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ON GALERKIN'S METHOD APPLICABLE TO THE PROBLEMS OF WATER WAVE SCATTERING BY BARRIERS

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The underlying mathematical idea behind Galerkin's method of determining approximate solution to a general operator equation $Lf=l$ alongwith approximation of the inner product $[f,l]$, is explained where L is a linear operator. Application of the method to a number of water wave scattering problems involving thin vertical barriers arising in the linearised theory of water wave, is mentioned.

Key Words: Galerkin's Method; Water Wave Scattering Problems; Oblique Wave Scattering; Chebyshev Polynomials; Rectangular Barriers

1 Introduction

Water wave scattering problems involving thin vertical barriers have been well studied in the literature of linearised theory of water waves. However, explicit solutions could be obtained only in a few simple cases. For example, when a single barrier is present in deep water and an incoming surface wave train arriving from a large distance is normally incident upon the barrier, the resulting reflected and transmitted waves and hence the related reflection and transmission coefficients, can be found explicitly. However, for oblique incidence and/or for water of uniform finite depth, such explicit solutions cannot be obtained. Usually certain approximate methods are employed to obtain the related reflection and transmission coefficients approximately. The Galerkin's method of approximate solution can be utilized successfully for this class of water wave problems to obtain numerical estimates for the reflection and transmission coefficients and its advantage lies in the fact that approximate results for these quantities of physical interest are obtained in terms of certain integrals which are rather easy to compute numerically.

In the present note we describe the underlying mathematical idea behind the Galerkin's method of approximate solutions as employed in the aforesaid class of water wave problems since this is not understood properly in many situations. A number of water wave scattering problems involving thin vertical barriers for which this method has been successfully employed is cited here. For some problems, '*single-term*' Galerkin approximations involving exact solutions of appropriate related problems, produce fairly accurate results. However, for some other problems, '*single-term*' approximations fail to produce the desired accuracy in the numerical results. For these problems, the method of '*multi-term*' Galerkin approximations needs to be utilized. The basis functions for these approximations are to be chosen suitably keeping in mind the appropriate physical requirement to be satisfied. For a number of problems, the basis functions are also cited here.

2 The Principal Method

Let us suppose that the major mathematical problem is to determine the solution of an operator equation in the form

$$(Lf)(x)=l(x) \quad x\in A \qquad \dots (1)$$

where L is a linear operator from a certain inner product space S to itself and A denotes a simply-connected domain in R^n, in standard notations. In certain problems of physical interest, along with the problem of solving the eq. (1), it is also desired to evaluate the inner product $[f,l]$ as defined by the integral relation

$$[f,l]=\int_A f(x)\, l(x)\,dx. \qquad \dots (2)$$

for the same $l\in S$, representing the forcing function in the eq. (1), when f and l are real.

Whenever the problem under consideration has both the parts as in eqs (1) and (2), it is possible to give a *special* meaning to the solution of the eq. (1), and this is described by the following defining idea.

Definition A real-valued function f is said to solve the operator eq. (1) if and only if

$$[Lf,\lambda]=[\lambda,Lf]=[\lambda,l] \qquad \dots (3)$$

for all real $\lambda\in S$.

If such a meaning is attributed to the solution of the eq. (1), then the evaluation of the quantity $[f,l]$ (see equation (2)) can be completed, at least approximately, by replacing f by a new real function F, say, where F is the solution of the equation (1) in an approximate sense, as described by,

$$[\lambda,LF]\approx[\lambda,l] \qquad \dots (4)$$

for all real $\lambda\in S$, the symbol '$\approx$' meaning *approximately equal*. Whenever the approximate relation as in eq. (4) holds good, we will regard F as the *approximate solution* of the operator equation (1).

The Galerkin's method of determining such an approximate solution of the equation (1) involves expressing the function F in the form of the *truncated series* as given by

$$F(x)=\sum_{j=1}^{n} a_j\,\phi_j(x) \qquad \dots (5)$$

where $\{\phi_j(x)\}_{j=1}^{n}$ denotes a set, contained in S, of n independent real-valued functions for $x\in A$, which is neither an orthogonal set nor is it necessary to be complete. Then, taking the inner product of both sides of the eq. (5) with $\lambda\in S$ and using the approximate identity eq. (4), we find that

$$\sum_{j=1}^{n} a_j\,[L\phi_j(x),\lambda(x)]\approx[l(x),\lambda(x)] \text{ for } \lambda\in S. \quad \dots (6)$$

In particular, choosing $\lambda(x)=\phi_k(x)$ for some fixed positive integer k, such that $1\le k\le n$, we obtain from the relation eq. (6), that

$$\sum_{j=1}^{n} a_j[L\phi_j(x),\phi_k(x)]=[l(x),\phi_k(x)],\ k=1,2,\cdots,n. \qquad \dots (7)$$

If we consider all the possible values of k in the eq. (7), we obtain exactly n linear equations for the determination of the n constants $a_1, a_2, \dots, a_n$, and these constants can be easily determined, once an approximate choice of the set of $\{\phi_j(x)\}_{j=1}^{n}$ has been made.

The approximation of f by F, where F is given by the n-term truncated series eq. (5) is usually termed as *multi-term Galerkin approximations*. Once the n constants $a_1, a_2, \dots a_n$ are determined by solving the linear system depicted in eq. (7), $[f, l]$, the quantity of our physical interest, is approximately evaluated as

$$[f,l]\approx\sum_{j=1}^{n} a_j[\phi_j,l] \qquad \dots (8)$$

where $[\phi_j,l]$, in the case when the inner product is defined by an integral of the form eq. (2), is evaluated by utilizing an approximate numerical quadrature formula.

3 Single-Term Galerkin Approximation

As a special circumstance, leading to a *single-term Galerkin approximation*, we take n=1 in the eq. (7) and find that

$$a_1=\frac{[l,\phi_1]}{[L\phi_1,\phi_1]} \qquad \dots (9)$$

producing the approximate solution for $f(x)$ as given by

$$F(x)=a_1\,\phi_1(x). \qquad \dots (10)$$

The approximate evaluation of the quantity $[f, l]$ in eq. (2), can then be completed by using the approximate relation

$$[f,l]\approx[F,l] \qquad \dots (11)$$

which takes up the value $a_1[\phi_1,l]$, if only a *single-term* Galerkin approximation is used, where the constant a_1 is given by the eq. (9).

We now make the following *very important* observations:

We have

(i) $[F,LF]\approx[F,l]$,

(ii) $[f,l]=[l,f]=[l,F]+[l,f-F]$,

(iii) $[l,f-F]=[l,f]-2[l,F]+[l,F]\approx[Lf,f]$
$-2[LF,F]+[F,LF]$ (by using (i))

(iv) $[f-F,L(f-F)]=[f,Lf]-[f,LF]-[F,Lf]$
$+[F,LF]\approx[LF,f]-2[LF,F]+[F,LF]$

if $[Lh_1,h_2]=[h_1,Lh_2]$ for all $h_1,h_2\in S$, i.e. if L is a *self-adjoint operator*.

Thus by using the results (iii) and (iv), we find that, if L is a self-adjoint operator, we have that

$$[l,f-F]\approx[f-F,L(f-F)]. \quad \text{... (12)}$$

If, further, we have either of the facts that

(a) L is positive semi-definite, i.e.,
$[h,Lh]\geq 0$ for all $h\in S$

and (b) L is negative semi-definite, i.e.,
$[h,Lh]\leq 0$ for all $h\in S$,

we find from the eq. (12), that, either

(I) $[l,F]\leq[l,f]$ in case (a), ... (13a)

or

(II) $[l,F]\geq[l,f]$ in case (b). ... (13b)

The above two facts (I) and (II) imply that the quantity $[l, F]$, computed out of the approximate solution F of the operator equation (1), provides a *lower bound* for the actual quantity $[l, f]$ in the cases where L represents a *positive semi-definite operator* whereas $[l, F]$ provides an *upper bound* for the actual quantity $[l, f]$ in the cases where L represents a *negative semi-definite operator.*

Several problems of water wave scattering arising in the linearised theory of water waves, can be resolved approximately in the sense as described above, and bounds for certain useful quantities of the type $[l, f]$ for known l, can be determined approximately where one has to work simultaneously with a pair of operators in this class of problems. In many cases it has been observed that the two bounds, when computed numerically, agrees upto 2 to 3 decimal places by employing the aforesaid single term approximations, and beyond six decimal places by employing multi-term approximations, so that their averages produce fairly accurate numerical estimates for the physical quantity $[l, f]$. This principle has been utilized in many water wave scattering problems involving barriers.

In section 4, we describe rather briefly the operator L arising in the study of a number of water wave scattering problems involving thin vertical barriers and give a list of exact solutions of appropriate related problems, which are used in the *single-term* approximations, while in section 5, a list of appropriate basis functions used in *multi-term* approximations are given.

4 Use of Single-Term Approximations

In this section, we describe a few water wave scattering problems for which *single-term* Galerkin approximations have been utilized successfully to obtain accurate numerical estimates for the reflection and transmission coefficients.

4.1 Oblique Wave Scattering By A Thin Vertical Barrier

I Deep Water

As mentioned in the introduction, the oblique water-wave scattering problems involving a plane vertical thin barrier in deep water cannot be solved explicitly unlike the case when the incoming surface wave train is normally incident on the barrier. The surface-piercing barrier was considered by Evans and Morris[1], who used the aforesaid single-term Galerkin approximation to obtain upper and lower bounds for the reflection and transmission coefficients. These bounds involve some definite integrals which are rather straightforward to compute numerically. It has been found that these bounds, when computed numerically for various values of the different parameters, coincide upto two to three decimal places and as such their averages produce fairly accurate numerical estimates for the reflection and transmission coefficients.

In the course of mathematical analysis for the problem of oblique wave scattering by a thin vertical barrier present in deep water, the following integral equations arise (see Evans and Morris[1], Mandal and Das[2])

$$\int_{\bar{L}} f(u)M(y,u)\,du=e^{-Ky},\ \ y\in\bar{L}, \quad \text{... (14a)}$$

$$\int_{L} g(u)N(y,u)\,du=e^{-Ky},\ \ y\in L \quad \text{... (14b)}$$

where L denotes an interval whose length is equal to the length of the wefted protion of the vertical barrier, $\bar{L}=(0,\infty)-L, \frac{2\pi}{K}$ is the wave length in deep water, $f(y)$ is proportional to the horizontal component of velocity in the gap above or below the barrier while $g(y)$ is proportional to the difference of velocity potential across the barrier so that $f(y)$ is required to have a square root singularity near an edge while $g(y)$ tends to zero as one approaches an edge, $M(y,u)$ and $N(y,u)$ are given by

$$M(y,u)=\int_0^\infty \frac{P(k,y)\,P(k,u)}{(k^2+v^2)^{1/2}(k^2+K^2)}dk, y,u\in\bar{L} \quad \dots (15a)$$

$$N(y,u)=\lim_{\epsilon\to 0+}\int_0^\infty \frac{(k^2+v^2)^{1/2}P(k,y)\,P(k,u)}{k^2+K^2} \times e^{-\epsilon k}\,dk,\ y,u\in L \quad \dots (15b)$$

with

$$P(k,y)=k\cos ky-K\sin ky, \quad v=K\sin\alpha, \quad \dots (16)$$

α being the angle of incidence of the wave train, the exponential term being introduced to ensure the convergence of the integral.

Thus $M(y,u)$ and $N(y,u)$ are real symmetric functions of y and u and M, N are positive semi-definite linear integral operators defined by

$$(Mf)(y)=\int_{\bar{L}} f(u)\,M(y,u)\,du, y\in\bar{L},$$

$$(Ng)(y)=\int_L g(u)\,N(y,u)\,du,\ y\in L.$$

Alongwith eqs. (14a) or (14b) we have that

$$\int_{\bar{L}} f(y)\,e^{-Ky}\,dy=C, \quad \dots (17a)$$

$$\int_L g(y)\,e^{-Ky}\,dy=\frac{1}{\pi^2 K^2 C} \quad \dots (17b)$$

where the real constant (unknown) C is related to the reflection and transmission coefficients (complex) R and T respectively be

$$C=\frac{(1-R)\cos\alpha}{i\pi R}=\frac{T\cos\alpha}{i\pi(1-T)}, \quad \dots (18)$$

α being the angle of incidence of a surface wave train incident upon the barrier from a large distance. The integral eqs. (14a & b) can be identified with the operator equation (1) while the integral eqs. (17a & b) can be identified with the inner product eq. (2), A denoting $\bar{L}$ or L and the inner product is simply the integral over A.

It so happens that for *normal incidence* of the incoming wave train, the integral equations corresponding to (14a & b) possess exact solutions. A single-term Galerkin approximation to $f(y)$ in terms of the corresponding exact solution $f_0(y)$, say, for normal incidence of the wave train provides a lower bound C_1 for C by noting the equality (17a) and using the inequality (13a) since M is a positive semi-definite linear operator. Similarly, a single-term Galerkin approximation to $g_0(y)$ in terms of the corresponding exact solution $g_0(y)$, say, for normal incidence of the wave train, provides a lower bound $\frac{1}{C_2}$ for $\frac{1}{C}$ and hence an upper bound C_2 for C by noting the equality (17b) and using the inequality (17a) again since N is also a positive semi-definite linear operator. Thus it is found that

$$C_1\le C\le C_2. \quad \dots (19)$$

Now we have from the eq. (18)

$$|R|=(1+\pi^2C^2\sec^2\alpha)^{-1/2},$$

$$|T|=\pi C(\pi^2C^2+\cos^2\alpha)^{-1/2}. \quad \dots (20)$$

It is found that R_1 and R_2 obtained from eq. (20) by using C_1 and C_2 respectively in place of C, provide upper and lower bounds for the true value $|R|$ of the reflection coefficient. Similarly the bounds T_1 and T_2 for the transmission coefficient $|T|$ are obtained. Since $|R|^2+|T|^2=1$ always, it is sufficient to consider $|R|$ only. At least for three configurations of the vertical barrier, viz. surface-piercing partially immersed barrier (Evans and Morris[1]), submerged vertical plate (Mandal and Das[2]), and thin wall with a submerged gap (Das *et al.*[3]) it has been observed that R_1 and R_2 agree within one or two decimal places when computed numerically for any wave number of some particular values of the different parameters. Thus averages of R_1 and R_2 produce fairly accurate estimates for $|R|$.

Four different configurations of the barrier are usually considered. For a surface-piercing partially immersed barrier,

$$L=(0,a),\ \bar{L}=(a,\infty),$$

for a submerged barrier extending infinitely downwards

$L=(a,\infty),\ \bar{L}=(0,a),$

for a submerged plate

$L=(a,c),\ \bar{L}=(0,a)+(c,\infty)$

and for a vertical wall with a submerged gap

$L=(0,a)+(c,\infty),\ \bar{L}=(a,c).$

We state below the functions $f_0(y)$ and $g_0(y)$ in terms of which the single-term Galerkin approximations of the integral equations (14a) and (14b) respectively are made for these four geometrical configurations.

(i) $L=(0,a),\quad \bar{L}=(a,\infty)$ (see Ursell[4])

$$f_0(y)=\frac{d}{dy}\left[e^{-Ky}\int_a^y \frac{ue^{Ku}}{(u^2-a^2)^{1/2}}du\right],\ y\in\bar{L}=(a,\infty), \quad \ldots (21a)$$

$$g_0(y)=e^{-Ky}\int_y^a \frac{ue^{Ku}}{(a^2-u^2)^{1/2}}du,\ y\in L=(0,a). \quad \ldots (21b)$$

(ii) $L=(0,\infty),\ \bar{L}=(0,a)$ (see Ursell[4])

$$f_0(y)=\frac{d}{dy}\left[e^{-Ky}\int_a^y \frac{e^{Ku}}{(u^2-a^2)^{1/2}}du\right],\ y\in\bar{L}=(0,a), \quad \ldots (22a)$$

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(iii) $L=(a,c),\ \bar{L}=(0,a)+(c,\infty)$ (see Mandal and Goswami[5], Mandal and Kundu[6])

$$f_0(y)=\begin{cases}\dfrac{d}{dy}\left[e^{-Ky}\displaystyle\int_a^y \frac{(d^2-u^2)e^{Ku}}{|\rho(u)|^{1/2}}du\right], & 0<y<a,\\[2ex] -\dfrac{d}{dy}\left[e^{-Ky}\displaystyle\int_a^y \frac{(d^2-u^2)e^{Ku}}{|\rho(u)|^{1/2}}du\right], & y>c\end{cases} \quad \ldots (23a)$$

$$g_0(y)=e^{-Ky}\int_a^y \frac{(d^2-u^2)e^{Ku}}{|\rho(u)|^{1/2}}du,\ y\in L=(a,c) \quad \ldots (23b)$$

where

$$\rho(u)=(u^2-a^2)(u^2-c^2). \quad \ldots (24)$$

$$d^2=\frac{\displaystyle\int_a^c \frac{u^2e^{Ku}}{|\rho(u)|^{1/2}}du}{\displaystyle\int_a^c \frac{e^{Ku}}{|\rho(u)|^{1/2}}du} \quad \ldots (25)$$

(iv) $L=(0,a)+(c,\infty),\ \bar{L}=(a,c)$ (see Porter[7], Mandal and Dolai[8])

$$f_0(y)=\frac{d}{dy}\left[e^{-Ky}\int_c^y \frac{ue^{Ku}}{|\rho(u)|^{1/2}}\left\{\delta-\frac{2}{\pi}F(u)du\right\}\right],\quad y\in\bar{L}, \quad \ldots (26a)$$

$$g_0(y)=\begin{cases}-e^{-Ky}\displaystyle\int_a^y \frac{ue^{Ku}}{|\rho(u)|^{1/2}}\left\{\delta-\frac{2}{\pi}F(u)\right\}du, & 0<y<a,\\[2ex] e^{-Ky}\displaystyle\int_c^y \frac{ue^{Ku}}{|\rho(u)|^{1/2}}\left\{\delta-\frac{2}{\pi}F(u)\right\}du, & y>c\end{cases} \quad \ldots (26b)$$

where

$$\delta=\frac{\dfrac{e^{Ka}}{K}+\dfrac{2}{\pi}\displaystyle\int_a^c \frac{ue^{Ku}}{|\rho(u)|^{1/2}}F(u)\,du}{\displaystyle\int_a^c \frac{ue^{Ku}}{|\rho(u)|}du}, \quad \ldots (27)$$

$$F(v)=\int_0^a \frac{|\rho(u)|}{v^2-u^2}du \quad \ldots (28)$$

and $\rho(u)$ is given by eq. (24)

As mentioned earlier, the oblique scattering problems corresponding to the first, third and fourth configurations have been considered respectively by Evans and Morris[1], Mandal and Das[2] and Das *et al.*[3] by employing the single-term Galerkin approximation. For the second configuration, Evans and Morris[1] reported that similar single-term approximations do not provide good results. For this reason, and to obtain more accurate results, multi-term Galerkin approximations in terms of suitable basis functions are required. This will be considered in section 5.

II *Finite Depth Water*

For water of uniform finite depth h, the oblique wave scattering problems involving thin vertical where L denotes an interval whose length is equal to the length of the wefted protion of the vertical barrier, $\bar{L}=(0,\infty)-L, \frac{2\pi}{K}$ is the wave length in deep water, $f(y)$ is proportional to the horizontal component of velocity in the gap above or below the barrier while $g(y)$ is proportional to the difference of velocity potential across the barrier so that $f(y)$ is required to have a square root singularity near an edge while $g(y)$ tends to zero as one approaches an edge, $M(y,u)$ and $N(y,u)$ are given by

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$$g_0(y)=e^{-Ky}\int_y^a \frac{ue^{Ku}}{(a^2-u^2)^{1/2}}du,\ y\in L=(0,a). \quad \ldots (21b)$$

(ii) $L=(0,\infty),\ \bar{L}=(0,a)$ (see Ursell[4])

$$f_0(y)=\frac{d}{dy}\left[e^{-Ky}\int_a^y \frac{e^{Ku}}{(u^2-a^2)^{1/2}}du\right], y\in\bar{L}=(0,a), \quad \ldots (22a)$$

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(iii) $L=(a,c),\ \bar{L}=(0,a)+(c,\infty)$ (see Mandal and Goswami[5], Mandal and Kundu[6])

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II *Finite Depth Water*

For water of uniform finite depth h, the oblique wave scattering problems involving thin vertical barriers, the following integral equations arise (Mandal and Dolai[8])

$$\int_{\bar{L}} f(u)\, M(y,u)\, du = \psi_0(y),\ y \in \bar{L}, \quad \ldots (29a)$$

$$\int_{L} g(u)\, N(y,u)\, du = \psi_0(y),\ y \in L \quad \ldots (29b)$$

alongwith

$$\int_{\bar{L}} f(y)\, \psi_0(y)\, dy = C, \quad \ldots (30a)$$

$$\int_{L} g(y)\, \psi_0(y)\, dy = \frac{1}{k_0^2 C} \quad \ldots (30b)$$

where

$$\psi_0(y) = \frac{\cosh k_0 (h-y)}{\cosh k_0 h}, \quad \ldots (31)$$

$$C = \frac{1-R}{iR}\cos\alpha = \frac{T}{i(1-T)}\cos\alpha \quad \ldots (32)$$

$L, \bar{L}$ are defined below depending upon the configurations of the barriers, $2\pi/k_0$ is the wave length in finite depth water where k_0 is the unique positive real root of the trancendental equation

$$k \tanh kh = K, \quad \ldots (33)$$

$$M(y,u) = \frac{\delta_0}{\cosh^2 k_0 h} \lim_{\varepsilon \to 0+}$$

$$\sum_{n=1}^{\infty} \frac{k_n \cos k_n (h-y)\, \cos k_n (h-u)}{s_n \delta_n} e^{-\varepsilon k_n},\ y, u \in \bar{L}, \quad \ldots (34a)$$

$$N(y,u) = \frac{\delta_0}{\cosh^2 k_0 h} \lim_{\varepsilon \to 0+}$$

$$\sum_{n=1}^{\infty} \frac{s_n k_n \cos k_n (h-y)\, \cos k_n (h-u)}{\delta_n} e^{-\varepsilon k_n},\ y, u \in \bar{L}, \quad \ldots (34b)$$

the exponential term being introduced to ensure convergence of the series. In the eqs. (34a) and (34b), k_n (n=1, 2, ...) are the positive roots of

$$k \tan kh + K = 0 \quad \ldots (35)$$

and

$$s_n^2 = k_n^2 + \nu^2,\ (\nu = k_0 \sin \alpha) \quad \ldots (36)$$

$$\delta_0 = 2k_0 h + \sinh 2k_0 h,$$
$$\delta_n = 2k_n h + \sin 2k_n h \quad \ldots (37)$$

Thus M (y, u) and N (y, u) are real and symmetric functions of y and u, and regarded as integral operators, M, N are linear and positive semi-definite.

Here $L = (0,a), \bar{L} = (a,h)$ for a partially immersed barrier; $L = (a,h), \bar{L} = (0,a)$ for a bottom-standing submerged barrier; $L = (a,c)$, $\bar{L} = (0,a) + (c,h)$ for a submerged plate with gaps above and below; and finally, $L = (0,a) + = (c,h), \bar{L} = (a,c)$ for a vertical wall with a submerged gap. For all these four types of barrier configurations Mandal and Dolai[8] employed single-term Galerkin approximations involving the corresponding exact solutions for *normally incident* waves in *deep water* to obtain good numerical estimates for the reflection coefficient $|R|$.

4.2 *Water Wave Scattering by Two Vertical Barriers*

I Deep Water

For the problem of water wave scattering by a pair of thin vertical identical barriers separated by a distance $2b$ when an incoming wave train is normally incident upon it from a large distance and the water is infinitely deep, the following pairs of integral equations arise in the mathematical analysis, after taking advantage of the geometrical symmetry,

$$\int_{\bar{L}} f^{s,a}(u) M^{s,a}(y,u) du = e^{-Ky},\ y \in \bar{L}, \quad \ldots (38a)$$

$$\int_{\bar{L}} g^{s,a}(u) N^{s,a}(y,u) du = e^{-Ky},\ y \in \bar{L}, \quad \ldots (38b)$$

alongwith

$$\int_{\bar{L}} f^{s,a}(y) e^{-Ky} dy = C^{s,a}, \quad \ldots (39a)$$

$$\int_{\bar{L}} g^{s,a}(y) e^{-Ky} dy = \frac{1}{\pi^2 K^2 C^{s,a}} \quad \ldots (39b)$$

where $L, \bar{L}$ are the same as in eqs. (14a) and (14b) respectively,

$$M^s(y,u) = \int_0^{\infty} \frac{(1+\coth kb)\, P(k,y) P(k,u)}{k(k^2+K^2)} dk, \quad \ldots (40a)$$

$$N^s(y,u) = \lim_{\varepsilon \to 0+} \int_0^\infty \frac{kP(k,y)\,P(k,u)}{(k^2+K^2)(1+\coth kb)} dk, \quad \dots (40b)$$

$M^a(y, u)$ and $N^a(y, u)$ are obtained from M^s and N^s respectively by replacing coth by tanh, $P(k,y)$ is the same as given by the eq. (16). The subscripts s and a correspond to the symmetric and antisymmetric case arising due to the splitting of the velocity potential into its symmetric and antisymmetric parts about $x = 0$. As before, $f^{s,a}(y)$ have square root singularity near an edge while $g^{s,a}(y)$ tend to zero there. Also, the real constant C^s is given by

$$C^s = \frac{-i(1-R^s)}{\pi\{(1+R^s) - i(1-R^s)\cot Kb\}} \quad \dots (41)$$

while C^a is the same as C^s with the superscript s replaced by a, and cot replaced by −tan, $R^{s,a}$ are related to the reflection and transmission coefficients R and T by

$$R, T = \frac{R^s \pm R^a}{2} e^{-2iKb}. \quad \dots (42)$$

For $L = (0, a)$ i.e. for two parallel surface piercing identical barriers immersed to a given depth a, Evans and Morris[9] used the single-term Galerkin approximations involving the corresponding exact solutions for the problem of waves normally incident on a single barrier. In this case $f_0(y)$ and $g_0(y)$ are given by eqs. (21a) and (21b) respectively. Once the bounds for $C^{s,a}$ are obtained, bounds for $|R|$ and $|T|$ are obtained by using the formulae

$$|R| = |1 - AB|(1 + A^2 + B^2 + A^2B^2)^{-1/2},$$

$$|T| = |A + B|(1 + A^2 + B^2 + A^2B^2)^{-1/2}, \quad \dots (43)$$

where

$$A = \cot Kb - \frac{1}{\pi C^s}, B = \cot Kb + \frac{1}{\pi C^a}. \quad \dots (44)$$

For other types of double barrier configurations such as $L = (a, \infty)$, $L = (a, c)$, $L = (0, a)$, $L = (c, \infty)$ similar single-term approximations involving corresponding exact solutions for an incoming wave train normally incident on a single barrier can be carried out. However, the results for these problems are not reported in the literature, perhaps due to poor accuracy achieved by using the single-term approximation.

For oblique incidence of the wave train on a pair of symmetrical barriers, the modifications in $M^{s,a}, N^{s,a}$ in eqs. (40) and $C^{s,a}$ in eq. (41) are obvious.

II. Finite Depth Water

For water of uniform finite depth, the problems of oblique wave scattering by two identical barriers separated by a distance $2b$ have been considered by Das *et al.*[10] and Kanoria and Mandal[11] by using single-term Galerkin approximation. In this case the following pairs of integral equations arise

$$\int_{\bar{L}} f^{s,a}(u) M^{s,a}(y,u)\,du = \psi_0(y), y \in \bar{L}, \quad \dots (45a)$$

$$\int_L g^{s,a}(u) N^{s,a}(y,u)\,du = \psi_0(y), y \in L, \quad \dots (45b)$$

alongwith

$$\int_{\bar{L}} f^{s,a}(y)\psi_0(y)\,dy = C^{s,a}, \quad \dots (46a)$$

$$\int_{\bar{L}} g^{s,a}(y)\psi_0(y)\,dy = \frac{1}{k_0^2 C^{s,a}}, \quad \dots (46b)$$

where L and $\bar{L}$ are the same as in eqs. (29b) and (29a) respectively, $\psi_0(y)$ is the same as in eq. (31) and

$$C^s = \frac{i(1-R^s)\cos\alpha}{1+R^s - i(1-R^s)\cot\mu b} \quad \dots (47)$$

with $\mu = k_0 \cos\alpha$, C^a is obtained from eq.(47) with R^s replaced by R^a and $\cot \mu b$ replaced by $-\tan \mu b$, $R^{s,a}$ are related to R and T through eq. (42). $M^s(y, a)$ and $N^s(y, u)$ are given by

$$M^s(y, u) = \frac{\delta_0}{\cosh^2 k_0 h} \frac{k_n(1+\coth s_n b)\cos k_n(h-y)}{} \times \lim_{\varepsilon \to 0+} \sum_{n=1}^{\infty} \frac{\cos k_n(h-y)\,\cos k_n(h-u)}{s_n \delta_n} e^{-\varepsilon k_n} \quad \dots (48a)$$

$$N^s(y, u) = \frac{\delta_0}{\cosh^2 k_0 h} \sum_{n=1}^{\infty} \frac{s_n k_n \cos k_n(h-y)\,\cos k_n(h-u)}{\delta_n(1+\coth s_n b)} \quad \dots (48b)$$

k_n, s_n, δ_n being defined earlier. $M^a(y,u)$ and $N^a(y,u)$ are obtained from M^s and N^s respectively by replacing coth by tanh. It is obvious that $M^{s,a}(y,u)$ and $N^{s,a}(y,u)$ are real and symmetric functions of y and u and the integral associated with them are semi-positive definite.

Four types of configurations of the two identical barriers, designated by type - I, type - II, type - III and type - IV corresponding to $L = (0, a)$, $L = (a,h)$, $L = (a,c)$, and $L = (0, a) + (c,h)$ $(0<a<c<h)$, respectively have been considered by Das *et al.*[10] (types - I, II, III) and by Kanoria and Mandal[11] (type - IV), and numerical estimates for $|R|$ in each case have been obtained by employing the single-term approximations involving the appropriate exact solutions for normal incidence of a wave train on a single barrier in deep water given by eqs. (23) to (26). It has been observed that in some cases, the estimates are not very accurate. Thus it is necessary to use multi-term Galerkin approximations for which the appropriate basis functions in terms of which the approximations will be made, are to be found. This will be given in the next section.

5 Use of Multi-Term Approximations

Banerjea *et al.*[12] employed the single-term Galerkin approximation to the problem of oblique wave scattering by a submerged thin vertical wall with a gap in finite depth water and its modification when another identical wall is introduced, the single-term approximation being made in terms of the corresponding exact solutions for normal incidence on a single submerged wall with a gap present in deep water obtained by Banerjea and Mandal[13]. It has been observed that for these two problems, the numerically computed bounds for $|R|$ are not very close, particularly for the double barrier problem, and as such, their averages cannot serve as good estimates for $|R|$. This type of difficulty associated with single-term Galerkin approximation has also been observed earlier by Evans and Morris[1] while considering the problem of oblique wave scattering by a vertical barrier submerged in deep water and extending infinitely downwards. For this reason, multi-term Galerkin approximations involving appropriate basis functions have to be utilized. This results in obtaining very accurate numerical estimates for the reflection and transmission coefficients.

The basis functions, for multi-term Galerkin approximations are to be chosen judiciously. The choice is made keeping in mind the satisfaction of appropriate physical conditions and the simplicity in the final forms. Porter and Evans[14] first utilized multi-term Galerkin approximations for some finite-depth single barrier and one double barrier problems (see Evans and Porter[15]). Banerjea *et al.*[12], Das *et al.*[10] used this method for oblique wave scatterings problems involving two identical barriers of various configurations. The basis functions for the multi-term expansions appropriate to various configurations of a single barrier or two identical barriers are given below.

(i) Surface Piercing Single or Two Identical Barriers

In this case $L = (0,a)$, $\bar{L} = (a,h)$ $(0<a<h)$. To choose the basic function for $f(y)$ $(a<y<h)$, we note that for flow close to a barrier-edge ((0,a) for a single barrier, (b,a) for two barriers),

$$f(y) \sim (y-a)^{-1/2} \text{ as } y \to a+0, \qquad \dots (49)$$

and since $\phi_y = 0$ at $y = h$ where $\phi(x,y)$ is the potential function, ϕ and hence $f(y) \propto \phi_x$ (at $x = 0$ for a single barrier, b for two barriers) can be continued across $y = h$ as an even function of y. Hence the even continuous function $\{(h-a)^2 - (h-y)^2\}^{1/2} f(y)$ can be expanded in (a, h) by a complete set of even degree Chebyshev polynomials of first kind. Hence we choose

$$f_n(y) = \frac{2(-1)^n}{\pi\{(h-a)^2-(h-y)^2\}^{1/2}} \times T_{2n}\left(\frac{h-y}{h-a}\right), a<y<h, \qquad \dots (50)$$

the constant factor being taken for convenience.

For the choice of the basis functions for the expansion of $g(y)$ $(0 < y < a)$ we have to keep in mind the free surface condition at $y = 0$ and the behaviour $g(y) = 0((a-y)^{1/2})$ as $y \to a-0$. Thus $g(y)$ satisfies

$$Kg(y) + g'(y) = 0, y = 0,$$
$$g(y) = 0, y = a.$$

Let

$$\hat{g}(y) = g(y) - K\int_y^a g(u)\,du, 0 < y < a,$$

then

$$\hat{g}'(y) = 0, \quad y = 0,$$

$\hat{g}(y)=0, \quad y=a.$

The first condition permits the extension of $\hat{g}(y)$ into $(-a, 0)$ as an even function of y, and because of the second condition, $(a^2-y^2)^{-1/2}\hat{g}(y)$ can be expanded in $(0, a)$ by a complete set of even degree Chebyshev polynomials of second kind. Thus we find

$$g_n(y)=\frac{d}{dy}\left[e^{-Ky}\int_y^a \hat{g}_n(u)e^{Ku}\,du\right], \quad 0<y<a \qquad \text{... (51)}$$

where $\hat{g}_n(y)$ is chosen as

$$\hat{g}_n(y)=\frac{2(-1)^n}{\pi(2n+1)h}(a^2-y^2)^{1/2}U_{2n}\left(\frac{y}{a}\right), 0<y<a \qquad \text{... (52)}$$

the constant factor being taken for convenience.

(ii) Bottom Standing Single or Two Identical Barriers

In this case $L=(a,h), \bar{L}=(0,a)(0<a<h)$.

To choose the basis functions for the expansion of $f(y)(0<y<a)$, we have to consider the free surface condition at $y = 0$ and the behaviour $f(y)\sim(a-y)^{-1/2}$ as $y \to a-0$ derived by considering the flow near the edge $y = a$. Thus $f(y)$ satisfies

$$Kf(y)+f'(y)=0 \quad y=0,$$
$$f(y)\sim(a-y)^{-1/2} \text{ as } y\to a-0.$$

Let

$$\hat{f}'(y)=f(y)-K\int_y^a f(u)du, 0<y<a$$

then the above conditions become

$$\hat{f}'(y)=0, y=0,$$
$$\hat{f}(y)\sim(a-y)^{-1/2} \text{ as } y\to a-0.$$

The first condition shows that $\hat{f}(y)$ can be continued as an even function of y into $(-a, 0)$, and because of the second condition, $(a^2-y^2)^{-1/2}\hat{f}(y)$ can be expanded in $(0, a)$ by a complete set of even degree Chebyshev polynomials of first kind. Thus we find

$$f_n(y)=-\frac{d}{dy}\left[e^{-Ky}\int_y^a e^{Ku}\hat{f}_n(u)du\right], 0<y<a \qquad \text{... (53)}$$

where $\hat{f}_n(y)$ is chosen as

$$\hat{f}_n(y)=\frac{2(-1)^n}{\pi(a^2-y^2)^{1/2}}T_{2n}\left(\frac{y}{a}\right), 0<y<a. \qquad \text{... (54)}$$

Similarly, it is not difficult to see that $g_n(y)$'s for $a<y<h$ are to be chosen as

$$g_n(y)=\frac{2(-1)^n}{\pi(2n+1)(h-a)h}\left\{(h-a)^2-(h-y)^2\right\}^{1/2} \times U_{2n}\left(\frac{h-y}{h-a}\right), a<y<h. \qquad \text{... (55)}$$

(iii) Totally Submerged Single or Two Identical Plates

In this case, $L=(a,c), \bar{L}=(0,a)+(c,h)$ $(a<c<h)$. We have to choose two sets of basis functions for $f(y)$ according as $0<y<a$ or $c<y<h$. For $0<y<a$, the choice of the basic functions for $f(y)$ is $f_n^{(1)}(y)$ which is given by eq. (53). For $c<y<h$, the choice is $f_n^{(2)}(y)$ which is given by eq. (50) with a replaced by c.

For choosing the basis functions $g_n(y)$ for $g(y)$, $(a<y<c)$ we must ensure the requirements that $g(y)\sim(y-a)^{1/2}$ as $y\to a+0$ and $g(y)\sim(c-y)^{1/2}$ as $y\to c+0$. For this we need the full set U_n of Chebyshev polynomials of second kind over (a, c). Thus we choose

$$g_n(y)=\frac{2\{(y-a)(c-y)\}^{1/2}}{\pi(n+1)(c-a)h}\times U_n\left(\frac{2y-a-c}{c-a}\right), a<y<c. \qquad \text{... (56)}$$

(iv) Single or Two Identical Walls with Submerged Gap

In this case, $L=(0,a)+(c,h)$, $\bar{L}=(a,c)$ $(a<c<h)$ Here $f(y)(a<y<c)$ has square root singularity near $y = a$ and c so that we have to expand the continuous functions $\{(y-a)(c-y)\}^{1/2} f(y)$ in terms of the full set of Chebyshev polynomials T_n of first kind over (a,c). Thus we choose

$$f_n(y)=\frac{1}{\pi\{(y-a)(c-y)\}^{1/2}}\times T_n\left(\frac{2y-a-c}{c-a}\right), a<y<c. \qquad \text{... (57)}$$

There exists two sets of basis functions to be chosen for $g(y)$ according as $0<y<a$ or $c<y<h$. For $0<y<h$, the choice of $g_n^{(1)}(y)$ which is given by eq. (51) and for $c<y<h$, the choice of $g_n^{(2)}(y)$ which is given by eq. (55) with a replaced by c.

(v) Submerged Single or Two Identical Walls with Gap

In this case, $L=(a,c)+(d,h)$, $\overline{L}=(0,a)+(c,d)$ $(a<c<d<h)$. The two sets of basis functions for $f(y)$ according as $0<y<a$ or $c<y<d$ are obtained as

$$f_n^{(1)}(y)=-\frac{d}{dy}\left[e^{-Ky}\int_y^a \hat{f}_n(u)e^{Ku}du\right]\ 0<y<a \quad \dots (58a)$$

with

$$\hat{f}_n(y)=\frac{2(-1)^n}{\pi(a^2-y^2)^{1/2}}\, T_{2n}\left(\frac{y}{a}\right),\ 0<y<a, \quad \dots (58b)$$

and

$$f_n^{(2)}(y)=\frac{1}{\pi\{(y-c)(d-y)\}^{1/2}}\times T_n\left(\frac{2y-c-d}{d-c}\right), c<y<d. \quad \dots (59)$$

Also, two sets of basis functions for $g(y)$ according as $a<y<c$ or $d<y<h$ are given by

$$g_n^{(1)}(y)=\frac{2\{(y-a)(c-y)\}^{1/2}}{\pi(n+1)(c-a)h}\times U_n\left(\frac{2y-a-c}{c-a}\right), a<y<c \quad \dots (60)$$

and

$$g_n^{(2)}(y)=\frac{2(-1)^n\{(h-d)^2-(h-y)^2\}^{1/2}}{(2n+1)(h-d)h}\times U_{2n}\left(\frac{h-y}{h-d}\right), d<y<h. \quad \dots (61)$$

It has been observed by Das *et al.*[10] and Banerjea *et al.*[12] that by taking only three to four terms in the multi-term approximations in the oblique wave scattering problems involving two identical barriers for the aforesaid five configurations, accuracy in the numerical estimates for $|R|$ beyond six decimal places have been achieved.

6 Conclusion

The mathematical idea behind Galerkin's method of approximate solution utilized in various water wave scattering problems involving thin vertical barriers is explained clearly. The basis functions for single-term as well as multi-term approximations used in a number of water wave scattering problems are listed. It may be noted that the *'single-term'* approximations as used here are not the same as *'one-term'* approximations in *'n-term'* approximations by putting n=1. The *'single-term'* approximations involve exact solutions for normal incidence in deep water.

For thick rectangular barriers, similar multi-term approximation method can be employed. However, these configurations give rise to only one set of integral equations for functions which are proportional to the horizontal velocity in the gap through the corner points of the thick barrier and as such upper and lower bounds for the reflection coefficients $|R|$ will not be obtained although very accurate numerical estimates for $|R|$ can be computed. The basis functions for these configurations involve ultraspherical Gegenbaue polynomials (see Evans and Fernyhough[16], Kanoria *et al*[17]).

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