Sankhyā : The Indian Journal of Statistics.
1977, Volume 39, Series B, Pt. 1, pp. 26–35.

ON RELATIVE EFFICIENCIES OF L-ESTIMATORS

By KESAR SINGH
Indian Statistical Institute

SUMMARY. In this paper we have studied a property of location estimators which are given as linear combination of order statistics. Intuitively, one would expect that the location estimators which give less weight to the extreme observations should be preferred for distributions having heavier tails. This feeling is made precise at different levels of generality introducing a new condition for heaviness of tail. Also, a method has been indicated to compute infimum of some relative efficiencies over the class of symmetric unimodal distributions.

1. Introduction

Throughout this paper, we restrict ourselves to the class C of strictly increasing absolutely continuous d.f. (distribution function). Let K be a d.f. on [0, 1], symmetric about $\frac{1}{2}$. Then following (Bickel and Lehmann, 1975), $\mu_K(F) = \int_0^1 F^{-1}(t)dK(t)$ is a location parameter. The most natural estimator of such a location parameter is $\hat{\mu}_K(F) = \int_0^1 F_n^{-1}(t)dK(t)$, where F_n denotes empirical d.f. and $F_n^{-1}(t)$ is defined in usual fashion. Asymptotic variance of such an estimator is given by (see Huber, 1969).

$$\sigma^2(\mu_K(F)) = \text{var}\,(U(K, F, T))$$

where T is uniform distribution on (0, 1) and

$$U(K, F, t) = \int_{1/2}^{t} dK(t)/f(F^{-1}(t)) \text{ with } f = dF(x)/dx.$$

By efficiency of one estimator w.r.t. other we mean the ratio of asymptotic variances. Whenever we replace $d(K(t))$ by $J(t)dt$, it is assumed that K permits a density which is given by $J(t)$. In general $e(K_2, K_1, F)$ denotes the relative efficiency of $\hat{\mu}_{K_2}(F)$ w.r.t. $\hat{\mu}_{K_1}(F)$. For brevity, we will also adopt the following notations.

(i) $e(T_\alpha, F)$ denotes the efficiency of α trimmed mean w.r.t. mean for d.f. F.

(ii) $e(W, \alpha, F)$ denotes the efficiency of α Winsorised mean w.r.t. mean for F.

(iii) $e(TW_\alpha, F)$ denotes the efficiency of α trimmed mean w.r.t. α Winsorised mean. For the sake of convenience, unless the contrary is precisely stated, we shall be assuming the distributions under consideration to be symmetric about zero which does not affect the generality of the results. Two distributions for which the efficiency is being compared are assumed to have same location parameter. We shall denote by D the subset of C consisting of all unimodal distributions.

Let F, $G \in C$ and suppose G has got heavier tail than F (A precise definition is given later). Suppose K_1 and K_2 are two d.f.'s on (0, 1) s.t. K_2 gives an estimator which is less sensitive for tails than that of K_1 (to be made precise later). Then we shall be proving results of the kind

$$e(K_2, K_1, F) \leqslant e(K_2, K_1, G).$$

In Bickel and Lehmann (1975) (also see Doksum, 1969), G is defined to have heavier tail than F if

$$G^{-1}(t)/F^{-1}(t) \text{ is non-decreasing in } (\tfrac{1}{2}, 1). \quad \dots \quad (1.1)$$

As long as weight functions are modification of the uniform distribution on [0, 1], (1.1) is enough. In order to achieve similar comparisons for a pair of estimators with general weight functions, a stronger condition (3.1) has been introduced for heaviness of tail. A consequence of Lemma (3.1) enables us to get lower bounds for certain relative efficiencies over the class D defined above.

A particular result of this kind is contained in Bickel and Lehmann (1975, Theorem 6) which proves that under above set up if (1.1) holds then $e(T_\alpha, F) \leqslant e(T_\alpha, G)$.

2. Linear weight functions

The first result which we are going to prove is essentially a generalisation of Theorem 6 in Bickel and Lehmann (1975). Denote $\frac{G^{-1}(t)}{F^{-1}(t)}$ by $r(t)$, $F^{-1}(t)^2$ by $p(t)$ and $G^{-1}(t)^2$ by $q(t)$.

Theorem 2.1 : *If G has heavier tail than F in the sense that for some $0 < \alpha < \frac{1}{2}$ we have the relation*

$$r(s) \leqslant r(1-\alpha) \leqslant r(t) \text{ for } \tfrac{1}{2} < s \leqslant 1-\alpha \leqslant t < 1,$$

then
$$e(T_\alpha, F) \leqslant e(T_\alpha, G). \quad \dots \quad (2.1)$$

For proof we need to strengthen Lemma 2b in Bickel and Lehmann (1975) as follows.

Lemma 2.1 : *Let H_1 be a continuous distribution function on $(0, \infty)$ and H_2 is obtained by truncating H_1 at the point u. Let $a(x)$, $b(x)$ be positive functions integrable w.r.t. H_1 s.t.*

(i) *$a(x)$ is non-decreasing;*

(ii) $$\frac{b(s)}{a(s)} \leqslant \frac{b(u)}{a(u)} \leqslant \frac{b(t)}{a(t)} \text{ for } 0 < s \leqslant u \leqslant t < \infty$$

then $$\frac{\int b(x)dH_1(x)}{\int a(x)dH_1(x)} \leqslant \frac{\int b(x)dH_2(x)}{\int a(x)dH_2(x)} .$$

Proof : Define $H_0(x) = H_1(x)$ in $(0, u)$

$$= \frac{1+H_1(x)}{2} \text{ in } [u, \infty).$$

Let $$H_1^*(t) = \int_0^t a(x)dH_1(x) / \int_0^\infty a(x)dH_1(x). \quad \ldots \quad (2.2)$$

Similarly we define H_2^* and H_0^* replacing H_1 by H_2 and H_0 respectively in (2.2) so that $\frac{dH_1^*}{dH_0^*}$ and $\frac{dH_2^*}{dH_0^*}$ exist. Set $\int_0^\infty a(x)dH_i(x) = \lambda_i$ for $i = 0, 1, 2$. Then the following are easy to verify :

$$\frac{dH_1^*}{dH_0^*} = \frac{\lambda_0}{\lambda_1} I(0, u) + \frac{2\lambda_0}{\lambda_1} I(u, \infty) \quad \ldots \quad (2.3)$$

$$\frac{dH_2^*}{dH_0^*} = \frac{\lambda_0}{\lambda_2} I(0, u) + \frac{2\lambda_0}{\lambda_2} I\{u\} \quad \ldots \quad (2.4)$$

where I stands for indicator function. Also $a(x)$ is non-decreasing $\Longrightarrow \lambda_1 \geqslant \lambda_2$. This fact together with condition (ii) of Lemma (2.1), (2.3) and (2.4) implies

$$\left[\frac{b(x)}{a(x)} - \frac{b(u)}{a(u)}\right]\left[\frac{dH_1^*}{dH_0^*} - \frac{dH_2^*}{dH_0^*}\right] \geqslant 0 \quad \forall u \geqslant 0. \quad \ldots \quad (2.5)$$

Integrating (2.5) w.r.t. H_0^* we get the required result.

Proof of Theorem 2.1 : Proof follows directly from above lemma by putting $u = 1-\alpha$, $a(t) = p(t)$, $b(t) = q(t)$ and H_1 as uniform distribution on $[\frac{1}{2}, 1]$.

Corollary 2.1 : *Let* $\alpha_0 = sup\{\alpha \leqslant \frac{1}{2} : r(t)$ *is non-decreasing in* $[1-\alpha, 1)\}$ *and* $r(s) \leqslant r(1-\alpha) \forall \frac{1}{2} < s \leqslant 1-\alpha_0$, *then* $\forall \alpha \leqslant \alpha_0$, (2.1) *holds. For* $\alpha_0 = \frac{1}{2}$, *Theorem 6 of Bickel and Lehmann* (1975) *is a special case of this corollary.*

Corollary 2.2 : *Let* $t_0 = inf\{t \leqslant \frac{1}{2} : r(t)$ *is non-decreasing in* $(\frac{1}{2}, 1-t)\}$ *and* $r(1-t_0) \leqslant r(s)$ *for* $(1-t_0) \leqslant s < 1$ *then* $\forall \alpha > t_0$, (2.1) *holds.*

Remark 2.1 : The following result shows that symmetry of F and G can also be relaxed to a little extent. Let F and G be s.t. $F^{-1}(t) = -F^{-1}(1-t)$, $G^{-1}(t) = G^{-1}(1-t)$ for $t \leqslant \alpha$ then under the conditions of Theorem (2.1) and zero mean (2.1) holds.

The statement follows easily from the following four inequalities which are true under above set up.

(i) $\left[\int_0^{\alpha} p(t)dt-\alpha p(\alpha)\right] \Big/ \int_0^{1/2} p(t)dt \leqslant$ similar expression for G.

(ii) $\left[\int_{1-\alpha}^{1} p(t)dt-\alpha p(1-\alpha)\right] \Big/ \int_{1/2}^{1} p(t)dt \leqslant$ similar expression for G.

(iii) $\left[\int_0^{\alpha} p(t)dt-\alpha p(\alpha)\right] \Big/ \int_{1/2}^{1} p(t)dt \leqslant$ similar expression for G.

(iv) $\left[\int_{1-\alpha}^{1} p(t)dt-\alpha p(1-\alpha)\right] \Big/ \int_0^{1/2} p(t)dt \leqslant$ similar expression for G.

The theorem proved below studies behaviour of efficiency (w.r.t. mean) of a class of location estimators which ignore tails to a lesser extent than trimmed mean. This result can be achieved, under stronger assumptions, as a particular case of Theorem (3.1) but the proof included here is of particular interest. In the statement which follow U stands for uniform distribution on [0, 1] which of course leads to mean. For convenience write

$$\int_{1-\alpha}^{1} p(t)dt = \mu_1, \ \int_{1-\alpha}^{1} q(t)dt = \mu_2, \ \int_{1/2}^{1} p(t)dt = \nu_1 \text{ and } \int_{1/2}^{1} q(t)dt = \nu_2.$$

Theorem 2.2 : *Let J be given as* $J = c$ *in* $[\alpha, 1-\alpha]$ *with* $1 < c < \dfrac{1}{1-2\alpha}$, $= d$ *otherwise s.t.* $\int_0^1 J(t)dt = 1$. *Assume* $r(t)$ *is non-decreasing in* $[1-\alpha, 1)$ *and for* $\frac{1}{2} < s \leqslant (1-\alpha)$, $r(s) \leqslant r(1-\alpha)$. *Then if K denotes distribution corresponding to J,* $e(K, U, F) \leqslant e(K, U, G)$.

Proof: After a little computation it follows that a sufficient condition for above result is

$$\frac{1}{\nu_1}\left[\mu_1-\int_{1-\alpha}^{1}(F^{-1}(1-\alpha)+\frac{d}{c}(F^{-1}(t)-F^{-1}(1-\alpha))^2dt\right]$$

$\leqslant$ similar expression for G ... (2.6)

using the conditions of above theorem it follows easily that $\mu_1/\mu_2 \leqslant \nu_1/\nu_2$. The fact that numerator in (2.6) is > 0 can be shown proving that $e(K, U, F) > \frac{1}{c^2}$. Notice that without strict inequality it straightaway follows from Theorem 5 in Bickel and Lehmann (1975). (2.6) follows if we prove

$$\left[\mu_1-\int_{1-\alpha}^{1}(F^{-1}(1-\alpha)+\frac{d}{c}(F^{-1}(t)-F^{-1}(1-\alpha))^2\right]\Big/\mu_1 \leqslant \text{similar expression for}$$

for G or equivalently

$$\mu_1\Big/\int_{1-\alpha}^{1}(F^{-1}(t)+K(F^{-1}(1-\alpha))^2dt \leqslant \text{similar expression for } G \text{ where } K > 0.$$

A sufficient condition for this is

$$\mu_1/\mu_2 \leqslant \int_{1-\alpha}^{1} F^{-1}(t)dt/r(1-\alpha)\int_{1-\alpha}^{1} G^{-1}(t)dt. \qquad \text{... (2.7)}$$

To establish (2.7) notice that $\mu_1 \leqslant \int_{1-\alpha}^{1} F^{-1}(t)G^{-1}(t)dt/r(1-\alpha)$. Hence it will be enough to show that

$$\int_{1-\alpha}^{1} G^{-1}(t)F^{-1}(t)dt\Big/\int_{1-\alpha}^{1} F^{-1}(t)dt \leqslant \int_{1-\alpha}^{1} q(t)dt\Big/\int_{1-\alpha}^{1} G^{-1}(t)dt \qquad \text{... (2.8)}$$

Since $G^{-1}(t)$ is an increasing function, (2.8) will follow from the argument of stochastic ordering if we show that $\forall 1-\alpha < t < 1$ one has $\int_{1-\alpha}^{t} F^{-1}(t)dt/\int_{1-\alpha}^{1} F^{-1}(t)dt \geqslant$ similar expression for G and this is clear from the fact that $r(t)$ is non-decreasing in $[1-\alpha, 1)$. This finishes the proof of Theorem 2.2.

Corollary 2.3 : *Let* $\alpha_0 = sup\,\{\alpha \leqslant \frac{1}{2} : r(t)$ *is non-decreasing in* $(1-\alpha, 1)\}$ *and* $r(s) \leqslant r(1-\alpha_0)\forall \frac{1}{2} \leqslant s \leqslant (1-\alpha_0)$ *then* $\forall\, \alpha \leqslant \alpha_0$. *the result of Theorem* 2.2 *holds.*

Remark 2.2 : Here also symmetry can be relaxed in the central part $[\alpha, 1-\alpha]$.

3. General weight functions permitting density

Let us write $h_1 = f(F^{-1}(t))$, $h_2(t) = g(G^{-1}(t))$ and $R(t) = h_2(t)/h_1(t)$ where f and g denote densities of F and G respectively. In this section we introduce a new condition for heaviness of tail, i.e., G has got heavier tail than F if

$$R(t) \text{ is non-increasing in } (\tfrac{1}{2}, 1). \qquad \ldots \quad (3.1)$$

It follows from Corollary 3.1 that this condition is stronger than (1.1). One also observes that condition (3.1) is transitive and invariant under scale transformation of one distribution or both. The following four distributions are arranged in increasing order of heaviness of tail according to criterion (3.1).

Example 3.1 : (i) Uniform, $f(F^{-1}(t)) = 1$

(ii) Triangular, $f(F^{-1}(t)) = [2(1-t)]^{\frac{1}{2}}$

(iii) Distribution with density $|x|e^{-x^2}$,

$f(F^{-1}(t)) = [-\log 2(1-t)]^{\frac{1}{2}}2(1-t)$

(iv) Double exponential, $f(F^{-1}(t)) = (1-t)$.

Now we prove a lemma which is of central importance in this section and also in next section.

Lemma 3.1 : *Let J denote density of a d.f. K on* [0, 1]. *If F and G satisfy* (3.1) *then $U(K, G, t)/U(K, F, t)$ is non-decreasing in* $(\frac{1}{2}, 1)$.

Proof : Set $\frac{1}{2} < S < S' < 1$. Using (3.1) observe that

$$\frac{U(K, F, S)}{U(K, G, S)} = \frac{\int_{1/2}^{s} R(t)[J(t)/h_2(t)]dt}{\int_{1/2}^{s} [J(t)/h_2(t)]dt} \geqslant R(S) \geqslant \frac{\int_{s}^{s'} R(t)[J(t)/h_2(t)]dt}{\int_{s}^{s'} [J(t)/h_2(t)]dt} .$$

From this the lemma follows easily.

Corollary 3.1 : *$R(t)$ is non-increasing $\Longrightarrow r(t)$ is non-decreasing in $(\frac{1}{2}, 1)$. Obviously, uniform distribution has got lightest tail in class D defined earlier according to criterion* (3.1) *and hence also according to* (1.1). *This fact enables us to compute infimum of many relative efficiencies over the class D. For example $e(T_\alpha, U) = 1/(1+4\alpha) = inf\ \{e(T_\alpha, F) : F \epsilon D\}$, a fact established by Bickel differently (see Bickel,* 1975). *Same procedure will work for $e(K, U, F)$ of Theorem* 2.2.

Using the lemma proved above we prove a theorem which is the most general result of the kind.

Theorem 3.1 : *Let J_1, J_2 be two symmetric densities of d.f.'s K_1 and K_2 s.t. $J_0 = J_1/J_2$ is non-decreasing in $[\frac{1}{2}, 1]$. Let F, G satisfy (3.1). Then $e(K_2, K_1, F) \leqslant e(K_2, K_1, G)$.*

Proof : In effect, we want to show

$$\int_{1/2}^{1} [U(K_1, F, t)/U(K_2, F, t)]^2 \, U^2(K_2, F, t)dt \Big/ \int_{1/2}^{1} U^2(K_2, F, t)dt$$

$\leqslant$ similar expression for G

we shall first prove

$$U(K_1, F, t)/U(K_2, F, t) \leqslant \text{similar expression for } G \quad \ldots \quad (3.2)$$

$$U(K_1, G, t)/U(K_2, G, t) \text{ is non-decreasing and} \quad \ldots \quad (3.3)$$

$$H_1(z) = \int_{1/2}^{z} U^2(K_2, F, t)dt \Big/ \int_{1/2}^{1} U^2(K_2, F, t)dt \geqslant \text{similar expression for } G$$

$$= H_2(z) \quad \ldots \quad (3.4)$$

$\forall z \epsilon (\frac{1}{2}, 1)$. Then using (3.2), (3.3) and stochastic ordering between H_1 and H_2 we conclude our desired result immediately. We shall indicate proof of (3.2), (3.3) and (3.4) briefly. (3.2) can be written as

$$\int_{1/2}^{t} J_0(t)[J_2(t)/h_1(t)]dt \Big/ \int_{1/2}^{t} [J_2(t)/h_1(t)]dt \leqslant \text{similar expression for } G$$

which is true observing that $J_0(t)$ is non-decreasing and $U(K_2, F, t)/U(K_2, G, t)$ is non-increasing.

Nextly for (3.3), set $\frac{1}{2} < S < S' < 1$. The result is a simple consequence of the following observation.

$$\frac{\int_{1/2}^{s} J_0(t)[J_2(t)/h_2(t)]dt}{\int_{1/2}^{s} [J_2(t)/h_2(t)]dt} \leqslant J_0(s) \leqslant \frac{\int_{s}^{s'} J_0(t)[J_2(t)/h_2(t)]dt}{\int_{s}^{s'} [J_2(t)/h_2(t)]dt} .$$

Finally (3.4) follows from The following inequalities. Writing $U^*(t) = [U(K_2, F, t)/U(K_2, G, t)]^2 \forall z \epsilon (\frac{1}{2}, 1)$

$$\frac{\int_{1/2}^{z} U^*(t)U^2(K_2, G, t)dt}{\int_{1/2}^{z} U^2(K_2, G, t)dt} \geqslant U^*(z) \geqslant \frac{\int_{z}^{1} U^*(t)U^2(K_2, G, t)dt}{\int_{z}^{1} U^2(K_2, G, t)dt} .$$

This finishes proof of Theorem 3.1

We can derive various interesting results similar to Theorems 2.1 and 2.2 for general weight functions permitting density and compute corresponding infimum of relative efficiencies over the class D of d.f.s.

4. Weight functions not permitting density

In this section we shall study related properties for the weight functions which give positive masses at few points. Following the notations adopted in Section 1,

$$[\mu_1+\alpha p(1-\alpha)]e(TW_\alpha, F) = (1-2\alpha)^2[\mu_1+\alpha(F^{-1}(1-\alpha)+\alpha/h_1(1-\alpha))^2].$$

So, obviously $\infty \geqslant e(TW_\alpha, F) \geqslant (1-2\alpha)^2$ with both ends sharp in c (i.e., both the bounds can be approached controlling $h_1(\alpha)$). The result proved below throws light on sensitivities of trimmed mean and Winsorised mean for the tail of underlying distribution. Also it enables us to find $\inf\{e(TW_\alpha, F) : F \in D\}$.

Theorem 4.1 : *If for two symmetric distributions F and G, $r(t)$ is non-decreasing in $(\frac{1}{2}, 1-\alpha]$ then $e(TW_\alpha, F) \leqslant e(TW_\alpha, G)$.*

Proof : Since the condition that $r(t)$ is non-decreasing and $e(TW_\alpha, F)$ are unaffected by scale transformation on F and G, it will be enough to prove that $e(TW_\alpha, F) \leqslant e(TW_\alpha, G^*)$ where $G^*(x) = G(r(1-\alpha)x)$. Write $g^* = dF^*/dx$, $h^*(t) = g^*(G^{*-1}(t))$ and $p^*(t) = G^{*-1}(t)^2$. An easy calculation shows that our theorem is true if we show

$$[(\alpha/h_1(1-\alpha))^2+2\alpha F^{-1}(1-\alpha)/h_1(1-\alpha)]/\int_{1/2}^{1-\alpha} p(t)dt+\alpha p(1-\alpha)$$

$\leqslant$ similar expression for G^*.

Above mentioned inequality is true under the light of following observations

(i) G^{*-1}/F^{-1} is non-decreasing in $[\frac{1}{2}, 1-\alpha]$ and $G^{*-1}(1-\alpha) = F^{-1}(1-\alpha)$ implies $G^{*-1} \leqslant F^{-1}$ in $(\frac{1}{2}, 1-\alpha]$.

(ii) $[d(G^{-1}(t)/F^{-1}(t))/dt]_{t=1-\alpha} \geqslant 0$ implies $h_1(1-\alpha) \geqslant h^*(1-\alpha)$.

This finishes proof of Theorem 4.2.

The theorem which follows is analogous to Theorem 3.1 for Winsorised type estimator. It is the most general result in this context.

Theorem 4.2 : *Let F and G satisfy (3.1) in $(\frac{1}{2}, 1-\alpha]$. Let J_1, J_2 be two symmetric densities of d.f.'s K_1 and K_2 respectively on $[0, 1]$ s.t. $J_0 = J_1/J_2$ is non-decreasing in $[\frac{1}{2}, 1]$. Define, $J_2^* = cJ_2$ in $[\alpha, 1-\alpha]$, $= 0$ O.W. where c is normalising constant and $J_1^* = J_1$ in $[\alpha, 1-\alpha]$, $= 0$ in $[\alpha, 1-\alpha]^c$. Let K_1^**

be the d.f. on [0, 1] whose absolutely continuous part is given by the density J_1^* *and it has two atomic points* α *and* $(1-\alpha)$ *which share rest of the mass equally. If* K_2^* *denotes d.f. corresponding to density* J_2^* *then* $e(K_2^*, K_1^*, F) \leqslant e(K_2^*, K_1^*, G)$.

We first prove briefly a lemma which is quite similar to Theorem 4.1.

Lemma 4.1 : *Let* $J_1^{**} = dJ_1$ *in* $[\alpha, 1-\alpha], = 0$ *O.W. where d is a normalizing constant. Let* K_1^{**} *denote d.f. corresponding to* J_1^{**} *then* $e(K_1^{**}, K_1^*, F) \leqslant e(K_1^{**}, K_1^*, G)$.

Proof: The argument is parallel to Theorem (4.1). If we apply the transformation $G^*(x) = G(x/a)$ then $U(K_1, G^*, t) = a\,U(K_1, G, t)$ and $\frac{dU(K_1, G, t)}{dt} = J_1(t)/h_1(t)$. For this result we apply the transformation $G^*(x) = G(U(K_1, G, 1-\alpha)x/U(K_1, F, 1-\alpha))$. Then supplying similar arguments one gets the lemma.

Proof of Theorem 4.2 : Notice that

$$\begin{aligned} e(K_2^*, K_1^*, F) &= e(K_2^*, K_1^{**}, F).e(K_1^{**}, K_1^*, F) \leqslant e(K_2^*, K_1^{**}, G).e(K_1^{**}, K_1^*, G) \\ &= e(K_2^*, K_1^*, G) \text{ (using Lemma 4.1 and a consequence of Theorem 3.1).} \end{aligned}$$

This establishes the theorem.

Remark 4.1 : $\infty \geqslant e(K_1^{**}, K_1^*) \geqslant 1/d^2$, with both ends sharp.

Remark 4.2 : Here we establish an interesting numerical result

$$\inf\{e(TW_\alpha, F) : F \in D\} = e(TW_\alpha, U) = (1-8\alpha^3+12\alpha^2)/(1+4\alpha)$$

For $\alpha = .05$, this quantity is about .86 while the universal lower bound is .81. Under the light of these facts one is safe in preferring trimmed mean to Winsorised mean.

Finally we turn to study of behaviour of relative efficiency of winsorised mean to mean. Such nice comparisons are not available in this case and as a consequence uniform distribution is not the least favourable distribution for this efficiency (also see Bickel 1965). The following result which is proved under a bit stringent condition throws some light on the behaviour of this efficiency.

Theorem 4.3 : *If* $r(t)$ *is constant in* $(\frac{1}{2}, 1-\alpha_0)$ *and non-decreasing in* $[1-\alpha_0, 1]$ *then* $e(W, \alpha, F) \leqslant e(W, \alpha, G)$ $\forall\, \alpha > \alpha_0$. *(*$e(W, \alpha, G)$ *is defined in Section* 1).

Proof: In effect we want to show that

$$\frac{\mu_1-\alpha[F^{-1}(1-\alpha)+\alpha/h_1(1-\alpha)]^2}{\nu_1-\mu_1+\alpha[F^{-1}(1-\alpha)+\alpha/h_1(1-\alpha)]^2} \leqslant \text{similar expression for } G. \quad \ldots \quad (4.1)$$

Let G^*, g^*, h^* and p^* be same as that defined in proof of Theorem 4.1. Then (4.1) can be concluded after replacing G by G^* (which is again good enough because everything is unaffected by scale transformation) from the following simple observations.

(i) $\int_{1/2}^{1-\alpha} p(t)dt =$ similar expression for G^*

(ii) $\int_{1-\alpha}^{1} p(t)dt \leqslant$ similar expression for G^*

(iii) $h(1-\alpha) = h^*(1-\alpha)$.

Example 4.1 : Following is an example of the situation described in Theorem 4.3. Let $F(\sigma x)$ have MLR in σ. $G(x) = F(x/\sigma_0)$ for $x \in (0, z)$ and $= \frac{1}{2}F(x/\sigma_1)+\frac{1}{2}F(x/\sigma_2)$ for $x \geqslant z$ where $\sigma_0, \sigma_1, \sigma_2$ are s.t. $F(z/\sigma_0) = (F(z/\sigma_1) + F(z/\sigma_2))/2$. Here $1-\alpha_0 = F(z/\sigma_0) = G(z)$. The required property follows from a theorem stated in Bickel and Lehmann (1975, pp. 1062).

Remark 4.3 : As $\alpha_0 \to \frac{1}{2}$ in Theorem 4.3, the condition of constant $r(t)$ tends to become void. Heuristically one can say that this is the reason why $e(W, \alpha, U) \to 1/3$ as $\alpha_0 \to \frac{1}{2}$ where $\frac{1}{3}$ is $\inf\{e(W, \alpha, F) : F \in D\}$ (Bickel, 1965).

Remark 4.4 : Theorem 4.3 admits the following generalisation. Let $R(t)$ be constant in $(\frac{1}{2}, 1-\alpha_0)$ and non-increasing after that. Let K be a d.f. on [0, 1] (having density) and K^* is obtained Winsorising K ay points α_0 and $1-\alpha_0$ then $\forall \alpha > \alpha_0$, $e(K^*, K, F) \leqslant e(K^*, K, G)$.

Acknowledgement

The author is greatful to Professor J. K. Ghosh for his guidance in this work. Thanks are also due to Dr. B. V. Rao and Mr. C. Srinivasan for their valuable suggestions during the preparation of this paper. Referee's valuable comments improved the presentation.

References

Bickel, P. J. (1965) : On some robust estimates of location. *Ann. Math. Statist.*, 36, 847-858.

Bickel, P. J. and Lehmann, E. L. (1975) : Descriptive statistics for non-parametric models—(I) Introduction, (II) Location. *Ann. Statist.*, 3, 1038-1069.

Doksum, K. (1969) : Starshaped transformations and the power of rank tests. *Ann. Math. Statist.*, 40, 1167-1176.

Huber, P. J. (1972) : Robust statistics : A review. *Ann. Math. Statist.*, 43, 1041-1067.

Paper received : July, 1976.

Revised : January, 1977.