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A CLASS OF SINGLE SAMPLING PLANS BASED ON FUZZY OPTIMISATION

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ABSTRACT

The problem of determining single sampling lot tolerance per cent defective (LTPD) inspection plans is considered under the fuzzy environment of satisfying the consumer's risk *closely*. A solution procedure, under Poisson conditions, for the above non-symmetrical fuzzy mathematical programming model involving minimisation of a crisp objective function with fuzzy solution space is developed. Numerical examples and a sensitivity analysis are included.

1. Introduction

We consider a producer's final inspection of a series of lots under the industrial production process where each lot retains its identity such as lots of electronic equipment for a large computer or a missile. The producer wants to minimize the average total inspection for product of process average quality and at the same time wants to be reasonably sure that lots of bad quality are not marketed.

In designing an attribute sampling plan for acceptance inspection, it is assumed that the producer knows his process average quality level p_1 under normal manufacturing conditions and that he occasionally produces lots of bad quality. The decision maker (DM) may then select lot tolerance fraction defective, p_2 (also called LTPD quality level), say $p_2 > p_1$ and a risk $P(p_2) = \beta$ of accepting the lots of this quality where $P(p)$ is the operating characteristic (OC) of the acceptance sampling plan.

In industrial applications, a single sampling plan (SSP) is generally employed. A SSP is the following: From each lot of size N, a random sample of size n is taken. If the number of defectives in the sample is less than or equal to the acceptance number c, the lot is accepted; otherwise the lot is rejected.

We consider the class of SSPs for which rejected lots are 100 per cent inspected. In the above class the classical model of Dodge Romig [6] LTPD SSP is to find the sample size n and the acceptance number c satisfying the following criteria:

$$\text{Minimize } I(p_1,n,c) = n + (N-n)\,(1-P(p_1)), \tag{1}$$

$$\text{subject to } P(p_2,n,c) \leqslant \beta, \tag{2}$$

$$\text{and} \qquad n,\ c \geqslant 0, \text{ integer}, \tag{3}$$

where $I(p_1,n,c)$ is the average total inspection (ATI).

We note that the above optimisation problem (1) through (3) is a non-linear integer programming (NLIP) problem and may be viewed as a non-linear integer knapsack problem (see Garfinkel and Nemhauser [7]). Since it is a NLIP problem, we have replaced the equality sign in (2), which does not change the solution. For applications of mathematical programming techniques in designing SSPs, see Chakraborty [2,4,5].

In industrial applications of sampling inspection plans, one faces imprecisions, for example, in the specification of LTPD quality, p_2, which is termed *rejectable* or *unsatisfactory* quality, and in the requirement of consumer's risk to be satisfied *as close as possible* to β (see Chakraborty [3] and Hald [8]). Now all imprecisions cannot be treated by randomness and modelled by using probability theory. From the above, we see that the imprecisions are often fuzzy in character and require to be modelled using fuzzy set theory. When the environment is really fuzzy in nature, modelling the problem with erroneous assumption of randomness produces sampling plans with appreciably larger ATI.

We consider the Dodge-Romig SSPs and assume that the DM wants a sampling plan with consumer's risk is to be satisfied *closely*. We model the problem as a Fuzzy Mathematical Programming (FMP) and derive a method for obtaining the solution. We may note that the problem is non-symmetrical (see Zimmermann [10,11]) since the objective function (1) is *crisp* and the constraint (2) is a fuzzy set.

We shall restrict our discussions for the industrial applications where $p_2 \leqslant 0.1$, $p_1/p_2 \leqslant 0.5$ and $n/N \leqslant 0.1$ such that Poisson distribution will provide sufficiently accurate approximation for $P(p_1)$ $(P(p_2))$ for which binomial (hypergeometric) distribution is involved. So $P(p) = G(c, np) = \sum_{x=0}^{c} e^{-np}\,(np)^x/x!$. For extension to the appropriate distributions involved, for notations and explanation of the terms, see Hald [8] and Chakraborty [2,3].

2. Fuzzy Mathematical Programming Model

The Dodge Romig [6] SSP when the DM wants to minimize the ATI at the process average p_1 subject to satisfying the consumer's risk *closely around* β is to find non-fuzzy non-negative integer pair (n,c) which minimizes

$$I(p_1,n,c) = n + (N-n)\,(1 - G(c,np_1)), \tag{4}$$

subject to $G(c,np_2) \lesssim \beta$, (5)

$$n,\ c \geqslant 0, \text{ integer.} \tag{6}$$

The symbol $\lesssim$ refers to fuzzified version of $\leqslant$ sign and means *approximately less than or equal to* or *essentially smaller than* or *closely around* stated value.

Clearly the model is a nonlinear integer fuzzy mathematical programming (NLIFMP) problem.For details of fuzzy set theory and *FMP* problem, see Zimmermann [9, 10] and references in Chakraborty [3].

An optimum value of a crisp function over a crisp domain attains at a precise point, called an optimum decision. The decision models involving fuzziness, the *optimum decision* according to Bellman and Zadeh [1, p. 150], is often considered to be a crisp set which contains those elements of fuzzy set *decision* attaining the maximum degree of membership. In the model (4) through (6), the objective function (4) is crisp which is to be minimised and it induces an order of the decision alternatives. The constraint (5) defines a fuzzy decision space. It is seen that the assumption of Bellman and Zadeh [1] about the symmetry of objective function and constraint sets are not valid in this model. This is a non-symmetrical case and following Zimmermann [10, 11], we will transform the model to a symmetrical case so that we may solve the problem by adapting the method given in Chakraborty [3].

Membership functions

Following Zimmermann [10, p. 231, 11, p. 104], we define:

Let R be a fuzzy feasible region, $S(R)$ support of R and R_1 α-level cut of R for $\alpha = 1$. The membership function of objective function (4) given solution space R is defined as

$$\mu_I(n,c) = \begin{cases} 1, & \text{if } I(p_1,n,c) \leqslant \inf\limits_{S(R)} I, \\ \dfrac{\inf\limits_{R_1} I - I(p_1,n,c)}{\inf\limits_{R_1} I - \inf\limits_{(SR)} I}, & \text{if } \inf\limits_{S(R)} I < I(p_1,n,c) < \inf\limits_{R_1} I, \\ 0, & \text{if } \inf\limits_{R_1} I \leqslant I(p_1,n,c). \end{cases} \tag{7}$$

Assuming that the *DM* specifies the upper tolerance limit of β as β_U, following Chakraborty [3] we define the fuzzy set corresponding to fuzzy constraint (5) by the membership function

$$\mu_1 = \begin{cases} 1, & \text{if } P(p_2,n,c) \leqslant \beta, \\ \dfrac{\beta_U - P(p_2,n,c)}{\beta_U - \beta}, & \text{if } \beta \leqslant P(p_2,n,c) \leqslant \beta_U, \\ 0, & \text{if } \beta_U \leqslant P(p_2,n,c). \end{cases} \tag{8}$$

Now *symmetry* is achieved between the objective function (4) and the constraint (5) and following Chakraborty [3] with *minimum operator* to aggregate the membership functions of fuzzy sets, we obtain the following model for the Dodge Romig problem:

$$\text{Maximise } [\min_{n,c \geqslant 0,\ \text{int}} \left\{ \frac{\inf\limits_{R_1} I - I(p_1,n,c)}{\inf\limits_{R_1} I - \inf\limits_{S(R)} I} = \lambda_1,\ \frac{\beta_U - P(p_2,n,c)}{\beta_U - \beta} = \lambda_2 \right\} = \lambda]. \tag{9}$$

subject to

$$\inf_{S(R)} I \leqslant I(p_1,n,c) \leqslant \inf_{R_1} I, \tag{10}$$

$$\beta \leqslant P(p_2,n,c) \leqslant \beta_U, \tag{11}$$

$$0 \leqslant \lambda \leqslant 1, \tag{12}$$

$$n,\ c \geqslant 0, \text{ integer}. \tag{13}$$

This problem can be rewritten in the equivalent optimisation problem: find n, c, λ, which maximises λ, subject to

$$\lambda \leqslant \frac{\inf\limits_{R_1} I - I(p_1,n,c)}{\inf\limits_{R_1} I - \inf\limits_{S(R)} I}, \tag{14}$$

$$\lambda \leqslant \frac{\beta_U - P(p_2, n, c)}{\beta_U - \beta} . \tag{15}$$

and inequalities (10) through (13).

Let $I_1 \equiv \inf_{R_1} I$ and $I_o \equiv \inf_{S(R)} I$.

Then the above optimisation problem (14), (15), (16), (10), (11), (12), (13) can be expressed as:

Find n, c, λ, which maximises λ, subject to

$$\lambda(I_1 - I_o) + I(p_1, n, c) \leqslant I_1, \tag{16}$$

$$\lambda(\beta_U - \beta) + P(p_2, n, c) \leqslant \beta_U, \tag{17}$$

$$I_o \leqslant I(p_1, n, c) \leqslant I_1. \tag{18}$$

and inequalities (11), (12) and (13).

This is a non-fuzzy mixed integer nonlinear programming (NFMINLP) problem. Noting that the values of I_1 and I_o can be found by solving two non-fuzzy optimisation problems, the above NFMINLP problem can be solved by standard methods.

3. Solution Procedure and Examples

It is obvious that the values of I_o and I_1 can be obtained by solving the following two NLIP problems respectively, viz.,

$$\text{Min } I(p_1, n, c) = n + (N - n)(1 - G(c, np_1)), \tag{19}$$

$$\text{subject to} \quad G(c, np_2) \leqslant \beta, \tag{20}$$

$$n, c \geqslant 0, \text{ integer}, \tag{21}$$

$$\text{and Min } I_1(p_1, n, c) = n + (N - n)(1 - G(c, np_1)), \tag{22}$$

$$\text{subject to} \quad \beta \leqslant G(c, np_2) \leqslant \beta_U, \tag{23}$$

$$n, c > 0, \text{ integer}. \tag{24}$$

Note that for a given c, $I(p,n,c)$ is an increasing function of n, since for a fixed c, $G(c,np)$ is a decreasing function of n. Thus for a fixed c, the minimum $I(p,n,c)$ will be for the minimum feasible n. Clearly the optimum n for the problem (19) through (21) for a given c is the smallest n satisfying

$G(c,np_2) \leqslant \beta$. Similarly the optimum n for the problem (22) through (24) for a given c is the smallest n satisfying $G(c,np_2) \leqslant \beta_L$.

Let $np = m$ and $m_\beta(c)$ be the β-fractile of the Poisson OC, $G(c,m)$. The values of $m_\beta(c)$ are given in Table 1 of Hald [8] for $c = 0(1)39$ for 11 values of β and in Table 1 of Chakraborty [3] for $c = 0(1)9$ for 28 values of β. It is easily seen that the minimum n satisfying $G(c,np) \leqslant \beta$ is given by

$$n(c) = \lceil m_\beta(c)/p \rceil \tag{25}$$

where for a real number y, $\lceil y \rceil$ is the smallest integer $\geqslant y$.

For each c, we can find $n(c)$ and from the nature of $I(p_1,n,c)$, the optimum solution for the problems (19) through (21) as well as (22) through (24) can be obtained by direct search method; see Hald [8, p. 101].

Let the optimal solution for the problem (19) through (21) be (n_o, c_o) with $I(p_1, n^o, c^o)_{opt} = I_1$ and that for the problem (22) through (24) be (n_o, c_o) with $I(p_1, n_o, c_o)_{opt} = I_o$.

Finally we consider the problem (16) through (18) together with (11) through (13). The solution procedure is similar with the difference that the set of feasible n's for a given c, is an interval instead of a singleton. For an arbitrary c, say c_o, we obtain the set of feasible n's satisfying (11)by using Table of $m_\beta(c)$. For each n of the set, we calculate $\lambda = \lambda_1$ for (16) and $\lambda = \lambda_2$ for (17) and obtain the minimum λ of the two, i.e., $\lambda = \min\{\lambda_1, \lambda_2\}$. The n which gives maximum of the minimum λ's is the optimum n for this $c = c_o$. The procedure is repeated for c's near $\{c_o, c^o\}$, and the SSP giving highest value of λ is the optimum SSP (n^*, c^*).

The search can be curtailed by noting, say, λ_o be the maximum λ corresponding to c_o and updating inequalities (16), (17) and (12), i.e.,

$$I(p_1,n,c) \leqslant I_o - \lambda_o(I_1 - I_o) = I_o', \tag{26}$$

$$P(p_2,n,c) \leqslant \beta_U - \lambda_o\ (\beta - \beta_U) = \beta_U', \tag{27}$$

$$\lambda_o \leqslant \lambda \leqslant 1. \tag{28}$$

This will result in only a small number of feasible c's. Also the cardinality of the set of feasible n's for each feasible c will be very small.

Numerical Examples

Example 1. We consider the example given in Hald [8, p. 101]: $N = 2{,}000$,

$p_1 = 0.02, p_2 = 0.10, \beta = 0.10$. In addition, let $\beta_U = 0.15$. The problem is to find the optimum LTPD SSP (n^*, c^*).

Solution. Solving the optimisation problems, we obtain $(n^o, c^o) = (93, 5)$ with $I_1 = 115.93$ and $(n_o, c_o) = (85, 5)$ with $I_o = 100.32$. For $c_o = 5$, we obtain the set of feasible sample size n's satisfying (11) as $[n_1, n_2] = [85, 92]$ and obtain the optimum n for this $c = c_o$ as $n = 98$ with $\lambda_o = 0.517$ by constructing Table 1. Using $\lambda_o = 0.517$, we find from (26), (27) and (28), $I(p_1,n,c) \leqslant 107.86$, $P(p_2,n,c) \leqslant 0.124$ and $0.517 \leqslant \lambda \leqslant 1$.

For $c = 4$, sample size n's satisfying (11) is $[n_1, n_2] = [73, 79]$.

For $c = 4$, the set of n's satisfying (11) and (27) is [77, 79].

However, for $c = 4$, the set of n's satisfying (11), (26) and (27) is empty. Similarly for $c = 6$, the set of sample size n's satisfying (11) is [98, 105]; satisfying (11) and (29) is [101, 105]. However, for $c = 6$, the set of feasible sample size is empty. Clearly the optimum LTPD SSP is $n^* = 89$, $c^* = 5$ with $I^*(p_1, n^*, c^*) = 107.86$ with $\lambda^* = 0.517$. Note also that $P(p_2, n^*, c^*) = 0.1219$

In this particular example, the DM takes an additional risk of 2% for a lot being rejected against a saving of 7% inspection effort per lot.

Example 2: Same problem as in Example 1 except $N = 5{,}000$.

TABLE 1—OPTIMAL SAMPLE SIZE FOR $c_o = 5$

n	$P(p_2)$	$I(p_1)$	$\lambda_1 = \frac{I_1 - I(p_1)}{I - I_o}$	$\lambda_2 = \frac{\beta_U - P(p_2)}{\beta_U - \beta}$	$\lambda = \min\{\lambda_1, \lambda_2\}$
85	.1496	100.32	1.000	.008	.008
88	.1284	105.92	0.641	.432	.432
89	.1219	107.86	0.517	.562	.517
90	.1157	109.82	.0391	.685	.391
92	.1041	113.86	0.133	.918	.133

TABLE 2 —OPTIMUM SAMPLE SIZE FOR DIFFERENT c

c	n	$P(p_2)$	$I(p_1)$	$\lambda_1 = \frac{I_1 - I(p_1)}{I_1 - I_o}$	$\lambda_2 = \frac{\beta_U - P(p_2)}{\beta_U - \beta}$	$\lambda = \min\{\lambda_1, \lambda_2\}$
	100	.1301	122.22	.712	.398	.398
6	<u>101</u>	.1240	124.41	.564	.520	<u>.520</u>
	102	.1180	126.66	.412	.640	.412
	112	.1301	122.79	.673	.398	.398
7	<u>113</u>	.1249	124.38	.566	.502	<u>.502</u>
	114	.1192	126.00	.456	.616	.456

Solution. Here $(n^o, c^o) = (118, 7)$ with $I_1 = 132.75$ and

$$(n_o, c_o) = (98,6) \text{ with } I_o = 117.96.$$

Proceeding as in Example 1, we search for plans with c's near {6, 7} and check for each feasible n corresponding to each c. From Table 2, the optimum LTPD SSP is $n^*=101$, $c^*=6$, $I^*=124.41$ and $\lambda^*=0.520$.

4. Sensitivity Analysis

Effect of Lot Size

Because of discreteness of c, the plan (n^*, c^*) which is optimum for N will also be optimum for some neighbouring values of N. The optimum SSP (89, 5) for the problem of Example 1, is optimum for all $N \in [1399, 2436]$. For a given set of parameters (p_1, p_2, β and β_U), it is seen that the optimum LTPD SSPs have the following properties: (*i*) c and n are increasing functions of N, (*ii*) ATI is increasing with N. A typical example is given in Table 3.

Effect of Tolerance Limit of Consumer's Risk

For a given set of parameters (N, p_1, p_2, β), as the tolerance limit β_U increases within reasonable limit, the optimum decision number remains same and the optimum sample size and ATI decreases slowly. However, if β_U is increased appreciably, the optimum decision number may also decrease by one unit. A typical example is given in Table 4.

TABLE 3—EFFECT OF LOT SIZE ON THE FUZZY LTPD SSPs

(p_1=0.02, p_2=0.10, β=0.10, β_U=0.15)

N	c^*	n^*	$P(p_2)$	$I(p_1)$	λ^*
500	3	64	0.1189	81.93	0.508
1000	4	76	0.1249	94.05	0.502
1399	5	89	0.1219	101.93	0.502
1500	5	89	0.1219	102.92	0.515
1800	5	89	0.1219	105.88	0.516
2000	5	89	0.1219	107.86	0.517
2200	5	89	0.1219	109.83	0.518
2436	5	89	0.1219	112.16	0.495
5000	6	101	0.1240	124.41	0.520
10000	7	113	0.1249	136.02	0.502

TABLE 4—EFFECT OF TOLERANCE LIMIT ON THE FUZZY LTPD SSPs

(N=2,000, p_1=0.02, p_2=0.10, β=0.10)

β_U	c^*	n^*	$P(p_2)$	$I(p_1)$	λ^*
0.11	5	92	0.1041	113.86	0.504
0.13	5	90	0.1157	109.82	0.477
0.15	5	89	0.1219	107.86	0.517
0.18	5	87	0.1352	104.02	0.560
0.20	5	86	0.1422	102.15	0.566

5. Concluding Remarks

In the classical Dodge Romig [6] model, one of the objectives of the DM of minimum consumer's risk is achieved by fixing it at a specified value of β and then the second objective of minimum average amount of inspection is achieved by minimizing this objective function subject to the above constraint. In the present model, the DM specifies an interval $[\beta, \beta_U]$ for the risk that may be taken for allowing the lot to be marketed with a view to reducing the amount of inspection per lot. This non-symmetric fuzzy model is able to scale the objective function with the constraint so that a compromise solution is obtained and thereby reduces the amount of inspection in comparison with the non-fuzzy model.

The model can be easily extended to the case when the DM can specify the *weight* or *cost* per unit risk per lot and that for reduced inspection per unit per lot. The DM may also consider addition operator to aggregate the fuzzy sets and design the optimum SSP by maximising the weighted fuzzy achievement function (see Chakraborty [3]).

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