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Concave fuzzy set: a concept complementary to the convex fuzzy set

B.B. Chaudhuri

Electronics and Communication Sciences Unit, Indian Statistical Institute, 203 B.T. Road, Calcutta 700035, India

Abstract

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The concept of concave fuzzy set in Euclidean space is developed in this note. Useful properties of the concave fuzzy set are established. Some associated concepts like concave hull, concave containment and concavity tree are also defined and their computational approaches are described. The concepts can be used to decompose or approximate a fuzzy set. It may also be useful in the development of fuzzy geometry of space.

Keywords. Fuzzy set, fuzzy concavity, fuzzy geometry, image processing, shape analysis.

1. Introduction

In this note we develop the concave fuzzy set, a concept complementary to the convex fuzzy set. The idea was introduced during our study of fuzzy geometrical shapes [1] where a formal definition of the concave fuzzy set was proposed. In the crisp case, no formal definition of concave set is known to us. The complement of a convex set can be viewed as a hole with convex border. When defined on a support of $\mathbb{R}^2$ plane, the concave fuzzy set may be viewed as a concave surface that is the fuzzy generalization of a hole.

Some interesting properties of concave fuzzy set are studied in this note. Also, the concave hull and concave containment of an arbitrary fuzzy set are defined. It is explained that concave containment is the natural counter-part of convex hull. An approach to compute concave containment of a bounded piecewise constant fuzzy set is described. Extension of the work to ortho-concavity is briefly studied.

Throughout the work it is assumed that the support of the fuzzy set is the Euclidean space. Most definitions and propositions are valid for support space of any dimensionality. Sometimes, however $\mathbb{R}^2$ has been considered although the propositions defined on $\mathbb{R}^2$ can be extended to higher dimensions. The convenience of considering $\mathbb{R}^2$ plane is that it is easy to visualize the fuzzy set as a hilly terrain, and we can draw a direct analogy between fuzzy set and two-dimensional image function.

2. Concave fuzzy set

Definition 1. A fuzzy set u in $\mathbb{R}^n$ is a *concave fuzzy set* if for all $p, q \in \mathbb{R}^n$ and all r on the line segment $\overline{pq}$, we have

$$u(r) \leqslant \max[u(p), u(q)]. \tag{1}$$

Definition 2. An *ortho-concave fuzzy set* is a fuzzy set for which relation (1) is true whenever p and q lie on a line parallel to the co-ordinate axes.

Clearly, a concave fuzzy set is ortho-concave but the converse is not generally true. If a fuzzy set u is bounded, $u=0$ outside the bounded region, say S. Note that S must be convex to define a concave fuzzy set on this region.

Proposition 1. *If u is concave, then its complement $\bar{u}=1-u$ is convex and vice versa.*

Proof. If u is concave, then for any two points p and q and any point r on the line segment $\overline{pq}$

$$u(r)\leqslant \max[u(p),u(q)],$$

i.e.,

$$\bar{u}(r)\geqslant 1-\max[1-\bar{u}(p),1-\bar{u}(q)].$$

If $1-\bar{u}(p)\geqslant 1-\bar{u}(q)$, then $\bar{u}(r)\geqslant\bar{u}(p)$, i.e.,

$$\text{if } \bar{u}(q)\geqslant\bar{u}(p), \quad \text{then } \bar{u}(r)\geqslant\bar{u}(p). \tag{a}$$

Similarly, if $1-\bar{u}(q)\geqslant 1-\bar{u}(p)$, then $\bar{u}(r)\geqslant\bar{u}(q)$, i.e.,

$$\text{if } \bar{u}(p)\geqslant\bar{u}(q), \quad \text{then } \bar{u}(r)\geqslant\bar{u}(q). \tag{b}$$

Relations (a) and (b) can be combined into

$$\bar{u}(r)\geqslant\min[\bar{u}(p),\bar{u}(q)]$$

which is the relation to state that $\bar{u}$ is a convex fuzzy set. The converse part of Proposition 1 can be proved in a similar manner. □

Note that a fuzzy set can be both concave and convex. A simple example is the support space mapped by a hyperplane where the height of the hyperplane at a point of the support space denotes the fuzzy membership of the point.

Proposition 2. *The union of two concave fuzzy sets is a concave fuzzy set.*

Proof. Let u and v be two concave fuzzy sets. Let $w=u\cup v$ so that $w(p)=\max[u(p),v(p)]$. We have to show that w is a concave fuzzy set.

Consider two points p and q, and a point r on the line segment $\overline{pq}$. Now, by definition

$$w(p)=\max[u(p),v(p)],$$

$$w(q)=\max[u(q),v(q)],$$

$$w(r)=\max[u(r),v(r)]. \tag{c}$$

Note that

$$\begin{aligned}\max[w(p),w(q)] &= \max[\max[u(p),v(p)],\max[u(q),v(q)]]\\ &= \max[u(p),v(p),u(q),v(q)]. \end{aligned}\tag{d}$$

In relation (c) let $u(r)\geqslant v(r)$ so that $w(r)=u(r)$. Since u is concave

$$\begin{aligned}u(r)&\leqslant\max[u(p),u(q)]\\ &\leqslant\max[u(p),v(p),u(q),v(q)],\end{aligned}$$

i.e.,

$$u(r)=w(r)\leqslant\max[w(p),w(q)]$$

by relation (d). Thus, if $u(r)\geqslant v(r)$, the proposition is true. It can be shown in a similar way that the proposition is true if $v(r)\geqslant u(r)$. □

It is easy to see that Proposition 2 holds for ortho-concave fuzzy sets as well.

Definition 3. For any line l and any point $p\in l$, let l_p be the line perpendicular to l at p. The *inf projection* u_l of a fuzzy set u in $\mathbb{R}^2$ is the mapping of each point $p\in l$ into $\inf\{u(r),r\in l_p\}$.

The inf projection is a complementary definition to sup projection [2].

Proposition 3. *If u is concave, then u_l is also concave.*

Proof. Let p,q,r be three points of l so that r lies on the line segment $\overline{pq}$. Given any $\varepsilon>0$, let p' and q' be points on lines l_p and l_q so that

$$u_l(p)>u(p')-\varepsilon \quad\text{and}\quad u_l(q)>u(q')-\varepsilon.$$

Let r' be the intersection of line segment $\overline{p'q'}$ with l_r. Since u is concave and $r'\in\overline{p'q'}$, we have

$$\begin{aligned}u(r')&\leqslant\max[u(p'),u(q')]\\ &<\max[u_l(p)+\varepsilon,u_l(q)+\varepsilon]\\ &=\max[u_l(p),u_l(q)]+\varepsilon.\end{aligned}$$

But by definition of inf projection $u(r')\geqslant u_l(r)$. Hence

$$u_l(r)\leqslant\max[u_l(p),u_l(q)]+\varepsilon.$$

Since ε is an arbitrary positive quantity, we have

$$u_l(r) \leqslant \max[u_l(p), u_l(q)].$$

So u_l is concave. □

The converse of the proposition is not in general true, i.e., if the inf projections of a fuzzy set u are concave, u is not necessarily concave.

The crisp analogy of concave fuzzy set is a set complement to the convex set which may be called a *hole with convex border*.

Definition 4. The *t-level set* of a fuzzy set u is the nonfuzzy set u_t given by

$$u_t = \{p; u(p) \geqslant t\}. \tag{2a}$$

Sometimes, u_t is defined as

$$u_t = \{p; u(p) > t\}. \tag{2b}$$

To distinguish the two types of definition, let relation (2a) be called *closed t-level set* and denoted as u_t^1 while relation (2b) is called *open t-level set* and denoted as u_t^2.

Proposition 4. *All (closed or open) t-level sets of a convex fuzzy set are nonfuzzy convex* [2].

An identical proposition for the concave fuzzy set is as follows.

Proposition 5. *If u is a concave fuzzy set, then all its (closed or open) level sets are holes with convex border.*

Proof. The complement of a hole with convex border is a convex (non-fuzzy) set. Hence, we have to show that the complements of level sets of u are all convex. We have, for the closed t-level set

$$(u_t^1) = \{p; u(p) \geqslant t\},$$

and for its complement

$$\begin{aligned}(\overline{u_t^1}) &= \{p; u(p) < t\} \\ &= \{p; 1 - \bar{u}(p) < t\} \\ &= \{p; \bar{u}(p) > 1 - t\} = \bar{u}_{1-t}^2\end{aligned}$$

which is the open $(1-t)$-level set of $\bar{u}$. By Proposition 1, $\bar{u}$ is convex if u is concave. Now, by Proposition 4, $\bar{u}_{1-t}^2$ is convex. Thus, the proposition is true if closed t-level sets are considered. Similarly, one can start with u_t^2 and come to the same conclusion. □

Definition 5. The *cross-section* of u by a line l is the restriction of u to l.

Definition 6. A fuzzy set u is *concavely star-shaped* from p if its cross-sections by lines through p are all concave.

Now, the following propositions are rather obvious.

Proposition 6. *u is concave iff all its cross-sections are concave.*

Proposition 7. *If u is concave, then it is concavely star-shaped from all points.*

3. Concave hull, concave containment and concavity tree

Definition 7. The *concave hull* u_{Hc} of a fuzzy set u is the smallest concave fuzzy set containing u.

The ortho-concave hull can be defined in an identical manner.

However, the computation of the concave hull from this definition is very complicated. Also, this definition may not lead to a unique hull. An alternative definition is proposed below.

Definition 8. The *concave containment* u_{c} of a fuzzy set is the largest concave fuzzy set contained in u. In other words

$$u_{\mathrm{c}} = \sup\{v \subseteq u;\ v \text{ is concave}\}.$$

The *ortho-concave containment* can be defined in a similar way. This definition does not satisfy the property of a hull because u_{c} does not contain u and hence we call it concave containment. However, it is the concave analogue of convex hull and is unique. Also, it can be computed conveniently from the level sets of u by a method similar to the computation of convex hull proposed in [3].

Let us assume that the fuzzy set u is bounded and has a finite number, say n, of *distinct membership values* (with respect to its level sets). Let us consider the closed type of level sets and denote it simply as u_t. Now, if $u_{t_1} \neq u_{t_2}$ and if both u_{t_1} and u_{t_2} are non-null, we say that t_1 and t_2 are two distinct membership values in u. These membership values can be ordered so that $t_1 > t_2 > \cdots > t_n$. For a level set u_{t_i}, find its complement $\bar{u}_{t_i}$ and its nonfuzzy convex hull $(\bar{u}_{t_i})_H$. Define the fuzzy set

$$u_{ci}(p) = \begin{cases} 0, & \text{if } p \in (\bar{u}_{t_i})_H, \\ t_i, & \text{otherwise.} \end{cases}$$

Then

$$u_c = \bigcup_{i=1}^{n} u_{ci}.$$

Note that u_c is unique since u_{ci} defined on the convex hull of $\bar{u}_{ci}$ is unique. It is also easy to see that u_c is contained in u. Finally, u_c can be computed if $(\bar{u}_{t_i})_H$ can be computed. This is possible if $\bar{u}_{t_i}$ can be represented by a finite number of points, say m. Then in $\mathbb{R}^2$, $O(m \log m)$ algorithms exist for convex hull computation. In this context, we can define convex containment as follows.

Definition 9. The *convex containment* u_{cc} of a fuzzy set is the largest convex set contained in u.

While the convex hull of a set is unique, the convex containment may not be so.

Representation of an object (i.e., non-fuzzy set in space) as a tree of convex hull and convex deficiency is available in literature [4]. In this representation, the root node of the tree denotes the convex hull A_H of the object A and its branches denote the components of the convex deficiency $A_H - A$. Each component can again be represented as its convex hull and the convex deficiency. The process is continued till no deficiency exists.

Fuzzy set extension of such a representation is possible both in terms of convex hull and concave containment. For objects in a gray-tone image mapped to fuzzy sets, this type of representation can be used. Possible applications of the representation are pattern recognition and data compression.

For example, the difference $u - u_c$ can be expressed as a few fuzzy sets (components) with bounded support. Each of them can represent the first level branch of the tree. Also, for each of the components we can find the concave containment and the difference, if any. The process is continued till at some level each component is concave. The tree may be called *fuzzy concavity tree*.

4. Discussion

The concept of fuzzy concave set in space is developed in this paper. One can define concave fuzzy objects as the complements of convex fuzzy objects. Thus fuzzy circle, ellipse and polygon have concave counterparts in circular, elliptic and polygonal holes. A concave fuzzy set is not connected by the usual definition of fuzzy connectivity, but it is complement connected.

The concept of concave set can be used for decomposition or partition of a fuzzy set. Also, to approximate a fuzzy set u by a smoother version, this concept can be applied. If u_H and u_c denote the convex hull and concave containment, respectively, then a measure on $u_H - u$ can be larger than that on $u - u_c$, so that u_c is accepted as a better approximation. It is hoped that researchers on fuzzy sets will find the concave fuzzy set a useful concept in other applications.

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