, \|\phi\|_H) < 1$ such that*

$$\mu\{x \mid \|\varepsilon x - \phi\| < \delta\} \geqslant \mu\{x \mid \|x - (\phi/\varepsilon)\| < \delta\}$$

$$\geqslant exp\,[-(2\varepsilon^2)^{-1}(\|\phi\|_H^2 + h)]$$

for all $0 < \varepsilon \leqslant \varepsilon_0$. The lower bound is uniform with respect to ϕ such that $\|\phi\|_H \leqslant const. < \infty$.

Proof: Let μ_ϕ be a Gaussian measure defined by $\mu_\phi(A) = \mu(A - \phi)$, $A \in \mathcal{B}$. Then μ and μ_ϕ are mutually absolutely continuous relative to $\mathcal{B}$ and the Radon-Nikodym derivative $d\mu_\phi/d\mu$ is given by

$$d\mu_\phi/d\mu = \exp\,[(\phi, x)^\sim - (1/2)\,\|\phi\|_H^2],$$

where $(\phi, x)^\sim$ is a Gaussian random variable, corresponding to ϕ, with mean 0 and variance $\|\phi\|_H^2$. Then

$$\mu\{x \mid \|x - (\phi/\varepsilon)\| < \delta/\varepsilon\} = \mu_{-\phi/\varepsilon}\{x \mid \|x\| < \delta/\varepsilon\}$$

$$= \exp\,[-(2\varepsilon^2)^{-1}\|\phi\|_H^2] \cdot \int_A \exp\,[-(\phi/\varepsilon, x)^\sim]\mu(dx),$$

where $A = \{x \mid \|x\| < \delta\}$. Put $a = \mu\{A\}$, which is known to be strictly positive, and let M be a number such that $M > (2 \log (2/a))^{\frac{1}{2}} \cdot \|\phi\|_H$. Put $D = \{x \mid (-\phi/\varepsilon, x)^\sim > -M/\varepsilon\}$. Then $\mu\{D\} = 1 - \mu\{x \mid (\phi, x)^\sim > M\}$ $> 1 - \exp\,[-(1/2)\, M^2/\|\phi\|_H^2] > 1 - (a/2)$ and so $\mu\{A \cap D\} > a/2$. Hence

$$\mu\{x \mid \|x - (\phi/\varepsilon)\| < \delta\} \geqslant \exp\,[-(2\varepsilon^2)^{-1}\|\phi\|_H^2 - M/\varepsilon].\mu\{A \cap D\}$$

$$\geqslant \exp\,[-(2\varepsilon^2)^{-1}(\|\phi\|_H^2 + 2M\varepsilon + 2 \log (2/a).\varepsilon^2)].$$

Given $h > 0$, choose ε_0 small enough so that $2M\varepsilon + 2 \log (2/a). \cdot \varepsilon^2 < h$ for all $\varepsilon \leqslant \varepsilon_0$. The proof is complete.

Theorem 2: *Let* $K_r = \{\phi \in H \mid \|\phi\|_H \leqslant r\}$ *and* $d(x, K_r) = \inf\{\|x-\phi\| \mid \phi \in K_r\}$. *Then, for any positive numbers* δ, h *and* r, *there is a positive number* $\varepsilon_0 = \varepsilon_0(\delta, h, r)$ *such that*

$$\mu\{x \mid d(\varepsilon x, K_r) > \delta\} \leqslant \exp[-(2\varepsilon^2)^{-1}(r^2-h)]$$

for all $0 < \varepsilon \leqslant \varepsilon_0$. *The upper bound is uniform in r such that* $r \leqslant$ *const.* $< \infty$.

Proof: For any n,

$$\mu\{x \mid d(\varepsilon x, K_r) > \delta\} \leqslant \mu\{x \mid x_n \notin (1/\varepsilon)K_r\} + \mu\{x \mid \|x-x_n\| > \delta/\varepsilon\}.$$

By (1), we can choose N such that

$$\mu\{x \mid \|x-x_N\| > \delta/\varepsilon\} \leqslant \text{const.} \cdot \exp[-(2\varepsilon^2)^{-1} r^2].$$

Noting that $x_N \in H$, we obtain

$$\mu\{x \mid x_N \notin (1/\varepsilon)K_r\} = \mu\{x \mid \|x_N\|_H^2 > r^2/\varepsilon^2\}$$

$$= \mu\left\{x \mid \sum_{j=1}^{N} <\phi_j, x>^2 > r^2/\varepsilon^2\right\}$$

$$\leqslant \text{const.} \cdot (r^2/\varepsilon^2)^{(N/2)-1} \cdot \exp[-(2\varepsilon^2)^{-1} r^2].$$

Hence

$$\mu\{x \mid d(\varepsilon x, K_r) > \delta\} < M(r/\varepsilon)^N \cdot \exp[-(2\varepsilon^2)^{-1} r^2]$$

$$= \exp[-(2\varepsilon^2)^{-1}\{r^2-(2\varepsilon^2)\log(M(r/\varepsilon)^N)\}],$$

where M is a constant. For a given $h > 0$ there is a number $\varepsilon_0 = \varepsilon_0(\delta, h, r)$ such that $(2\varepsilon^2)\log(M(r/\varepsilon)^N) < h$ for all $0 < \varepsilon \leqslant \varepsilon_0$ The proof is complete.

Theorems 1 and 2 can be extended to a Gaussian measure on $(B, \mathcal{B})$ with zero mean and covariance function $(Sy_1, y_2)_H$, $y_1, y_2 \in B^*$, where S is a bounded, self-adjoint, positive, invertible, linear operator on H. We first remark that any such operator S determines a Gaussian measure on $(B, \mathcal{B})$.

LEMMA 1: *Let S be a bounded, self-adjoint, positive, invertible, linear operator on H. Then there is a Gaussian measure (call it ν) on $(B, \mathcal{B})$ with zero mean and covariance function*

$$R_\nu(y_1, y_2) = \int_B < y_1, x > < y_2, x > \nu(dx) = (Sy_1, y_2)_H, \; y_1, y_2 \in B^*.$$

Proof: Define a new norm $|||\cdot|||$ on H by $|||x||| = \|S^{1/2}x\|$, $x \in H$. Then $|||\cdot|||$ is a measurable norm on H (cf. Gross, 1962). Let B' be the $|||\cdot|||$-completion of H and let μ' be the σ-additive extension to B' of the canonical normal distribution on H. Denote elements of B' by primes. Suppose that $\{x_n\}$ $(x_n \in H)$ converges in $|||\cdot|||$ to $x' \in B'$. Then $\|S^{1/2}x_m - S^{1/2}x_n\| = |||x_m - x_n||| \to 0$ as $m, n \to \infty$, and hence there is $x \in B$ such that $\|S^{1/2}x_n - x\| \to 0$ as $n \to \infty$. Define the map J from B' to B by $Jx' = x$. J is linear and continuous, for $\|Jx'\| = \lim_{n\to\infty} \|S^{1/2}x_n\| = \lim_{n\to\infty} |||x_n||| = |||x'|||$, and the restriction of J to H is $S^{1/2}$. Let J^* be the transpose of J, i.e., the map from B^* to the dual of B' such that $< y, Jx' > = < J^*y, x' >$, $x \in B'$, $y \in B^*$. If $x' \in H$, then $(S^{1/2}y, x')_H = (y, S^{1/2}x') = < y, Jx' > = (J^*y, \; x')_H$ and hence $J^*y = S^{1/2}y$ for $y \in B^*$. Since J is continuous, it is measurable and induces a probability measure $\nu = \mu' J^{-1}$ on $(B, \mathcal{B})$. It remains to show that $R_\nu(y_1, y_2) = (Sy_1, y_2)_H$. We have

$$\begin{aligned} R_\nu(y_1, y_2) &= \int_B < y_1, x > < y_2, x > \mu' J^{-1}(dx) \\ &= \int_B < y_1, Jx' > < y_2, Jx' > \mu'(dx') \\ &= \int_B < J^*y_1, x' > < J^*y_2, x' > \mu'(dx') \\ &= \int_B < S^{1/2}y_1, x' > < S^{1/2}y_2, x' > \mu'(dx') \\ &= (S^{1/2}y_1, S^{1/2}y_2)_H = (Sy_1, y_2)_H. \end{aligned}$$

The proof is complete.

Let ν be the Gaussian measure on $(B, \mathcal{B})$ with zero mean and covariance function $(Sy_1, y_2)_H$, $y_1, y_2 \in B^*$, where S is a bounded, self-adjoint, positive, invertible, linear operator on H, Theorem 3 : *Let $\phi \in H$. Then, for any positive numbers δ and h, there is a positive number $\varepsilon_0 = \varepsilon_0(\delta, h, \|S^{-1}\phi\|_H)$ such that*

$$\nu\{x \,|\, \|\varepsilon x-\phi\| < \delta\} \geqslant \nu\{x \,|\, \|x-(\phi/\varepsilon)\| < \delta\}$$
$$\geqslant exp\,[-(2\varepsilon^2)^{-1}(\|S^{-1}\phi\|_H^2+h)]$$

for all $0 < \varepsilon \leqslant \varepsilon_0$.

Proof: Let μ' and J be the Gaussian measure and the map introduced in the proof of Lemma 1. Since J is continuous and $J^{-1}\phi = S^{-1}\phi$ for $\phi \in H$, there is a $\delta' = \delta'(\delta) > 0$ such that

$$\nu\{x \,|\, \|x-(\phi/\varepsilon)\| < \delta\} > \mu'\{x' \,|\, |||\, x'-(S^{-1}\phi/\varepsilon)\,||| < \delta'\}.$$

Applying Theorem 1 to μ', we obtain the conclusion.

Theorem 4 : *Let $L_r = \{\phi \in H \,|\, \|S^{-1}\phi\|_H \leqslant r\}$ and $d(x, L_r) = inf\,\{\|x-\phi\| \,|\, \phi \in L_r\}$. Then, for any $\delta, h, r > 0$, there is a positive number $\varepsilon_0 = \varepsilon_0(\delta, h, r)$ such that*

$$\nu\{x \,|\, d(\varepsilon x, L_r) > \delta\} \leqslant exp[-(2\varepsilon^2)^{-1}(r^2-h)]$$

for all $0 < \varepsilon \leqslant \varepsilon_0$.

Proof: Note that $J^{-1}L_r = S^{-1}L_r = K_r$. Since J is linear and continuous, there is a $\delta' > 0$ such that

$$\nu\{x \,|\, d(\varepsilon x, L_r) > \delta\} < \mu'\{x' \,|\, d'(\varepsilon x', K_r) > \delta'\}$$

where d' is the distance in $|||\cdot|||$-norm in B'. The conclusion is obtained by applying Theorem 2 to μ'.

Define a real functional $I_\nu(\phi)$ on B by

$$I_\nu(\phi) = \begin{cases} (S^{-1}\phi, \phi) = \|S^{-1}\phi\|_H, & \text{if } \phi \in H, \\ \infty, & \text{if } \phi \notin H. \end{cases}$$

$I_\nu(\phi)$ may be called the *action functional* or the *energy functional* corresponding to the Gaussian measure ν. We shall write $I(\phi)$ for $I_\mu(\phi)$. The following properties of $I(\phi)$ are useful in applications.

(a) The set $K_r = \{\phi \in H \mid I(\phi) \leqslant r^2\} (0 < r < \infty)$ is compact in B.

(b) $I(\phi)$ is lower semi-continuous on H with respect to $\|\cdot\|$-norm convergence, i.e., if $\|\phi_n - \phi\| \to 0$, $\phi_n, \phi \in H$, then $I(\phi) \leqslant \liminf_{n\to\infty} I(\phi_n)$.

The property (a) seems to be widely known. For a proof, see, e.g., Kuelbs and LePage, (1973). For a proof of (b), we note that if $\|\phi_n - \phi\| \to 0$, then $\langle y, \phi_n \rangle = (y, \phi_n)_H \to \langle y, \phi \rangle = (y, \phi)_H$ for any $y \in B^*$. Since B^* is dense in H, $\{\phi_n\}$ converges weakly to ϕ in H, and hence $I(\phi) = \|\phi\|_H^2 \leqslant \liminf_{n\to\infty} \|\phi_n\|_H^2 = \liminf_{n\to\infty} I(\phi_n)$.

We note also that $I_\nu(\phi)$ has the same properties :

(a') The set $L_r = \{\phi \in H \mid I_\nu(\phi) \leqslant r^2\}$ is compact in B, and

(b') $I_\nu(\phi)$ is lower semi-continuous on H with respect to $\|\cdot\|$-norm convergence.

An expression for $I_\nu(\phi)$. Recall that μ and ν are mutually absolutely continuous relative to $\mathcal{B}$ if and only if S is a bounded, self-adjoint, positive, invertible, linear operator on H of the form $S = I - T$, where I is the identity operator and T is a Hilbert-Schmidt operator. In this case, if π is a maximal chain of orthoprojectors on H, then S^{-1} admits the following factorization relative to π

$$S^{-1} = (I - L^*)(I - L),$$

where L is a Hilbert-Schmidt operator and L^* is the adjoint operator of L (see Kallianpur and Oodaira, 1973 for the details). Since $(S^{-1}\phi, \phi)_H = \|(I-L)\phi\|_H^2 = \|\phi - L\phi\|_H^2$, $I_\nu(\phi)$ can be expressed in terms of L as follows

$$I_\nu(\phi) = \begin{cases} \|\phi - L\phi\|_H^2, & \text{if } \phi \in H, \\ \infty \; , & \text{if } \phi \notin H. \end{cases} \qquad \ldots \quad (2)$$

In particular, if π does not have gaps, then L is a Volterra operator (see Kallianpur and Oodaira, 1973).

3. Specialization to spaces of continuous functions

Let $T_p = [0, 1]^p \, (p \geqslant 1)$ be the p-dimensional cube and $C = C(T_p)$ be the Banach space of real continuous functions x on T_p with the supremum norm $\|\cdot\|_\infty$. Let μ be a Gaussian measure on the σ-field $\mathcal{B}(C)$ of Borel sets of C with zero mean and continuous covariance function $R(t, s)$, where $t, s \in T_p$. Let

$H = H(R)$ be the reproducing kernel Hilbert space (RKHS) with reproducing kernel (r.k.) $R(\underline{t}, \underline{s})$. Then $H \subset C$ and $\|x\|_\infty \leqslant \sup_{\underline{t} \in T_p} R^{\frac{1}{2}}(\underline{t}, \underline{t}) \cdot \|x\|_H$ for $x \in H$, where $\|\cdot\|_H$ is the norm of H. It is easy to see that in order that Theorems 1-4 hold for μ and ν on $(C, \mathcal{B}(C))$, it suffices to verify that $\|\cdot\|_\infty$ is a measurable norm on H. The latter seems to be a reasonably widely known fact, which can be deduced from a result of Dudley-Feldman-LeCam (1970) as follows.

Assume for simplicity that the completion of H with respect to $\|\cdot\|_\infty$ coincides with C. Only slight changes are needed in the proof if this is not the case. Let m be the canonical normal distribution on H, and define a cylinder set measure $\tilde{\mu}$ on C by

$$\tilde{\mu}\{x \in C \mid (< y_1, x > , \ldots, < y_n, x >) \in E\}$$
$$= m\{x \in H \mid ((y_1, x)_H, \ldots, (y_n, x)_H) \in E\},$$

where $y_i \in C^*$ $(i = 1, \ldots, n)$ and E is a Borel set in R^n. Since $\tilde{\mu}$ and μ agree on all cylinder sets of the form $\{x \in C \mid (x(\underline{t}_1), \ldots, x(\underline{t}_n)) \in E\}$ and μ is a given σ-additive Gaussian measure on $\mathcal{B}(C)$, $\tilde{\mu}$ has a σ-additive extension to $\mathcal{B}(C)$, i.e., μ itself. The conclusion follows from the following result of Dudley-Feldman-LeCam : $\tilde{\mu}$ has a σ-additive extension to $\mathcal{B}(C)$ if and only if $\|\cdot\|_\infty$ is a measurable norm on H.

Thus Theorems 1-4 hold for Gaussian measures μ and ν on $(C, \mathcal{B}(C))$.

Remark : Consider the following important special case : $C = C[0, 1]$ and $\mu =$ the standard Wiener measure on $(C, \mathcal{B}(C))$. In this case H consists of absolutely continuous functions ϕ on $[0, 1]$ vanishing at the origin and having square integrable derivatives $\dot{\phi}$. Further $\|\phi\|_H^2 = \int_0^1 (\dot{\phi}(t))^2 dt = \|\dot{\phi}\|_2^2$ and H is isometrically isomorphic to $L^2[0, 1]$ by the map $\phi \to \dot{\phi}$. Assume that a Gaussian measure ν on $(C, \mathcal{B}(C))$ is absolutely continuous with respect to μ. Choose a maximal chain π in $L^2[0, 1]$ consisting of orthoprojectors P_t with range $\{f \in L^2[0, 1] \mid f(s) = 0 \text{ a.e. for } t \leqslant s \leqslant 1\}$. Then, by (2), using the isometry from H to $L^2[0, 1]$, we see that for $\phi \in H$

$$I_\nu(\phi) = \|\dot{\phi} - \tilde{L}\dot{\phi}\|_2^2 = \int_0^1 \{\dot{\phi}(t) - \int_0^t \tilde{L}(t, s)\dot{\phi}(s) ds\}^2 dt, \qquad \ldots \quad (3)$$

where $\tilde{L}$ is the Volterra operator on $L^2[0, 1]$ corresponding to L on H and $\tilde{L}(t, s)$ is the corresponding square integrable Volterra kernel on $L^2[0, 1]^2$. Note that the formula (3) bears close resemblance to the action functional which occurs in the estimates of Wentzell-Freidlin (1970).

4. Application to high level occupation times of path continuous Gaussian processes

In this section we apply Theorems 1 and 2 to obtain some asymptotic estimates for the probabilities of high level occupation times of path continuous Gaussian processes. Let μ be a Gaussian measure on $C = C(T_p)$ with zero mean and covariance function $R(\underset{\sim}{s}, \underset{\sim}{t})$, $\underset{\sim}{s}, \underset{\sim}{t} \in T_p$, and let H be the RKHS associated with μ.

Theorem 5 : *Let D be a subset of C such that $D^\circ \cap H \neq \phi$, where D° denotes the interior of D. Then*

(a) $\liminf_{\alpha \to \infty} (1/\alpha^2) \cdot \log \mu\{\alpha D\} \geqslant -(1/2) \cdot \inf\{\|\phi\|_H^2 \mid \phi \in D^\circ \cap H\}$,

(b) $\limsup_{\alpha \to \infty} (1/\alpha^2) \cdot \log \mu\{\alpha D\} \leqslant -(1/2) r_0^2$,

where $r_0 = \sup\{r \mid K_r \cap \overline{D} = \phi\}$ and $\overline{D}$ is the closure of D.

Remark : It is known (Kallianpur, 1971) that the support of μ coincides with the closure of H in C, and so the condition $D^\circ \cap H \neq \phi$ is not restrictive.

Proof : (a) If $\phi \in D^\circ \cap H$, then there is a $\delta > 0$ such that $\{x \in C \mid \|x-\phi\|_\infty < \delta\} \subset D^\circ$, and so, by Theorem 1 with $\varepsilon = 1/\alpha$,

$$\mu\{\alpha D\} \geqslant \mu\{x \mid \|(x/\alpha)-\phi\|_\infty < \delta\} \geqslant \exp[-(\alpha^2/2)(\|\phi\|_H^2 + h)]$$

for any $h > 0$, if α is sufficiently large. Hence

$$\liminf_{\alpha \to \infty} (1/\alpha^2) \cdot \log \mu\{\alpha D\} \geqslant -(1/2)\|\phi\|_H^2$$

for all $\phi \in D^\circ \cap H$, and (a) follows.

(b) The case $r_0 = 0$ is obvious. Let $0 < r < r_0$. Then there is a $\delta > 0$ such that $d(K_r, D) > \delta$, since K_r is compact in C, and hence, by Theorem 2 with $\varepsilon = 1/\alpha$,

$$\mu\{\alpha D\} \leqslant \mu\{x \mid d(x/\alpha, K_r) > \delta\} \leqslant \exp[-(\alpha^2/2)(r^2 - h)]$$

for any $h > 0$, if α is sufficiently large. Therefore

$$\limsup_{\alpha \to \infty} (1/\alpha^2) \cdot \log \mu\{\alpha D\} \leqslant -(1/2)r^2$$

for all $r < r_0$, and (b) follows. The proof is complete.

Theorem 6: *Suppose that $D^0 \cap H \neq \phi$ and $\inf \{\|\phi\|_H^2 \mid \phi \in \bar{D} \cap H\} = \inf \{\|\phi\|_H^2 \mid \phi \in D^0 \cap H\} = b^2$, say. Then*

$$\lim_{\alpha \to \infty} (1/\alpha^2) \cdot \log \mu\{\alpha D\} = -(1/2)b^2$$

and $b^2 = \sup \{r^2 \mid K_r \cap \bar{D} = \phi\}$.

Proof: For any $r < b$, $K_r \cap \bar{D} = \phi$, and so $r_0 = b$. The assertion follows from Theorem 5.

Theorem 7: *Let*

$$D_\beta = \{x \in C \mid \int_{T_p} I_{\{x(\underline{t}) > 1\}}(\underline{t})\, d\underline{t} > \beta\}$$

$$= \{x \in C \mid \lambda\{\underline{t} \mid x(\underline{t}) > 1\} > \beta\}$$

for $0 \leqslant \beta < 1$, where I_A is the indicator function of a set A and λ is Lebesgue measure on T_p. Then

$$\lim_{\alpha \to \infty} (1/\alpha^2) \cdot \log \mu\{x \in C \mid \lambda\{\underline{t} \mid x(\underline{t}) > \alpha\} > \beta\} = -(1/2)b_\beta^2,$$

where $\quad b_\beta^2 = \inf \{\|\phi\|_H^2 \mid \phi \in D_\beta \cap H\} = \sup \{r^2 \mid K_r \cap \bar{D}_\beta = \phi\}$.

Proof: Note that D_β is open and

$$\bar{D}_\beta = \begin{cases} \{x \mid \lambda\{\underline{t} \mid x(\underline{t}) \geqslant 1\} \geqslant \beta\} & \text{for } \beta > 0 \\ \{x \mid x(\underline{t}) > 1 \text{ for some } \underline{t} \in T_p\} & \text{for } \beta = 0. \end{cases}$$

It is readily seen that the conditions of Theorem 6 are satisfied, and the conclusion follows from Theorem 6.

Remark: Theorem 7 is an extension of Theorem 1.1 of Marlow (1973) to path continuous p-parameter Gaussian processes. In the case of $p = 1$, Marlow shows that if $R(s, t)$, $0 \leqslant s, t \leqslant 1$, is non-decreasing in s for each t, then $b_\beta^2 = 1/R(1-\beta, 1-\beta)$. This can easily be seen as follows. It is known and easy to see that, for each $t \in [0, 1]$, $\sup\{|\phi(t)| \mid \phi \in K_r\} = rR^{\frac{1}{2}}(t, t)$ and the supremum is attained by $rR(\cdot, t)/R^{\frac{1}{2}}(t, t)$. Hence $\bar{D}_\beta \cap K_r = \phi$ or $\neq \phi$ according as $r <$ or $> 1/R^{\frac{1}{2}}(1-\beta, 1-\beta)$, and so $b_\beta^2 = 1/R(1-\beta, 1-\beta)$. The more general case $D = \{x \mid \lambda\{t \mid x(t) > f(t)\} > \beta\}$ with $0 < f \in C$ can be treated in the same way.

5. Application to tail probabilities of real continuous fuctionals of Gaussian processes

Let F be a real continuous functional on $C = C(T_p)$. Then, from Theorems 1 and 2 it follows that if $\phi \in H$, then, for any $\delta, h > 0$,

$$\mu\{x \mid |F(x/\alpha)-F(\phi)| < \delta\} \geqslant \exp[-(\alpha^2/2)(\|\phi\|_H^2+h)],$$

and

$$\mu\{x \mid \inf_{\phi \in K_r} |F(x/\alpha)-F(\phi)| > \delta\} \leqslant \exp[-(\alpha^2/2)(r^2-h)],$$

if α is sufficiently large. Suppose F is homogeneous, i.e., $F(cx) = c^q F(x)$ with some $q > 0$ for any positive constant c and $x \in C$, and let D be a subset of R^1 such that $D^0 \cap F(H) \neq \phi$ and $\inf\{\|\phi\|_H \mid F(\phi) \in \bar{D} \cap F(H)\} = \inf\{\|\phi\|_H \mid F(\phi) \in D^0 \cap F(H)\}$. Then, by the reasoning used in Section 4, we obtain

$$\lim_{\alpha \to \infty} (1/\alpha^{2/q}) \cdot \log \mu\{x \mid F(x) \in \alpha D\} = -(1/2)b^2,$$

where $b^2 = \inf\{\|\phi\|_H^2 \mid F(\phi) \in D^0 \cap F(H)\} = \sup\{r^2 \mid \bar{D} \cap F(K_r) = \phi\}$. From this, putting $D = (1, \infty)$, we immediately obtain the following asymptotic estimate for the tail probabilities of F.

Theorem 8 : *Let F be a real continuous functional on C such that $F(cx) = c^q F(x)$ with $q > 0$ for any positive constant c and $F(\phi) > 0$ for some $\phi \in H$. Then*

$$\lim_{\alpha \to \infty} (1/\alpha^{2/q}) \cdot \log \mu\{x \mid F(x) > \alpha\} = -(1/2)b^2,$$

where

$$b^2 = \inf\{\|\phi\|_H^2 \mid F(\phi) > 1\} = \sup\{r^2 \mid F(K_r) < 1\}.$$

Remark : Marlow (1970) has obtained, by a different method, a similar asymptotic formula for tail probabilities of uniformly Hölder continuous, asymptotically homogeneous functionals of Gaussian processes.

Note that $1 = \sup F(K_b) = b^q \cdot \sup F(K_1)$ and hence the value of b^2 can be determined by solving the extremal problem $\sup F(K_1)$. In what follows we consider several examples for which b^2 can be explicitly given by evaluating $\sup F(K_1)$.

Example 1 : Let μ be a Gaussian measure on $C[0, 1]$ with zero mean and covariance function $R(s, t)$, and let $F(x) = \int_0^1 x^2(t)dt = \|x\|_2^2$. Then

$$\lim_{a\to\infty} (1/\alpha) \cdot \log \mu \left\{ \int_0^1 x^2(t)dt > \alpha \right\} = -(2\lambda_1)^{-1},$$

where λ_1 is the largest eigenvalue of the covariance operator R with kernel $R(s, t)$ on $L^2[0, 1]$.

This is a known result, and so we give only a sketch of the proof. Let $\{\lambda_i\}$ and $\{\psi_i\}$ be the eigenvalues and the corresponding normalized eigenfunctions of R. Then $\{\phi_i = \lambda_i^{1/2} \psi_i\}$ is a complete orthonormal system in H. It can be shown that $\|\phi\|_2^2 \leqslant \lambda_1 \|\phi\|_H^2$ for any $\phi \in H$. Hence $\sup F(K_1) \leqslant \lambda_1$. Since $\|\phi_1\|_2^2 = \lambda_1$, we have $\sup F(K_1) = \lambda_1$ and hence the result.

Example 2 : Let μ be the standard Wiener measure on $C[0, 1]$ and let $F(x) = \int_0^1 |x(t)|^q dt$, $q \geqslant 1$. The RKHS H associated with the standard Wiener measure is the space of all absolutely continuous functions ϕ vanishing at the origin and having square integrable derivarives $\dot{\phi}$, and $(\phi, \psi)_H = \int_0^1 \dot{\phi}\dot{\psi}dt$. Strassen (1964, p. 220) proved that $\sup F(K_1) = c(q)$, where

$$c(q) = 2(q+2)^{(q/2)-1}/(\int (1-t^q)^{-\frac{1}{2}} dt)^q q^{q/2}.$$

We thus obtain

$$\lim_{a\to\infty} (1/\alpha^{2/q}) \cdot \log \mu \left\{ \int_0^1 |x(t)|^q dt > \alpha \right\} = -(1/2)(c(q))^{-2/q}, \quad q \geqslant 1.$$

In particular, $c(1) = 3^{-1}$ and $c(2) = 4/\pi^2$. The case $q = 1$ has been previously obtained by Marlow (1970) by a different method, and the case $q = 2$ is of course a particular case of Example 1. If q is an integer, then the same formula holds for $F(x) = \int_0^1 x^q(t)\, dt$.

Example 3 : Let μ be the standard Wiener measure on $C[0, 1]$ and let

$$F(x) = \int_0^1 |x(t)|^2 \, dt / \int_0^1 |x(t)| \, dt.$$

Then $\sup F(K_1) = 2s_0$, where $0 < s_0 < 1$ is the largest solution of

$$(1-s)^{\frac{1}{2}} \sin ((1-s)^{\frac{1}{2}}/s) + \cos ((1-s)^{\frac{1}{2}}/s) = 0$$

(see Strassen, 1964, p. 222). Hence

$$\lim_{\alpha \to \infty} (1/\alpha^2) \cdot \log \mu \left\{ \int_0^1 |x(t)|^2 \, dt / \int_0^1 |x(t)| \, dt > \alpha \right\} = -(8s_0)^{-2}.$$

Example 4 : Let μ be the Gaussian measure on $C[0, 1]$ induced by a Brownian bridge, i.e., the Gaussian measure with zero mean and covariance function $R(s, t) = s(1-t)$ for $0 \leqslant s \leqslant t \leqslant 1$, and let $F(x) = \int_0^1 |x(t)|^q \, dt$, $q \geqslant 1$. Then we have

$$\lim_{\alpha \to \infty} (1/\alpha^{2/q}) \cdot \log \mu \left\{ \int_0^1 x(t) | \, |^q \, dt > \alpha \right\} = -2(c(q))^{-2/q}, \quad q \geqslant 1,$$

where $c(q)$ is the same as in Example 2.

The RKHS with r.k.R is the space of all absolutely continuous functions ϕ on $[0, 1]$ such that $\phi(0) = \phi(1) = 0$ and $\dot{\phi} = d\phi/dt \in L^2[0, 1]$, and $(\phi, \psi)_H = \int_0^1 \dot{\phi}\dot{\psi} dt$. Hence $\sup F(K_1) = \sup \left\{ \int_0^1 |\phi(t)|^q \, dt \,|\, \phi(0) = \phi(1) = 0 \text{ and } \|\dot{\phi}\|_2 \leqslant 1 \right\}$. It can be shown that $\sup F(K_1) = 2^{-q} c(q)$, by the classical methods of the calculus of variations in a similar way as in Strassen's proof for the Wiener measure case. We omit the details of the proof.

If q is an integer, the above $F(x)$ can be replaced by $\int_0^1 x^q(t) dt$. Note that the above result $F(K_1) = 2^{-q} c(q)$ can be used to obtain an iterated logarithm result for the functional F of empirical distributions (cf. Finkelstein 1971). Finkelstein discusses only the case $q = 2$, which is also a particular case of Example 1.

6. An asymptotic formula for integrals on abstract Wiener spaces

A LaPlace asymptotic formula for integrals in one dimension is of the form

$$\lim_{\lambda \to 0} \left\{ \int_{-\infty}^{\infty} G(x) \exp [-\lambda^{-2} F(x)] \, dx / \int_{-\infty}^{\infty} \exp [-\lambda^{-2} F(x)] dx \right\} = G(\xi),$$

where F is a continuous function having a unique global minimum at ξ and G is continuous at ξ.

Schilder (1966) proved the following analogue of the above formula for integrals with respect to Wiener measure μ on $C = C[0, 1]$:

$$\lim_{\lambda \to 0} \left\{ \int_C G(\lambda x) \exp[-\lambda^{-2}F(\lambda x)]\mu(dx) / \int_C \exp[-\lambda^{-2}F(\lambda x)]\mu(dx) \right\} = G(\phi),$$

where F and G are real functionals on C satisfying certain conditions and ϕ is a minimizing point at which the functional $F(x)+(1/2)\int_0^1 (dx/dt)^2\, dt$ attains a global minimum over the space of absolutely continuous functions on [0, 1] vanishing at the origin and having square integrable derivatives, i.e., the RKHS associated with Wiener measure. Pincus (1968) generalized the above result of Schilder to a large class of Gaussian processes and showed a close connection with Hammerstein integral equations. Also, in a recent paper Marcus (1974) indicated that the results of Schilder and Pincus can be obtained by the method of that paper.

In this section we extend the asymptotic formula of Schilder and Pincus to integrals on abstract Wiener spaces, applying Theorem 1 and the arguments used in Section 2.

Theorem 9 : *Suppose F and G are real measurable functionals on B satisfying the following conditions* :

(a) $F(\psi)+(1/2)\|\psi\|_H^2$, $\psi \in H$, *attains its unique global minimum over H at $\phi \in H$,*

(b) $F(x) > -a_1-a_2\|x\|^2$ *for all $x \in B$, where a_1 is any positive constant, $a_2 < (4\sigma^2)^{-1}$, and $\sigma = \|i\|$ (see Section 2),*

(c) *F is uniformly continuous on the set $E = \{x \in B \mid \|x\| < 2r\sigma\}$, where $r^2 > 4\{a_1+F(\phi)+(1/2)\|\phi\|_H^2\}$,*

(d) $|G(x)| < b_1 \exp(b_2\|x\|^2)$ *for all $x \in B$, where b_1 and b_2 are any positive constants, and G is continuous at ϕ. Then*

$$\lim_{\lambda \to 0} \{ \int_B G(\lambda x) \exp[-\lambda^{-2}F(\lambda x)]\mu(dx) / \int_B \exp[-\lambda^{-2}F(\lambda x)]\, \mu(dx)\} = G(\phi). \quad \ldots (4)$$

Remark : It follows from (b) and (c) that $r^2 > \|\phi\|_H^2 \geqslant 0$. Note also that $\phi \in E$, since $\|\phi\| \leqslant \sigma\|\phi\|_H < \sigma r$.

Proof: (i) By condition (d), for any $\varepsilon > 0$, there is a $\delta > 0$ such that $\|x-\phi\| < \delta$ implies $|G(x)-G(\phi)| < \varepsilon/2$. Hence

$$|\int_B G(\lambda x)\exp[-\lambda^{-2}F(\lambda x)]\mu(dx)/\int_B \exp[-\lambda^{-2}F(\lambda x)]\mu(dx)-G(\phi)| < \varepsilon/2$$

$$+\int_{\|\lambda x-\phi\|>\delta}|G(\lambda x)-G(\phi)|\exp[-\lambda^{-2}F(\lambda x)]\mu(dx)/\int_B \exp[-\lambda^{-2}F(\lambda x)]\mu(dx).$$

We shall show that the second term is $<\varepsilon/2$ for all sufficiently small λ.

(ii) Consider first the denominator $\int_B \exp[-\lambda^{-2}F(\lambda x)]\ \mu(dx)$. By condition (c) and the remark, for any $\eta > 0$, there is a $\delta' > 0$ such that $\|\lambda x-\phi\| < \delta'$ implies $|F(\lambda x)-F(\phi)| < \eta/2$ and hence $F(\lambda x) < F(\phi)+\eta/2$. Therefore, using Theorem 1, we obtain

$$\int_B \exp[-\lambda^{-2}F(\lambda x)]\mu(dx) \geqslant \int_{\|\lambda x-\phi\|<\delta'}\exp[-\lambda^{-2}F(\lambda x)]\ \mu(dx)$$

$$\geqslant \exp\{-\lambda^{-2}(F(\phi)+\eta/2)\}\cdot\mu\{\|\lambda x-\phi\| < \delta'\}$$

$$\geqslant \exp[-\lambda^{-2}(F(\phi)+\eta/2)]\cdot\exp[-(1/2)\lambda^{-2}(\|\phi\|_H^2+\eta)]$$

$$= \exp[-\lambda^{-2}\{F(\phi)+(1/2)\|\phi\|_H^2+\eta\}]$$

for any $\eta > 0$, if λ is sufficiently small.

(iii) Since K_r is compact and the set $\{x|\,\|x-\phi\| \geqslant \delta/2\}$ is closed,

$$D = D(r,\delta) = K_r \cap \{x|\,\|x-\phi\| \geqslant \delta/2\}$$

is compact in B. Since $D\subseteq K_r\subseteq E$ and $I(\psi) = \|\psi\|_H^2$ is lower semi-continuous, $F(\psi)+(1/2)\|\psi\|_H^2$ is lower semi-continuous. By condition (a), ϕ is the unique minimizing point, and hence there is an $\eta' = \eta'(r,\delta) > 0$ such that

$$\min_{\psi\in D}\{F(\psi)+(1/2)\|\psi\|_H^2\} > F(\phi)+(1/2)\|\phi\|_H^2+\eta'.$$

Choose $\delta'' > 0$ small enough so that $\delta'' < \min(\delta/2,\ r\sigma)$ and if $\|x-y\| < \delta''$, $x, y\in E$, then $|F(x)-F(y)| < \eta'/3$, which is possible by condition (c).

Let N be an integer such that $(2\sigma_N^2)^{-1} > \gamma = r^2(2(\delta'')^2)^{-1}$, where $\sigma_N^2 = \sup_{y\in E^*}\int_B |<y, x-x_N>|^2\mu(dx)$ and $x_N = \sum_{j=1}^N <\phi_j, x>\phi_j$ (see Section 2).

Put

$$A_1 = \{x|\,\|\lambda x-\phi\| \geqslant \delta,\ \|\lambda x-\lambda x_N\| < \delta'',\ \|\lambda x_N\|_H \leqslant r\}$$

and

$$A_2 = \{x|\,\|\lambda x-\lambda x_N\| \geqslant \delta''\}\cup\{x|\,\|\lambda x_N\|_H > r\}.$$

Then $\{x \mid \|\lambda x-\phi\| \geqslant \delta\} \subset A_1 \cup A_2$.

(iv) By condition (b), (d) and Schwarz's inequality, we have

$$\int_{A_2} |G(\lambda x)| \exp[-\lambda^{-2}F(\lambda x)]\mu(dx)$$

$$\leqslant \int_{A_2} |b_1 \exp(b_2\|\lambda x\|^2) \cdot \exp[\lambda^{-2}(a_1+a_2\|\lambda x\|^2)]\, \mu(dx)$$

$$\leqslant b_1 \exp(a_1\lambda^{-2})\{ \int_B \exp[2(b_2\lambda^2+a_2)\|x\|^2]\mu(dx)\}^{\frac{1}{2}} \cdot (\mu\{A_2\})^{\frac{1}{2}}.$$

By condition (b), $2(b_2\lambda^2+a_2) < (2\sigma^2)^{-1}$ for all sufficiently small λ and hence

$$\int_B \exp[2(b_2\lambda^2+a_2)\|x\|^2]\, \mu(dx) \leqslant \text{const.} < \infty.$$

Let now $h>0$ be small enough so that $r^2 > 4\{a_1+F(\phi)+(1/2)\|\phi\|_H^2\}+5h$. Then, by condition (c),

$$\mu\{A_2\} \leqslant \mu\{\|x-x_N\| > \delta/\lambda\}+\mu\{\|x_N\|_H > r/\lambda\}$$

$$\leqslant \text{const} \cdot \exp[-\gamma(\delta'')^2\lambda^{-2}]+\text{const.} \cdot \exp[-(1/2)\lambda^{-2}(r^2-h)]$$

$$< \text{const.} \cdot \exp[-2\lambda^{-2}\{a_1+F(\phi)+(1/2)\|\phi\|_H^2+h\}].$$

Hence we have, for all sufficiently small λ,

$$\int_{A_2} |G(\lambda x)|\, \exp[-\lambda^{-2}F(\lambda x)]\,\mu(dx)$$

$$\leqslant \text{const.} \cdot \exp[-\lambda^2\{F(\phi)+(1/2)\|\phi\|_H^2+h\}].$$

Similarly we have

$$\int_{A_2} |G(\phi)|\, \exp[-\lambda^{-2}F(\lambda x)]\,\mu(dx)$$

$$\leqslant \text{const.} \cdot \exp[-\lambda^{-2}\{F(\phi)+(1/2)\|\phi\|_H^2+h\}].$$

(v) Let $x \in A_1$. Then $\lambda x_N \in K_r \subset E$ and $\|\lambda x\| \leqslant \|\lambda x-\lambda x_N\|+\|\lambda x_N\| < \delta''+r\sigma < 2r\sigma$. Since $\|\lambda x-\lambda x_N\| < \delta''$ and $\lambda x, \lambda x_N \in E$, $F(\lambda x) > F(\lambda x_N)-\eta'/3$. Furthermore, since $\|\lambda x_N-\phi\| > \|\lambda x-\phi\|-\|\lambda x-\lambda x_N\| > \delta-\delta'' > \delta/2$, $F(\lambda x_N) > -(1/2)\|\lambda x_N\|_H^2+F(\phi)+(1/2)\|\phi\|_H^2+\eta'$, and hence

$$F(\lambda x) > -(1/2)\|\lambda x_N\|_H^2+F(\phi)+(1/2)\|\phi\|_H^2+(2/3)\eta'.$$

Let β be a number such that $0<\beta<(2/3)\eta' r^{-2}$. Then $(1/2)\beta\|\psi\|_U^2<\eta'/3$ for all $\psi\in K_r$. Therefore, $(1/2)\beta\|\lambda x_N\|_U^2<\eta'/3$ and hence

$$F(\lambda x)>-(1/2)(1-\beta)\|\lambda x_N\|_U^2+F(\phi)+(1/2)\|\phi\|_U^2+\eta'/3.$$

Thus

$$\begin{aligned}&\int_{A_1}|G(\lambda x)|\exp[-\lambda^{-2}F(\lambda x)]\,\mu(dx)\\&<b_1\exp[b_2\lambda^2\cdot 4\sigma^2r^2]\cdot\int_{A_1}\exp[-\lambda^{-2}F(\lambda x)]\,\mu(dx)\\&<\text{const.}\cdot\exp[-\lambda^{-2}\{F(\phi)+(1/2)\|\phi\|_U^2+\eta'/3\}]\cdot\\&\int_B\exp[(1/2)(1-\beta)\|x_N\|_U^2]\,\mu(dx).\end{aligned}$$

But

$$\int_B\exp[(1/2)(1-\beta)\|x_N\|_U^2]\,\mu(dx)=(2\pi)^{-N/2}\cdot\prod_{j=1}^{N}\int_{-\infty}^{\infty}\exp[-(\beta/2)s^2]ds<\infty.$$

Therefore

$$\begin{aligned}&\int_{A_1}|G(\lambda x)|\exp[-\lambda^{-2}F(\lambda x)]\,\mu(dx)\\&\qquad<\text{const.}\cdot\exp[-\lambda^{-2}\{F(\phi)+(1/2)\|\phi\|_U^2+\eta'/3\}].\end{aligned}$$

Similarly we have

$$\begin{aligned}&\int_{A_1}|G(\phi)|\exp[-\lambda^{-2}F(\lambda x)]\,\mu(dx)\\&\qquad<\text{const.}\cdot\exp[-\lambda^{-2}\{F(\phi)+(1/2)\|\phi\|_U^2+\eta'/3\}].\end{aligned}$$

(vi) From (iv) and (v) we get

$$\begin{aligned}&\int_{\|\lambda x-\phi\|>\delta}|G(\lambda x)-G(\phi)|\exp[-\lambda^{-2}F(\lambda x)]\,\mu(dx)\\&\qquad<\text{const}\cdot\{\exp[\lambda^{-2}\{F(\phi)+(1/2)\|\phi\|_U^2+\eta'/3\}]\\&\qquad+\exp[-\lambda^2\{F(\phi)+(1/2)\|\phi\|_U^2+h\}]\}.\end{aligned}$$

Choose $\eta>0$ (in (ii)) small enough so that $\eta<\min(\eta'/3,h)$. Then

$$\begin{aligned}&\int_{\|\lambda x-\phi\|>\delta}|G(\lambda x)-G(\phi)|\exp[-\lambda^{-2}F(\lambda x)]\mu(dx)/\int_B\exp[-\lambda^{-2}F(\lambda x)]\,\mu(dx)\\&\qquad<\text{const.}\cdot\{\exp[-\lambda^{-2}(\eta'/3-\eta)]+\exp[-\lambda^{-2}(h-\eta)]\}<\varepsilon/2\end{aligned}$$

for all sufficiently small λ. This completes the proof.

For the integrability of $G(\lambda x)\exp[-\lambda^{-2}F(\lambda x)]$ it is enough to assume $a_2 < (2\sigma^2)^{-1}$, and it is desirable to weaken the restriction $a_2 < (4\sigma^2)^{-1}$ in condition (b). This can be done if condition (c) is replaced by a stronger condition.

Theorem 10 : *Assume conditions* (a) *and* (d) *of Theorem* 9 *and the following conditions :*

(b′) $F(x) > -a_1 - a_2\ \|x\|^2$ *for all* $x \in B$, *where* a_1 *is a positive constant;* $a_2 < (2\sigma^2)^{-1}$ *and* $\sigma = \|i\|$, *and*

(c′) *F is uniformly continuous on any bounded set in B.*
Then the asymptotic formula (4) *holds.*

Proof: The condition $a_2 < (4\sigma^2)^{-1}$ is used only in the step (iv) of the proof of Theorem 9, and hence it suffices to make the following slight changes in the arguments. Choose a number $p > 1$ close enough to 1 so that $pa_2 < (2\sigma^2)^{-1}$ and put $q = p/(p-1)$. Let r be a number such that

$$r^2 > 2q\cdot\{a_1 - F(\phi) + (1/2)\ \|\phi\|_H^2\} + (2q+1)h$$

for some $h > 0$. Use Hölder's inequality instead of Schwarz's in the step (iv) of the proof. Then

$$\int_{A_2} |G(\lambda x)|\ \exp[-\lambda^{-2}F(\lambda x)]\,\mu(dx)$$

$$\leqslant b_1 \exp(a_1\lambda^{-2}) \cdot \{\int_B \exp[p(b_2\lambda^2 + a_2)\|x\|^2]\,\mu(dx)\}^{1/p} \cdot (\mu\{A_2\})^{1/q}.$$

Since $p(b_2\lambda^2 + a_2) < (2\sigma^2)^{-1}$ for all sufficiently small λ,

$$\int_B \exp[p(b_2\lambda^2 + a_2)\|x\|^2]\,\mu(dx) < \text{const.} < \infty,$$

and by the same reasoning as in (iv) we get, for sufficiently small λ,

$$\mu\{A_2\} < \text{const.} \cdot \exp[-q\lambda^{-2}\{a_1 + F(\phi) + (1/2)\|\phi\|_H^2 + h\}].$$

Hence, for all sufficiently small λ,

$$\int_{A_2} |G(\lambda x)|\exp[-\lambda^{-2}F(\lambda x)]\,\mu(dx) \leqslant \text{const.} \cdot \exp[-\lambda^{-2}\{F(\phi) + (1/2)\|\phi\|_H^2 + h\}].$$

The rest of proof does not require any changes.

Remark : If F is continuous on B, then the condition (b') (and (b)) implies that $F(\psi)+(1/2)\|\psi\|_H^2$ attains its infimum over H at a point in H, and hence we need only assume the uniqueness of its minimizing point ϕ in H. Indeed, since

$$F(\psi)+(1/2)\|\psi\|_H^2 > -a_1-a_2\|\psi\|^2+(1/2)\|\psi\|_H^2$$
$$> -a_1-(a_2\sigma^2-(1/2))\|\psi\|_H^2 \geqslant -a_1,$$

$-a_1 < \inf_{\psi\in H}\{F(\psi)+(1/2)\|\psi\|_H^2\} = d_0$, say, and, given any $d > 0$, if $\|\psi\|_H^2 > (a_1-d_0-d)/((1/2)-a_2\sigma^2) = c^2$, then $F(\psi)+(1/2)\|\psi\|_H^2 > d_0+d$. Since $F(\psi)+(1/2)\|\psi\|_H^2$ is lower semi-continuous and K_c is compact, $F(\psi)+(1/2)\|\psi\|_H^2$ attains its infimum on K_c, and hence on H, in $K_c \subset H$.

In (ii) of the proof of Theorem 9 it has been shown that

$$\int_B \exp[-\lambda^{-2}F(\lambda x)]\,\mu(dx) > \exp[-\lambda^{-2}\{F(\phi)+(1/2)\|\phi\|_H^2+\eta\}]$$

for any $\eta > 0$, if λ is sufficiently small. In fact, the arguments used in the proof yield the following asymptotic estimate for $\int_B \exp[-\lambda^{-2}F(\lambda x)]\,\mu(dx)$.

Theorem 11 : *Suppose F satisfies conditions* (a), (b) *and* (c) (*or* (a), (b') *and* (c')). *Then*

$$\lim_{\lambda\to 0} \lambda^2\cdot\log\int_B \exp[-\lambda^{-2}F(\lambda x)]\,\mu(dx) = -\{F(\phi)+(1/2)\|\phi\|_H^2\}.$$

Proof : In view of the above result in (ii), it suffices to show that

$$\int_B \exp[-\lambda^{-2}F(\lambda x)]\,\mu(dx) < \exp[-\lambda^{-2}\{F(\phi)+(1/2)\|\phi\|_H^2-\epsilon\}]$$

for any $\epsilon > 0$, if λ is sufficiently small. Put

$$A = \{x\,|\,\|\lambda x-\lambda x_N\| < \delta,\ \|\lambda x_N\|_H \leqslant r\},$$

where $\delta > 0$ is a number to be specified later and N is an integer such that $(2\sigma_N^2)^{-1} > r^2(2\delta^2)^{-1}$ (see (iii)). Then, in exactly the same way as in the proofs of Theorems 9 and 10, it can be shown that, for any $\delta > 0$,

$$\int_{B-A} \exp[-\lambda^{-2}F(\lambda x)]\,\mu(dx)$$
$$< \text{const.}\cdot\exp[-\lambda^{-2}\{F(\phi)+(1/2)\|\phi\|_H^2+h\}]$$
$$< \exp[-\lambda^{-2}\{F(\phi)+(1/2)\|\phi\|_H^2+h'\}]$$

for a sufficiently small $h' > 0$, if λ is sufficiently small.

Given $\varepsilon > 0$, choose $\delta > 0$ small enough so that if $x \in A$, λx, $\lambda x_N \in E$, and if $\|\lambda x - \lambda x_N\| < \delta$, then $F(\lambda x) > F(\lambda x_N) - \varepsilon/3$. Since $F(\lambda x_N) \geqslant F(\phi) + (1/2)\|\phi\|_H^2 - (1/2)\|\lambda x_N\|_H^2$, we have

$$\int_A \exp[-\lambda^{-2}F(\lambda x)]\,\mu(dx)$$

$$< \exp[-\lambda^{-2}\{F(\phi) + (1/2)\|\phi\|_H^2 - \varepsilon/3\}] \cdot$$

$$\int_A \exp[(1/2)\lambda^{-2}\|\lambda x_N\|_H^2]\,\mu(dx).$$

Just as in (v) of the proof of Theorem 9, let $\beta > 0$ be a number such that $\beta < (2/3)\,\varepsilon\, r^{-2}$. Then

$$\int_A \exp[(1/2)\lambda^{-2}\|\lambda x_N\|_H^2]\,\mu(dx)$$

$$\leqslant \exp[(\varepsilon/3)\lambda^{-2}] \cdot \int_A \exp[(1/2)(1-\beta)\|x_N\|_H^2]\,\mu(dx)$$

$$< \text{const.} \cdot \exp[(\varepsilon/3)\lambda^{-2}] < \exp[(2/3)\varepsilon\lambda^{-2}]$$

for sufficiently small λ. Hence

$$\int_A \exp[-\lambda^{-2}F(\lambda x)]\,\mu(dx) < \exp[-\lambda^{-2}\{F(\phi) + (1/2)\|\phi\|_H^2 - \varepsilon\}]$$

if λ is sufficiently small. The proof is complete.

Consider now the special case $B = C = C(T_p)$ and $H =$ the RKHS with r.k. $R(\underline{t}, \underline{s})$, $\underline{t}, \underline{s} \in T_p$, associated with a Gaussian measure μ on C (Section 3)

Assume that the Gateau differential $DF(x; \cdot)$ of F at $x \in C$ in the direction $\psi \in H$ exists for any $\psi \in H$, i.e.,

$$\lim_{\varepsilon \to 0} \varepsilon^{-1}\{F(x + \varepsilon\psi) - F(x)\} = DF(x; \psi)$$

exists for all $\psi \in H$ and $DF(x; \cdot)$ is a bounded linear functional on H for all $x \in C$. Assume further that $DF(x; \cdot)$ is continuous in the $L^2(T_p)$-topology. Then, by Riesz's theorem, there is a $\delta F(x)(\underline{t})$ in $L^2(T_p)$ such that

$$DF(x; \psi) = \int_{T_p} \delta F(x)(\underline{s})\psi(\underline{s})d\underline{s}.$$

$\delta F(x)$ is called the Fréchét-Volterra derivative of F at x.

Suppose that F satisfies the conditions of Theorem 9 or Theorem 10 and has a Fréchét-Volterra derivative in a neighborhood N of ϕ. Then, for any $\psi \in H$,

$$O = DF(\phi; \psi) - (\phi, \psi)_H = \int_{T_p} \delta F(\phi)(\underline{s})\psi(\underline{s})d\underline{s} - (\phi, \psi)_H.$$

Take $\psi(\cdot) = R(\cdot, \underline{t})$. Then

$$\phi(\underline{t}) - \int_{T_p} \delta F(\phi)(\underline{s}) R(\ , \underline{t}) d_{\underline{s}} = 0.$$

Thus ϕ satisfies the following Hammerstein integral equation

$$x(\underline{t}) - \int_{T_p} \delta F(x)(\underline{s}) R(\underline{s}, \underline{t}) d\underline{s} = 0, \ x \in N,$$

(for Hammerstein equations, (see e.g. Vainberg, 1973), and Theorems 9 and 10 show that a solution ϕ of the above nonlinear integral equation is given in the form

$$\lim_{\lambda \to 0} \{ \int_C \lambda x \cdot \exp[-\lambda^{-2} F(\lambda x)] \mu(dx) / \int_C \exp[-\lambda^{-2} F(\lambda x)] \mu(dx) \} = \phi.$$

This is an extension of Pincus' result (Pincus, 1968) on the solution of a Hammerstein equation for functions of many variables.

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