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A SUBSIDY-SURPLUS MODEL AND THE SKOROKHOD PROBLEM IN AN ORTHANT

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We consider the deterministic Skorokhod problem in an orthant of the form

$$Zw(t) = w(t) + \int_0^t b(u, Yw(u), Zw(u))\,du + \int_0^t R(u, Yw(u-), Zw(u-))\,dYw(u)$$

with $(Yw)_i(\cdot)$ nondecreasing, and $(Yw)_i(\cdot)$ not increasing while $(Zw)_i(\cdot) > 0$. This can be viewed as a subsidy-surplus model in an interdependent economy. Existence of a unique solution is established under fairly general conditions (viz. with $R(\cdot,\cdot,\cdot)$ satisfying a uniform spectral radius condition). Comparison result for (SP) vis-a-vis the usual partial order on the orthant is studied; we show that the more "inward looking" the reflection vectors and the drift, the larger the values of Yw will be but the values of Zw will be smaller. In addition to showing that the Leontief-type output is a feasible subsidy, connection between (SP) and "minimality" of feasible subsidies is discussed (consequently it is suggested that (SP) may be taken as a continuous time feedback-form analogue of open Leontief model).

In the stochastic case, $(Y(t), Z(t))$ turns out to be a strong Markov process if $w(\cdot)$ arises from aLevy process. Relevance of the comparison result to recurrence/positive recurrence of $Z(\cdot)$ process is pointed out.

1. Introduction. The Skorokhod problem in nonsmooth domains has been an object of intensive study ever since reflected Brownian motion in an orthant with oblique reflection was suggested as a heavy traffic limit for tandem queues; see, for example, Harrisonand Reiman (1981), Dupuis and Ishii (1991a, 1993); Shashiashvili (1994), Mandelbaum and Pats (1998). Prior to these Tanaka (1979) has investigated the Skorokhod problem with normal reflection in convex (not necessarily smooth) domains. The deterministic Skorokhod problem itself attracted attention because, in the case of Brownian motion, one can get the reflected process in a path-by-path fashion once the deterministic problem is solved; in addition to the references given above see Reiman (1984), Bernhard and El Kharroubi (1991), Mandelbaum (1989).

Recently in an interesting paper, Chen and Mandelbaum (1991) have proposed the deterministic Skorokhod problem in an orthant as a continuous time Leontief type input-output model (this paper is the direct motivation for the present investigation). They assume that the reflection matrix (or input-output/consumption matrix in the language of economics) is constant and that the off diagonal elements are negative. They show in particular that the y-part of the solution of (SP) (that is, the function whose coordinates are nondecreasing) is minimal (or efficient) among all feasible production plans. (They interpret $y_i(t)$ as a feasible cumulative production of commodity i over $[0, t]$.) This minimality result is inspired by a similar result of Reiman (1984) in the context of queueing

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networks. (In queueing theory $y_i(t)$ is interpreted as a measure of cumulative potential output/demand lost over $[0,t]$ due to buffer i being empty.)

In the present work we consider the Skorokhod problem in an orthant G given by

$$(1.1)\qquad Zw(t) = w(t) + \int_0^t b(u, Yw(u), Zw(u))\,du + \int_0^t R(u, Yw(u-), Zw(u-))\,d(Yw)(u),$$

$$(1.2)\qquad (Yw)_i(t) = \int_0^t I_{\{0\}}((Zw)_i(s))\,d(Yw)_i(s),$$

where w is a given r.c.l.l. $\mathbb{R}^d$ valued function, b, R, respectively, are given $\mathbb{R}^d$ valued, matrix valued functions on $[0,\infty) \times \bar{G} \times \bar{G}$; see §2.

We argue that (SP) in an orthant can be considered also as a subsidy-surplus model in an interdependent economy; the y-part is taken as cumulative subsidy and the z-part as the current surplus; so $y(\cdot)$ can be considered as a measure of the subsidy needed to have a surplus (≥ 0) in each sector of the economy. In such a set up, $R(\cdots) = ((r_{ij}(\cdots)))$ can be taken as a policy of allocation of funds/subsidies among various sectors of the economy. It is natural to assume that the policy of allocation depends not only on time and current surplus *but also on the cumulative subsidy*. For example the policy can be that a sector which has received a greater subsidy up to the present is to be allotted less, or should play a greater role in mobilising subsidy for other sectors. So the r_{ij}'s are functions of y in addition to t, z. We also introduce a drift $b(\cdots)$, called innovations; this again can depend on t, y, z. (It may be mentioned that, in all the papers cited above, the drift and the reflection matrix are either constant or state dependent; to our knowledge, the coefficients depending on y have not been considered before.)

Also $r_{ij}(\cdots)$ being positive or negative, for $i \neq j$, has a natural interpretation in our setup. $\int_0^t r_{ij}^+(\cdots)\,dy_j(\cdot)$ can be taken as the part of the subsidy mobilised for Sector j but spent actually on Sector i over $[0,t]$; such a situation is not meaningless. Next, $\int_0^t r_{ij}^-(\cdots)\,dy_j(\cdot)$ can be taken as the part of the subsidy for Sector j raised from Sector i. Many obvious examples can be given to illustrate this situation.

These considerations make it necessary to distinguish between "the subsidy mobilised for Sector j" and "the subsidy given to Sector j." Also the difference between "the subsidy mobilised for Sector j" and "subsidy mobilised for Sector j from other sectors" can be taken as "the subsidy mobilised from external sources." Our uniform spectral radius condition (A3) would mean that the subsidy mobilised from external sources is nonzero; so this would be an "open system" in the jargon of input-output analysis. The normalisation $r_{ii} \equiv 1$ just means that a nonzero subsidy is actually given to Sector i from the subsidy mobilised for Sector i.

Similar interpretations apply to $b^+(\cdots)$, $b^-(\cdots)$; see §2.

We now briefly outline the contents of the paper. In §2 we describe the subsidy-surplus model in terms of the deterministic Skorokhod problem. In §3 we prove the existence of a unique solution. For this we use a generalization of the contraction mapping argument of Harrison and Reiman (1981); however in our approach we simultaneously approximate y, z-parts, using the variation norm for the y-part and the sup norm for the z-part. Besides an estimate on the variational distance of maximal functions due to Shashiashvili (1994), the key to our approach is an a priori estimate for the standard subsidy (that is, the y-part of the solution of (SP)) in terms of a Leontief-type output function, and the fact that the set of subsidies bounded by the Leontief type output is invariant under the map T. The maps T and S are defined by (3.15)–(3.17).

In §4 we establish some comparison results, a key assumption being $r_{ij}(\cdots)$, $i \neq j$ are nonpositive. This involves a detailed bare-hands analysis of the map (T,S). The results are intuitive in the sense that the more "inward looking" the reflection vectors (that is, the greater the internal mobilization of subsidies) the larger the values of y will be but smaller the values of z will be. As far as we know there has been no previous work on the comparison result for Skorokhod problem even when the coefficients have no y-dependence.

The next section deals with feasibility and minimality: As can be expected, a subsidy y is called feasible if the solution of the corresponding vector integral equation (see (5.1)) is nonnegative. It is shown that the Leontief-type output function is a feasible subsidy. Various natural notions of minimality are defined. Among other things, we prove a generalization of the minimality result of Reiman (1984) and Chen and Mandelbaum (1991) alluded to above.

Section 6 deals briefly with the stochastic case when the exogeneous flow $w(\cdot)$ is driven by a Levy process. It is shown that under our assumptions the solution $\{(Y(t),Z(t)): t \geq 0\}$ of the stochastic (SP) is a strong Markov process. $Z(\cdot)$ itself will be Markov if $b(\cdots)$, $R(\cdots)$ are independent of y. When the exogeneous flow is Brownian, results of §4 enable us to say something about recurrence, positive recurrence of the $Z(\cdot)$-processes corresponding to different reflection fields — viz. a more inward looking process is recurrent/positive recurrent if a less inward looking one is. In the last section we mention a few open problems.

To sum up, the subsidy-surplus model outlined in §2 can be taken as the continuous-time, feedback form analogue of the open Leontief model. In the continuous time case, when $R(\cdots)$ is not constant, it is not clear in general if optimal solutions (or efficient plans) exist under the usual type of minimality criteria (of the sort considered in §5). In this context note that the condition (2.1) of the Skorokhod problem, viz. standard subsidy for Sector i can be mobilised only when Sector i has no surplus, is essentially a minimality condition. This minimality condition (which is also quite natural, but perhaps less obvious than other minimality conditions) proves to be mathematically more tractable. See Paragraph 3 in §7.

We now fix some notations.

$$G = \{x \in \mathbb{R}^d : x_i > 0,\ i = 1,2,\ldots,d\};$$

$D([0,\infty):\mathbb{R}^d)$ denotes the space of all $\mathbb{R}^d$-valued right continuous functions on $[0,\infty)$ having left limits at every $t > 0$; for a function w on $[0,\infty)$, $w(t-) := \lim_{s\uparrow t} w(s)$ denotes the left limit at t.

$$D([0,\infty):\bar{G}) = \{w \in D([0,\infty):\mathbb{R}^d): w(t) \in \bar{G}\ \forall t\};$$

$$D_\uparrow([0,\infty):\bar{G}) = \{y \in D([0,\infty):\bar{G}): y_i(\cdot) \text{ is nondecreasing function for each } i\};$$

$$D_{\uparrow 0}([0,\infty):\bar{G}) = \{y \in D_\uparrow([0,\infty):\bar{G}): y(0) = 0\}.$$

Similarly $D([0,t]:\mathbb{R}^d), D([0,t]:\bar{G})$, $D_\uparrow([0,t]:\bar{G})$, $D_{\uparrow 0}([0,t]:\bar{G})$, can be defined. Whenever the context is clear we shall write $D_\uparrow \times D$ for $D_\uparrow([0,\infty):\bar{G}) \times D([0,\infty):\bar{G})$.

2. The model. Consider a class of d interdependent sectors in an economic system. The system is known to evolve in a certain fashion if no action is imposed on it; (sometimes called "exogeneous" evolution). We propose introducing a set of "innovations" and a "policy of allocation of funds/subsidies" on the system. The situation may be described by the model below.

Let $\bar{G} = \{x \in \mathbb{R}^d : x_i \geq 0, 1 \leq i \leq d\}$ denote the nonnegative d dimensional orthant. We have $w \in D([0,\infty):\mathbb{R}^d)$, $b: [0,\infty)\times\bar{G}\times\bar{G} \to \mathbb{R}^d$, $R: [0,\infty)\times\bar{G}\times\bar{G} \to \mathbb{M}_d(\mathbb{R})$; here $\mathbb{M}_d(\mathbb{R})$

denotes the space of $d \times d$ matrices with real entries (we often denote $w(t) = (w_1(t), \ldots, w_d(t))$, $b(u,y,z) = (b_1(u,y,z), \ldots, b_d(u,y,z))$, $R(u,y,z) = ((r_{ij}(u,y,z))_{1 \le i,j \le d})$. We seek Yw, Zw such that the following hold:

(i) $Zw \in D([0,\infty): \bar{G})$;

(ii) $Yw \in D_{\uparrow}([0,\infty): \bar{G})$;

(iii) for $1 \le i \le d$, $t \ge 0$,

$$\int_0^t I_{(0,\infty)}((Zw)_i(s))\, d(Yw)_i(s) = 0; \tag{2.1}$$

(iv) for $1 \le i \le d$, $t \ge 0$,

$$\begin{aligned}(Zw)_i(t) = w_i(t) &+ \int_0^t b_i(u, Yw(u-), Zw(u-))\, du \\ &+ (Yw)_i(t) + \sum_{j \ne i} \int_0^t r_{ij}(u, Yw(u-), Zw(u-))\, d(Yw)_j(u).\end{aligned} \tag{2.2}$$

We have taken $Yw(0-) = 0 = Zw(0-)$, and suitably normalised so that $r_{ii}(\cdot) \equiv 1$ for all i. Observe that there are two unknowns; (2.2) is called the *Skorokhod equation.* Any (Yw, Zw) satisfying the above is said to solve SP(w,b,R) or the *Skorokhod problem* ((SP) for short) corresponding to (w,b,R).

The following interpretation can be given. We assume that the (effects of) innovations, subsidies, surplus etc. have been quantified in terms of some currency in the discussion.

$w_i(t)$ = net of cumulative *exogeneous* amount of income less expenditure in Sector i over the period $[0,t]$; so $w(\cdot)$ represents the "net evolution" of the system when no action is imposed.

In expressions of the form $b_i(u, y(u-), z(u-))$, $r_{ij}(u, y(u-), z(u-))$ we take u as the time parameter, $y(u) = (y_1(u), \ldots, y_d(u))$ as the *cumulative subsidy given* to the various sectors over the period $[0,u]$, and $z(u)$ as the *current surplus* at time u; $b(\cdot,\cdot,\cdot)$ is considered the *innovations* introduced, and $R(\cdot,\cdot,\cdot)$ is interpreted as the *policy of allocation/mobilization of subsidies.* If $y(\cdot)$, $z(\cdot)$ are respectively the cumulative subsidy and current surplus functions, then

$$\int_s^t b_i^+(u, y(u-), z(u-))\, du = \int_s^t b_i^+(u, y(u), z(u))\, du$$

= amount of Sector i *produced* over $[s,t]$ due to innovations (but excluding the subsidy for Sector i);

$$\int_s^t b_i^-(u, y(u-), z(u-))\, du = \int_s^t b_i^-(u, y(u), z(u))\, du$$

= amount of Sector i *consumed* over $[s,t]$ due to innovations (but excluding that for the purpose of subsidies);

$$\int_s^t r_{ij}^+(u, y(u-), z(u-))\, dy_j(u)$$

= amount of subsidy mobilised for Sector j which is actually used in Sector i (but not as a subsidy in Sector i) over $[s,t]$;

$$\int_s^t r_{ij}^-(u, y(u-), z(u-))\, dy_j(u)$$

= amount of subsidy for Sector j mobilised from Sector i over $[s,t]$.

It is possible that part of the subsidy for other sectors mobilised from Sector i could be from the subsidy of Sector i. For example such a situation can arise if $z_i(\cdot)=0$, $z_j(\cdot)=0$, $r_{ij}(\cdot)\leq 0$, $i\neq j$ over a certain period; see Example 3.8.

If (Yw,Zw) solve SP(w,b,R), then we call Yw the *standard subsidy* and Zw the *standard surplus* corresponding to the exogeneous evolution w, innovations b, and subsidy allocation policy R. The condition (2.1) means that *the standard subsidy for Sector j can be mobilised only when Sector j is "empty"* (that is *only when Sector j has no surplus*; mathematically $(Yw)_j(\cdot)$ can increase only when $(Zw)_j(\cdot)=0$). Consequently observe that

$$\begin{aligned}(2.3)\qquad &\sum_i\int_0^t r_{ij}^+(u,Yw(u-),Zw(u-))\,d(Yw)_j(u)\\ &=(Yw)_j(t)+\sum_{i\neq j}\int_0^t r_{ij}^+(u,Yw(u-),Zw(u-))\,d(Yw)_j(u)\\ &=\text{cumulative standard subsidy } \textit{mobilised}\\ &\quad\text{for Sector } j \text{ over } [0,t].\end{aligned}$$

And, of the above, $(Yw)_j(t)$ is actually given to Sector j (so our normalisation $r_{jj}(\cdot)\equiv 1$ basically means that the fraction of the subsidy given to Sector j from that mobilised for Sector j is bounded away from zero under the continuity assumption (A2) of the next section; this is quite a natural assumption). Also observe that

$$\begin{aligned}(2.4)\qquad &\sum_i\int_0^t r_{ij}^-(u,Yw(u-),Zw(u-))\,d(Yw)_j(u)\\ &=\sum_{i\neq j}\int_0^t r_{ij}^-(u,Yw(u-),Zw(u-))\,d(Yw)_j(u)\\ &=\text{cumulative standard subsidy mobilised for Sector } j\\ &\quad\text{from other sectors of the system over } [0,t].\end{aligned}$$

Therefore (2.3) minus (2.4) gives the cumulative (standard) subsidy mobilised for Sector j from *external sources* over $[0,t]$.

REMARKS.

(1) The coefficients b_i, r_{ij} being functions of y is perhaps new, but quite reasonable. For example, increased food subsidy in times of drought (when food surplus is zero) can have positive effects on other sectors. As extreme examples one can consider public health, education, research and development, government, etc.; such welfare sectors can in principle be "no surplus" sectors.

(2) The reason for taking $R(u,Yw(u-),Zw(u-))$ as the infinitesimal element, rather than the more natural $R(u,Yw(u),Zw(u))$ is technical; in the latter case there is no uniqueness even in certain simple situations; see Shashiashvili (1994). In the context of the stochastic problem in §6, this would ensure predictability of the processes.

(3) While considering queueing networks and reflected diffusions, it is customary to take $w(0)\in\bar{G}$ and hence $Yw(0)=0$. However, in our discussion such a restriction may not be warranted. For example a particular sector may be in the red to start with and would need a subsidy right away. Also $Yw(0-)=0=Zw(0-)$ is a very natural convention to make.

REMARK 2.1. Suppose $b\equiv 0$, r_{ij} are nonpositive constants for $i\neq j$, $r_{ii}\equiv 1$, and $w_i(\cdot)=$ negative constant for all i. This situation is formally the classical *Leontief input-output system*. One can write $R=I-V$ with I denoting the identity matrix and V being a nonnegative matrix with zero diagonal entries; R is sometimes called a Minkowski-Leontief matrix; see Karlin (1959). If the spectral radius of V is less than unity, the system is generally known as the *open Leontief model.* In such a case $(-w)$ is interpreted as "final demand," the nonnegative solution $y=(1-V)^{-1}(-w)$ is taken as "the output needed to meet the demand without any surplus/profit," and r_{ij}'s are input coefficients ($|r_{ij}|$ being proportional to the amount of ith good needed to produce one unit of the jth good). For more information concerning the Leontief model see Karlin (1959), Nikaido (1968). The matrix $(I-V)^{-1}$ is called the *Leontief inverse* in economics literature.

REMARK 2.2. The case $b\equiv 0$, $r_{ij}=$ nonpositive constant, $i\neq j$, and $r_{ii}=1$ has been considered by Harrison and Reiman (1981), Reiman (1984), Chen and Mandelbaum (1991), just to mention a few. Harrison and Reiman investigate a heavy traffic model for queueing networks in terms of reflected/regulated Brownian motion; R is then called the reflection matrix. Chen and Mandelbaum (1991) consider a continuous time analogue of the Leontief model. In the context of RBM and SDE's with boundary conditions, when r_{ij} are not necessarily negative, and the coefficients depend on z, there have been many studies; see Bernard and El Kharroubi (1991), Dupuis and Ishii (1991a, 1993), Shashiashvili (1994), Mandelbaum and Pats (1998).

3. Existence and uniqueness. To establish the existence of a unique solution to the deterministic Skorokhod problem formulated in the preceding section, we make the following hypotheses on b, R. The continuity assumptions below are with respect to the Euclidean norm.

ASSUMPTION (A1). *For $1\leq i\leq d$, b_i are bounded continuous; also $(y,z)\mapsto b_i(t,y,z)$ are Lipschitz continuous, uniformly in t.*

ASSUMPTION (A2). *For $1\leq i,\ j\leq d$, r_{ij} are bounded continuous; $(y,z)\mapsto r_{ij}(t,y,z)$ are Lipschitz continuous, uniformly in t. Moreover $r_{ii}\equiv 1$ for all i.*

ASSUMPTION (A3). *For $i\neq j$ there exist constants v_{ij} such that $|r_{ij}(t,y,z)|\leq v_{ij}$. Set $V=((v_{ij}))$ with $v_{ii}=0$; we assume that $\sigma(V)<1$ where $\sigma(V)$ denotes the spectral radius of V.*

REMARK 3.1. As $\sigma(V)=\sigma(V^{\dagger})$, where $V^{\dagger}$ is the transpose of V, by (A3) note that there exist constants $a_i>0$, $1\leq i\leq d$, and $0<\alpha<1$ such that

$$\sum_{j\neq i} a_j\,|r_{ji}(t,y,z)|\leq \sum_{j\neq i} a_j v_{ji}\leq \alpha a_i \tag{3.1}$$

for all $1\leq i\leq d$, $t\geq 0$, $y,z\in\bar{G}$; see p. 160 of Dupuis and Ishii (1991b) for a proof. □

For our purpose it will be convenient to have the following metric on $D_{\uparrow}([0,M]:\bar{G})\times D([0,M]:\bar{G})$ where $M>0$.

Let a_i's be as in Remark 3.1. Let $c_1>0$, $c_2>0$ be arbitrary but fixed numbers (these will be chosen suitably later). For $(y(\cdot),z(\cdot))$, $(\hat{y}(\cdot),\hat{z}(\cdot))\in D_{\uparrow}\times D$ set

$$d_M((y,z),(\hat{y},\hat{z}))=c_1\sum_{i=1}^{d} a_i\varphi_M(y_i-\hat{y}_i)+c_2\sum_{i=1}^{d} a_i\psi_M(z_i-\hat{z}_i), \tag{3.2}$$

where $\varphi_M(\cdots)$ denotes the total variation over $[0,M]$ and $\psi_M(\cdots)$ denotes the supremum norm over $[0,M]$. Note that $(D_{\uparrow}\times D, d_M)$ is a complete metric space.

For $w\in D([0,\infty):\mathbb{R}^d)$ and $b(\cdot)$ satisfying (A1), set

$$h_i(t)=\sup_{0\leq s\leq t}\max\{0,-w_i(s)\}=\sup_{0\leq s\leq t} w_i^{-}(s), \tag{3.3}$$

$$\beta_i(t)=\int_0^t \sup\{b_i^-(s,y,z):(y,z)\in\bar{G}\times\bar{G}\}\,ds, \tag{3.4}$$

for $t\geq 0,\ 1\leq i\leq d$. Define $(h+\beta)(\cdot)\in D_\uparrow$ by setting

$$(h+\beta)_i(t)=h_i(t)+\beta_i(t),\qquad t\geq 0,\ 1\leq i\leq d. \tag{3.5}$$

Also to simplify notation, given w, b, R, for $(y(\cdot),z(\cdot))\in D_\uparrow([0,\infty):\bar{G})\times D([0,\infty):\bar{G})$, write

$$\begin{aligned}(X(y,z;w,b,R))_i(t)&\equiv(X(y,z;w))_i(t)\\ &=w_i(t)+\int_0^t b_i(u,y(u-),z(u-))\,du\\ &\quad+\sum_{j\neq i}\int_0^t r_{ij}(u,y(u-),z(u-))\,dy_j(u)\end{aligned} \tag{3.6}$$

for $i=1,2,\dots,d,\ t\geq 0$.

We now give an a priori bound for the standard subsidy.

Proposition 3.2. *Assume* (A1)–(A3). *Suppose* (y,z) *solves the Skorokhod problem corresponding to* w, b, R. *Then*

$$y\leq(I-V)^{-1}(h+\beta) \tag{3.7}$$

in the sense that

$$y_i(t)\leq((I-V)^{-1}(h+\beta))_i(t),\qquad 1\leq i\leq d,\ t\geq 0. \tag{3.8}$$

Proof. By assumption, for fixed i, (y_i,z_i) solves the one dimensional (normal) Skorokhod problem in $[0,\infty)$ for $(X(y,z;w))_i$. Therefore we have

$$y_i(t)=\sup_{0\leq s\leq t}\max\{0,-(X(y,z;w))_i(s)\}. \tag{3.9}$$

By (A3), (3.3), (3.4), (3.6) note that

$$-(X(y,z;w))_i(s)\leq h_i(s)+\beta_i(s)+\sum_{j\neq i}v_{ij}y_j(s).$$

Consequently as h_i,β_i,y_j are nondecreasing, by (3.9) we get

$$y_i(t)\leq h_i(t)+\beta_i(t)+\sum_{j\neq i}v_{ij}y_j(t)$$

for $1\leq i\leq d,\ t\geq 0$. That is

$$((I-V)y(t))_i\leq(h+\beta)_i(t). \tag{3.10}$$

Observe that V is a nonnegative matrix; by (A3) we have that

$$(I-V)^{-1}=I+V+V^2+\cdots \tag{3.11}$$

is a well-defined nonnegative matrix. So the conclusion (3.7) is now immediate from (3.10). □

Remark 3.3. Note that h_i and β_i respectively can be taken as the maximum possible demands on Sector i due to "exogeneous consumption" and "consumption on account of innovation." So the preceding proposition states that the standard subsidy need not exceed the Leontief-output $(I-V)^{-1}(h+\beta)$ corresponding to the worst possible demand $(h+\beta)$. □

From the preceding proof the following is clear.

PROPOSITION 3.4. *Assume* (A1), (A2), *and that* $r_{ij}(\cdots)\geq 0$ *for all* i,j. *Let* (y,z) *be as in Proposition* 3.2. *Then* $y\leq(h+\beta)$. □

Before proceeding further the following comment concerning the Lipschitz condition may be in order.

Let a_i's be as in Remark 3.1. For $x\in\mathbb{R}^d$, put $\|x\|=\sum_{i=1}^d a_i|x_i|$. As $a_i>0$ for all i, this gives a norm equivalent to the Euclidean norm. So we may assume (under (A1), (A2)) without loss of generality that there is $\hat{K}>0$ such that

$$|f(t,y,z)-f(t,\hat{y},\hat{z})|\leq\hat{K}(\|y-\hat{y}\|+\|z-\hat{z}\|) \tag{3.12}$$

with $f=b_i$ or $r_{ij}, 1\leq i,\ j\leq d,\ t\geq 0,\ (y,z),(\hat{y},\hat{z})\in\bar{G}\times\bar{G}$.

A semiconstructive method of solution. The method consists of several steps. Assume (A1)–(A3).

Step 1. As $Yw(0-)=0=Zw(0-)$, at $t=0$, solving (SP) amounts to solving the following Linear Complementarity problem (LCP): given $w(0)\in\mathbb{R}^d$ and matrix $R(0,0,0)$, to find $Zw(0),Yw(0)\in\bar{G}$ such that

$$Zw(0)=w(0)+R(0,0,0)Yw(0), \tag{3.13}$$

$$(Zw)_i(0)\cdot(Yw)_i(0)=0, \qquad 1\leq i\leq d. \tag{3.14}$$

We can write $R(0,0,0)=I+\hat{R}(0,0,0)$ with spectral radius of $\hat{R}(0,0,0)$ less than 1; to see this use (3.1) and the remark on p. 160 of Dupuis and Ishii (1991b). So by standard results on LCP (see Cottle, et al. 1992 or Berman and Plemmons 1979) a unique solution exists. This can be constructed by the fixed point argument given on pp. 176–177 of Shashiashvili (1994); such a method can also be gleaned from the analysis given below in a more general context. See also Step 1 of §4.

Step 2. We claim that it is enough to consider the case when $w(0)\in\bar{G}$. To see this consider $Zw(0)+w(\cdot)-w(0), Yw(\cdot)-Yw(0), (u,y,z)\mapsto b(u,y+Yw(0),z), (u,y,z)\mapsto R(u,y+Yw(0),z)$ respectively in the place of w, Yw, b, R, and use Step 1.

Step 3. Let $w\in D([0,\infty):\mathbb{R}^d)$ be *fixed* with $w(0)\in\bar{G}$. In such a case note that $Yw(0)=0$. We now define a map (T,S) on $D_{\uparrow 0}\times D$ (depending on w as a parameter) as follows:

$$(T,S)(y,z)\equiv(T(y,z;w,b,R),S(y,z;w,b,R))\equiv(T(y,z;w),S(y,z;w)), \tag{3.15}$$

$$(T(y,z;w))_i(t)=\sup_{0\leq s\leq t}\max\{0,-(X(y,z;w))_i(s)\}, \tag{3.16}$$

$$(S(y,z;w))_i(t)=(X(y,z;w))_i(t)+(T(y,z;w))_i(t), \tag{3.17}$$

for $i=1,2,\ldots,d,\ t\geq 0,\ y\in D_{\uparrow 0}([0,\infty):\bar{G}),\ z\in D([0,\infty):\bar{G})$, where $X(y,z;w)$ is given by (3.6). Observe that any solution to the (SP) will be a fixed point for the map (T,S) (this has been partly indicated in the proof of Proposition 3.2 as well). It is also clear from (3.16), (3.17) that $(T(y,z;w))_i(\cdot)$ can increase only when $(S(y,z;w))_i(\cdot)$ is zero. The map T as a function of y is similar to the one considered by Harrison and Reiman (1981) and by Mandelbaum, Massey and Pats (1995). As $(S(y,z;w))_i(t)\geq 0$ is clear from (3.16), (3.17) the following proposition is immediate.

PROPOSITION 3.5. *(T,S) is a map from $D_{\uparrow 0} \times D$ into itself with $T(y,z;w) \in D_{\uparrow 0}([0,\infty):\bar{G})$ and $S(y,z;w) \in D([0,\infty):\bar{G})$. If w is continuous then $C_{\uparrow 0}([0,\infty):\bar{G}) \times C([0,\infty):\bar{G})$ is invariant under (T,S).* □

Another invariant set is provided by the following result, which enables the fixed point approach to work.

PROPOSITION 3.6. *Assume (A1)–(A3); let $w(0) \in \bar{G}$ and $(h+\beta)$ be as in (3.5). Let $y \in D_{\uparrow 0}([0,\infty):\bar{G})$ be such that $y \le (I-V)^{-1}(h+\beta)$ in the sense of (3.8). Then*

$$T(y,z;w) \le (I-V)^{-1}(h+\beta) \tag{3.18}$$

for any $z \in D([0,\infty):\bar{G})$; that is $(T(y,z;w))_i(t) \le ((I-V)^{-1}(h+\beta))_i(t)$ for $1 \le i \le d$, $t \ge 0$, $z \in D([0,\infty):\bar{G})$.

PROOF. Observe that $(h+\beta)$ and $(I-V)^{-1}(h+\beta)$ are in $D_{\uparrow 0}$. Next note that

$$\begin{aligned} -\sum_{j\neq i}\int_0^s r_{ij}(u,y(u-),z(u-))\,dy_j(u) &\le \sum_{j\neq i} v_{ij}y_j(s) \\ &\le \sum_{j\neq i} v_{ij}((I-V)^{-1}(h+\beta))_j(s) = (V(I-V)^{-1}(h+\beta))_i(s) \\ &= ((I-V)^{-1}(h+\beta))_i(s) - (h_i(s)+\beta_i(s)) \end{aligned} \tag{3.19}$$

whence it follows that

$$-(X(y,z;w))_i(s) \le ((I-V)^{-1}(h+\beta))_i(s). \tag{3.20}$$

This completes the proof. □

Step 4. Set

$$\zeta(t) = t + \sum_{j=1}^{d}((I-V)^{-1}(h+\beta))_j(t). \tag{3.21}$$

Let $(y,z),(\hat{y},\hat{z}) \in D_{\uparrow 0} \times D$ with $y \le (I-V)^{-1}(h+\beta)$, $\hat{y} \le (I-V)^{-1}(h+\beta)$. For convenience denote $T(y,z;w), T(\hat{y},\hat{z};w), S(y,z;w), S(\hat{y},\hat{z};w)$ respectively by $Ty, T\hat{y}, Sz, S\hat{z}$; also write $K = \hat{K}(\sum_{i=1}^d a_i)$ where $\hat{K}$ is the Lipschitz constant given by (3.12) and the a_i's are as in Remark 3.1. By the lemma of Shashiashvili (1994 pp. 170–175) concerning variational distance between maximal functions, and using (3.12), (3.21), (A3) we get

$$\begin{aligned} &\varphi_t((Ty)_i - (T\hat{y})_i) \\ &\quad \le \varphi_t((X(y,z;w))_i(\cdot) - (X(\hat{y},\hat{z};w))_i(\cdot)) \\ &\quad \le \int_0^t |b_i(u,y(u),z(u)) - b_i(u,\hat{y}(u),\hat{z}(u))|\,du \\ &\qquad + \sum_{j\neq i}\int_0^t |r_{ij}(u,y(u-),z(u-)) - r_{ij}(u,\hat{y}(u-),\hat{z}(u-))|\,dy_j(u) \\ &\qquad + \varphi_t\left(\sum_{j\neq i}\int_0^{\cdot} r_{ij}(u,\hat{y}(u-),\hat{z}(u-))\,d(y_j-\hat{y}_j)(u)\right) \end{aligned} \tag{3.22}$$

$$\leq \hat{K}\int_0^t [\|y(u)-\hat{y}(u)\| + \|z(u)-\hat{z}(u)\|]\,du$$

$$+\hat{K}\int_0^t [\|y(u-)-\hat{y}(u-)\| + \|z(u-)-\hat{z}(u-)\|]\left(\sum_{j\neq i} dy_j(u)\right)$$

$$+\sum_{j\neq i} v_{ij}\varphi_t(y_j-\hat{y}_j)$$

$$\leq \hat{K}\int_0^t \left[\sup_{0\leq u\leq s}\left(\sum_{\ell=1}^d a_\ell|z_\ell(u)-\hat{z}_\ell(u)|\right)\right]\left(ds+\sum_{j\neq i} dy_j(s)\right)$$

$$+\hat{K}\int_0^t \left[\sup_{0\leq u\leq s}\left(\sum_{\ell=1}^d a_\ell|y_\ell(u)-\hat{y}_\ell(u)|\right)\right]\left(ds+\sum_{j\neq i} dy_j(s)\right)$$

$$+\sum_{j\neq i} v_{ij}\varphi_t(y_j-\hat{y}_j)$$

$$\leq \hat{K}\zeta(t)\left(\sum_{j=1}^d a_j\psi_t(z_j-\hat{z}_j)\right)$$

$$+\hat{K}\zeta(t)\left(\sum_{j=1}^d a_j\varphi_t(y_j-\hat{y}_j)\right)+\sum_{j\neq i} v_{ij}\varphi_t(y_j-\hat{y}_j).$$

Multiplying (3.22) by a_i, adding and using (3.1) we obtain for any $t\geq 0$,

$$\sum_{i=1}^d a_i\varphi_t((Ty)_i-(T\hat{y})_i) \tag{3.23}$$

$$\leq [\alpha+K\zeta(t)]\left(\sum_{j=1}^d a_j\varphi_t(y_j-\hat{y}_j)\right)+K\zeta(t)\left(\sum_{j=1}^d a_j\psi_t(z_j-\hat{z}_j)\right)$$

Next using (3.17), (3.1), (3.12), (3.23) we can similarly get

$$\sum_{i=1}^d a_i\psi_t((Sz)_i-(S\hat{z})_i)\leq 2\quad \{\text{r.h.s. of } (3.23)\}. \tag{3.24}$$

Consequently for any $t>0$, with d_t defined by (3.2) we have

$$d_t((T,S)(y,z),(T,S)(\hat{y},\hat{z})) \tag{3.25}$$

$$\leq \left[\frac{c_1+2c_2}{c_2}K\zeta(t)\right]\left[c_2\sum_{j=1}^d a_j\psi_t(z_j-\hat{z}_j)\right]$$

$$+\left[\frac{c_1+2c_2}{c_1}(\alpha+K\zeta(t))\right]\left[c_1\sum_{j=1}^d a_j\varphi_t(y_j-\hat{y}_j)\right].$$

Therefore if we choose c_1, c_2 such that

$$\frac{c_1}{c_2}>\frac{(1+2\alpha)}{(1-\alpha)}, \tag{3.26}$$

and $t > 0$ such that

$$\zeta(t) \le \frac{\alpha c_2}{K(c_1 - c_2)} \tag{3.27}$$

(such a t exists as $\zeta(t) \downarrow 0$ as $t \downarrow 0$ since $w(0) \in \bar{G}$), then (T,S) is a contraction map on the complete metric space

$$\hat{D}_t := \{(y,z) \in (D_{\uparrow 0} \times D, d_t): y \le (I - V)^{-1}(h + \beta)\}. \tag{3.28}$$

So there exists a unique fixed point in $(\hat{D}_t, d_t)$. However, any fixed point (y,z) in $D_{\uparrow 0} \times D$ is a solution for (SP), and hence $y \le (I - V)^{-1}(h+\beta)$ by Proposition 3.2. Therefore (T,S) has a unique fixed point in $(D_{\uparrow 0}([0,t]: \bar{G}) \times D([0,t]: \bar{G}), d_t)$ and the fixed point lies in $\hat{D}_t$. Thus (SP) is uniquely solved on $[0,t]$.

Step 5. Observe that c_1, c_2 are chosen so that (3.26) holds and is fixed. Let

$$t_1 = \sup\left\{ t > 0: \zeta(t) \le \frac{\alpha c_2}{K(c_1 - c_2)} \right\}. \tag{3.29}$$

Note that $t_1 > 0$ as $w(0) \in \bar{G}$. If $\zeta(t_1) \le \alpha c_2/K(c_1 - c_2)$, then by the preceding step the problem is solved uniquely on $[0, t_1]$. If $\zeta(t_1) > \alpha c_2/K(c_1 - c_2)$ then ζ has a jump at t_1 (which can happen only if some of the w_i's have negative jumps of sufficient magnitude at t_1). As $\zeta(t) < \alpha c_2/K(c_1 - c_2)$ for all $t < t_1$, the problem has a unique solution over $[0,t]$ for every $t < t_1$. Consequently $Yw(t_1-)$, $Zw(t_1-)$ are well defined and are at our disposal. Now to extend the solution to $[0,t_1]$ we need to find $Yw(t_1), Zw(t_1) \in \bar{G}$ such that the following hold:

$$\begin{aligned}(Zw)(t_1) = (Zw)(t_1-) + w(t_1) - w(t_1-)& \\ + (R(t_1, Yw(t_1-), Zw(t_1-))\Delta Yw(t_1)&,\end{aligned} \tag{3.30}$$

$$(Zw)_i(t_1) \cdot \Delta(Yw)_i(t_1) = 0, \qquad 1 \le i \le d, \tag{3.31}$$

where $\Delta Yw(t_1) = Yw(t_1) - Yw(t_1-)$. This is again a linear complementarity problem with $Zw(t_1)$, $\Delta Yw(t_1)$ as the unknowns. Since

$$R(t_1, Yw(t_1-), Zw(t_1-)) = I + \hat{R}(t_1, Yw(t_1-), Zw(t_1-))$$

with the spectral radius of $\hat{R}(t_1, Yw(t_1-), Zw(t_1-))$ less than 1, as in Step1, we have a unique solution (which again can be constructed in principle). Thus (SP) has been uniquely solved on the interval $[0, t_1]$.

Step 6. Set

$$\bar{w}(s) = (Zw)(t_1) + w(s + t_1) - w(t_1), \qquad (\bar{Y}w)(s) = Yw(s + t_1) - Yw(t_1),$$

$$\bar{Z}w(s) = Zw(s + t_1), \qquad \bar{b}(s, y, z) = b(s + t_1, y + Yw(t_1), z),$$

$$\bar{R}(s, y, z) = R(s + t_1, y + Yw(t_1), z), \qquad s \ge 0, \; y, z \in \bar{G}.$$

To solve SP(w, b, R) on $[t_1, \infty)$ it is enough to solve SP$(\bar{w}, \bar{b}, \bar{R})$ on $[0, \infty)$ where $\bar{w}(0) \in \bar{G}$ and $\bar{b}$, $\bar{R}$ satisfy (A1)–(A3). Using the analysis as in Steps 3–5, this problem can be uniquely solved, say, on $[0, t_2 - t_1]$. This means that the original (SP) is uniquely solved on $[0, t_2]$. Repeating this procedure one gets $0 < t_1 < t_2 < \cdots < t_n < \cdots$ such that the Skorokhod problem is uniquely solved on $[0, t_n]$ for each n.

Step 7. Let $t_\infty = \sup\{t: \text{(SP) corresponding to } (w, b, R) \text{ has a unique solution on } [0,s]\ \forall 0 \le s \le t\}$. If $t_\infty < \infty$, then there exists $t_n \uparrow t_\infty$ such that the problem is well posed on $[0, t_n]$ for each n. So by Proposition 3.2 we have for $1 \le i \le d$,

$$(Yw)_i(t_\infty-) = \lim_{n \to \infty} (Yw)_i(t_n) \le ((I - V)^{-1}(h + \beta))_i(t_\infty-) < \infty. \tag{3.32}$$

Consequently by (2.2) and (A3) we get $(Zw)_i(t_\infty-) < \infty$ for all i. Thus $Zw(t_\infty-)$ and $Yw(t_\infty-)$ are well defined and the problem is uniquely solvable on $[0,t]$ for any $t < t_\infty$. Now, as in Step 5 the solution can be uniquely extended on $[0,t_\infty]$; and then, as in Step 6 the solution can be uniquely extended to $[0,\hat{t}]$ for some $\hat{t} > t_\infty$. This contradicts the maximality of t_∞.

Thus we have proved:

THEOREM 3.7. *Assume* (A1)–(A3). *Then there is a unique solution* (Yw, Zw) *for the Skorokhod problem corresponding to* (w,b,R). *Moreover*

$$(Yw)_i(t) \le ((I-V)^{-1}(h+\beta))_i(t), \tag{3.33}$$

$$(Zw)_i(t) \le w_i^+(t) + \gamma_i(t) + ((I+V)(I-V)^{-1}(h+\beta))_i(t), \tag{3.34}$$

where V *is as in* (A3), h, β *are given by* (3.3), (3.4) *and*

$$\gamma_i(t) = \int_0^t \sup\{b_i^+(s,y,z) : (y,z) \in \bar{G} \times \bar{G}\}\, ds.$$

Also, if w *is continuous then so are* Yw, Zw. □

Note that (3.34) follows from (3.33) and (2.2).

One can explicitly solve the Skorokhod problem in the following situation:

EXAMPLE 3.8. Suppose $b_i(\cdot,\cdot,\cdot) \le 0$ for all i, $r_{ij}(\cdot,\cdot,\cdot) \le 0$ for $i \ne j$, $w_i(\cdot) = -h_i(\cdot)$, $1 \le i$, $j \le d$ where h_i are nonnegative nondecreasing functions. Since $w_i(t) + \int_0^t b_i(s,\cdot,\cdot)\,ds$ in a nonincreasing negative function for each i, the nature of the Skorokhod problem suggests that one may try out $Zw \equiv 0$. In such a case the Skorokhod equation (2.2) gives for $1 \le i \le d$, $0 \le s \le t$,

$$\begin{aligned}(Yw)_i(t) - (Yw)_i(s) + \sum_{j\ne i} \int_{(s,t]} r_{ij}(u, Yw(u-), 0)\, d(Yw)_j(u) \\ = h_i(t) - h_i(s) + \int_{(s,t]} b_i^-(u, Yw(u-), 0)\, du.\end{aligned} \tag{3.35}$$

Or, equivalently,

$$R(s, Yw(s-), 0)\, d(Yw)(s) = dh(s) + b^-(s, Yw(s-), 0)\, ds, \tag{3.36}$$

where $h(\cdot) = (h_1(\cdot), \dots, h_d(\cdot))$, $b^-(\cdot,\cdot,\cdot) = (b_1^-(\cdot,\cdot,\cdot), \dots, b_d^-(\cdot,\cdot,\cdot))$. Thus Yw can be taken as the solution of the vector integral equation

$$\begin{aligned}y(t) = h(0) + \int_0^t R^{-1}(s, y(s-), 0)\, dh(s) \\ + \int_0^t R^{-1}(s, y(s-), 0) b^-(s, y(s-), 0)\, ds.\end{aligned} \tag{3.37}$$

In addition to (A2), (A3) assume that $R^{-1}(s,y,0)$ is Lipschitz continuous in y uniformly in s. So by a Picard iteration scheme, the integral equation (3.37) has a unique solution. As $r_{ij}(\cdot,\cdot,\cdot) \le 0$ for $i \ne j$ by (A3) and an analogue of (3.11) note that $R^{-1}(\cdot,\cdot,\cdot)$ is a nonnegative matrix valued function. Thus $y_i(\cdot)$ given by (3.37) is nonnegative and nondecreasing. (It is easily verified that if we take Yw as the solution of (3.37) and $Zw \equiv 0$, then they solve SP(w,b,R).) This class of examples admits the following interpretation. The equation (3.37) gives the standard subsidy needed just to keep a system afloat when there is no production (due to exogeneous evolution or innovations) but there is an ever-increasing demand due to consumption (due to exogeneous evolution, innovation and allocation of subsidies). Such a situation need not be as bad as it may sound! If one considers, for example, only the collection of all welfare/no surplus sectors then (3.37) gives the standard subsidy needed to be mobilised from other sectors/external sources. □

In view of the lemma of Shashiashvili (1994, pp. 170–175) concerning the variational distance between maximal functions, and Proposition 3.2, the following result is easily proved proceeding as in Step 4 above.

PROPOSITION 3.9. *Let $w, \hat{w} \in D([0,\infty) : \mathbb{R}^d)$ be such that $w - \hat{w}$ is of bounded variation over every finite interval. Let (Yw, Zw) (resp. $(Y\hat{w}, Z\hat{w})$) be the solution of (SP) corresponding to (w, b, R) (resp. $(\hat{w}, b, R)$). If $\hat{w}(\cdot) \to w(\cdot)$ in the variation norm over $[0,t]$, then $Y\hat{w}(\cdot) \to Yw(\cdot)$ in the variation norm and $Z\hat{w}(\cdot) \to Zw(\cdot)$ in the sup norm over $[0,t]$. In particular for $x \in \mathbb{R}^d, w \in D([0,\infty): \mathbb{R}^d)$ denote $w_x(\cdot) = w(\cdot) + x$; then for any $t > 0$, $Yw_x(\cdot) \to Yw(\cdot)$ in the variation norm and $Zw_x(\cdot) \to Zw(\cdot)$ in the sup norm over $[0,t]$, as $x \to 0$.* □

REMARK 3.10. Assumptions (A1), (A2) are standard; (3.1) has been assumed by Shashiashvili (1994) when R depends on z alone. Note that (A3) is a sort of uniform Hawkins-Simon condition known in input-output analysis. If we assume that $R(s,y,z) = I + \hat{R}(s,y,z)$ with spectral radius of $\hat{R}(s,y,z)$ less than 1 for each s, y, z, then (3.1) holds with a_i, α depending on s, y, z (this is precisely condition (3.8) on p. 557 of Dupuis and Ishii (1993) when R depends on z alone). Using continuity of R it is not difficult to show that for any s, y, z one can find a neighbourhood around s, y, z such that (3.1) holds with the same a_i, α in that neighbourhood. So by a localisation argument one can establish the existence of a unique solution to (SP) in such a case (but analogue of (3.33) can be asserted only locally).

If r_{ij}, b_i are functions only of time t, one can obtain an estimate on $\sum_i \varphi_t((Ty)_i - (T\hat{y})_i)$ in terms of $\sum_{i=1}^d \varphi_t(y_i - \hat{y}_i)$ in (3.22); it is enough to consider only T rather than (T,S), and the contraction map argument goes through without any need for restriction to $\hat{D}_t$; see Mandelbaum, Massey and Pats (1995). □

REMARK 3.11. In our approach, only Step 7 is not constructive (this explains the adjective in the subsectional title). To make it fully constructive one has to show that given $t > 0$ there exists n such that $t < t_n$ where t_j's are as in Step 6. It may be noted that both Dupuis and Ishii (1993) and Shashiashvili (1994) use a compactness argument (invoking the Ascoli theorem) to prove existence of a solution. □

4. Comparison result. The objective in this section is to prove a comparison result for the Skorokhod problem. Comparison results for ordinary differential equations and stochastic differential equations are well known; see Birkhoff and Rota (1978), Ikeda and Watanabe (1981); such results for one dimensional s.d.e.'s with (normal) reflection at 0 have been obtained recently by Zhang (1994).

In our context we will be dealing with a partial order, viz. $x \leq \tilde{x} \Leftrightarrow x_i \leq \tilde{x}_i$ for all i. This partial order has been found very useful in queueing networks, LCP, (SP) and concerning RBM in the orthant (for example, the notion of completely-$\mathcal{S}$ property plays a crucial role in the semimartingale formulation of RBM; see Bernard and El Kharroubi (1991), Williams (1995) and the references therein).

The main result of this section is

THEOREM 4.1. *Let b, R (resp. $\tilde{b}, \tilde{R}$) satisfy* (A1)–(A3). *Assume that for $t \geq 0, 1 \leq i, j \leq d$,*

$$\tilde{b}_i(t, \tilde{y}, \tilde{z}) \leq b_i(t, y, z), \tag{4.1}$$

$$\tilde{r}_{ij}(t,\tilde{y},\tilde{z}) \le r_{ij}(t,y,z) \le 0, \tag{4.2}$$

whenever $\tilde{y} \ge y$, $\tilde{z} \le z$. *Let* $w, \tilde{w}$ *be such that*

$$\tilde{w}_i(t_2) - \tilde{w}_i(t_1) \le w_i(t_2) - w_i(t_1) \tag{4.3}$$

for any $0 \le t_1 \le t_2$, *and* $\tilde{w}_i(0) \le w_i(0)$ *for* $i = 1,2,\ldots,d$. *Let* (Yw, Zw) (*resp.* $(\tilde{Y}\tilde{w}, \tilde{Z}\tilde{w})$) *solve the Skorokhod problem corresponding to* (w,b,R) (*resp.* $(\tilde{w},\tilde{b},\tilde{R})$). *Then*

$$(Yw)_i(t) \le (\tilde{Y}\tilde{w})_i(t), \tag{4.4}$$

$$(Yw)_i(t_2) - (Yw)_i(t_1) \le (\tilde{Y}\tilde{w})_i(t_2) - (\tilde{Y}\tilde{w})_i(t_1), \tag{4.5}$$

$$(Zw)_i(t) \ge (\tilde{Z}\tilde{w})_i(t), \tag{4.6}$$

for any $t \ge 0$, $0 \le t_1 \le t_2$, $1 \le i \le d$. □

Proof of the theorem basically involves a comparison of the map (T,S) defined by (3.15)–(3.17) and the corresponding map $(\tilde{T},\tilde{S})$ for $(\tilde{w},\tilde{b},\tilde{R})$. For this we initially use the following simplified set-up.

Set-up I: 1. $w,\tilde{w}$ are as in Theorem 4.1; also $b(\cdot) \equiv \tilde{b}(\cdot) \equiv 0$.

2. $R(t) = ((r_{ij}(t))), \tilde{R}(t) = ((\tilde{r}_{ij}(t)))$ are r.c.l.l. functions of t alone, satisfy (A3), $r_{ii}(\cdot) \equiv 1 \equiv \tilde{r}_{ii}(\cdot)$, and

$$\tilde{r}_{ij}(t) \le r_{ij}(t) \le 0, \qquad i \ne j,\ t \ge 0. \tag{4.7}$$

3. $y, \tilde{y} \in D_{\uparrow 0}([0,\infty):\bar{G})$ and

$$y_i(t_2) - y_i(t_1) \le \tilde{y}_i(t_2) - \tilde{y}_i(t_1) \tag{4.8}$$

for all $0 \le t_1 \le t_2$, $1 \le i \le d$.

Now define for $t \ge 0$, $1 \le i \le d$,

$$(Xy)_i(t) = w_i(t) + \sum_{j \ne i} \int_0^t r_{ij}(u-)\, dy_j(u), \tag{4.9}$$

$$(\tilde{X}\tilde{y})_i(t) = \tilde{w}_i(t) + \sum_{j \ne i} \int_0^t \tilde{r}_{ij}(u-)\, d\tilde{y}_j(u), \tag{4.10}$$

$$(Ty)_i(t) = \sup_{0 \le s \le t} \max\{0, -(Xy)_i(s)\}, \tag{4.11}$$

$$(\tilde{T}\tilde{y})_i(t) = \sup_{0 \le s \le t} \max\{0, -(\tilde{X}\tilde{y})_i(s)\}, \tag{4.12}$$

$$(Sy)_i(t) = (Xy)_i(t) + (Ty)_i(t), \tag{4.13}$$

$$(\tilde{S}\tilde{y})_i(t) = (\tilde{X}\tilde{y})_i(t) + (\tilde{T}\tilde{y})_i(t). \tag{4.14}$$

Note that $Ty, \tilde{T}\tilde{y} \in D_{\uparrow 0}$ and $Sy, \tilde{S}\tilde{y} \in D([0,\infty):\bar{G})$.

LEMMA 4.2. *Under set-up* I, *for* $1 \le i \le d$, $t \ge 0$,

$$(Ty)_i(t) \le (\tilde{T}\tilde{y})_i(t). \tag{4.15}$$

PROOF. Easily verified. □

LEMMA 4.3. *Assume set-up* I. *If* $(\hat{T}\tilde{y})_i(t_1)=(\hat{T}\tilde{y})_i(t_2)$ *for some* $1\le i\le d$, $0\le t_1\le t_2$ *then* $(Ty)_i(t_1)=(Ty)_i(t_2)$.

PROOF. If $(\hat{T}\tilde{y})_i(t_1)=(\hat{T}\tilde{y})_i(t_2)=0$, then the result follows by the preceding lemma. So we may assume that $(\hat{T}\tilde{y})_i(t_1)>0$. Observe that

$$(\hat{T}\tilde{y})_i(t_2)=\max\left\{(\hat{T}\tilde{y})_i(t_1), \sup_{t_1\le s\le t_2}\{-(Xy)_i(s)\}\right\}. \tag{4.16}$$

So there exists $\bar{s}_1\le t_1$ such that one of the following two cases obtains, viz. for all $s\in[t_1,t_2]$,

$$(\hat{T}\tilde{y})_i(t_1)=-(\hat{X}\tilde{y})_i(\bar{s}_1)\ge -(\hat{X}\tilde{y})_i(s), \tag{4.17}$$

or

$$(\hat{T}\tilde{y})_i(t_1)=-(\hat{X}\tilde{y})_i(\bar{s}_1-)\ge -(\hat{X}\tilde{y})_i(s). \tag{4.18}$$

We consider the case when (4.18) holds; the case when (4.17) holds can be similarly dealt with. By (4.3), the inequality in (4.18), (4.7), (4.8) we get

$$\begin{aligned} w_i(s)-w_i(\bar{s}_1-) &\ge \hat{w}_i(s)-\hat{w}_i(\bar{s}_1-)\\ &\ge \sum_{j\ne i}\int_{[\bar{s}_1,s)}(-\tilde{r}_{ij}(u-))\,d\tilde{y}_j(u)\\ &\ge \sum_{j\ne i}\int_{[\bar{s}_1,s)}(-r_{ij}(u-))\,dy_j(u) \end{aligned} \tag{4.19}$$

for all $t_1\le s\le t_2$. From (4.19) it follows that $-(Xy)_i(\bar{s}_1-)\ge -(Xy)_i(s)$ for all $t_1\le s\le t_2$. The required conclusion is now clear. □

LEMMA 4.4. *Under set-up* I, *for* $1\le i\le d$, $0\le t_1\le t_2$,

$$(Ty)_i(t_2)-(Ty)_i(t_1)\le(\hat{T}\tilde{y})_i(t_2)-(\hat{T}\tilde{y})_i(t_1). \tag{4.20}$$

PROOF. We may assume that the left side of (4.20) is >0, for otherwise there is nothing to prove. This forces the right side of (4.20) to be >0 also by the preceding lemma. So by (4.16) there exists the *least* number $\bar{s}_2$ in $[t_1,t_2]$ such that one of the following holds, viz.

$$(\hat{T}\tilde{y})_i(t_1)<-(\hat{X}\tilde{y})_i(\bar{s}_2)=(\hat{T}\tilde{y})_i(\bar{s}_2)=(\hat{T}\tilde{y})_i(t_2), \tag{4.21}$$

or

$$(\hat{T}\tilde{y})_i(t_1)<-(\hat{X}\tilde{y})_i(\bar{s}_2-)=(\hat{T}\tilde{y})_i(\bar{s}_2)=(\hat{T}\tilde{y})_i(t_2). \tag{4.22}$$

Note that $\bar{s}_2$ could depend on i, but i is fixed for this discussion. Because of the strict inequality in the above and right continuity of the functions concerned, we have $\bar{s}_2>t_1$. Similarly by the analogue of (4.16) for Ty, there exists the *least* number s_2 in $[t_1,t_2]$ such that one of the following holds, viz.

$$(Ty)_i(t_1)<-(Xy)_i(s_2)=(Ty)_i(s_2)=(Ty)_i(t_2), \tag{4.23}$$

or

$$(Ty)_i(t_1)<-(Xy)_i(s_2-)=(Ty)_i(s_2)=(Ty)_i(t_2). \tag{4.24}$$

Again, as before $s_2 > t_1$. By the last equality in (4.21) or (4.22) and the preceding lemma we get $(Ty)_i(t_2)=(Ty)_i(\tilde{s}_2)$. Therefore by minimality of s_2 it follows that

$$t_1 < s_2 \le \tilde{s}_2 \le t_2. \tag{4.25}$$

Now if $(\tilde{T}\tilde{y})_i(t_1)=0$ the result follows by Lemma 4.2. So assume $(\tilde{T}\tilde{y})_i(t_1)>0$. Let $\tilde{s}_1$ denote the least element in $[0,t_1]$ such that

$$0 < -(\tilde{X}\tilde{y})_i(\tilde{s}_1) = (\tilde{T}\tilde{y})_i(\tilde{s}_1) = (\tilde{T}\tilde{y})_i(t_1), \tag{4.26}$$

or

$$0 < -(\tilde{X}\tilde{y})_i(\tilde{s}_1-) = (\tilde{T}\tilde{y})_i(\tilde{s}_1) = (\tilde{T}\tilde{y})_i(t_1). \tag{4.27}$$

Observe that

$$0 \le \tilde{s}_1 \le t_1 < s_2 \le \tilde{s}_2 \le t_2. \tag{4.28}$$

We consider only the case when (4.22), (4.24), (4.27) hold (the other seven cases can be treated analogously). We have

$$\begin{aligned} (\tilde{T}\tilde{y})_i(t_2) - (\tilde{T}\tilde{y})_i(t_1) &= (\tilde{T}\tilde{y})_i(\tilde{s}_2) - (\tilde{T}\tilde{y})_i(\tilde{s}_1) \\ &\ge (\tilde{T}\tilde{y})_i(s_2) - (\tilde{T}\tilde{y})_i(\tilde{s}_1) \\ &\ge [-(\tilde{X}\tilde{y})_i(s_2-)] - [-(\tilde{X}\tilde{y})_i(\tilde{s}_1-)] \\ &= -[\tilde{w}_i(s_2-) - \tilde{w}_i(\tilde{s}_1-)] + \sum_{j\ne i} \int_{[\tilde{s}_1,s_2)} (-\tilde{r}_{ij}(u-))\, d\tilde{y}_j(u) \\ &\ge -[w_i(s_2-) - w_i(\tilde{s}_1-)] + \sum_{j\ne i} \int_{[\tilde{s}_1,s_2)} (-r_{ij}(u-))\, dy_j(u) \\ &= (Ty)_i(t_2) - [-(Xy)_i(\tilde{s}_1-)] \\ &\ge (Ty)_i(t_2) - (Ty)_i(t_1), \end{aligned} \tag{4.29}$$

where we have used (4.28), (4.3), (4.7), (4.8), definitions of $Ty, \tilde{T}\tilde{y}$, besides (4.22), (4.24), (4.27). This completes the proof. □

LEMMA 4.5. *Under set-up* I, *for* $1 \le i \le d,\ t \ge 0$,

$$(Sy)_i(t) \ge (\tilde{S}\tilde{y})_i(t). \tag{4.30}$$

PROOF. Fix i, t. Consider first the case when $(\tilde{T}\tilde{y})_i(t)=0$. Then $(Ty)_i(t)=0$ by Lemma 4.2. So

$$\begin{aligned} (Sy)_i(t) - (\tilde{S}\tilde{y})_i(t) &= w_i(t) - \tilde{w}_i(t) + \sum_{j\ne i}\int_0^t (r_{ij}(u-) - \tilde{r}_{ij}(u-))\, dy_j(u) \\ &\quad + \sum_{j\ne i}\int_0^t (-\tilde{r}_{ij}(u-))\, d(\tilde{y}_j - y_j)(u) \\ &\ge 0 \end{aligned} \tag{4.31}$$

by (4.7), (4.8) and as $\tilde{w}_i(\cdot) \le w_i(\cdot)$.

Now suppose $(\tilde{T}\tilde{y})_i(t) > 0$. Then there exists $t_0 \le t$ such that

$$(\tilde{T}\tilde{y})_i(t) = -(\tilde{X}\tilde{y})_i(t_0) > 0, \tag{4.32}$$

or

$$(\tilde{T}\tilde{y})_i(t) = -(\tilde{X}\tilde{y})_i(t_0-) > 0. \tag{4.33}$$

We consider only the case when (4.33) holds. Since $[-(Xy)_i(t_0-)]\le (Ty)_i(t)$, by (4.3), (4.7), (4.8), (4.13), (4.14), (4.33) we get

$$
\begin{aligned}
(4.34)\quad (Sy)_i(t)-(\tilde{S}\tilde{y})_i(t)
&\ge [w_i(t)-w_i(t_0-)]-[\tilde{w}_i(t)-\tilde{w}_i(t_0-)]\\
&\quad+\sum_{j\ne i}\int_{t_0}^{t}(r_{ij}(u-)-\tilde{r}_{ij}(u-))\,dy_j(u)+\sum_{j\ne i}\int_{t_0}^{t}(-\tilde{r}_{ij}(u-))\,d(\tilde{y}_j-y_j)(u)\\
&\ge 0,
\end{aligned}
$$

completing the proof. □

REMARK 4.6. Let w, $\tilde{w}$, b, $\tilde{b}$, R, $\tilde{R}$ be as in Theorem 4.1. For $y\in D_{\uparrow 0}([0,\infty):\bar{G})$, $z\in D([0,\infty):\bar{G})$ let $T(y,z;w,b,R)$, $S(y,z;w,b,R)$, $T(y,z;\tilde{w},\tilde{b},\tilde{R})$, $S(y,z;\tilde{w},\tilde{b},\tilde{R})$ be defined using (3.6), (3.15)–(3.17). For notational convenience put

$$T^{(1)}(y,z)=T(y,z;w,b,R),\qquad \tilde{T}^{(1)}(y,z)=T(y,z;\tilde{w},\tilde{b},\tilde{R}),$$

$$S^{(1)}(y,z)=S(y,z;w,b,R),\qquad \tilde{S}^{(1)}(y,z)=S(y,z;\tilde{w},\tilde{b},\tilde{R}),$$

and for $n=2,3,\dots$, set

$$
\begin{aligned}
T^{(n)}(y,z)&=T(T^{(n-1)}y,S^{(n-1)}z;w,b,R),\\
S^{(n)}(y,z)&=S(T^{(n-1)}y,S^{(n-1)}z;w,b,R),\\
\tilde{T}^{(n)}(y,z)&=T(\tilde{T}^{(n-1)}y,\tilde{S}^{(n-1)}z;\tilde{w},\tilde{b},\tilde{R}),\\
\tilde{S}^{(n)}(y,z)&=S(\tilde{T}^{(n-1)}y,\tilde{S}^{(n-1)}z;\tilde{w},\tilde{b},\tilde{R}).
\end{aligned}
$$

Under the hypotheses of Theorem 4.1, with Lemmas 4.4 and 4.5 applied repeatedly we obtain for $1\le i\le d$, $t\ge 0$, $0\le t_1\le t_2$,

$$(4.35)\qquad (T^{(n)}(y,z))_i(t)\le(\tilde{T}^{(n)}(y,z))_i(t),$$

$$(4.36)\qquad (T^{(n)}(y,z))_i(t_2)-(T^{(n)}(y,z))_i(t_1)\le(\tilde{T}^{(n)}(y,z))_i(t_2)-(\tilde{T}^{(n)}(y,z))_i(t_1),$$

$$(4.37)\qquad (S^{(n)}(y,z))_i(t)\ge(\tilde{S}^{(n)}(y,z))_i(t),$$

for $n=1,2,3,\dots$. □

PROOF OF THEOREM 4.1. The proof is in a few steps roughly paralleling the steps in the proof of Theorem 3.7.

Step 1. We consider (SP) at $t=0$, so the set up is as in Step 1 of §3. Define $\tau:\bar{G}\to\bar{G}$, $\sigma:\bar{G}\to\bar{G}$ by

$$(\tau y)_i=\max\left\{0,-\left\{w_i(0)+\sum_{j\ne i}r_{ij}(0,0,0)y_j\right\}\right\},$$

$$(\sigma y)_i=w_i(0)+\sum_{j\ne i}r_{ij}(0,0,0)y_j+(\tau y)_i.$$

Similarly $\tilde{\tau}$, $\tilde{\sigma}$ are defined by

$$(\tilde{\tau} y)_i=\max\left\{0,-\left\{\tilde{w}_i(0)+\sum_{j\ne i}\tilde{r}_{ij}(0,0,0)y_j\right\}\right\},$$

$$(\tilde{\sigma} y)_i=\tilde{w}_i(0)+\sum_{j\ne i}\tilde{r}_{ij}(0,0,0)y_j+(\tilde{\tau} y)_i.$$

Set $\tau^{(n)}y=\tau(\tau^{(n-1)}y)$, $\sigma^{(n)}y=\sigma(\sigma^{(n-1)}y)$, $\tilde{\tau}^{(n)}y=\tilde{\tau}(\tilde{\tau}^{(n-1)}y)$, $\tilde{\sigma}^{(n)}y=\tilde{\sigma}(\tilde{\sigma}^{(n-1)}y)$, for $n=2,3,\ldots$, with of course $\tau^{(1)}y=\tau y$, $\sigma^{(1)}y=\sigma y$, $\tilde{\tau}^{(1)}y=\tilde{\tau}y$, $\tilde{\sigma}^{(1)}y=\tilde{\sigma}y$.

Observe that τ, $\tilde{\tau}$ are the analogues of T, $\tilde{T}$ of (4.11), (4.12) when w, $\tilde{w}$, R, $\tilde{R}$ are constants and y is degenerate. As R, $\tilde{R}$ satisfy (A3), by a contraction mapping argument it is easily seen that $\tau^{(n)}y\to Yw(0)$, $\tilde{\tau}^{(n)}y\to\tilde{Y}\tilde{w}(0)$, $\sigma^{(n)}y\to Zw(0)$, $\tilde{\sigma}^{(n)}y\to\tilde{Z}\tilde{w}(0)$, as $n\to\infty$. By the proofs of Lemmas 4.2–4.5 and Remark 4.6 it follows that $\tau^{(n)}y\le\tilde{\tau}^{(n)}y$, $\sigma^{(n)}y\ge\tilde{\sigma}^{(n)}y$ componentwise for each n. Hence $Yw(0)\le\tilde{Y}\tilde{w}(0)$ and $Zw(0)\ge\tilde{Z}\tilde{w}(0)$.

Step 2. Now the set up is as in Steps 2–4 of §3. Let $T^{(n)}$, $S^{(n)}$, $\tilde{T}^{(n)}$, $\tilde{S}^{(n)}$ be as in Remark 4.6. By Step 4 of §3 there is $t>0$ such that $T^{(n)}(y,z)(\cdot)+Yw(0)\to Yw(\cdot)$ in variation norm over $[0,s]$, $S^{(n)}(y,z)(\cdot)\to Zw(\cdot)$ uniformly over $[0,s]$ for all $s<t$ for a sufficiently large class of (y,z) (for example one can take $y(\cdot)\equiv 0$, $z(\cdot)\equiv 0$; this is because (T,S) is a contraction on $\hat{D}_s$). Similarly there is $\tilde{t}>0$ such that $\tilde{T}^{(n)}(y,z)(\cdot)+\tilde{Y}\tilde{w}(0)\to\tilde{Y}\tilde{w}(\cdot)$ in variation norm and $\tilde{S}^{(n)}(y,z)(\cdot)\to\tilde{Z}\tilde{w}(\cdot)$ uniformly over $[0,s]$ for all $s<\tilde{t}$ for a large class of (y,z). Take $t_0\equiv t\wedge\tilde{t}$. By Step 1 and Remark 4.6 it is now seen that the theorem holds on $[0,t_0)$.

Step 3. With the theorem holding on $[0,t_0)$ we want to show that it holds on $[0,t_0]$; see Step 5 of §3. Analogous to Step 1, define τ, $\tilde{\tau}$ on $\bar{G}$ by

$$(\tau y)_i=\max\left\{0,-\left[(Zw)_i(t_0-)+w_i(t_0)-w_i(t_0-)+\sum_{j\neq i}r_{ij}(t_0,Yw(t_0-),Zw(t_0-))y_j\right]\right\},$$

$$(\tilde{\tau}y)_i=\max\left\{0,-\left[(\tilde{Z}\tilde{w})_i(t_0-)+\tilde{w}_i(t_0)-\tilde{w}_i(t_0-)+\sum_{j\neq i}\tilde{r}_{ij}(t_0,\tilde{Y}\tilde{w}(t_0-),\tilde{Z}\tilde{w}(t_0-))y_j\right]\right\}.$$

Similarly σ, $\tilde{\sigma}$ can be defined with obvious modifications. As in Step 1, $\tau^{(n)}y:=\tau(\tau^{(n-1)}y)\to Yw(t_0)-Yw(t_0-)$, $\sigma^{(n)}y:=\sigma(\sigma^{(n-1)}y)\to Zw(t_0)$, $\tilde{\tau}^{(n)}y:=\tilde{\tau}(\tilde{\tau}^{(n-1)}y)\to\tilde{Y}\tilde{w}(t_0)-\tilde{Y}\tilde{w}(t_0-)$, $\tilde{\sigma}^{(n)}y:=\tilde{\sigma}(\tilde{\sigma}^{(n-1)}y)\to\tilde{Z}\tilde{w}(t_0)$ as $n\to\infty$. By an argument as in Step 1 it is seen that the theorem holds on $[0,t_0]$.

Step 4. To complete the proof, in view of Step 7 of §3, it just remains to show that the theorem holds on $[0,t_0+\hat{t})$ for some $\hat{t}>0$ if it holds on $[0,t_0]$. For this as in Step 6 of §3 we consider the functions

$$b(u+t_0,y+Yw(t_0),z),\qquad \tilde{b}(u+t_0,y+\tilde{Y}\tilde{w}(t_0),z),$$

$$R(u+t_0,y+Yw(t_0),z),\qquad \tilde{R}(u+t_0,y+\tilde{Y}\tilde{w}(t_0),z),$$

as functions of (u,y,z). By the preceding step and our hypotheses, note that the situation is very similar to the one in Step 2; hence the proof can be completed proceeding as in Step 2. □

REMARK 4.7. In addition to the hypotheses of Theorem 4.1, assume that $w_i(\cdot)=-h_i(\cdot)$, $\tilde{w}_i(\cdot)=-\tilde{h}_i(\cdot)$, where h_i, $\tilde{h}_i$ are nonnegative nondecreasing functions, and that strict inequality holds in the first inequality of (4.2) or in (4.3). Then strict inequality holds in (4.4) for all $t>0$ and in (4.5) for all $0\le t_1<t_2$; this is easily seen from the proof. In such a case note that $Zw\equiv 0\equiv\tilde{Z}\tilde{w}$. □

Following the terminology in the theory of ordinary differential equations, one can say that SP(w,b,R) is *stable* if Zw is attracted to the origin; that is, for every $\varepsilon>0$ there is an $M<\infty$ such that $|Zw(t)|<\varepsilon$ for all $t\ge M$. The next corollary is immediate from Theorem 4.1.

COROLLARY 4.8. *Assume the notation and hypotheses of Theorem* 4.1. *If SP(w,b,R) is stable then so is SP$(\tilde{w},\tilde{b},\tilde{R})$.* □

Another immediate corollary from (4.4), (4.5) is

COROLLARY 4.9. *Under the hypotheses of Theorem* 4.1, *for* $i=1,2,\ldots,d$ *the measure* $d(Yw)_i$ *is dominated by* $d(\tilde{Y}\tilde{w})_i$; *that is, for any Borel set* $A\subseteq[0,\infty)$,

$$\int_A d(Yw)_i(t)\leq\int_A d(\tilde{Y}\tilde{w})_i(t),$$

and the Radon-Nikodym derivative $d(Yw)_i/d(\tilde{Y}\tilde{w})_i\leq 1$. □

REMARK 4.10. In addition to the hypotheses of Theorem 4.1, suppose $b_i(\cdot,\cdot,\cdot)\leq 0$, $w_i(\cdot)=-h_i(\cdot)$, $\tilde{w}_i(\cdot)=-\tilde{h}_i(\cdot)$ where $h_i(\cdot)$, $\tilde{h}_i(\cdot)$ are nonnegative nondecreasing functions for $1\leq i\leq d$. Note that $dh_i(\cdot)\leq d\tilde{h}_i(\cdot)$, $1\leq i\leq d$. Let $dYw(t)/d\tilde{Y}\tilde{w}$ (resp. $dh(t)/d\tilde{h}$) denote the $(d\times d)$ diagonal matrix whose ith diagonal entry is $d(Yw)_i(t)/d(\tilde{Y}\tilde{w})_i$ (resp. $dh_i(t)/d\tilde{h}_i$). Then by (3.36), (3.37) in Example 3.8 one can see that

(4.38)
$$\begin{aligned}\int_s^t \frac{dYw}{d\tilde{Y}\tilde{w}}(u)\,d\tilde{Y}\tilde{w}(u)&=Yw(t)-Yw(s)\\ &=\int_s^t R^{-1}(u,Yw(u-),0)\,\frac{dh}{d\tilde{h}}(u)\tilde{R}(u,\tilde{Y}\tilde{w}(u-),0)\,d\tilde{Y}\tilde{w}(u)\\ &\quad+\int_s^t R^{-1}(u,Yw(u-),0)\left[b^-(u,Yw(u-),0)-\frac{dh}{d\tilde{h}}(u)\tilde{b}^-(u,\tilde{Y}\tilde{w}(u-),0)\right]du.\end{aligned}$$

In particular if $b(\cdots)\equiv 0\equiv\tilde{b}(\cdots)$, and $R(u,y,0)$ is independent of y, then

$$\frac{dYw}{d\tilde{Y}\tilde{w}}(s)=R^{-1}(s,0,0)\frac{dh}{d\tilde{h}}(s)\tilde{R}(s,\tilde{Y}\tilde{w}(s-),0)\tag{4.39}$$

for almost all s w.r.t. $d\tilde{Y}\tilde{w}(\cdot)$. □

What happens if the condition of nonpositivity of $r_{ij},\tilde{r}_{ij}$, $i\neq j$ in (4.2) is dropped? The following examples indicate that one cannot expect very general results.

EXAMPLE 4.11. Let $d=2$ and $0<q<\hat{q}<\alpha<1<\alpha^{-1}$; set $r_{11}=1=r_{22}$, $r_{12}=q=r_{21}$, $\hat{r}_{11}=1=\hat{r}_{22}$, $\hat{r}_{12}=\hat{q}=\hat{r}_{21}$, $w_1(t)=-f(t)$, $w_2(t)=-\alpha f(t)$, $t\geq 0$, where $f(\cdot)$ is a nonnegative nondecreasing function, $b(\cdots)\equiv 0\equiv\hat{b}(\cdots)$, $\hat{w}(\cdot)=w(\cdot)$. Then SP$(w,0,R)$ is solved by $Yw(\cdot)=y(\cdot)$, $Zw\equiv 0$ where

$$y_1(t)=\frac{(1-q\alpha)}{(1-q^2)}f(t),\qquad y_2(t)=\frac{(\alpha-q)}{(1-q^2)}f(t).$$

Similarly SP$(w,0,\hat{R})$ is solved by $\hat{Y}w(\cdot)=\hat{y}(\cdot)$, $\hat{Z}w\equiv 0$ where

$$\hat{y}_1(t)=\frac{(1-\hat{q}\alpha)}{(1-\hat{q}^2)}f(t),\qquad \hat{y}_2(t)=\frac{(\alpha-\hat{q})}{(1-\hat{q}^2)}f(t).$$

It is easily seen that $y_1(t)\geq\hat{y}_1(t)\Leftrightarrow\alpha\geq(q+\hat{q})/(1+q\hat{q})$ and $y_2(t)\geq\hat{y}_2(t)\Leftrightarrow\alpha^{-1}\geq(q+\hat{q})/(1+q\hat{q})$. Observe that $(q+\hat{q})/(1+q\hat{q})<1$. Consequently, if $\hat{q}<\alpha<(q+\hat{q})/(1+q\hat{q})$ then $y_2(\cdot)>\hat{y}_2(\cdot)$ but $y_1(\cdot)<\hat{y}_1(\cdot)$. If $(q+\hat{q})/(1+q\hat{q})\leq\alpha\leq 1$ then $y_i(\cdot)\geq\hat{y}_i(\cdot)$, $i=1,2$. □

EXAMPLE 4.12. Let $d=3$, q, $\hat{q}$, α be as in Example 4.11, $0<q_0<1$ and $f(\cdot), g(\cdot)$ be nonnegative nondecreasing functions. Set

$$R=\begin{pmatrix}1 & q & 0\\ q & 1 & 0\\ q_0 & 0 & 1\end{pmatrix}, \qquad \hat{R}=\begin{pmatrix}1 & \hat{q} & 0\\ \hat{q} & 1 & 0\\ q_0 & 0 & 1\end{pmatrix}, \qquad w(t)=\begin{pmatrix}-f(t)\\ -\alpha f(t)\\ g(t)\end{pmatrix}.$$

Then $\mathrm{SP}(w,0,R)$ is solved by $Yw(\cdot)=(y_1(\cdot),y_2(\cdot),0)$, $Zw(\cdot)=(0,0,z_3(\cdot))$, and $\mathrm{SP}(w,0,\hat{R})$ is solved by $\hat{Y}w(\cdot)=(\hat{y}_1(\cdot), \hat{y}_2(\cdot),0), \hat{Z}w(\cdot)=(0,0,\hat{z}_3(\cdot))$ where y_1, y_2, $\hat{y}_1$, $\hat{y}_2$ are as in Example 4.11 and $z_3(\cdot)=g(\cdot)+q_0y_1(\cdot), \hat{z}_3(\cdot)=g(\cdot)+q_0\hat{y}_1(\cdot)$. If $\hat{q}<\alpha<(q+\hat{q})/(1+q\hat{q})$ then $\hat{z}_3(t)>z_3(t)$ for all t; but if $(q+\hat{q})/(1+q\hat{q})<\alpha\le 1$, then $\hat{z}_3(t)<z_3(t)$ for all t. □

5. Feasibility and minimality. Recall that the standard subsidy Yw can be mobilised only when the corresponding sector is empty. There can of course be situations when it may be advisable not to have such a restriction. This leads to the following natural definition.

Let (w,b,R) be as in §2. Let $(y,z)\in D_{\uparrow}([0,\infty)\colon \bar{G})\times D([0,\infty)\colon \bar{G})$ be such that

$$z_i(t)=w_i(t)+\int_0^t b_i(u,y(u-),z(u-))\,du+y_i(t) + \sum_{j\neq i}\int_0^t r_{ij}(u,y(u-),z(u-))\,dy_j(u) \tag{5.1}$$

is nonnegative for $1\le i\le d$, $t\ge 0$; then y is called a *feasible subsidy*, z a *feasible surplus*, (y,z) a *feasible solution* corresponding to (w,b,R).

If b,R satisfy (A1), (A2), for a given $w\in D([0,\infty)\colon \mathbb{R}^d), y\in D_{\uparrow}([0,\infty)\colon \bar{G})$ the system of integral equations (5.1) has a unique solution in $D[0,\infty)$; this can be seen by a Picard iteration (moreover if w,y are continuous then so is the solution of (5.1)). For feasibility we demand that the solution be nonnegative in each coordinate. Clearly the solution to the Skorokhod problem is a feasible solution.

The next result shows that the Leontief output (corresponding to the worst possible demand) forms a feasible subsidy.

THEOREM 5.1. *Let $w\in D([0,\infty)\colon \mathbb{R}^d)$; let b,R satisfy* (A1)–(A3); *let V be as in* (A3); *let h,β be given respectively by* (3.3), (3.4). *Then $(I-V)^{-1}(h+\beta)$ is a feasible subsidy for (w,b,R).*

PROOF. Define $\hat{R}(u,y,z)=((\hat{r}_{ij}(u,y,z)))$ by $\hat{r}_{ij}(\cdot,\cdot,\cdot)=r_{ij}(\cdot,\cdot,\cdot)$, $i\neq j, \hat{r}_{ii}(\cdots)\equiv 0$. Then we can write for any u,y,z,

$$R(u,y,z)=(I-V)+(V+\hat{R}(u,y,z)). \tag{5.2}$$

Note that by (A3), $V+\hat{R}(u,y,z)$ is a nonnegative matrix.

Let $z(\cdot)$ be the unique solution for the vector integral equation

$$z(t)=w(t)+\int_0^t b(u,y(u-),z(u-))\,du + \int_0^t R(u,y(u-),z(u-))\,dy(u), \tag{5.3}$$