Quasipotential analysis for deriving the multidimensional Sagdeev potential equation in multicomponent plasma

Rajkumar Roychoudhury[a)]
Physics & Applied Mathematics Unit, Indian Statistical Institute, Calcutta 700035, India

G. C. Das
Plasma Physics Division, Institute of Advanced Study in Science & Technology, Guwahati 781022, Assam, India

Jnanjyoti Sarma
Department of Mathematics, R. G. Baruah College, Guwahati 781025, India

(Received 3 December 1998; accepted 12 March 1999)

The exact multidimensional Sagdeev potential is derived in a multicomponent plasma consisting of negative ions wherein a fraction of electrons is trapped in the potential well developed in the plasma. More precisely, the Sagdeev potential equation revisits the results stemming from the Kadomtsev–Petviashvili (K–P) equation deduceable by applying the reductive perturbation technique in plasma-acoustic wave dynamics. In the study we show that the multidimensional Sagdeev potential derived here yields the formation and propagation of solitons, as well as double layers in plasma, by using a new approach known as the tanh-method to solve out the soliton phenomena. It is seen that different ordering in ϕ, the electrical potential, yields different solitary wave solutions that agree with earlier observations.

I. INTRODUCTION

The theoretical observations on soliton dynamics derived through the augmentation of the Korteweg-deVries (K-dV) equation was probably first achieved by using the reductive perturbation technique in fluid dynamics. Later, in the same decade, it was extended to plasma dynamics[1] and took its place with the other approaches for studying the nonlinear wave phenomena such as through the Sagdeev potential equation,[2] Nonlinear Schrodinger (NLS) equation, sine-Gordon equation,[3] and Burger equation.[4] However, a new milestone was reached when the study was extended to a multicomponent plasma, especially plasma with negative ions, and became a boon to bridging the theoretical and experimental observations in plasmas (see Ref. 5 and references therein). The study of ion-acoustic soliton dynamics in multicomponent plasma with negative ions by Das[6] and the work by Das and Tagare,[7] who extended the earlier study to generalized multicomponent plasmas, have had considerable impact in laboratory plasmas.[4] However, these observations have dealt mainly with the unidirectional soliton phenomena, while such a study on the soliton dynamics has been extended to space plasmas through the derivation of the Kadomtsev–Petviashvili (K–P) equation,[8] classified as

$$\frac{\partial}{\partial \xi}\left[\frac{\partial \phi^{(1)}}{\partial \tau}+F(\phi^{(1)})\,\frac{\partial \phi^{(1)}}{\partial \xi}+B\,\frac{\partial^3 \phi^{(1)}}{\partial \xi^3}\right]+D\left[\frac{\partial^2 \phi^{(1)}}{\partial \eta^2}+\frac{\partial^2 \phi^{(1)}}{\partial \zeta^2}\right]=0, \quad (1)$$

where $\phi^{(1)}$ is the first order perturbed potential and $F(\phi^{(1)})$ recognizes the contribution of the nonlinear effect arising from the plasma configuration. Later, the configuration was extended to a plasma with some electrons trapped in the potential well. However, the general form of plasma describes the similar soliton phenomena from the solution of the Sagdeev potential equation,[2] garnering the formation and propagation of the solitons and double layers (see Refs. 8–10 and also the references therein). But, previously, the Sagdeev potential was derived mostly for the unidirectional soliton propagation in the plasma. Our present aim is to derive a multimensional Sagdeev potential equation in the form of an energy integral equation analogous to the nature of unidirectional particle motion observed first by Davis *et al.*[11] while studying the nonlinear phenomena in plasmas. In the course of this study, the nonlinear wave equation has been derived, under certain conditions, in a space coordinate system. Afterwards, the simple wave solution technique[12–14] is modified to the so-called tanh-method,[15,16] which is applied to recover the earlier results on unidirectional soliton propagation, as well as to obtain some new findings.

II. BASIC EQUATIONS AND FORMULATION OF THE SAGDEEV POTENTIAL EQUATION

To study the nonlinear ion-acoustic wave phenomena, we have considered an unmagnetized plasma consisting of negative ions. The electrons are of a free nature, but a fraction of them moves into the potential well, losing energy continuously, and as a result of which, electrons bounce back and forth within the potential well and ultimately are trapped therein. The trapped electrons are found to change the features of plasma-acoustic waves experimentally,[17,18] a fact supported by many theoretical observations,[19–21] as well

a)Electronic mail: raj@www.isical.ac.in

The distribution of the trapped electrons, following Schamel,[19,20] is given by the form of the electron density, which is as follows:

$$n_e(\phi)=\exp(\phi)\mathrm{erfc}(\sqrt{\phi})+\frac{1}{\sqrt{\beta}}[\exp(\beta\phi)\mathrm{erf}\sqrt{|\beta\phi|}],$$

where

$$\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt,$$

and

$$\mathrm{erfc}(x)=1-\mathrm{erf}(x); \tag{2}$$

where β indicates the temperature ratio between free and trapped electrons, respectively. Expanding Eq. (2) as a Taylor series in ϕ, we get the electron density $n_e(\phi)$ as a linear combination of free and trapped electron effects, as shown below,

$$n_e(\phi)=1+\phi-\frac{4}{3}b_1\phi^{3/2}+\frac{1}{2}\phi^2-\frac{8}{15}b_2\phi^{5/2}+\frac{1}{6}\phi^3+\cdots, \tag{3}$$

where the potential ϕ and density $n_e(\phi)$ are normalized to kT_{eff}/e (defined later on) and unperturbed density n_0, respectively. Other constants are defined as

$$b_1=\frac{1-\beta}{\sqrt{\pi}},\quad b_2=\frac{1-\beta^2}{\sqrt{\pi}};\quad \text{where}\quad \beta=1,0$$

correspond to the plasmas having the Maxwellian and flat topped distribution, respectively. In the isothermal plasma $\beta=1$, implying $b_1=0$ and $b_2=0$, while for the nonisothermal plasma we have the following relations: viz., $0<b_1<1/\sqrt{\pi}$ and $0<b_2<1/\sqrt{\pi}$.

The basic equations governing the plasma, under the fluid descriptions, include the equation of continuity and the equation of motion, in the following nondimensional form:[22,23]

$$\frac{\partial n_\alpha}{\partial t}+\nabla\cdot(n_\alpha\mathbf{v}_\alpha)=0, \tag{4}$$

$$\frac{\partial \mathbf{v}_\alpha}{\partial t}+(\mathbf{v}_\alpha\cdot\nabla)\mathbf{v}_\alpha+q_\alpha\mu_\alpha\nabla\phi=0. \tag{5}$$

These equations are closed and linked to the charged particles through the Poisson equation, given as

$$\nabla^2\phi=n_e-\sum_\alpha q_\alpha n_\alpha, \tag{6}$$

where $\alpha=i,j$ represent, respectively, positive and negative ions with $\mu_\alpha=m_i/m_\alpha$. $\mathbf{v}_\alpha$ is the normalized velocity of the α particle normalized to the ion-acoustic speed $c_s=(kT_{\mathrm{eff}}/m_i)^{1/2}$, and n_α is the density with $q_\alpha=\pm1$, respectively, for $\alpha=i$ and j. Space and time are normalized to the Debye length, $\lambda_D=(kT_{\mathrm{eff}}/4\pi n_0e^2)^{1/2}$ and ion plasma frequency, $\Omega_i=(4\pi n_0e^2/m_i)^{1/2}$, respectively, where $T_{\mathrm{eff}}=T_hT_c/(n_cT_h+n_hT_c)$, T_h,T_c being the temperatures of the nonidentical electrons whose initial densities are n_h,n_c, respectively, satisfying $n_h+n_c=1$. To derive the multidimensional Sagdeev potential wave equation, we introduce the usual linear transformation as

$$\eta=\gamma[(l,m,n)(x,y,z)-Vt], \tag{7}$$

where (l,m,n) is the direction cosine of the plasma-acoustic wave propagation. Now, using Eq. (7) along with the appropriate boundary conditions: $n_\alpha\rightarrow n_\alpha^{(0)}$, $v_\alpha^{(0)}\rightarrow0$, and $\phi\rightarrow0$, Eq. (4) reduces to

$$-V\frac{dn_\alpha}{d\eta}+l\frac{d}{d\eta}(n_\alpha v_\alpha)+m\frac{d}{d\eta}(n_\alpha v_\alpha)+n\frac{d}{d\eta}(n_\alpha v_\alpha)=0, \tag{8}$$

from which the density n_α is obtained as follows:

$$n_\alpha=\frac{Vn_\alpha^{(0)}}{V-\mathbf{L}\cdot\mathbf{v}_\alpha}. \tag{9}$$

Again from Eq. (5), the following set of differential equations are obtained:

$$-V\frac{dv_{\alpha x}}{d\eta}+lv_{\alpha x}\frac{dv_{\alpha x}}{d\eta}+mv_{\alpha y}\frac{dv_{\alpha x}}{d\eta}+nv_{\alpha z}\frac{dv_{\alpha x}}{d\eta}+q_\alpha\mu_\alpha l\frac{d\phi}{d\eta}=0, \tag{10}$$

$$-V\frac{dv_{\alpha y}}{d\eta}+lv_{\alpha x}\frac{dv_{\alpha y}}{d\eta}+mv_{\alpha y}\frac{dv_{\alpha y}}{d\eta}+nv_{\alpha z}\frac{dv_{\alpha y}}{d\eta}+q_\alpha\mu_\alpha m\frac{d\phi}{d\eta}=0, \tag{11}$$

$$-V\frac{dv_{\alpha z}}{d\eta}+lv_{\alpha x}\frac{dv_{\alpha z}}{d\eta}+mv_{\alpha y}\frac{dv_{\alpha z}}{d\eta}+nv_{\alpha z}\frac{dv_{\alpha z}}{d\eta}+q_\alpha\mu_\alpha n\frac{d\phi}{d\eta}=0. \tag{12}$$

From Eq. (10) and Eq. (12), we have, after some algebra,

$$(V-\mathbf{L}\cdot\mathbf{v}_\alpha)^2=V^2-2q_\alpha\mu_\alpha L^2\phi, \tag{13}$$

where $\mathbf{L}=(l,m,n)$.

Again from Eq. (13),

$$\mathbf{L}\cdot\mathbf{v}_\alpha=V-\sqrt{V^2-2q_\alpha\mu_\alpha L^2\phi}. \tag{14}$$

Using the transformation Eq. (7), the Poisson equation (6) reduces to

$$\gamma^2\frac{d^2\phi}{d\eta^2}=\frac{1}{L^2}\left(n_e-\sum_\alpha q_\alpha n_\alpha\right)=-\frac{d\psi}{d\phi}, \tag{15}$$

where

$$\psi=-\frac{1}{L^2}\int n_e(\phi)d\phi+\frac{1}{L^2}\int\sum_\alpha q_\alpha n_\alpha d\phi$$
$$=\psi_e+\sum_\alpha\psi_\alpha(\phi), \tag{16}$$

$$\psi_e(\phi)=-\frac{1}{L^2}\left[e^{\phi}(1-\mathrm{erf}\sqrt{\phi})-1+\frac{1}{\beta\sqrt{\beta}}e^{\beta\phi}\right.$$
$$\left.\times\mathrm{erf}(\sqrt{\beta\phi})+\frac{2}{\beta\sqrt{\pi}}\phi^{1/2}(\beta-1)\right]. \quad (17)$$

Expanding $\psi_e(\phi)$ in powers of ϕ, we have

$$\psi_e=-\frac{1}{L^2}\left(\phi+\frac{\phi^2}{2}-\frac{8b_1}{15}\phi^{5/2}+\frac{1}{6}\phi^3-\frac{16b_2}{105}\phi^{7/2}+\cdots\right), \quad (18)$$

and also

$$\sum\psi_\alpha(\phi)=\frac{V}{L^4}\frac{\Sigma n_\alpha^{(0)}(V-\sqrt{V^2-2L^2q_\alpha\mu_\alpha\phi})}{\mu_\alpha}. \quad (19)$$

Expanding, we get

$$\sum\psi_\alpha=\sum_\alpha\frac{n_\alpha^{(0)}}{L^2\mu_\alpha}\left[\mu_\alpha q_\alpha\phi+\frac{L^2q_\alpha^2\mu_\alpha^2\phi^2}{2V^2}\right.$$
$$\left.-\frac{L^4q_\alpha^3\mu_\alpha^3\phi^3}{2V^4}\right]+\cdots. \quad (20)$$

Now, in order to relate to the earlier observations, we assume $V=l\lambda+U$, where U is small compared to λ, and we have

$$\frac{1}{V^2}=\frac{1}{(l\lambda+U)^2}\simeq\frac{1}{l^2\lambda^2}\left(1-\frac{2U}{l\lambda}\right). \quad (21)$$

Now, assuming $U\ll\lambda$ and the quasineutrality condition, the coefficient of ϕ^2 derives the phase velocity of the plasma-acoustic wave as

$$\lambda^2=\sum\mu_\alpha n_\alpha^{(0)}, \quad (22)$$

which is exactly the same as that derived by Das and Sen,[22] while studying the soliton dynamics through the augmentation of the K–P equation by using the reductive perturbation technique.

Now balancing the coefficient of ϕ^2 from the Sagdeev potential equation, one gets the U dependent part as

$$\frac{-\Sigma_\alpha\mu_\alpha q_\alpha^2n_\alpha U}{l^3\lambda^3}=-\frac{U}{2Bl^3}\quad\text{with}\quad B=\frac{\lambda}{2}. \quad (23)$$

Similarly, the coefficients of ϕ^3 give

$$-\frac{1}{6L^2}+\frac{\Sigma_\alpha L^2\mu_\alpha^2q_\alpha^3n_\alpha^{(0)}}{2\lambda^4l^4}$$
$$\simeq\frac{A}{6Bl^2}\quad\text{with}\quad A=\frac{\lambda}{2}\left(\frac{\Sigma3(q_\alpha\mu_\alpha^2n_\alpha^{(0)})}{\lambda^4}-1\right). \quad (24)$$

So whatever the power in ϕ is taken to be, the term could be simplified to take into account the effect of nonlinearity in a manner similar to the earlier derivation of the K–P equation.[22–23] It can be shown, up to the third order, that it is same as what was exactly derived earlier and could be continued to any number of terms. Now the potential equation could be reduceable to the following form (assuming γ to be 1):

$$\frac{1}{2}\left(\frac{d\phi}{d\eta}\right)^2=\frac{1}{2}\phi^2[A_0+A_1\phi^{1/2}+A_2\phi+A_3\phi^{3/2}+\cdots], \quad (25)$$

where

$$A_0=\frac{Dl^2+Ul-D}{Bl^4},\quad A_1=-\frac{8}{15}\frac{C}{Bl^2},$$

$$A_2=-\frac{1}{3}\frac{A}{Bl^2},\quad A_3=\frac{8}{35}\frac{C'}{Bl^2},$$

etc. Here $D=B=\lambda/2$, $C=\lambda(1-\beta)/\sqrt{\pi}$, and $C'=2\lambda(1-\beta^2)/3\sqrt{\pi}$.

Here we have considered the effect of different ordering in ϕ. While doing so, we neglect other small effects, like Landau damping, viscosity, collision, etc., which might also play important roles, since our interest was to see only the ordering effect in ϕ in isolation. We may consider the same elsewhere to show the totality with other possible interactions in plasma-acoustic waves. However, our present aim is to derive soliton phenomena from the multidimensional Sagdeev potential wave equation, employing a new approach known as the tanh-method, and thence highlight the earlier results along with the present new findings, as well.

III. SOLUTION OF THE SAGDEEV POTENTIAL EQUATION

Now we will proceed step by step to solve the Sagdeev equation with approximated ϕ and, accordingly, take the number of the nonlinear terms to show the features of solitons. First, we assume $|\phi|\ll1$ and neglect the square and the higher order terms from the nonlinear coefficient and the Sagdeev potential equation [using (18) and (20)] is written as

$$\gamma^2\frac{d^2\phi}{d\eta^2}=A_1\phi-A_2\phi^{3/2}, \quad (26)$$

where $A_1=1/L^2[1-\Sigma_\alpha n_\alpha q_\alpha^2\mu_\alpha/V^2]$ and $A_2=4b_1/3L^2$, subjected to the boundary condition given as $\psi(0)=\psi'(0)=0$, $\psi''<0$, and $\psi(\phi_0)=0$ for arbitrarily chosen ϕ_0 along with $\psi(\phi_0)<0$ for $0<|\phi|<|\phi_0|$, where $|\phi_0|$ is the amplitude of the soliton profile. To find the soliton solution from Eq. (26), we use a hyperbolic transformation; $z=\tanh(\eta)$ and $W(z)=\phi(\eta)$. Equation (26) then transforms as

$$\gamma^2(1-z^2)^2\frac{d^2W}{dz^2}-2\gamma^2z(1-z^2)\frac{dW}{dz}-A_1W+A_2W^{3/2}=0. \quad (27)$$

It is obvious that Eq. (27) is a Fuchsian-like nonlinear ordinary differential equation and thus could be assumed to have a Frobenius series solution, as follows:

$$W(z)=\sum_{r=0}^{\infty} a_r z^{\rho+r}. \tag{28}$$

Here ρ determines the number and nature of the solution. But following Das *et al.*,[15,24] the series is truncated to a finite one, viz., $W(z)=\Sigma_{r=0}^{N} a_r z^r$. Thereafter, if one substitutes the series in Eq. (27), the leading order of nonlinear terms balancing the order of the differential equation yields $N=4$, i.e., the series $W(z)$ should have five terms. Again, the nature of the differential equation enables one to take the series with even order terms, only whence $W(z)$ is found to be of the form

$$W(z)=a_0-2a_2z^2+a_4z^4, \tag{29}$$

where the relations among a_0, a_2, and a_4 are used to express all the parameters in terms of a_0. Consider the recurrence relation,

$$4\gamma^2(5z^2-1)-A_1+A_2a_0^{1/2}(1-z^2)=0. \tag{30}$$

From this recurrence relation, the unknowns a_0 and γ are determined as $a_0=(5A_1/4A_2)^2$ and $\gamma=(A_1/16)^{1/2}$.

Correspondingly, the solution is obtained as

$$\phi=\left(\frac{5A_1}{4A_2}\right)^2 \operatorname{sech}^4\left(\frac{\eta}{\delta}\right), \quad \text{with } \delta=\sqrt{\frac{5}{4}}, \tag{31}$$

which yields a compressive solitary wave feature derived from the Sagdeev potential wave equation under the condition $|\phi|\ll 1$. From the Sagdeev potential equation, it is clear that the nonisothermality is introduced through the term A_2, and the soliton solution for an isothermal plasma is not possible directly from the solution, as the case $A_2\to 0$ breaks the solution. The case for an isothermal plasma has to be derived from the basic equations governing the plasma along with different stretching co-ordinates and the perturbation scheme, as well. So the lowest order in ϕ evaluates only the compressive soliton profile. If one includes the next higher order term, the Sagdeev potential equation reduces to

$$\gamma^2\frac{d^2\phi}{d\eta^2}=A_1\phi-A_2\phi^{3/2}+A_3\phi^2=-\frac{d\psi}{d\phi}. \tag{32}$$

After integrating Eq. (32), we get

$$\frac{1}{2}\left(\gamma\frac{d\phi}{d\eta}\right)^2+\psi(\phi)=0, \tag{33}$$

where $\psi(\phi)=-(A_1/2)\phi^2+(2A_2/5)\phi^{5/2}-(A_3/3)\phi^3$. Substituting $\phi=\Phi^2$, Eq. (33) transforms as

$$2\gamma^2\left(\frac{d\Phi}{d\eta}\right)^2=\frac{A_1}{2}\Phi^2-\frac{2A_2}{5}\Phi^3+\frac{A_3}{3}\Phi^4=-\psi(\Phi). \tag{34}$$

To get the double layer solution, the modified Sagdeev potential function $\psi(\Phi)$ must satisfy the following boundary conditions: $\psi(\Phi)=d\Phi/d\eta=0$ at $\Phi=0$ (and at $\Phi=\Phi_m$) and $d\psi/d\Phi=0$ at $\Phi=0$ (and at $\Phi=\Phi_m$) and also the condition $[d^2\psi/d\Phi^2]_{\Phi=\Phi_m}<0$.

Following the above boundary conditions, we have $A_1=\frac{4}{5}A_2\Phi_m-\frac{2}{3}A_3\Phi_m^2$ and $A_1=\frac{6}{5}A_2\Phi_m-\frac{4}{3}A_3\Phi_m^2$.

From the above two relations, A_1, A_2 are evaluated as $A_2=\frac{5}{3}A_3\Phi_m$, $A_1=\frac{2}{3}A_3\Phi_m^2$, and $25A_1A_3=6A_2^2$. Inserting the values of A_1, A_2, Eq. (34) could be modified as

$$\gamma\frac{d\Phi}{d\eta}=k\Phi(\Phi_m-\Phi); \quad \text{with } k=\pm\left(\frac{A_3}{6}\right)^{1/2}. \tag{35}$$

Now we again apply the tanh-method to Eq. (35), which now reduces to

$$\gamma(1-z^2)\frac{dW}{dz}-k\Phi_mW+kW^2=0. \tag{36}$$

The process of the tanh-method, as described earlier, derives $N=1$, and we get the series as

$$W(z)=a_0+a_1z. \tag{37}$$

Using Eq. (36) we get from Eq. (37) the following recurrence relations:

$$-\gamma a_1+ka_1^2=-ka_1\Phi_m+2ka_0a_1=-\gamma-k\Phi_ma_0+ka_0^2=0, \tag{38}$$

from which the unknowns are determined as $a_1=\frac{1}{2}\Phi_m$, $a_0=\pm\frac{1}{2}\Phi_m$, and $\gamma=\frac{1}{2}k\Phi_m$. Correspondingly, the soliton solution is found to be

$$\Phi(\eta)=\frac{1}{4}\Phi_m\left[1\pm\tanh\left(\frac{\chi}{\delta}\right)\right]^2;$$
$$\text{with } \chi=lx+my+nz-Vt \quad \text{and } \delta=\left(\tfrac{1}{2}k\Phi_m\right)^{-1}, \tag{39}$$

which represents the profile of double layers in the plasma-acoustic wave.

To study further the solitary wave solution from Eq. (35), we write Eq. (34),

$$4\gamma^2\frac{d^2\Phi}{d\eta^2}=A_1\Phi-\frac{6}{5}A_2\Phi^2+\frac{4}{3}A_3\Phi^3. \tag{40}$$

Using a linear transformation of the form $\Phi=\mu F+\nu$ with $\mu=1$; $\nu=\frac{3}{10}(A_2/A_3)$, Eq. (40) reduces to the form

$$4\gamma^2\frac{d^2F}{d\eta^2}-MF-\frac{4}{3}A_3F^3=0, \tag{41}$$

where the relations $M=A_1-\frac{12}{5}A_2\nu+4A_3\nu^2$ and $A_1=\frac{9}{125}(A_2^3/A_3^2)$ are used. Now, employing the tanh-method, viz., $z=\tanh(\eta)$, $W(z)=F(\eta)$, Eq. (41) reduces to a Fuchsian-like nonlinear ordinary differential equation as

$$4\gamma^2(1-z^2)^2\frac{dW^2}{dz^2}-8\gamma^2(1-z^2)z\frac{dW}{dz}-MW-\frac{4}{3}A_3W^3=0. \tag{42}$$

Now the Frobenius series solution method, as described earlier, finds the number of the terms in the series, N equals 1, failing to evaluate the proper soliton solution as such. In this case, an infinite series of the form $F(z)=\Sigma_{\gamma=0}^{\infty}a_rz^r$ is desirable. This series reduces, after some algebra, to the following:

$$F(z)=k(1-z^2)^{1/2}, \tag{43}$$

k and γ can be obtained from the the recurrence relation

$$4\gamma^2(2z^2-1)-M-\tfrac{4}{3}A_3k^2(1-z^2)=0, \tag{44}$$

from which we get k and γ as $k=\sqrt{-3M/4A_3}$, $\gamma=\sqrt{M}/2$, and, finally, the soliton solution is found to be

$$\phi=\left[\frac{3}{5}\frac{A_2}{A_3}\pm\left(-\frac{3}{2}\frac{M}{A_3}\right)^{1/2}\operatorname{sech}\left(\frac{\chi}{\delta}\right)\right]^2, \quad \text{with } \delta=\sqrt{\frac{4}{M}}. \tag{45}$$

However, Eq. (36) derives the solution as

$$\Phi(\eta)=\left[\frac{2}{5}\frac{A_2}{A_1}\pm\left(\frac{4}{25}\frac{A_2^2}{A_1^2}-\frac{2}{3}\frac{A_3}{A_1}\right)^{1/2}\cosh\left(\frac{A_1}{4}\eta\right)\right]^{-2}. \tag{46}$$

The solution yields the possible coexistence of a shock-wave structure of the Sagdeev potential equation and when $\frac{4}{25}(A_2^2/A_1^2)-\frac{2}{3}(A_3/A_1)<0$, otherwise, the case $\frac{4}{25}(A_2^2/A_1^2)-\frac{2}{3}(A_3/A_1)\geqslant 0$ derives the soliton phenomena. As a degenerate case, we can derive two limiting cases: viz., $A_2\ll A_3$ and $A_2\gg A_3$. The former case reads the soliton profile given by

$$\Phi(\eta)=\frac{3}{2}\frac{A_1}{A_3}\operatorname{sech}^2\left(\frac{A_1}{4}\eta\right), \tag{47}$$

which represents the rarefactive soliton because of the assumption $A_2\ll A_3$. The case $A_3=0$ leads to the explosion or collapse of the solitary waves depending on the conservation of the energy therein. Until now it has been shown that the multidimensional Sagdeev potential yields, under different approximations, features of soliton phenomena the same as those derived earlier[24,25] by the augmentation of the K–P equation. But since the Sagdeev potential derived here is exact, we can expand it to any order in ϕ. The wave equation for the next order in ϕ can be written as

$$\gamma^2\frac{d^2\phi}{d\eta^2}=A_1\phi-A_2\phi^{3/2}+A_3\phi^2-A_4\phi^{5/2}\equiv-\frac{d\psi}{d\phi}. \tag{48}$$

Now, the integration of Eq. (48) with a transformation $\phi=\Phi^2$, as well as the boundary conditions $d\Phi/d\eta\to 0$, $\Phi\to 0$ at $\eta\to\infty$, leads to

$$2\gamma^2\left(\frac{d\Phi}{d\eta}\right)^2=\frac{1}{2}A_1\Phi^2-\frac{2}{5}A_2\Phi^3+\frac{1}{3}A_3\Phi^4-\frac{2}{7}A_4\Phi^5, \tag{49}$$

which is again simplified by differentiating it with respect to η as

$$4\gamma^2\frac{d^2\Phi}{d\eta^2}=A_1\Phi-\frac{6}{5}A_2\Phi^2+\frac{4}{3}A_3\Phi^3-\frac{10}{7}A_4\Phi^4. \tag{50}$$

To use the tanh-method, we, as before, transform Eq. (50) to a standard form for which we use the linear transformation $\Phi=\mu F+\nu$ with $\mu=1$ and $\nu=\frac{7}{30}(A_3/A_4)$. Equation (50) then reduces to

$$4\gamma^2\frac{d^2F}{d\eta}-PF+\frac{10}{7}A_4F^4=0, \tag{51}$$

wherein the following relations are derived: $A_2=\frac{7}{18}(A_3^2/A_4)$, $A_1=\frac{49}{900}(A_3^3/A_4^2)$, and $P=A_1-\frac{12}{5}A_2\nu+4A_3\nu^2-\frac{40}{7}A_4\nu^3$. Now, to use the tanh-method, we take the transformation $W(z)=F(\eta)$ with $z=\tanh(\eta)$ and Eq. (51) is then reduced to the Fuchsian-like ordinary differential equation as follows:

$$4\gamma^2(1-z^2)^2\frac{d^2W}{dz^2}-8\gamma^2z(1-z^2)\frac{dW}{dz}-PW+\frac{10}{7}A_4W^4=0, \tag{52}$$

and, consequently, the Frobenius series solution method, similar to the earlier procedure, derives the solution as

$$W(z)=k(1-z^2)^{1/2}. \tag{53}$$

Substituting Eq. (53) in Eq. (52), we get

$$\tfrac{8}{9}\gamma^2(5z^2-3)-P+\tfrac{10}{7}A_4k^3(1-z^2)=0. \tag{54}$$

Equation (54) determines the unknowns k and a_0. Finally, the solution of Eq. (51) is given by

$$F(\eta)=\pm\left(\frac{7}{4}\frac{P}{A_4}\right)^{1/3}\operatorname{sech}^{2/3}(\eta), \tag{55}$$

and reduces in the original coordinates as

$$\phi(x,y,z,t)=\left[\frac{7}{30}\frac{A_3}{A_4}\pm\left(\frac{7}{4}\frac{P}{A_4}\right)^{1/3}\times\operatorname{sech}^{2/3}\left(\frac{lx+my+nz-Vt}{\delta a}\right)\right]^2; \tag{56}$$

where the width

$$\delta a=\left(\frac{9P}{16}\right)^{-1/2}.$$

Equation (48) and Eq. (49) for some other modes could be studied by transforming the equations as

$$\left(\frac{d\Phi}{d\eta}\right)^2=a_1\Phi^2(\phi_0-\Phi)^3, \tag{57}$$

where $a_1=A_4/7$, $\phi_0=\frac{7}{18}(A_3/A_4)$, and $A_2=\frac{35}{108}(A_3^2/A_4)$, and $A_1A_4=\frac{7}{270}A_2A_3$ are used. Equation (57) can be solved for the soliton profile, and the solution $\phi_S(\eta)$ can be obtained only as an implicit function of η in the following way:

$$\phi_S(\eta)=\phi_0^2\operatorname{sech}^4\left[\left(\frac{\phi_0}{\phi_0-\sqrt{\phi_S(\eta)}}\right)^{1/2}\pm\frac{1}{2}\sqrt{a_1\phi_0^3}(\eta-\eta_0)-C_1\right], \tag{58}$$

where $C_1=(\phi_0/(\phi_0-\sqrt{\phi_m}))^{1/2}-\operatorname{sech}^{-1}(\sqrt{\phi_m}/\phi_0)^{1/2}$, and ϕ_m is the optimal amplitude of the acoustic mode. Note that $\phi_s(\eta)$ occurs on both left and right hand sides of Eq. (58). The solution [Eq. (58)] gives a profile of a spiky solitary wave defined in the region $0<\phi(\eta)<\sqrt{\phi_0}$. While for the other region defined as $\phi<0$, the soliton solution can be obtained in a similar manner, and is given by

$$\phi_E(\eta)=\phi_0^2\,\mathrm{cosech}^4\left[\left(\frac{\phi_0}{\phi_0-\sqrt{\phi_E(\eta)}}\right)^{1/2}\pm\frac{1}{2}\sqrt{a_1\phi_0^3}(\eta-\eta_0)-C_2\right], \quad (59)$$

where $C_2=(\phi_0/(\phi_0-\sqrt{\phi_m}))^{1/2}-\mathrm{cosech}^{-1}(\sqrt{\phi_m}/\phi_0)^{1/2}$, and this is to be recognized as the explosive solitary wave in the plasma-acoustic dynamics. Again, from the same equation, it is possible to get the double layer solution following the usual procedure, given by Das and Sen.[22] The double layer solution is of the form

$$\Phi=\phi_0\tanh^2(\kappa\eta), \quad (60)$$

which, when expressed in the original coordinate system, looks like

$$\phi_D(\eta)=\phi_0^2\tanh^4[\pm\tfrac{1}{2}\sqrt{a_1\phi_0^3}\phi_D^{1/2}(\eta)\times(\sqrt{\phi_0}-\sqrt{\phi_D(\eta)})\eta]. \quad (61)$$

Thus, one can proceed, taking the nonlinear term to any order in ϕ, and could derive different natures of the solitary waves under different approximations. Here it may be mentioned that during the ordering in ϕ, especially in the case of higher order nonlinearity, some other effects such as Landau damping, the collisional effect, and viscosity may play vital roles as well, but since we are presently interested in finding only the ordering effect of ϕ in isolation on the existence and the behaviors of the solitary acoustic waves in the plasma under consideration, we did not take into account other effects, as mentioned above.

IV. CONCLUSION

The exact form of the Sagdeev pseudopotential is derived for a multicomponent plasma consisting of negative ions, wherein a fraction of electrons is trapped in the potential well. A simplified wave solution technique, known as the tanh-method, is applied to find the formation and characteristic behavior of the soliton dynamics in plasma. It is found that the rarefactive solitons derived from the pseudopotential are the same as those obtained earlier by the argumentation of the K–P equation using the reductive perturbation technique. The tanh-method transforms the equation of motion obtained for ϕ, the electric potential, to a Fuchsian-like ordinary differential equation, and, finally, the Frobenius series solution is employed to find the different soliton solutions, like compressive and rarefactive solitary waves and double layers, etc.

In the case of higher order nonlinearity, both explosive solitary waves, where the energy in the soliton is conserved, and the collapse of the soliton, where the energy is not conserved in the wave profile, were found. Moreover, as the exact Sagdeev's potential is obtained, one can expand it up to any order in ϕ. To obtain solitary waves with arbitrary amplitude, without any approximations, one has to have recourse to numerical analysis to solve Eq. (15), taking into account Eq. (17) and Eq. (19).

ACKNOWLEDGMENTS

Dr. G. C. Das and Sri J. Sarma thank the Director, Indian Statistical Institute (ISI) Calcutta and the scientists and workers in Physics and Applied Mathematics Unit for warm hospitality during their stay in ISI, Calcutta. The authors are also grateful to the referee for helpful comments and suggestions.

RR is grateful to the Department of Science and Technology (DST) Government of India, for financial help.

[1] H. Washimi and T. Taniuti, Phys. Rev. Lett. **17**, 996 (1966).
[2] R. Z. Sagdeev, Plasma Phys. **4**, 23 (1966).
[3] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Phys. Rev. Lett. **50**, 1262 (1973).
[4] G. C. Das, C. B. Dwivedi, M. Talukdar, and J. Sarma, Phys. Plasmas 4, 4236 (1997).
[5] K. E. Lonngren, Plasma Phys. Controlled Fusion **25**, 943 (1983).
[6] G. C. Das, IEEE Plasma Sciences **3**, 68 (1975).
[7] G. C. Das and S. G. Tagare, Plasma Phys. **17**, 1025 (1975).
[8] S. Torven, Phys. Rev. Lett. **47**, 1053 (1981); S. Torven, Phys. Scr. **33**, 262 (1986).
[9] B. B. Kadomtsev and V. I. Petviashvili, Sov. Phys. Dokl. **15**, 539 (1970).
[10] M. A. Raadu, Phys. Rep. **178**, 25 (1989).
[11] L. Davis, R. Lust, and A. Schluter, Z. Naturforsch. A **13a**, 916 (1958).
[12] N. N. Ianenko, Dokl. Akad. Nauk SSSR **109**, 44 (1956).
[13] Y. T. Chiu, Phys. Fluids **13**, 2950 (1970).
[14] G. C. Das, Acta Math. Acad. Sci. Hung. **34**, 279 (1979).
[15] G. C. Das, J. Sarma, and C. Uberoi, Phys. Plasmas **4**, 2095 (1997).
[16] B. Tian and Y. T. Gao, Comput. Phys. Commun. **95**, 139 (1996).
[17] W. D. Jones, A. Lee, S. M. Gleeman, and H. J. Doucet, Phys. Rev. Lett. **35**, 1349 (1975).
[18] Y. Nishida and T. Nagasawa, Phys. Fluids **29**, 345 (1986).
[19] H. Schamel, J. Plasma Phys. **7**, 1 (1972).
[20] H. Schamel, J. Plasma Phys. **9**, 377 (1973).
[21] G. C. Das and K. M. Sen, Contrib. Plasma Phys. **31**, 13 (1991).
[22] G. C. Das and K. M. Sen, Planet. Space Sci. **42**, 41 (1994).
[23] G. C. Das and K. M. Sen, Contrib. Plasma Phys. **31**, 647 (1991).
[24] G. C. Das and J. Sarma, Phys. Plasmas **4**, 4236 (1997).
[25] G. C. Das, J. Sarma, and M. Talukdar, Phys. Plasmas **5**, 63 (1998).

The geometry and symmetries of magnetohydrodynamic turbulence: Anomalies of spatial periodicity

David C. Montgomery and Jason W. Bates[a)]
Department of Physics and Astronomy, Dartmouth College, Hanover, New Hampshire 03755

(Received 25 January 1999; accepted 8 April 1999)

It has become common to formulate theories and computations of magnetohydrodynamic turbulent effects in rectangular periodic boundary conditions, proceeding by analogy with what is seen as a useful framework for Navier–Stokes fluid turbulence. It is shown here that because of certain features of Maxwell's equations for electrodynamics, it is inconsistent to invoke three-dimensional, rectangular, periodic boundary conditions and symmetry at the same time that the displacement current is neglected. The difficulty does not arise in the two-dimensional case. In three dimensions, the difficulty can be remedied by a reformulation in cylindrical geometry, imposing symmetry in the azimuthal and axial directions, but not in the radial one; a geometry that is closer to laboratory possibilities than the wholly three-dimensional periodic assumption. The reformulation seems particularly necessary in cases with a net flux of magnetic field and/or electric currents through the system. These cases no longer seem discontinuous from those without net magnetic fluxes or currents. The price paid is a loss of some possibilities for dimensional analysis and identification of similarity variables. [S1070-664X(99)02907-9]

I. INTRODUCTION

This article offers a reconsideration of the mathematical framework in which magnetohydrodynamic (MHD) turbulence is approached. Heretofore, the most widely-used setting has been the now classical "homogeneous turbulence" formulation of Kolmogorov,[1–3] Batchelor,[4] and others[5,6] for Navier–Stokes (NS) fluids. It has seemed particularly convenient to assume rectangular periodic boundary conditions in space, wherein the system is imagined as repeating itself an infinite number of times in all directions.[7–13] For some theoretical purposes, the volume is then allowed to become infinite; for others (and for computational ones) it remains "large," but finite. This has several advantages. The Fourier transformation immediately converts spatial derivatives into multiplications by wave number components and also provides a simple way of classifying excitations ("eddies") by wave number in order of their spatial scales. Dimensional analyses (in the case of isotropy) become transparent. Finally, recurrently troubling problems[14] associated with enforcing realistic mechanical boundary conditions at material rigid walls are apparently avoided.

It will be argued here that certain features of electromagnetic theory,[15] ones which have no analogs for NS fluids, render spatially periodic boundary conditions for some MHD cases less than satisfactory, and can lead to basic inconsistencies. There seems to be no problem associated with what has come to be called two-dimensional (2D) MHD turbulence theory.[16–22] There also seems to be no problem with approaching three-dimensional (3D) MHD turbulence by invoking spatial periodicity in two out of three dimensions in cylindrical geometry. The difficulty arises in attempting periodic symmetry on all three (Cartesian) rectangular spatial coordinates. It becomes conspicuous when a mean dc magnetic field is present.[23] The difficulties may be resolved in a natural way by considering the case of an infinite straight cylinder with material walls at a finite radius, a situation closer by far to situations in which MHD turbulence may appear in the laboratory; the resolution may seem to be more than a fortunate coincidence.

In Sec. II, an example introduces the essential difficulties and shows why the complete neglect of the displacement current leads to difficulties with 3D MHD turbulence when subjected to a 3D periodic symmetry requirement. In Sec. III, it is shown why the difficulty is unimportant for 2D MHD. In Sec. IV, the 3D case is again considered and it is shown how to by-pass the difficulty by going to cylindrical geometry and giving up any periodicity in the radial direction. Section V summarizes the results and speculates on a possible generalization of MHD that includes the displacement current.

The following assumptions will be made throughout: (1) When the full set of Maxwell's equations[15] can be shown to disagree significantly with an approximation to them, they must take precedence over the approximation. (2) In any electromagnetic application, an "infinite" system must be at least imaginable and visualizable as a limit of some bounded system, as in the useful elementary fictions of an "infinite parallel-plate capacitor" or an "infinitely-long, current-carrying, straight wire." (3) The validity will be taken for granted of several other standard approximations of incompressible MHD which are not being scrutinized here, such as local charge neutrality, uniform mass density and transport coefficients, nonrelativistic mechanical responses, Ohm's

---

a)Present address: Hydrodynamic Methods Group (X-HM), Applied Theoretical and Computational Physics Division, MS D413, Los Alamos National Laboratory, Los Alamos, New Mexico 87545.