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A Marketing Decision Problem in Single-Period Stochastic Inventory Model

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Abstract

This article deals with a marketing decision problem in the classical single-period stochastic inventory model, where the level of marketing effort decides on the extent of demand. Specifically, it is assumed that mean demand is an increasing concave function of the level of marketing effort. The problem under consideration is optimal determination of both order quantity as well as level of marketing effort. Results are presented to describe the solution procedure for general demand distribution. Optimal decision rules are derived for some particular demand distributions.

Key Words

Single-period, stochastic, inventory, marketing effort.

1. Introduction

Marketing operations are vital to business management. The budgetary allocation of the companies towards marketing is growing at a very rapid rate. However, the decisions involved in marketing are taken generally on ad hoc basis, and therefore, the effectiveness of such decisions remains in doubt. This leads us to analyze the marketing decision problems in more objective way.

The marketing environment of most of the companies is characterized by keen competition, an over-supply of goods/services, high rate of product/service innovation and so on. These make the overall marketing process very complex to analyze in general. In fact, without some knowledge of the functional relationship between sales and marketing effort, it becomes very difficult to know how much to spend on marketing.

For quite sometime, the researchers have been studying the inventory models when (higher) demand is induced by different kind of marketing strategies. A major focus on such strategy has been the pricing-policy /quantity-discount. For instance, see Kotler [4], Ladany and Sternlieb [5], Lal and Staelin [6], Jucker and Rosenblatt [3], Shah and Jha [7], Eliashberg and Steinberg [2], Bhunia and Maiti [1] etc.

It is well known that marketing in the form of advertisement induces an increase in demand and sale. However, it may be observed that as the marketing effort, say frequency of advertisement, gradually increases demand is also expected to grow. The rate of this growth is generally decreasing in nature. Further, we note that the true demand is never known in practice. In order to analyze an inventory system, we therefore attach probability to different possible demand values. That is, demand is treated as random variable (or stochastic). In this article, we study the classical single-period stochastic inventory model where the (stochastic) demand can be controlled by marketing effort. It is assumed that with higher marketing effort, the mean demand can be increased but with a diminishing rate. The problem under consideration is to find simultaneously the optimal marketing effort as well as order quantity.

We formulate the problem in Section 2 as maximization of the *expected profit.* Section 3 contains the main results that describe the solution approach. We then derive the optimal decision rules for some special cases of demand distribution in Section 4.

2. Problem Formulation

Consider the classical single-period stochastic inventory model, which can be described as follows. The decision is to be made on the number of units (q) of an item to be procured, at a cost of $\$c$/unit, for inventory at the beginning of a period. The period represents the duration of the planning horizon. The demand (X) for the item during the period is a random variable. The sale price of the item is $\$s$/unit ($s > c$). The units that remain unsold at the end of the period, can be disposed of at the rate $\$v$/unit ($v<c$). However, if there is a shortage, it results in an opportunity loss of $\$p$/unit. We intend to maximize the expected profit over a given planning horizon.

In the above, we assume that the random demand (X) can be influenced by marketing effort, so that, demand is dependent on the level of marketing effort. Let us denote the level of marketing effort (for example, the number units of advertisements) by m and its unit cost by $\$r$.

It is natural that demand is likely to grow with an increase in marketing effort, but with a diminishing rate. Therefore, we assume that the mean demand, denoted by μ_m (with $\mu_0 > 0$) is an increasing concave function of m (See Figure 1), that is, μ_m increases with m, but the rate of increase is non-increasing (diminishing). Mathematically,

$$(i)\ \mu'_m = \frac{d\mu_m}{dm} > 0, \tag{2.1}$$

$$\text{and } (ii)\ \mu_m'' = \frac{d^2\mu_m}{dm^2} \le 0. \tag{2.2}$$

For example, the function

$$\mu_m = A - B\rho^m, \quad \text{for } A > B > 0,\ 0 < \rho < 1$$

possesses both the above properties given by equations (2.1) and (2.2).

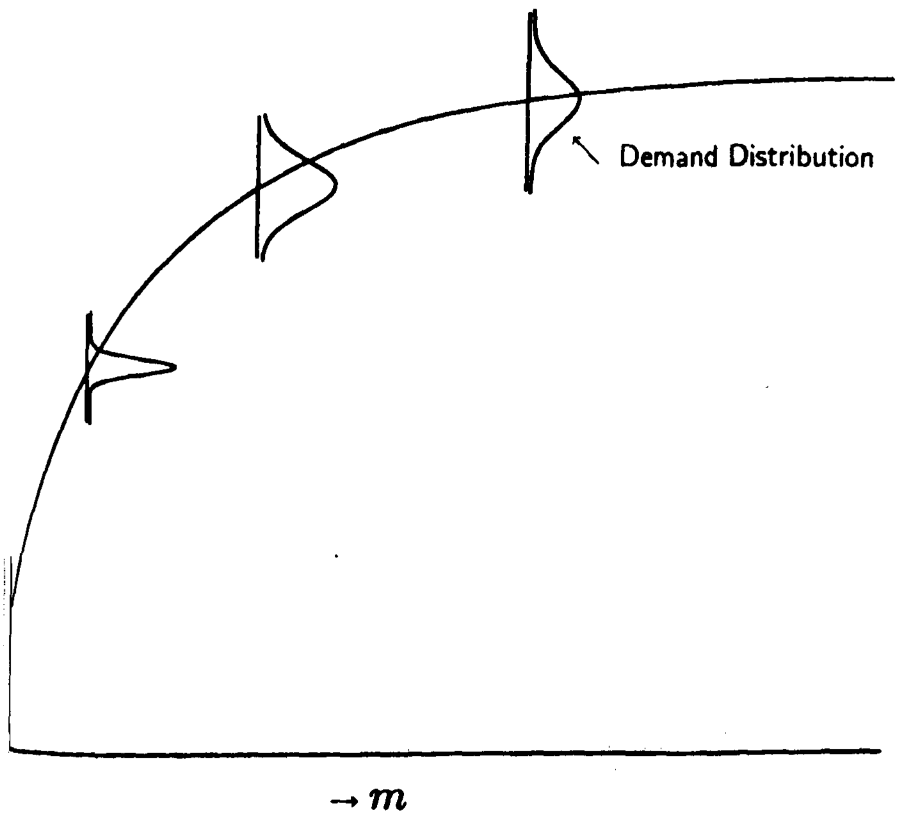

Figure 1 Relationship between m and μ_m

Further, it is assumed that X is non-negative continuous random variable. When the discrete demand values are quite large, it is customary to treat the same as continuous in various modelling situations. Denote its probability density function and cumulative distribution function by $f(x;\mu_m)$ and $F(x;\mu_m)$ respectively, when the level of marketing effort is m. Therefore,

$$\mu_m = \int_0^\infty xf(x;\mu_m)dx.$$

In order to write the mathematical expression for profit, we suppose that (q,m) is specified, and the demand is given as x. Obviously, $q \geq 0$ and $m \geq 0$. Then, the profit function

$$\pi(q,m \mid x) = \begin{cases} sx + v(q-x) - cq - rm & \text{if } x \leq q \\ sq - p(x-q) - cq - rm & \text{if } x > q \end{cases} \tag{2.3}$$

and hence, the expected profit denoted by $Z(q,m)$ can be written as

$$Z(q,m) = vqF(q;\mu_m) + (s-v)\int_0^q xf(x;\mu_m)dx$$
$$+ (s+p)q\left[1 - F(q;\mu_m)\right] - p\int_q^\infty xf(x;\mu_m)dx - cq - rm$$
$$= (v-c)q + (s-v)\mu_m - rm - (s+p-v)\int_q^\infty (x-q)f(x;\mu_m)dx \tag{2.4}$$

The problem under consideration is to find (q^*, m^*) such that

$$Z(q^*, m^*) = \max_{(q,m)} Z(q,m) \tag{2.5}$$

3. Main Results

In this section, we discuss the approach for derivation of optimal solution (q^*, m^*) with general forms of μ_m and $f(x;\mu_m)$.

Let k $(0 < k < 1)$ be any constant. If $F(q;\mu_m) = k$ holds, then we write $q = H(k;\mu_m)$, or simply $q = H(\mu_m)$. ($H(.)$ may be viewed as inverse function of $F(.)$.) We shall use this notation throughout the rest of the article.

Theorem 1 : An optimal solution (q^*, m^*) is given by :

(i) q^* is the solution for q in

$$F(q;\mu_{m^*}) = \frac{s+p-c}{s+p-v} \tag{3.1}$$

where m^* is described below.

(ii) m^* is the value of m that maximizes

$$z(m) = (s-v)\mu_m - (s+p-v)\int_{H(\mu_m)}^{\infty} xf(x;\mu_m)dx - rm \tag{3.2}$$

where $q = H(\mu_m)$ is obtained from the equation (3.1) by putting $m^* = m$.

Proof: For fixed m, we get from the equation (2.4),

$$\frac{dZ(q,m)}{dq} = (v-c) + (s+p-v)[1 - F(q;\mu_m)] \tag{3.3}$$

$$\text{and} \quad \frac{d^2Z(q,m)}{dq^2} = -(s+p-v)f(q;\mu_m) \tag{3.4}$$

Using the equations (3.3) and (3.4), one can see that

$$q = H\left(\frac{s+p-c}{s+p-v};\mu_m\right) = H(\mu_m) \tag{3.5}$$

maximizes the expected profit for any given m.

we use the relation (3.5) to eliminate q from the expression of expected profit, which in turn becomes a function of m only. Let us denote it by $z(m)$. On simplification, we get its expression as given in the equation (3.2). Consequently, we get m^* by maximizing $z(m)$. This completes the proof.

Therefore, according to the above result, we find m^* by maximizing $z(m)$ given by the equation (3.2), and then obtain q^* using the relation (3.1).

With regard to maximization of $z(m)$, we observe that

$$\frac{dz(m)}{dm} = \mu'_m[(s-v)-(s+p-v)\int_{H(\mu_m)}^{\infty} x\frac{df(x;\mu_m)}{d\mu_m}dx$$

$$+(s+p-v)\frac{dH(\mu_m)}{d\mu_m}\{H(\mu_m).f(H(\mu_m);\mu_m)\}]-r = \mu'_m.T(m)-r \quad \text{(say)},$$

and $$\frac{d^2z(m)}{dm^2} = \mu''_m.T(m)+\mu'_m.\frac{dT(m)}{dm}.$$

It must be noted that the function $z(m)$ depends on μ_m, $f(x;\mu_m)$ and the constant parameters -- c, s, v, p and r. A closed form solution for m^* looks very difficult with general forms of μ_m and $f(x;\mu_m)$. However, the following result gives the values of m^* under two possible situations.

Theorem 2 : (i) $m^* = 0$ if for every m, $T(m) \le 0$ or $\mu'_m T(m) < r$ holds, and (ii) $m^* = \infty$ if $\mu'_m T(m) > r \quad \forall m$.

Proof : Suppose that for every m, $T(m) \le 0$ or $\mu'_m T(m) < r$ holds. Since $\mu'_m > 0$, we then have $\frac{dz(m)}{dm} < 0$ for all m. It implies that $z(m)$ is decreasing monotone function of m. Consequently, optimal value of m is given by $m^* = 0$. Similarly, it can observed that $z(m)$ is increasing monotone function of m if $\mu'_m T(m) > r \quad \forall m$. Therefore, $m^* = \infty$ in this situation.

4. Special Cases

We now consider some special cases of demand distribution -- $f(x;\mu_m)$, and discuss the solution procedure even with the general form of the function μ_m. These cases are Exponential, Uniform, Normal and Lognormal. We assume throughout this section that the two situations described in Theorem 2 do not hold good, as otherwise, the solution is known. Therefore, we suppose that

(a) $\mu'_m T(m) \ge r$ for some m, and (b) $\mu'_m T(m) \le r$ for some m. (4.1)

Case I (Exponential) : For any m, let $X \sim Exp\ (\mu_m)$, that is,

$$f(x;\mu_m) = \frac{1}{\mu_m} e^{-x/\mu_m} \quad \text{for } x > 0.$$

Consequently,

$$F(x;\mu_m) = 1 - e^{-x/\mu_m}, \qquad q = H(\mu_m) = \mu_m . \ln\left(\frac{s+p-v}{c-v}\right),$$

$$z(m) = \mu_m \left[(v-c)\ln\left(\frac{s+p-v}{c-v}\right) + (s-c) \right] - rm,$$

and $T(m) = (v-c)\ln\left(\dfrac{s+p-v}{c-v}\right) + (s-c).$

Therefore, we have the following result describing the procedure to obtain optimal solution.

Corollary 1 : An optimal solution (q^*, m^*) is given by :

$$q^* = \mu_{m^*} \ln\left(\frac{s+p-v}{c-v}\right) \tag{4.2}$$

where m^* is the solution of

$$\mu'_m \left[(v-c)\ln\left(\frac{s+p-v}{c-v}\right) + (s-c) \right] = r. \tag{4.3}$$

Proof : This follows from Theorem 1 by observing that

(a) $T(m)$ is independent of m,

(b) the equation (4.3) has always a solution because of the assertion (4.1), and

(c) $\dfrac{d^2 z(m)}{dm^2} = \mu''_m . T(m) \le 0$, since $\mu''_m \le 0$ and $T(m) > 0$ hold.

<u>Case II (Uniform)</u> : For any m, let $X \sim U(0, 2\mu_m)$, so that,

$$f(x;\mu_m) = \frac{1}{2\mu_m} \quad \text{for } x \in (0, 2\mu_m).$$

This implies that

$$F(x;\mu_m) = \frac{x}{2\mu_m}, \qquad q = H(\mu_m) = 2\mu_m \cdot \left(\frac{s+p-c}{s+p-v} \right),$$

$$z(m) = \mu_m \left[\frac{(s+p-c)^2}{s+p-v} - p \right] - rm,$$

and $$T(m) = \frac{(s+p-c)^2}{s+p-v} - p.$$

Hence, optimal solution is derived as follows.

Corollary 2 : An optimal solution (q^*, m^*) is given by :

$$q^* = 2\mu_{m^*} \cdot \left(\frac{s+p-c}{s+p-v} \right) \tag{4.4}$$

where m^* is the solution of

$$\mu'_m \left[\frac{(s+p-c)^2}{s+p-v} - p \right] = r. \tag{4.5}$$

Proof : The argument is exactly same as in the proof of Corollary 1.

<u>Case III (Normal)</u> : For any m, let $X \sim N(\mu_m, \sigma^2)$, so that,

$$f(x;\mu_m) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}\left(\frac{x-\mu_m}{\sigma} \right)^2 \right\} \quad \text{for } -\infty < x < \infty.$$

It must be noted that even though $-\infty < x < \infty$, the use of Normal distribution has been extensive in all most all kinds of management decision problems (e.g. quality control, inventory, queues etc.) in order to describe the variation pattern of positive random variables. Furthermore, it can be observed that the expression for $Z(q,m)$ in the equation (2.4) remains unaltered, and the results of the previous section hold good for this distribution.

We then have

$$F(x;\mu_m)=\Phi\left(\frac{x-\mu_m}{\sigma}\right),$$

$$q=H(\mu_m)=\mu_m+\sigma\ \Phi^{-1}\left(\frac{s+p-c}{s+p-v}\right),$$

$$z(m)=(s-c)\mu_m-\frac{(s+p-v)\sigma}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left\{\Phi^{-1}\left(\frac{s+p-c}{s+p-v}\right)\right\}^2\right]-rm,$$

and $T(m)=(s-c)$, where $\Phi(.)$ is the cumulative distribution function of Standard Normal variable.

Therefore, we have the following result that can be proved by the arguments used for earlier corollaries.

Corollary 3 : An optimal solution (q^*,m^*) is given by :

$$q^*=\mu_{m^*}+\sigma\ \Phi^{-1}\left(\frac{s+p-c}{s+p-v}\right) \quad (4.6)$$

where m^* is the solution of

$$(s-c)\mu_m'=r. \quad (4.7)$$

Case IV (Lognormal) : For any m, let $X\sim LN(\xi,\sigma^2)$,, so that,

$$f(x;\mu_m)=\frac{1}{x\sigma\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{\ln x-\xi}{\sigma}\right)^2\right\} \quad \text{for } x>0,$$

where $\mu_m = E(X) = \exp(\xi + \sigma^2/2)$, that is, $\ln \mu_m = \xi + \sigma^2/2$. Or in other words, μ_m is a function of both the parameters of the distribution that implies μ_m is influenced by m through ξ or σ. We assume σ to be fixed, and derive optimal solution. Now, it may be observed that

$$F(x;\mu_m) = \Phi\left(\frac{\ln(x/\mu_m) + \sigma^2/2}{\sigma}\right), \quad q = H(\mu_m) = \mu_m \exp\left[-\frac{\sigma^2}{2} + \sigma\ \Phi^{-1}(k)\right],$$

$$z(m) = \mu_m\left[(s+p-v)\Phi\left\{\Phi^{-1}(k) - \sigma\right\} - p\right] - rm,$$

and $T(m) = (s+p-v)\Phi\left\{\Phi^{-1}(k) - \sigma\right\} - p$, where $\Phi(.)$ is the cumulative distribution function of Standard Normal variable and $k = \dfrac{s+p-c}{s+p-v}$.

Hence, optimal solution can be obtained from the following result.

Corollary 4 : An optimal solution (q^*, m^*) is given by :

$$q^* = \mu_{m^*} \exp\left[-\frac{\sigma^2}{2} + \sigma\Phi^{-1}(k)\right] \tag{4.8}$$

where m^* is the solution of

$$\mu'_m\left[(s+p-v)\Phi\left\{\Phi^{-1}(k) - \sigma\right\} - p\right] = r. \tag{4.9}$$

Before we conclude this section, it is important to note the following. In all the special cases considered above, we have noticed that $T(m)$ is independent of m. However, this is not true in general. For instance, in the case of lognormal demand distribution with fixed ξ, $T(m)$ is indeed a function of m. A simple decision rule for the same is not quite apparent.

5. Conclusion

We have studied the classical single-period stochastic inventory model wherein demand is under the influence of marketing effort. The mean demand is assumed to grow with an increase in marketing effort, but at a diminishing rate. The relationship used is in generic form, that is, it represents a class of functions. The problem under consideration is optimal determination of both order quantity as well as level of marketing effort. Results are presented to describe the solution procedure for general demand distribution. We also illustrate the same for some particular demand distributions -- Exponential, Uniform, Normal and Lognormal.

It is assumed throughout that demand is continuous random variable. However, we observe that the same approach can be adopted for discrete situation as well.

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7. References

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