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SHORT COMMUNICATIONS

THE OPTIMUM CONTINUOUS SAMPLING PLAN CSP-2 WITH $k = i$ THAT MINIMISES THE AMOUNT OF INSPECTION

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ABSTRACT

Several combinations of (i, f) are possible that will ensure the same AOQL under the continuous sampling plan CSP-2 wilh $k=i$. A procedure is developed here to find a unique (i, f) that will achieve the AOQL requirement and also minimise the amount of inspection for a given process average $\bar{p}$.

1. Introduction

Dodge [1] introduced a random order continuous sampling plan CSP-1 and thereafter Dodge and Torrey [2] offered additional continuous sampling plans CSP-2 and CSP-3. The CSP plans provide for corrective inspection with a view to having a *limiting average outgoing quality* AOQL which will not be exceeded no matter what quality is submitted. Plan CSP-2 differs from plan CSP-1 in that once sampling inspection is started, 100% inspection is not invoked immediately when a defect is found but is invoked only if a second defect occurs in the next k or less sample units.

The factor k may be theoretically assigned any value. Since it is difficult to find analytically (i, f, k) that would ensure a given AOQL no matter how the incoming quality varies. Dodge and Torrey [2] studied elaborately the CSP-2 plan with $k=i$ only and obtained several combinations of (i, f) that would ensure a desired AOQL. For the sake of convenience, hereafter throughout the text, we should use the term CSP-2 to denote CSP-2 plan with $k=i$ only.

It is needless to say that if the choice of one of the plans (either of two types namely CSP-1 or CSP-2 or one out of many possible for either type) is not judiciously made, one has to undertake unnecessary extra inspection. Ghosh [3] worked out the optimum CSP-1 plan to minimise the amount of inspection for a known distribution of incoming quality p. However, the major problem in studying the CSP-2 plan was that unlike CSP-1 plan, it was not possible to find analytically f for a given choice of i for a required AOQL. Dodge and Torrey [2] obtained the (i, f) combination by a tedious trial and error procedure. Ghosh [4] found an algebraic relation connecting i, f and AOQL making selection of i and f much easier than before and eventually leading to the development of a near optimum CSP-2 plan. In the present paper we have worked out the exact optimum CSP-2 plan.

Continuous sampling plan relates to Statistical Quality Control. However, the problem of finding an optimum CSP plan, in general, is a problem of mathematical programming and our approach and analysis of this problem will be of interest to practitioners in the field of Operations Research as well.

2. Notations

p_L : limiting average outgoing quality AOQL

p_A : average outgoing quality AOQ

p_1 : the quality level for which AOQL is reached

$\bar{p}$: the process average

p : the incoming proportion defective

and $q = 1-p$.

The symbols i, k, f, u and v have their usual meanings.

From the results of Dodge and Torrey [2], we have

$$u = \frac{1-q^i}{pq^i}, \tag{1}$$

$$fv = \frac{1}{p} + \frac{q^k}{1-q^k}\left(\frac{1}{p} + k\right) + \frac{1}{p} - \frac{kq^k}{1-q^k}$$
$$= \frac{2-q^k}{p(1-q^k)}. \tag{2}$$

$$F \text{ (amount of inspection)} = \frac{u+fv}{u+v} = \frac{\dfrac{1-q^i}{pq^i} + \dfrac{2-q^k}{p(1-q^k)}}{\dfrac{1-q^i}{pq^i} + \dfrac{2-q^k}{fp(1-q^k)}}$$
$$= \frac{f}{f+(1-f)q^i(2-q^i)} \text{ for } k=i, \tag{3}$$

and
$$p_A = p(1-F). \tag{4}$$

3. On Some Properties of i, p_1 and f

The determination of the value of f which will ensure a desired AOQL for a given i under CSP-2 inspection has been discussed in Section 3 of [4] and the relationship between i and f is stated below:

$$f = \frac{q_1^{i+1}}{ip_L . \frac{s_1}{r_1^2} + q_1^{i+1}}, \tag{5}$$

where $q_1 = 1-p_1$, $s_1=2-2q_1^i$, $r_1 = 2-q_1^i$ and p_1 is the value of incoming quality p for which the desired AOQL is attained.

To find f it is necessary to determine p_1 for a given i from the equation

$$p_1 = \frac{ip_L + \frac{r_1}{s_1}}{i + \frac{r_1}{s_1}}. \tag{6}$$

Since p_1 appears in the expression of r_1 and s_1 in the r.h.s., it is not possible to obtain p_1 directly. It has to be determined numerically by following an iterative procedure, the basis for which is provided by the following lemmas.

LEMMA 1. *For a given i,* $\frac{r}{s}$ which is a function of p decreases as p *increases.*

Proof.
$$\frac{d\left(\frac{r}{s}\right)}{dp} = \frac{d}{dp}\left[\frac{2-(1-p)^i}{2-2(1-p)^i}\right]$$
$$= \frac{-2i\ (1-p)^{i-1}}{\{2-2(1-p)^i\}^2}$$
< 0 for all p in the range $0 \leqslant p \leqslant 1$ and for all $i > 0$.

Hence the result follows.

LEMMA 2. *For a given i the expression* $\frac{ip_L + \frac{r}{s}}{i + \frac{r}{s}}$ is a decreasing *function of p bounded by* 1 *and* $\frac{ip_L + 1}{i+1}$ *for* $0 \leqslant p \leqslant 1$.

Proof. We have $\dfrac{i\,p_L + \frac{r}{s}}{i + \frac{r}{s}} = \dfrac{\left(i + \frac{r}{s}\right)p_L}{i + \frac{r}{s}} + \dfrac{\frac{r}{s}\left(1-p_L\right)}{i + \frac{r}{s}}$

$$= p_L + (1-p_L)\left[1 - \frac{i}{i+\frac{r}{s}}\right].$$

For $p = 0$, $1 - \dfrac{i}{i+\frac{r}{s}} \to 1$ and for $p = 1$, $\dfrac{r}{s} = 1$. In view of Lemma 1,

for a given i, $\left[1 - \dfrac{i}{i+\frac{r}{s}}\right]$ decreases with increase of p.

Hence, the whole expression is a decreasing function of p lying between 1 and $\dfrac{ip_L + 1}{i+1}$ for $0 \leqslant p \leqslant 1$.

LEMMA 3. *For a given i and p_L, there exists an unique p, called p_1, for which equation (6) is satisfied.*

Proof. The left hand side is a strictly increasing function of p ranging from $0 \leqslant p \leqslant 1$. The righthand side is a strictly decreasing function of p ranging from

$1 \leqslant \dfrac{ip_L + \frac{r}{s}}{i + \frac{r}{s}} \leqslant \dfrac{ip_L + 1}{i+1}$ for $0 \leqslant p \leqslant 1$. Since $p_L < 1$, $\dfrac{i\,p_L + 1}{i+1} < 1$.

As p increases from 0 to 1, $p - \dfrac{i\,p_L + \frac{r}{s}}{i + \frac{r}{s}}$ changes progressively from negative to positive value. Hence the result follows.

LEMMA 4. *For a given p_L, the value of proportion defective p_1 at which the AOQL is attained, decreases as i incrases.*

Proof. We have $\dfrac{i\,p_L + \frac{r}{s}}{i + \frac{r}{s}} = p_L + (1-p_L)\, G\,(i,p),$

where
$$G(i,p) = \frac{\dfrac{2-q^i}{2-2q^i}}{i+\dfrac{2-q^i}{2-2q^i}}.$$

Treating G as a function of i alone for a given p, we have

$$\frac{dG}{di} = \frac{\dfrac{2i\,q^i \log q}{(2-2q^i)^2} - \dfrac{2-q^i}{2-2q^i}}{\left\{i+\dfrac{2-q^i}{2-2q^i}\right\}^2} < 0,$$

as $0 < q < 1$ and $\dfrac{2-q^i}{2-2q^i} > 0.$

Hence $\dfrac{i\,p_L + \dfrac{r}{s}}{i+\dfrac{r}{s}}$ is a decreasing function of i for a given p and

$$G(i+1,\, p) < G(i,p) \quad \text{for all } p \tag{7}$$

Since $p_1(i)$ satisfies equation (6), $p_1(i)$ is the abscissa of the intersection of $y = G(i.p)$ and $y = p$. Hence $p_1\,(i+1) < p_1(i)$ in view of (7) and the result follows.

Finally, in order to determine p_1, we express equation (6) as a function of p and solve numerically the equation

$$f(p) = 0, \tag{8}$$

where $f(p) = ip + \dfrac{2-q^i}{2-2q^i}\,(p-1) - i\,p_L.$

$$= p\left(i+\frac{r}{s}\right) - i\,p_L - \frac{r}{s}.$$

Differentiating w.r.t. p, we get

$$f'(p) = i + \frac{2-q^i}{2-2q^i} + (p-1).\,\frac{-2i(1-p)^{i-1}}{\{2-2(1-p)^i\}^2}$$

$$= i + \frac{r}{s} + \frac{i(2-s)}{s^2}.$$

Using Newton–Raphson method, we have

$$p_{(n+1)} = p_{(n)} - \frac{f(p_{(n)})}{f'(p_{(n)})}$$

$$= p_{(n)} - \frac{p_{(n)}\left(i+\frac{r_{(n)}}{s_{(n)}}\right) - i\,p_L - \frac{r_{(n)}}{s_{(n)}}}{\left(i+\frac{r_{(n)}}{s_{(n)}}\right) + \frac{i(2-s_{(n)})}{s^2_{(n)}}}$$

$$= \frac{s^2_{(n)}\left(i\,p_L + \frac{r_{(n)}}{s_{(n)}}\right) + i p_{(n)}\ (2-s_{(n)})}{s^2_{(n)}\left(i+\frac{r_{(n)}}{s_{(n)}}\right) + i\,(2-s_{(n)})}. \tag{9}$$

As discussed in [4], $p = \frac{ip_L + 1}{i+1}$ provides a reasonably good approximate solution to (6) particularly when p is not too small and i is moderately large.

The iterative procedure can therefore be started with

$$p_{(o)} = \frac{i\,p_L + 1}{i+1}, \; r_{(o)} = 2 - q^i_{(o)}, \; s_{(o)} = 2 - 2q^i_{(o)},$$

and terminated to give a solution p at any desired level of accuracy.

Taking this solution value as p_1 for a given i, f can be determined readily from equation (5).

4. To Determine Optimum (i, f) that Minimises the Amount of Inspection for a Given $\bar{p}$.

The procedure is developed on the basis of the following theorem:

THEOREM 1. *For a given p_L, there exists a pair (i, f) which minimises the amount of inspection at $\bar{p}$ under CSP-2 scheme of inspection provided $\bar{p} > p_L$. Furthermore, the minimum is attained for that value of i for which $p_1(i) = \bar{p}$.*

Proof. Let $F_{\bar{p}}\,(i)$ denote the amount of inspection at $\bar{p}$ for a given (i, f). Let $p_1(i_1)$, $p_1(i)$ and $p_1(i_2)$ be respectively the values of p at which the desired AOQL is attained for $i_1 < i < i_2$. Let $p_1(i)$ be equal to $\bar{p}$.

It is known that $p_1(i_1) > p_1(i) = \bar{p} > p_1(i_2)$. Hence we have from the property of AOQ curve

$$\bar{p}(1 - F_{\bar{p}}\,(i_1)) < p_L,$$

$$\bar{p}(1 - F_{\bar{p}}\,(i)) = p_L,$$

and $$\bar{p}(1 - F_{\bar{p}}\,(i_2)) < p_L,$$

which implies that $F_{\bar{p}}(i)$ is the minimum of the three. Since this is true for all $i_1 < i < i_2$, $F_{\bar{p}}(i)$ is the minimnm over all values of i.

Since for an optimal plan $p_1(i)=\bar{p}$, it follows from (6) that

$$\bar{p}\left(i+\frac{2-\bar{q}^i}{2-2\bar{q}^i}\right)-ip_L-\frac{2-\bar{q}^i}{2-2\bar{q}^i}=0$$

$$\Rightarrow \frac{i}{i+\dfrac{2-\bar{q}^i}{2-2\bar{q}^i}}=\frac{1-\bar{p}}{1-p_L}.$$

For $\bar{p} < p_L$, $\dfrac{1-\bar{p}}{1-p_L} > 1$ but the left hand side is less than 1 since $\dfrac{2-\bar{q}^i}{2-2\bar{q}^i} > 1$. Thus for $\bar{p} < p_L$ we can not find any optimum plan as it can be easily shown that the amount of inspection goes on decreasing as i increases. Hence the result follows.

The algorithm to find the optimum (i, f) can, therefore, be stated as follows :

Step 1. Compute $p_1(i)$ for successive integer values of i starting from 1 and identify i such that $p_1(i) \geqslant \bar{p} > p_1(i+1)$.

Step 2. If $p_1(i)=\bar{p}$, take $i_o=i$ and compute f_o using (5).

Step 3. Otherwise compute values of f corresponding to both i and $i+1$ and evaluate $F_{\bar{p}}(i)$ and $F_{\bar{p}}(i+1)$.

Step 4. Take i_o as i or $i+1$ according as $F_{\bar{p}}(i)$ is less than or greater than $F_{\bar{p}}(i+1)$ and take the corresponding f as f_o.

The pair (i_o, f_o) is the optimum choice of (i, f).

Remark 1. It is easy to see that if $p_1(i_o)=\bar{p}$, then the choice of (i, f) is unique. Otherwise it can not be said that the optimum (i, f) combination is unique. However, our experience shows that even in such cases the choice of (i, f) turns out to be unique.

Remark 2. The search effort for i_o can be reduced considerably by determining p_1 only for $i \geqslant \dfrac{1-\bar{p}}{1-p_L}$, as it can be proved that $i_o \geqslant \dfrac{1-\bar{p}}{1-p_L}$ It may be recalled that $i\left(=\dfrac{1-\bar{p}}{1-p_L}\right)$ is the optimum choice for the CSP-1 plan (Refer Sec. 4 of [3]).

5. Comparison of Optimum CSP-2 Plan with Optimum CSP-1 Plan

A table has been prepared showing the optimum values of (i, f) and the minimum amount of inspection for a given $\bar{p}$ for a wide choice of combinations of $(p_L, \bar{p})$. This may be obtained from the author on request.

The minimum amount of inspection at $\bar{p}$ for a given p_L was found to be same for both CSP-1 and CSP-2. The conclusions reached in [4] about the performance of near optimal CSP-2 plan in comparison to CSP-1 were found to be valid for the exact optimum plan also. To sum up, the optimum CSP-2 plan with $k=i$ has little more to offer than the optimum CSP-1 plan. Since a CSP-2 plan is more difficult to execute, the use of optimum CSP-1 plan alone is recommended.

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