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CONSTRUCTION OF (M, S)-OPTIMAL DESIGN FOR BLOCK SIZE 3

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Abstract: (M, S)-optimal designs are constructed for block size three when the number of treatments is of the form $6t+3$.

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1. Introduction

Shah (1960) introduced an optimality criterion which was called S-optimality by Kiefer (1974) and was subsequently extended by Eccleston and Hedayat (1974). They introduced a new optimality criterion which they call (M, S)-optimality. Shah (1960) indicated why this criterion should lead to designs with high efficiency with respect to A, D or E optimality. John and Mitchell (1977) further exploited this and conjectured that a class of S-optimal designs called regular graph designs (RGD) are A, D and E optimal. Jacroux and Seely (1980) derived some sufficient conditions for establishing this (M, S)-optimality. They showed that if a binary design with v treatments and b blocks has an incidence matrix $N_{v\times b}$ such that the diagonal elements of NN' are either r or $r+1$, for some r, and the off-diagonal elements of NN' are λ or $\lambda+1$, for some λ, then that design is (M, S)-optimal. We look at the case where all blocks are of size 3. By direct construction techniques we show that if the number of treatments $v\neq 3 \pmod 6$ then such designs exist for any b.

2. Blocks of size 3

Before considering the case $k=3$, it is necessary to recall something about Steiner Triple System (STS) and Kirkman Triple System (KTS). For information on Steiner and Kirkman triple systems, the reader is referred to the extensive bibliography by Doyen and Rosa [1].

An STS(v) is a BIBD with v treatments, block size 3 and $\lambda = 1$.

A KTS(v) is an STS(v) which is resolvable, that is, the triples can be divided into r classes, called resolution classes, such that every treatment occurs exactly once in each resolution class; here r is the replication number of the STS which is equal to $(v-1)/2$.

It is well known that a necessary and sufficient condition for the existence of STS(v) is $v \equiv 1$ or 3 (mod 6).

A necessary and sufficient condition for the existence of KTS(v) is $v \equiv 3$ (mod 6).

Given any v and b, our problem is to construct b blocks of size 3 each such that each treatment is replicated r or $r+1$ times and each pair of treatments occurs λ or $\lambda + 1$ times. The values of r and λ are uniquely determined from b and v.

We claim that if $v \equiv 1$ or 3 (mod 6), it is enough to consider $\lambda = 0$.

If $\lambda > 0$, let $B = \binom{v}{2}/3$, the number of blocks in STS(v) and let $b = \lambda B + c$, $0 \le c < B$.

First construct a set of c triples in which each treatment is replicated r' or $r'+1$ times and each pair occurs at most once; then add λ copies of STS(v) so that each pair now occurs λ or $\lambda + 1$ times and every element is replicated r or $r+1$ times for some r. In fact, in each STS(v), the replication number is $(v-1)/2$ so that $r = \lambda(v-1)/2 + r'$.

Thus for $v \equiv 1$ or 3 (mod 6) it is enough to consider $\lambda = 0$. As a consequence it is enough to consider $b < B$.

Theorem. *If* $v \equiv 3$ (mod 6) *and if b is any integer in the range* $0 \le b \le \binom{v}{2}/3$, *then there exists a set of b triplets of some v-set such that each treatment is replicated r or $r+1$ times and each pair of treatments occurs in at most one triple; furthermore, $r = [3b/v]$, the integer part of $3b/v$.*

Proof. For $v = 6t + 3$, $t \ge 0$, there exists a KTS(v). The replication number for such a KTS(v) is $(v-1)/2 = 3t + 1$.

So, all the triples can be divided into $3t+1$ resolution classes, say $R_1, R_2, \ldots, R_{3t+1}$ such that each R_i contains $2t+1$ triples and each treatment is in exactly one triple of R_i.

Now let $b = r(2t+1) + l$, $0 \le l < 2t+1$, $r \le 3t+1$. Then for our design, take all the triples from $R_1, \ldots, R_r$ and any l triples from R_{r+1}. Clearly each treatment is replicated r or $r+1$ times and, of course, each pair occurs at most once. □

Note that when $l = 0$ the design is a regular graph design as defined by John and Mitchell. Jacroux (1978) proved that there exists a connected (M, S)-optimal design iff $bk \ge v + b - 1$. According to that the design described is connected if $r \ge 2$ or $r \ge 1$ and $l \ge t$.

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