A POSSIBLE USE OF CERTAIN MEASUREMENTS IN CONTROL BY GAUGING

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1. Of Shewhart's quality control charts, the p-chart is specially convenient on account of the simplicity with which the requisite observations can be made. Depending on whether the tolerances specify a superior limit, an inferior limit, or both, to the measure of the characteristic concerned, a single "go" gauge or "no go" gauge or the two together will serve the purpose. In the sequel, we shall be concerned with the last case; it will be comparatively simple to work out our suggestions with regard to the first two cases.

The population of measures of the characteristic will be assumed to be normal. If the sample sizes are constant, say n, points representing the number of "defectives" in samples of n falling beyond the control limits at $np \pm 3\sqrt{npq}$ (p being the probability of a defective under controlled conditions) indicate lack of control.

One drawback of the ordinary p-chart is that an out-of-control point does not indicate whether the population mean or the population standard deviation (s.d.) has altered. Usually, a change in the mean points to the need of resetting the machine, while a change in the s.d. points to a change of process, which may mean a change of the operator, or the material, or the condition of temperature etc. or a change of machine characteristics. It is therefore necessary to distinguish between the two types of changes.

Among attempts directed to overcome this drawback is an elaborate paper by Stevens (1948), where it was suggested that one "go" gauge and another "no go" gauge (not necessarily adjusted to the tolerance limits of manufacture, but suited solely to purposes of control) should be used. If c and a be the numbers of articles which fail the two gauges respectively in a sample of n, then Stevens recommends $c-a$ as an 'indicator' of change of the population mean and $c+a$ that of a change of the population s.d.

Here it is desired to examine how measurements, instead of mere enumeration of the two categories of "defectives", defined by the "go" gauge and "no go" gauges respectively, could be used for the purpose of distinguishing between the two types of changes.

2. In practice, the p-chart is always preceded by $\bar{X}$ and R charts for a duration until controlled condition is attained. We shall therefore assume that the mean μ and the s.d. σ of the (normal) population have been satisfactorily estimated. Without loss of generality, we may take the mean as zero.

Simplifying the problem further, pending a more complete treatment in future, we shall suppose that the gauges are adjusted to the two quartiles. These quartiles divide the

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population into three sections which we designate as A, B and C. The notations used are defined as follows:

a, b, c = numbers of observations in a sample of n falling in the sections A, B, C respectively,

x_1 = semi-inter-quartile range,

$z_C = x - x_1$, for $x > x_1$,

$z_A = x + x_1$, for $x < -x_1$,

$$\mu'_{1C} = \int_{x_1}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx \Big/ \int_{x_1}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx = \frac{4\sigma}{\sqrt{2\pi}} e^{-\frac{x_1^2}{2\sigma^2}} = \sigma D \text{ say,}$$

$$\mu'_{1A} = \int_{-\infty}^{-x_1} x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx \Big/ \int_{-\infty}^{-x_1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx = -\sigma D,$$

$$\mu'_{2A} = \mu'_{2C} = \int_{x_1}^{\infty} x^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx \Big/ \int_{x_1}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx$$

= the value when the integrals are taken between the limits $-\infty$ and $-x_1$

$$= 2\sigma^2 \left[1 - \frac{2}{\sqrt{\pi}} \int_0^{x_1^2/2\sigma^2} t^{\frac{1}{2}} e^{-t} dt \right] = \sigma^2 F \text{ say.}$$

The value of F can be found by a reference to the Incomplete Γ-Function Tables.

3. Let us first consider changes in the population mean. The essential point in Stevens' indicator $c-a$ is that each observation is given the same weight. We shall construct a similar "indicator" by giving different weights, namely $|z_C|$ or $|z_A|$ according as the observation falls in section C or A. This new indicator X corresponding to Stevens' $c-a$, is then given by

$$X = \sum_C |z_C| - \sum_A |z_A| = \sum_C x + \sum_A x - (c-a)x_1$$

since z_A's are negative and z_C's positive.

It is likely that X, based on actual measures, will yield better results than Stevens' indicator $c-a$, but a comparison of the 'powers' of the two will not be undertaken.

In calculating the mean and variance of X we utilise the fact that the random variables x are independent of the random variables c and a, since a measure x, falling in the section C say, does not depend on the total number c of them falling in C. Further, x's are independent of one another. We therefore have

$$\begin{aligned} E(X) &= E(c)\mu'_{1C} + E(a)\mu'_{1A} - x_1 E(c-a) \\ &= (\mu'_{1C} - x_1)E[(c-a)]. \end{aligned}$$

If the probabilities of x falling in the sections A, B, C be p, q, r respectively $(p+q+r = 1)$,

$$E(c-a) = n(r-p), \text{ and } \mathrm{Var}\,(c-a)=n[(p+r)-(p-r)^2].$$

In our case $p = r = \frac{1}{4}$, and hence $E(X) = 0$.

The variance of X is given by

$$V(X) = \sigma_X^2 = E\Big[\sum_C x+\sum_A x-(c-a)x_1\Big]^2$$

$$= E\Big[\sum_C x^2+\sum_{\substack{C\\(i\neq j)}} x_i x_j+\sum_A x^2+\sum_{\substack{A\\(i\neq j)}} x_i x_j+(c-a)^2 x_1^2-$$

$$-2x_1(c-a)\sum_C x-2x_1(c-a)\sum_A x+2\sum_C x_C \sum_A x_A\Big]$$

$$= \frac{n}{2}\mu'_{2C}+\frac{n}{2}\Big(\mu'_{1C}-x_1\Big)^2-\frac{n}{2}\mu'^2_{1C}$$

$$= n\sigma^2 G$$

where $G = \frac{1}{2}\Big[F+\frac{x_1^2}{\sigma^2}-\frac{2x_1}{\sigma}D\Big]$, a constant independent of n and σ.

4. Similarly, corresponding to Stevens' indicator $c+a$ for the purpose of detecting changes in the population s.d., we may frame another indicator Y as follows :

$$Y = \sum_C |z_C|+\sum_A |z_A| = \sum_C x-\sum_A x-(c+a)x_1.$$

Y is thus the sum of the deviations from the mean. The mean deviation is a recognized measure for the spread of a population and Y differs from it in being the sum instead of the mean. Stevens' indicator $c+a$ is none other than the proportion defective of the ordinary chart where the defectives are marked off by the two gauge-limits. Looked from this point of view, an out-of-control point on a $(c+a)$-chart may not be so indicative of a change of population s.d. as Y is.

Proceeding exactly as before

$$E(Y) = \frac{n}{2}(\mu'_{1C}-x_1) = \frac{n\sigma}{2}\Big(D-\frac{x_1}{\sigma}\Big) = n\sigma H \text{ say.}$$

$$E(Y^2) = E(c)\mu'_{2C}+E\{c(c-1)\}\mu'^2_{1C}+E(a)\mu'_{2A}$$

$$+E\{a(a-1)\}\mu'^2_{1A}+x_1^2E(c+a)^2-2x_1\mu'_{1C}E(c+a)^2+2\mu'^2_{1C}E(ca).$$

$$= \frac{n}{2}\mu'_{2C}+(\mu'_{1C}-x_1)^2\Big(\frac{n}{4}+\frac{n^2}{4}\Big)-\frac{n}{2}\mu'^2_{1C}.$$

Therefore, $\mathrm{Var}(Y)= \sigma_Y^2 = \frac{n}{2}\Big[\sigma^2F-\sigma^2D^2+\frac{1}{2}(\sigma D-x_1)^2\Big] = n\sigma^2K$ say.

5. It should be noted that the quantities D, F, G, H and K are all independent of n and σ, while σ_X, $E(Y)$ and σ_Y have variable factors depending on n and σ. Suitable tables for σ_X, $E(Y)$ and σ_Y can therefore be prepared for selected values of n and σ.

From the point of view of practical applications, it is unnecessary to maintain a chart for X or Y. We may prepare tables of values of $3\sigma_X$ and $3\sigma_Y$ for selected values of n and σ, and calculate $|X|$ and $|Y-E(Y)|$ *only when a point falls outside the control limits of the ordinary p-chart* i.e. the $(c+a)$-chart, which is kept as a matter of routine. The value of p for such a chart corresponding to the adjustment of the gauges to quartiles is obviously ½ and thus the control limits are $\frac{n}{2} \pm \frac{3}{2}\sqrt{n}$.

If $|X|$ exceeds the limit $3\sigma_X$ given in the tables, we conclude that the population mean has shifted towards the positive or negative direction according as X is positive or negative. The computation of Y may be avoided altogether if we can assume that whenever X is significant, *only* the population mean has changed, and when X is not significant the population s.d. has changed.

The author hopes to undertake a more complete study of the problem in future

REFERENCE

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