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SOME RESULTS ON THE DISTRIBUTION OF ADDITIVE ARITHMETIC FUNCTIONS I

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SUMMARY. It has been conjectured by P. Erdös that if f is an additive arithmetic function and I is a bounded interval such that $f^{-1}(I)$ has positive natural density, then f has a distribution. This conjecture was solved partially by E.M. Paul in 1967. Our object of this paper is an attempt to answer the following question. Suppose $\{m : f(F(m)) \in I\}$ has positive density. Then is it true that $f(F(m))$ has a distribution (where, I is a bounded interval and F is an integral polynomial) ?

1. Introduction

It has been conjectured long ago by P. Erdös if f is an additive arithmetic function such that $f^{-1}(I)$ has positive natural density for some bounded interval I, then f has a distribution. Partial solution to this conjecture was given by E. M. Paul (1967). In this paper some more necessary and sufficient conditions for f to have a distribution are given. Let $F(m)$ be an integral valued polynomial such that $F(m) > 0$ for $m = 1, 2, \ldots$ and $F(m)$ is not divisible by square of any irreducible polynomial. Suppose density of $\{m : f(F(m)) \epsilon I\}$ exists and is positive for some bounded interval I. Then is it true the $f(F(m))$ has a distribution ? Partial answer to this question is given in the positive direction. It is also shown, under very general conditions, that $f(F(m))$ has a distribution, then the spectrum of the distribution is the closure $\{f(F(m)) : m \geqslant 1\}$.

2. Notations and definitions

Let $F(m) = a_k m^k + \ldots + a_0$ be a polynomial which is not divisible by square of any irreducible polynomial, where $a_0, a_1, \ldots, a_k$ are integers and $a_k \neq 0$. Let $F(m) > 0$, for $m = 1, 2, \ldots$. Let f be a real-valued additive arithmetic function. Let $r(d)$ denote the number of incongruent solutions of the congruence relation $F(m) \equiv 0 \pmod{d}$. Let $p, p_1, p_2, \ldots$ denote prime numbers.

Throughout this paper, if $k \geqslant 2$, we assume that f satisfies the condition $f(p^t)r(p^t) \to 0$ as $p \to \infty$ for $t = 1, \ldots, k-1$. If $k = 1$ we do not put any such restrictions on f.

Let $N_n\{\cdots\}$ denote the number of positive integers $m \leqslant n$ satisfying the conditions $\{\cdots\}$. A real-valued arithmetic function g is said to have a distribution if there is a distribution function Q on the real line such that, $\lim_{n \to \infty} n^{-1} N_n \{g(m) < c\}$ exists and equals $Q(c)$ for all continuity points c of Q.

We write

$$f'(p^t) = \begin{cases} f(p^t) & \text{if} |f(p^t)| < 1, \\ 1 & \text{if} |f(p^t)| \geqslant 1. \end{cases}$$

$$A(v, n) = \sum_{v < p \leqslant n} \frac{1}{p} f'(p) r(p)$$

$$B(v, n) = \Big[\sum_{v < p \leqslant n} \frac{1}{p} (f'(p))^2 r(p) \Big]^{1/2}$$

$$A(n) = A(0, n)$$

$$B(n) = B(0, n).$$

Main results.

Theorem 1 : *If* $\limsup_{n \to \infty} \frac{1}{n} N_n\{f(F(m)) = a\} > 0$ *for some real number* a, *then* $f(F(m))$ *has a distribution.*

Theorem 2 : *If* $f(F(m))$ *has a distribution on a bounded non degenerate interval* I *and if this distribution is not uniform, then* $f(F(m))$ *has a distribution.*

Here, by a distribution on a bounded interval we mean a finite countably additive measure μ on I such that whenever a and b are interior points of I and $\mu(a) = \mu(b) = 0$, then

$$\lim_{n \to \infty} \frac{1}{n} N_n\{f(F(m)) \in (a, b)\} \text{ exists and equals } \mu(a, b).$$

Theorem 3 : *If* $f(F(m))$ *has a distribution, then* $f(F(1)), f(F(2)), \ldots$ *all belong to the spectrum of the distribution.*

Theorem 4 : *If* $\lim_{n \to \infty} \frac{1}{n} N_n\{f(Fm)) \in I\} = 1$ *for some bounded interval then* $f(F(m))$ *has a distribution.*

Proof of this theorem, when $F(m) = m$, was supplied by Professor E. M. Paul during one of our discussions.

Let $M_L(S)$ and $M_U(S)$ denote the lower and the upper magnifications of S, where S is any set of positive integers. For these notations and definitions see Paul (1962). Let $\lambda(S)$ denote the logarithmic density of S, whenever it exists. Let $\overline{P}$ denote the P-measure on Paul's space introduced in Paul (1962).

Theorem 5 : *The following are equivalent.*

(i) *There is a real number a such that for all $\varepsilon > 0$*

$$\overline{P}\{M_L(m:f(m) \in (a-\varepsilon,\ a+\varepsilon))\} > 0.$$

(ii) *For all* $\varepsilon > 0$, $\overline{P}\left\{\bigcup_{n=1}^{\infty} A_{n,\varepsilon}\right\} > 0$,

where $$A_{n,\varepsilon} = \left\{x : \left|\sum_{r<t\leq} f(p_t^{\prime t})\right| < \varepsilon, \quad \text{for all} \quad r, l \geqslant n\right\}$$

(iii) *f has a distribution.*

Preliminary results. We need the following lemmas.

Lemma 1 : *If $B(n) \to \infty$ and $f(p)r(p) = o(B(p))$, then*

$$\lim_{n\to\infty} \frac{1}{n} N_n\{f(F(m)) - A(n) < xB(n)\} = G(x)$$

where $$G(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy.$$

This lemma follows from Halberstam's theorem (Halberstam, 1956).

Let $\mathfrak{S}$ be the ring of all polyadic numbers containing the ring of integers as a dense subspace. See Novoselov (1966) for a description of $\mathfrak{S}$. Let P be the completion of the normalised Haar measure on $\mathfrak{S}$. Let $F(x)$ be the extension of $F(m)$ to $\mathfrak{S}$, that is

$$F(x) = a_k x^k + \ldots + a_0.$$

Let

$$f(p, x) = \sum_{t=0}^{\infty} f(p^t) W(p^t, x)$$

where $$W(p^t, x) = \begin{cases} 1 & \text{if} \quad p^t \| F(x), \\ 0 & \text{otherwise.} \end{cases}$$

Note that there exists a p_0 such that the sequence $\{f(p, x)\}_{p>p_0}$ is independent. (See proof of Theorem 1 of Jogesh Babu, 1972).

Lemma 2 : *Suppose* $k \geqslant 2$. *Then for each* $\varepsilon > 0$ *there exists* $v_0 = v_0(\varepsilon)$ *such that for all* n

$$\sum_{m=1}^{n} \Big(\sum_{p > v} \bar{f}(p, m) - A(v, n)\Big)^2 \leqslant CnB^2(v, n) + \varepsilon n$$

for all $v > v_0$, *where* C *is a constant and*

$$\bar{f}(p^t) = \begin{cases} f(p^t) & \text{if } \ t \leqslant k-1 \\ 0 & \text{otherwise.} \end{cases}$$

The proof of this lemma is similar to that of Turan-Kubilius inequality (Lemma 3.1 of Kubilius, 1964, p. 31).

Lemma 3 : *Suppose* $\sup B(n)$ *is finite. Then there exists a distribution function* $Q(c)$ *such that*

$$\lim_{n \to \infty} \frac{1}{n} N_n\{f(F(m)) - A(n) < c\} = Q(c)$$

for all continuity points c *of* Q.

Further, if the sequence $\{A(n)\}$ is bounded, then for any $\theta \in [\theta_1, \theta_2]$, $\{f(F(m)) + \theta : m \geqslant 1\}$ is contained in the spectrum of Q, where

$$\theta_1 = \liminf_{n \to \infty} A(n) \text{ and } \theta_2 = \limsup_{n \to \infty} A(n).$$

Proof: Since $\{B(n)\}$ is a convergent sequence, by Kolmogorov's three-series theorem applied to the sequence of random variables

$$\Big\{ f(p, x) - \frac{1}{p} f'(p) r(p) \Big\}_{p > p_0}$$

we have convergence of

$$\sum_{p} \Big(f(p, x) - \frac{1}{p} f'(p) r(p) \Big) \qquad \text{a.e.}[P].$$

Let

$$g(x) = \begin{cases} \sum_{p} \Big(f(p, x) - \frac{1}{p} f'(p) r(p) \Big) \text{ if this converges,} \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise.} \end{cases} \qquad \ldots \quad (1)$$

Let $Q(c) = P\{x: g(x) < c\}$. Let δ and $\varepsilon > 0$ and let r be a sufficiently large number such that

$$P\left\{ x: \left| \sum_{p>} \left(f(p, x) - \frac{1}{p} f'(p) r(p) \right) \right| > \delta \right\} < \varepsilon.$$

Applying Lemma 2 or Turan-Kuiblius inequality for cases $k \geqslant 2$ and $k = 1$, we have for sufficiently large n,

$$\begin{aligned} N_n\{f(F(m)) - A(n) < c\} &\leqslant N_n\{ \sum_{p \leqslant r} f(p, m) - A(r) < c + \delta\} \\ &\quad + N_n\{| \sum_{r < p} f(p, m) - A(n) + A(r) | > \delta\} \\ &\leqslant nP\{x: \sum_{p \leqslant r} f(p, x) - A(r) < c + \delta\} + \varepsilon n \\ &\leqslant nP\{x: g(x) < c + \delta\} + 2\varepsilon n = nQ(c + \delta) + 2\varepsilon n. \quad \ldots \quad (2) \end{aligned}$$

Similarly we can show that, for sufficiently large n

$$nQ(c - \delta) - 2\varepsilon n \leqslant N_n\{f(F(m)) - A(n) < c\}. \quad \ldots \quad (3)$$

From (2) and (3) we have

$$\frac{1}{n} N_n\{f(F(m)) - A(n) < c\} \to Q(c)$$

as $n \to \infty$, for all continuity points c of Q. This proves the first part of the lemma. Another proof of this part of the lemma can be found in Katai, 1969.

To prove the second part of the lemma, since $\frac{1}{p} f'(p) r(p) \to 0$ as $p \to \infty$, $\theta \epsilon [\theta_1, \theta_2]$ implies that θ is a limit point of the sequence $\{A(n)\}$. Let $\theta \epsilon [\theta_1, \theta_2]$. Let $\{n_r\}$ be a subsequence such that $A(n_r) \to \theta$ as $r \to \infty$. Hence we have, from (2) and (3), that

$$\lim_{r \to \infty} n_r^{-1} N_{n_r}\{f(F(m)) < c\} = Q(c + \theta).$$

Now fix $\varepsilon > 0$ and $m \geqslant 1$. Let $F(m) = p_1^{x_1} \ldots p_n^{x_n}$, where $p_1, p_2, \ldots, p_n$ are prime numbers. By Egoroff's theorem applied to the sequence $\{g(x) - \theta - \sum_{p < n_r} f(p, x)\}$, we can find, for any $\delta > 0$, a measurable set H and a r_0 such that $P(H) > 1 - \delta$ and $H \subset \{x : |g(x) - \theta - \sum_{p < n_r} f(p, x)| < \varepsilon$, for all $r \geqslant r_0\}$.

Now let $n_{r_1} \geqslant \max(p_0, p_1, \ldots, p_n, n_{r_0})$ and let

$$R = \{x \epsilon H : x = F(m)y \text{ and } p \leqslant n_{r_1} \Longrightarrow p \text{ does not divide } y\}.$$

Hence $P\{x : g(x) \epsilon (f(F(m))+\theta-\varepsilon, f(F(m))+\theta+\varepsilon)\}$

$$\geqslant P\{x \epsilon R : |g(x)-\theta-\sum_{p \leqslant n_{r_1}} f(p.x)| < \varepsilon\}$$

$$\geqslant \eta(1-\delta) \prod_{i=1}^{n} \left(\frac{r(p_i^{x_i})}{p_i^{x_i}} - \frac{r(p_i^{x_i+1})}{p_i^{x_i+1}}\right) > 0,$$

where
$$\eta = \prod_{\substack{p \neq p_1, \ldots, p_n \\ p \leqslant n_{r_1}}} \left(1-\frac{r(p)}{p}\right).$$

So $f(F(m))+\theta$ is in the spectrum of $Q(c)$. This completes the proof of the lemma

Lemma 4 (P. Levy) : *Let $X_1, X_2, \ldots$ be a sequence of independent, purely discrete random variables such that $\sum_i X_i$ converges almost everywhere. Then the distribution of $\sum_i X_i$ is purely discrete, purely continuous singular or purely absolutely continuous. Moreover, if*

$$d_i = \sup_b P\{X_i = b\},$$

then the distribution of $\sum_i X_i$ is continuous if and only if

$$\prod_i d_i = 0.$$

Proofs of the main results.

Proofs of Theorems 1 and 2 : From the hypotheses it follows, as in (Winter and Erdös (1939, pp. 717-718), that $\sum_{|f(p)| \geqslant 1} p^{-1} < \infty$. Now by Lemma 1 and the hypotheses it follows, in either case, that $\sup_n B(n) < \infty$. From the proof of Lemma 3, it follows that

$$\lim_{n \to \infty} \frac{1}{n} N_n\{f(F(m)) < A(n)+c\} = P\{x : g(x) < c\} = Q(c).$$

In case of Theorem 1, since $\limsup_{n \to \infty} \frac{1}{n} N_n\{f(F(m)) = a\} > 0$, we can find a sequence $\{n_r\}$ of positive integers such that $A(n_r)$ tends to a limit θ as $r \to \infty$ and

$$0 < \limsup_{n \to \infty} \frac{1}{n} N_n\{f(F(m)) = a\} = \lim_{r \to \infty} n_r^{-1} N_{n_r}\{f(Fm)) = a\}. \quad \ldots \quad (4)$$

So, for every positive ε for which $a-\theta-\varepsilon$ and $a-\theta+\varepsilon$ are continuity points of Q, the right hand side of (4) is not less than

$$\lim_{r\to\infty} n_r^{-1}N_{n_r}\{f(F(m))-A(n_r)\epsilon(a-\theta-\varepsilon,\ a-\theta+\varepsilon)\} = P\{x:|g(x)+\theta-a| < \varepsilon\}.$$

Hence $P\{x:g(x)+\theta = a\} > 0$. By Lemma 4, the distribution of $g(x)$ is discrete and hence

$$\sum_{f(p)r(p)\neq 0} \frac{1}{p}\, r(p) < \infty.$$

Thus $\{A(n)\}$ converges. Now Theorem 1 follows from Theorem 1 of Jogesh Babu (1972).

In case of Theorem 2, we have $\liminf\limits_{n\to\infty} \frac{1}{n} N_n\{f(F(m))\epsilon J\} > 0$, for some bounded interval J. It follows that $\{A(n)\}$ is a bounded sequence. Let θ_1 and θ_2 be as in Lemma 3. If $\theta_1 = \theta_2$, then there is nothing to prove. If $\theta_1 < \theta_2$, from the proof of Lemma 3, it follows that every $\theta\epsilon[\theta_1, \theta_2]$ is a limit point of $\{A(n)\}$. If a and b are in the interior of I and are continuity points of the distribution of $f(F(m))$ on I, then for all $\theta\epsilon[\theta_1, \theta_2]$, we have

$$\lim_{n\to\infty} \frac{1}{n} N_n\{f(F(m))\epsilon(a, b)\} = P\{x : g(x)+\theta\epsilon(a, b)\}.$$

By changing θ continuously we get uniform distribution on I, which contradicts the hypothesis. Hence $\{A(n)\}$ converges. So, again by Theorem 1 of Jogesh Babu (1972) it follows that $f(F(m))$ has a distribution. This completes the proofs of Theorems 1 and 2.

Theorem 3 follows from Lemma 3.

Proof of Theorem 4 : As in the proofs of Theorems 1 and 2, we have $\sup\limits_n B(n) < \infty$ and $\{A(n)\}$ is bounded. From Lemma 3, it follows that $\{f(F(m)): m \geqslant 1\}$ is contained in the closure of the bounded interval I, i.e. for all $m \geqslant 1$, $|f(F(m))| < M$, for some $M > 0$. Now we define a new additive arithmetic function g by

$$g(p^l) = \begin{cases} |f(p^l)| & \text{if} \quad r(p^l) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $g(F(m))$ is non-negative for all $m \geqslant 1$ and $g(F(m)) < 2M$ for all $m \geqslant 1$. Hence by Theorem 3 of Jogesh Babu (1972), we have

$$\sum_{p} \frac{1}{p} g'(p)r(p) < \infty,$$

which implies convergence of $\sum_{p} \frac{1}{p} f'(p)r(p)$. Now the result follows from Theorem 1 of Jogesh Babu (1972).

Proof of Theorem 5 : Suppose f has a distribution. Then from Theorem 2 of Paul (1963) and from the proof of Theorem 1 Paul (1963), it follows that

$$\lambda\{m : f(m) \epsilon (c, d)\} = \overline{P}\{M_L(m : f(m) \epsilon (c, d))\}.$$

By Theorem 1 of Paul (1967) we have for every $\varepsilon > 0$

$$D\{m: f(m)| < \varepsilon\} > 0.$$

Hence,

$$\overline{P}\{M_L(m : |f(m)| < \varepsilon) > 0 \quad \text{for all } \varepsilon > 0.$$

This proves the implication (iii) $\Longrightarrow$ (i). (i) $\Longrightarrow$ (ii) is clear because

$$M_L(m : f(m) \epsilon (a-\varepsilon, a+\varepsilon)) \subset \bigcup_{n=1}^{\infty} A_{n, 2\varepsilon}.$$

To prove (ii) $\Longrightarrow$ (iii), assume (ii), then there exists an N such that $\overline{P}(A_{N,\varepsilon}) > 0$. Hence,

$$\overline{P}\{x : | \sum_{N < i \leqslant t} f(p_i^{x_i})| < \varepsilon, \text{ for all } \quad t > N\} > 0.$$

So,

$$\limsup_{t \to \infty} \overline{P}\{x : | \sum_{N < i \leqslant t} f(p_i^{x_i})| < \varepsilon\} > 0.$$

Hence there exists a sequence $\{d_i\}$, such that

$$\sum_{i=1}^{\infty} (f(p_i^{x_i}) + d_i) \text{ converges } \quad \text{a.e. } [\overline{P}]$$

(see Doob, 1953; p. 121). Let $\delta > 0$. Find an n_1 such that

$$\overline{P}(A_{n_1, \delta/2}) = \eta > 0.$$

By Egoroff's theorem choose a measurable set H such that $P(H) > 1-\eta/2$ and $\sum_{i=1}^{\infty} (f(p_i^{\prime t})+d_i)$ converges uniformly on H. Hence there exists $n > n_1$ such that

$$\bar{P}\{x : \text{ for all } t, r \geqslant n, |\sum_{r<i\leqslant t} (f(p_i^{\prime t})+d_i)| < \delta\} > 1-\eta/2.$$

Hence there exists an x such that for all $r, t \geqslant n$, we have

$$\left|\sum_{r<i\leqslant t} f(p_i^{\prime t}) < \frac{\delta}{2} \text{ and } \left|\sum_{r<i\leqslant t} (f(p_i^{\prime t})+d_i)\right| < \frac{\delta}{2}.\right.$$

Consequently for all $r, t \geqslant n$, $|\sum_{r<i\leqslant t} d_i| < \delta$.

So Σd_i converges, hence $\sum_{i=1}^{\infty} f(p_i^{\prime t})$ converges almost everywhere. Hence f has a distribution (see Paul, 1963). This completes the proof of Theorem 5.

Theorem 2, for $F(m) = m$, together with the following remarks, supports Erdös conjecture, stated in the introduction.

Remarks : Suppose f has a distribution with characteristic function h. Then for some u, $h(u) = 0$ if and only if $\frac{u}{\pi} f(2^t)$ is an odd integer for all positive integers t. This can be proved as follows. It is well known (Kubilius, 1964) that for all real u

$$h(u) = \prod_p \left(1-\frac{1}{p}\right)\left(1+\sum_{s=1}^{\infty} p^{-s} e^{iuf(p^s)}\right).$$

Note that

$$\prod_{p\geqslant 3} \left(1-\frac{1}{p}\right)\left(1+\sum_{s=1}^{\infty} p^{-s} \exp(iuf(p^s))\right) \neq 0$$

for all real u, since, if $p \geqslant 3$

$$\left|\sum_{s=1}^{\infty} p^{-s} \exp(iuf(p^s))\right| \leqslant \sum_{s=1}^{\infty} p^{-s} = \frac{1}{p-1} \leqslant \frac{1}{2}.$$

Now $\sum_{s=1}^{\infty} 2^{-s} \exp(i\, u\, f(p^s)) = -1$ if and only if $\frac{u}{\pi} f(2^s)$ is an odd integer for all $s \geqslant 1$. This proves our assertion.

From this remark it is easy to prove that an additive arithmetic function cannot have uniform distribution. This result was stated without proof by Erdös (1956).

Suppose on the contrary, the distribution of f is uniform, then there exists a non-zero real number a such that $h(2\pi n/a) = 0$ for all integers n, where h is the characteristic function of the distribution of f. From this it follows that $\frac{2}{a} f(2^s)$ is an odd integer for every $s \geqslant 1$. Hence

$$\sum_{s=1}^{\infty} 2^{-s} \exp\left(i\, 2\pi\, \frac{2}{a}\, f(2^s)\right) = 1.$$

So $h(2\pi.2/a) \neq 0$, which is a contradiction. This proves our assertion.

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References

Doob, J. L. (1953): *Stochastic Processes*, John Wiley and Sons, New York.

Erdös, P. (1956): On additive arithmetical functions and applications of probability to number theory. *Proc. International Congress of Math.* (Amsterdam, 1954), Vol. III, 13-19. Noordhoff, Groningen.

Halberstam, H. (1956): On the distribution of additive number theoretical functions II. *The Journal of the Lond. Math. Soc.*, 31, No. 121, 1-14.

Jogesh Babu, G. (1972): On the distribution of additive arithmetical functions of integral polynomials. *Sankhyā*, A, 34, 323-334.

Katai, I. (1969): On the distribution of arithmetical functions. *Acta. Math. Acad. Sci. Hung.*, T. 20, 69-87.

Kubilius, J. (1961): Probabilistic methods in the theory of numbers. *Translations of Math. Mono.*, 11, A.M.S.

Novoselov, E. V. (1966): A new method in probabilistic number theory. *A.M.S. Translations*, Series 2, 52.

Paul, E. M. (1962): Density in the light of probability theory I. *Sankhyā*, A, 24, 103-114.

——— (1963): Density in the light of probability theory III. *Sankhyā*, A, 25, 273-280.

——— (1967): Some properties of additive arithmetic functions. *Sankhyā*, A, 29, 279-282.

Wintner, A and Erdös, P. (1939): Additive arithmetical functions and statistical independence. *American J. of Math.*, 61, 713-721.

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