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PROBABILITIES OF MODERATE DEVIATIONS FOR SOME STATIONARY STRONG-MIXING PROCESSES

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SUMMARY. Asymptotic expression for probabilities of moderate deviations of the sample mean from the population mean are obtained for stationary strong mixing processes under suitable mixing condition.

1. Introduction

Let $\{X_n : n \geqslant 1\}$ be a strictly stationary sequence of r.v.'s (random-variables). Let M_a^b denote the σ-field generated by $X_n (a \leqslant n \leqslant b)$. The process $\{X_n\}$ is called strong mixing if there exists a sequence $\{\alpha(n)\}$ s.t.

$$1 \geqslant \alpha(1) \geqslant \alpha(2), \ldots, \lim_{n \to \infty} \alpha(n) = 0$$

and

$$\sup_{k \geqslant 1} \sup_{B \in M_1^k} \sup_{A \in M_{k+n}^{\infty}} |P(A \cap B) - P(A)\, P(B)| \leqslant \alpha(n).$$

The process is called ϕ-mixing if

$$\sup_{k \geqslant 1} \sup_{B \in M_1^k} \sup_{A \in M_{k+n}^{\infty}} |P(A \mid B) - P(A)| \leqslant \phi(n), \;\; P(B) > 0$$

where

$$1 \geqslant \phi(1) \geqslant \phi(2) \geqslant, \ldots, \phi(n) \to 0 \text{ as } n \to \infty.$$

Let

$$S_n = \sum_1^n X_i, \; \mu = E(X_1)$$

and

$$\sigma^2 = V(X_1) + 2 \sum_{j=1}^{\infty} \text{cov}\,(X_1, X_{1+j}).$$

We adopt the convention of denoting by $K_1 (> 0)$, $K_2 (> 0)$, $\gamma (> 0)$, etc., generic constants. $a_n \sim b_n$ will mean $a_n / b_n \to 1$ as $n \to \infty$.

Recently Ghosh and Babu (1977) obtained exact asymptotic expression for $P(n^{-1}S_n - \mu > c\,\sigma\, (\log n/n)^{\frac{1}{2}})$ for ϕ-mixing processes under the conditions (a) $E|X_1|^{c^2+2+\delta} < \infty$ for some $\delta > 0$, (b) $\Sigma\, \phi^{\frac{1}{2}}(j) < \infty$ and (c) $\sigma^2 \neq 0$. In this paper, we extend the result for strong mixing processes to establish the following theorem.

Theorem 1.1 : *Let* $\{X_n\}$ *be a strong mixing process with* $\alpha(n) = O(e^{-\lambda n})$ *for some* $\lambda > 0$. *If* $\sigma^2 > 0$ *and* $E|X_1|^{2+c^2+\delta} < \infty$ *for* $c > 0$ *and* $\delta > 0$, *then*

$$P(n^{-1}S_n - \mu > c\,\sigma(\log n/n)^{\frac{1}{2}}) \sim (2\pi c^2 \log n)^{-\frac{1}{2}} n^{-\frac{1}{2}c^2} \quad \dots \quad (1.1)$$

and

$$P(|n^{-1}S_n - \mu| > c\,\sigma(\log n/n)^{\frac{1}{2}}) \sim 2(2\pi c^2 \log n)^{-\frac{1}{2}} n^{-\frac{1}{2}c^2} \quad \dots \quad (1.2)$$

Clearly (1.2) follows with the help of (1.1).

Since our proof is on the same lines as that of the theorem in Ghosh and Babu (1977), we give only the essential part of the proof and omit the rest. We shall adopt the notations of Ghosh and Babu (1977) without any further explanation.

2. Preliminaries

Lemma 2.1 : *Let* $\{X_n\}$ *be a strong mixing stationary sequence. Let* r, s, h *be such that* $r^{-1}+s^{-1}+h^{-1} = 1$. *Suppose* X *and* Y *are r.v.'s measurable with respect to* $M^{v}_{-\infty}$ *and* M^{∞}_{v+n} *respectively for some integer* v *and further assume that* $||X||_r < \infty$ *and* $||Y||_s < \infty$ *where* $||X||_r = E^{1/r}|X|^r$. *Then*

$$|E(XY) - E(X)E(Y)| \leqslant 10(\alpha(n))^{1/h} ||X||_r ||Y||_s. \quad \dots \quad (2.1)$$

One or both of r and s can be taken to be $+\infty$ for bounded r.v.'s. This lemma is due to Davydov (1970). For a proof see Deo (1973).

The following lemma is essentially contained in Lemma 2.5 of Babu and Singh (1977), but for the sake of reference we include the proof.

Lemma 2.2 : *Let* $\{Y_i\}$ *be a sequence of identically distributed r.v.'s such that* $0 < Y_i \leqslant A$ *for some real number* $A > 0$. *Let* Y_i *be measurable with respect to the* σ-*field* $M^{i(l+m)+m}_{i(l+m)}$ *for some positive integers* l *and* m. *Then, for any positive integer* a, *we have*

$$E\left(\prod_{1}^{a} Y_i\right) \leqslant [g + 10(A^a+1)(\alpha(l))^{1/(a+1)}]^a \quad \dots \quad (2.2)$$

where

$$g = \sup_{0 \leqslant u \leqslant a} E(Y_1^u).$$

Proof : Taking $r = \infty$, $s = 1 + \frac{1}{a}$ and $h = 1+a$ in Lemma 2.1, we get

$$E\left(\prod_{1}^{a} Y_i\right) \leqslant E(Y_1)\, E\left(\prod_{1}^{a-1} Y_i\right) + 10A\alpha^{1/h}(l) E^{1/s}\left(\sum_{1}^{a-1} Y_i^s\right).$$

By Hölder's inequality we obtain

$$E\left(\prod_1^a Y_i\right) \leqslant [E(Y_1)+10A\alpha^{1/h}(l)]E^{1/s}\left(\prod_1^{a-1} Y_i^s\right).$$

Since, $1 \leqslant s^i \leqslant e$ for $i = 1, 2, \ldots, a$ and g in the lemma is not less than 1, a repeated application of the above recurrence relation gives that

$$E\left(\prod_{i=1}^a Y_i\right) \leqslant \prod_{j=0}^{a-1} E(Y_1^{s^j})+10(A^e+1)\alpha^{1/h}(l)$$

$$\leqslant [g+10(A^e+1)\alpha^{1/h}(l)]^a.$$

This completes the proof of the lemma.

Without loss of generality we assume $\mu = 0$, $\sigma = 1$. We prove the following lemma as a substitute for Lemma 2 in Ghosh and Babu (1977).

Lemma 2.3 : *For $1 \leqslant u \leqslant n$, n sufficiently large and for any positive integer $m \geqslant 2$,*

$$E|S_u'|^m \leqslant K_1(uR(m)+u^{m/2})\,(\log n)^{K_2} \qquad \ldots \quad (2.3)$$

where

$$R_m = R_n(m) = n^{\frac{1}{2}(m-c^2-2-\delta)} \text{ or } 0$$

according as $m >$ or $\leqslant c^2+2+\delta$. K_1 and K_2 may depend on m but not on u or n.

Proof : For $m = 2$, the corresponding proof in Lemma 2 of Ghosh and Babu (1977) goes through except that in this case we estimate $|E(X_1'X_{1+j}')-E(X_1')E(X_{1+j}')|$ with the help of Lemma 2.1 taking $r = s = 2+\delta/2$ and $h = (4+\delta)/\delta$.

To prove the Lemma by induction, assume

$$C(u, m) = E|S_u'|^m \leqslant K_1(u\,R(m)+u^{m/2})(\log n)^{K_2} = D(u, m). \qquad \ldots \quad (2.4)$$

To prove a similar inequality for $C(u, m+1)$, we shall only show that

$$E|S_u'+S_{u,l}'|^{m+1} \leqslant (2+K_1(\log n)^{-1}C(u, m+1)+D(u, m+1) \ldots \quad (2.5)$$

for sufficiently large n and $l = (\log n)^{(m+1)/2}$. The rest of the arguments are same as in the proof of Lemma 2 of Ghosh and Babu (1977) with some straightforward modifications.

To prove (2.5), first observe that

$$E|S_u'+S_{u,l}'|^{m+1} = 2C(u, m+1)+\sum_{j=1}^{m}\binom{m+1}{j}E|S_u'|^{m+1-j}.\,E|S_{u,l}'|^j$$

We now estimate $E|S'_u|^{m+1-j}|S'_{u,t}|^j$. Taking $\varepsilon = 1/\log n$, we observe that

$$E(|S'_u|^{m+1-j}|S'_{u,t}|^j) \leqslant e^2E(|S'_u|^{m+1-j-\varepsilon}|S'_{u,t}|^j)$$
$$\leqslant e^2|E|S'_u|^{m+1-j-\varepsilon}|S'_{u,t}|^j$$
$$-E|S'_u|^{m+1-j-\varepsilon}E|S'_{u,t}|^j|$$
$$+e^2E|S'_u|^{m+1-j-\varepsilon}E|S'_{u,t}|^j. \qquad \ldots (2.6)$$

An application of Lemma 2.1 in the first term $\left(\text{with } r = \frac{m+1}{m+1-j-\varepsilon},\ s = \frac{m+1}{j} \text{ and } h = \frac{m+1}{\varepsilon}\right)$ and of Hölder's inequality in the second term yield

$$\text{L.H.S. of (2.6)} \leqslant K_1 \exp(-\lambda t/(m+1)\log n)\,(E|S'_u|^{m+1})^{1-\frac{\varepsilon}{m+1}}$$
$$+K_1(E|S'_u|^m)^{\frac{m+1-\varepsilon}{m}} \leqslant K_1\Big[(\log n)^{-1}E|S'_u|^{m+1}+(E|S'_u|^m)^{\frac{m+1}{m}}+1\Big].$$

Now using the facts that

$$K_1\left[(E|S'_u|^m)^{\frac{m+1}{m}}+1\right] \leqslant D(u, m+1)$$

for sufficiently large n, one concludes (2.5).

3. Proof of theorem 1.1

We display the essential modifications in the proof of the theorem in Ghosh and Babu (1977) in the form of following lemmas. Rest of the arguments go through with some easily conceivable changes. Interestingly enough, the only properties of α and β that we make use of here is that $\frac{1}{2} < \beta < \alpha < 1$.

Lemma 3.1 : *Let $w = c_n((\log k)/k)^{\frac{1}{2}}$, then*

$$\left|E\left(\exp\left(W\sum_{i=1}^{k}\xi_{i0}\right)\right)-f_n^k\right| = O(n^{-\gamma}),\ \gamma > 0. \qquad \ldots (3.1)$$

Proof: Clearly,

$$\text{L.H.S. of (3.1)} \leqslant \sum_{j=1}^{k-1} f_n^{j-1}\left|E\left(\exp\left(w\sum_{i=1}^{k-j+1}\xi_{i0}\right)\right)-f_n E\left(\exp\left(w\sum_{i=1}^{k-j}\xi_{i0}\right)\right)\right|$$

$$\leqslant 10\alpha^{\frac{1}{2}}(q)e^{c_n}\sum_{j=1}^{k-1} f_n^{j-1}\,E^{\frac{1}{2}}\left(\exp\left(2W\sum_{i=1}^{k-j}\xi_{i0}\right)\right)$$

$$\leqslant 10\alpha^{\frac{1}{2}}(q)e^{c_n}k[g'+10(1+e^{2c_0 n})\alpha^{1/(k+1)}(q)]^k$$

where

$$g' = \sup_{0 \leqslant u \leqslant 2c} E(\exp(u\, w\, \xi_{10})).$$

The second inequality above follows from Lemma 2.1 and the last from Lemma 2.2.

Taylor's expansion yields

$$g' = 1 + O(\log k/k).$$

(3.1) is now immediate using the facts that $\alpha(q) = e^{-\lambda q}$, $\lambda > 0$, and $q/k = n^{\gamma}$ for some $\gamma > 0$.

Lemma 3.2 : $\sup_{z} |\Pi_{k,c_n}(z) - \Phi(z)| = O((\log n)^{-3/2})$

Proof : To validate the corresponding proof in Ghosh and Babu (1977), we need to prove the following

$$|\int e^{itz}\, d \prod_{k,c_n}(z) - [\int \exp(z - m_n)k^{-1}\sigma_n^{-1})dH_n(z)]^k| = O(n^{-\gamma}). \quad \ldots \quad (3.2)$$

Some easy calculations show that (3.2) is equivalent to

$$\left| \frac{g_k(w+it)}{g_k(w)} - \left(\frac{f_n(w+it)}{f_n}\right)^k \right| = O(n^{-\gamma}),\ \gamma > 0 \quad \ldots \quad (3.3)$$

where

$$g_k(\theta) = E\left(\exp \theta \sum_{i=1}^{k} \xi_{i0}\right),$$

for some complex θ, and

$$f_n(w+it) = E(\exp(w+it)\xi_{10}).$$

Since $|g_k(w+it)| \leqslant g_k(w)$,

we have

$$\text{L.H.S. of (3.3)} \leqslant f_n^{-k}\,|g_k(w+it) - f_n^k(w+it)| + |g_k(w+it)|\,|g_k^{-1}(w) - f_n^{-k}|$$

$$\leqslant f_n^{-k}\,|g_k(w+it) - f_n^k(w+it)| + |g_k(w) - f_n^k|.$$

The second term above has been estimated in Lemma 3.1 and the first term can be shown to be $O(n^{-\gamma})$, $\gamma > 0$, on the same lines. This, with the fact that $f_n^k \sim n^{c^2/2}$ (see (3.31) of Ghosh and Babu, 1977) establishes (3.3) proving the lemma.

Lemma 3.3 : $P(R_n > \zeta_n (n \log n)^{\frac{1}{2}}) = O(n^{-c^2})$

where $\zeta_n = (\log n)^{-1-\nu}$, $\nu > 0$, *for* $n \geqslant 3$.

Proof : As in Lemma 4 of Ghosh and Babu (1977), it is enough to show

$$P\Big(\sum_{1}^{k} \eta_{i0} > \zeta_n (n \log n)^{\frac{1}{2}} = O(n^{-c^2})\Big). \qquad \dots \quad (3.4)$$

By Markov's inequality and Lemma 2.2, we have

$$\text{L.H.S. of } (3.4) \leqslant e^{-c^2 \log n}\, E\Big(\exp\Big(c^2 n^{-\frac{1}{2}} (\log n)^{3/2+\nu} \sum_{1}^{k} \eta_{i0} \Big)\Big)$$

$$\leqslant n^{-c^2} (g^* + K_1 \alpha^{1/(k+1)}(p))^k$$

where

$$g^* = \sup_{0 \leqslant u \leqslant c} E(\exp(u\, c^2 n^{-\frac{1}{2}} (\log n)^{3/2+\nu} \eta_{10})).$$

Now since

$$g^* = 1 + O(n^{-1/2} (\log n)^{3/2+\nu} E\,|\,\eta_{10}\,|\,) + O(n^{-1} (\log n)^{3+2\nu}\; E\eta_{10}^2)$$

and

$$k\alpha^{1/(k+1)}(p) = O(n^{-K_1}),$$

it follows with the help of (3.4) of Ghosh and Babu (1977) that

$$k \log (g + K_1 \alpha^{1/(k+1)})(p)) = 0(1)$$

and this proves the lemma.

It has already been mentioned that the above lemmas prove the theorem as in Ghosh and Babu (1977).

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