TABLES OF TWO-SIDED 5% AND 1% CONTROL LIMITS FOR INDIVIDUAL OBSERVATIONS OF THE *r-th* ORDER

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L. I. Braginsky of the Ordzyonikize Institute of Economics and Engineering, published a booklet in 1951 in which he introduced his own method viz. the 'method of individual value'. His heuristic method of approach is interesting but it is accurate enough only when the size of the sample is large. The tables given in this paper are valid for small samples. Let the rth order statistic in a sample of size n be x_r and the distribution function of the original variate be $F(x)$. Then the lower limit L and the upper limit U of $F(x_r)$ are given by

$$\int_0^L F^{r-1}(1-F)^{n-r}dF/B(r, n-r+1) = \alpha/2 = \sum_{x=r}^{n}\binom{n}{x} L^x(1-L)^{n-x} \quad \dots \quad (1)$$

and

$$\int_U^1 F^{r-1}(1-F)^{n-r}dF/B(r, n-r+1) = \alpha/2 = 1-\sum_{x=r}^{n}\binom{n}{x} U^x(1-U)^{n-x} \quad \dots \quad (2)$$

where $1-\alpha$ is the confidence coefficient.

The upper and lower limits of x_r depend naturally upon its original distribution function $F(x)$ but U and L are independent of $F(x)$, provided that $F(x)$ is continuous. K_L and K_U in Tables 1 and 2 are the upper and lower limits respectively of x_r in the case of the standardized normal distribution $N(0, 1)$.

We assume that $\alpha = 0.05$ and 0.01. If $\alpha = 0.00270$ is preferred, we should refer to Koller's table in which L is given in a nomogram. We note that L coincides with the lower limit of the fraction r/n with confidence coefficient $1-\alpha/2$ but U does not coincide with the upper confidence limit of the fraction r/n with confidence coefficient $1-\alpha/2$. However, comparing (1) with (2) we can see that the lower limit of x_r, say L_r, is equal to $1-U_{n-r+1}$, where U_{n-r+1} is the upper limit of x_{n-r+1}, α being the same. The mean of $F(x_r)$ is given by $r/(n+1)$.

TABLE 1. CONTROL LIMITS FOR INDIVIDUAL OBSERVATIONS, $\alpha = 0.05$

		any continuous distribution function		normal variate $N(0, 1)$	
n	r	L	U	K_L	K_U
3	1	0.0084038	0.70760	−2.391	0.5464
	2	0.094299	0.905701	−1.315	1.315
	3	0.29240	0.9915962	−0.5464	2.391
4	1	0.0063095	0.60236	−2.494	0.2595
	2	0.067586	0.80588	−1.494	0.8628
	3	0.19412	0.932414	−0.8628	1.494
	4	0.39764	0.9936905	−0.2595	2.494
5	1	0.0050508	0.52182	−2.572	0.05472
	2	0.052745	0.71642	−1.610	0.5722
	3	0.14663	0.85337	−1.031	1.0702
	4	0.28358	0.947255	−0.5722	1.610
	5	0.47818	0.9949492	−0.05472	2.572
6	1	0.0042107	0.45926	−2.635	−0.1023
	2	0.043272	0.64123	−1.714	0.3617
	3	0.11812	0.77722	−1.184	0.7628
	4	0.22278	0.88188	−0.7628	1.184
	5	0.35877	0.956728	−0.3617	1.714
	6	0.54074	0.9957893	+0.1023	2.635
8	1	0.0031587	0.36942	−2.731	−0.3334
	2	0.031851	0.52651	−1.854	0.06650
	3	0.085233	0.65086	−1.371	0.3877
	4	0.15701	0.75514	−1.007	0.6907
	5	0.24486	0.84299	−0.6907	1.007
	6	0.34914	0.914767	−0.3877	1.371
	7	0.47340	0.968149	−0.06650	1.854
	8	0.63058	0.9968413	+0.3334	2.731
10	1	0.0025286	0.30850	−2.803	−0.5001
	2	0.025198	0.44502	−1.957	−0.1383
	3	0.066731	0.55610	−1.501	0.1411
	4	0.12156	0.65245	−1.167	0.3919
	5	0.18709	0.73762	−0.8887	0.6360
	6	0.26238	0.81291	−0.6360	0.8887
	7	0.34755	0.87844	−0.3919	1.167
	8	0.44390	0.933269	−0.1411	1.501
	9	0.55498	0.974802	+0.1383	1.957
	10	0.69150	0.9974714	+0.5001	2.803

TABLE 2. CONTROL LIMITS FOR INDIVIDUAL OBSERVATIONS, $\alpha = 0.01$

		any continuous distribution function		normal variate $N(0, 1)$	
n	r	L	U	K_L	K_U
3	1	0.0016695	0.82900	−2.929	0.9502
	2	0.041400	0.95860	−1.735	1.735
	3	0.17100	0.9983305	−0.9502	2.929
4	1	0.0012524	0.73409	−3.023	0.6252
	2	0.029443	0.88912	−1.889	1.222
	3	0.11088	0.970553	−1.222	1.889
	4	0.26591	0.9987476	−0.6252	3.023
5	1	0.0010020	0.65343	−3.090	0.3946
	2	0.022881	0.81490	−1.998	0.8061
	3	0.082829	0.917171	−1.386	1.386
	4	0.18510	0.977119	−0.8061	1.998
	5	0.34657	0.9989980	−0.3946	3.090
6	1	0.00083507	0.58648	−3.144	0.2185
	2	0.018721	0.74601	−2.081	0.6620
	3	0.066279	0.85640	−1.504	1.064
	4	0.14360	0.933721	−1.064	1.504
	5	0.25399	0.981279	−0.6620	2.081
	6	0.41352	0.99916493	−0.2185	3.144
8	1	0.00062617	0.48433	−3.226	−0.03932
	2	0.013733	0.63152	−2.205	0.3359
	3	0.047464	0.74217	−1.670	0.6501
	4	0.099867	0.83030	−1.282	0.9535
	5	0.16970	0.900133	−0.9535	1.282
	6	0.25783	0.952536	−0.6501	1.670
	7	0.36848	0.986267	−0.3359	2.205
	8	0.51567	0.99937383	+0.03932	3.226
10	1	0.00050113	0.41130	−3.290	−0.2242
	2	0.010845	0.54429	−2.296	0.1112
	3	0.037092	0.64820	−1.787	0.3805
	4	0.076765	0.73511	−1.427	0.6280
	5	0.12831	0.80908	−1.134	0.8745
	6	0.19092	0.87169	−0.8745	1.134
	7	0.26489	0.92323	−0.6280	1.427
	8	0.35180	0.962908	−0.3805	1.787
	9	0.45571	0.989155	−0.1112	2.296
	10	0.58870	0.99949887	+0.2242	3.290

NUMERICAL ILLUSTRATIONS

Case 1: *Normal population*

Let us suppose we want to find two limits K_U and K_L of x_r the *rth* order statistic for a sample of size n drawn from a normal population having zero mean and unit standard deviation such that Prob $(x \geqslant K_U) =$ Prob $(x \leqslant K_L) = \frac{\alpha}{2}$ when $\alpha = 0.05$;, $n = 6$ and $r = 4$.

In Table 1 for $\alpha = 0.05$, $n = 6$ and $r = 4$ we see at once that

$$U = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{K_U} e^{-\frac{1}{2}t^2} dt = 0.88188$$

$$L = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{K_L} e^{-\frac{1}{2}t^2} dt = 0.22278$$

which yield $K_U = 1.184$,

$K_L = -0.763$.

Case 2: *Exponential population*

From Table 1 we can also find out the limits K_U and K_L of x_r for a sample drawn from an exponential population.

For example let us take $\alpha = 0.05$ $n = 6$ and $r = 4$ as in the previous case.

Then $$U = \int_0^{K_U} e^{-t} dt = 1 - e^{-K_U} = 0.88188$$

and $$L = \int_0^{K_L} e^{-t} dt = 1 - e^{-K_L} = 0{\cdot}22278$$

which give $K_U = -\log_e 0.11812 = 2.136$,

$K_L = -\log_e 0.77722 = 0.252$.

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MODIFIED MEAN SQUARE SUCCESSIVE DIFFERENCE WITH AN EXACT DISTRIBUTION

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1. Introduction

Let x_i $(i = 1, 2, ..., n)$ denote a sample sequence from a normal population with a variable mean μ_i and a constant variance σ^2. If $\mu_i = \mu$, then the best estimator of the variance is of course $s^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2/(n-1)$. Very often however we come across situations in which μ_i may have a 'slow-moving' continuous trend. In such situations s^2 suffers from a heavy bias and hence alternative estimators of the dispersion which are comparatively free from bias have been considered. One such estimator viz.

$$\delta^2 = \sum_{i=1}^{n-1} (x_i - x_{i+1})^2/(n-1)$$

was proposed by Von Neumann, Kent, Bellinson, and Hart (1941) who have also given in their paper a brief historical review of this problem with reference to situations arising in ballistics and astronomy. Since the exact distribution of δ^2 could not be found, Von Neumann *et al* have given an approximation based on a Pearson Type VI curve fitted to the first three moments of δ^2.

We have *not* been able to obtain an exact distribution of δ^2 but as an alternative, in this paper, we

(a) propose the use of a modified form of the mean square successive difference with an exact distribution and whose efficiency as an estimate of σ^2 is very close to that of δ^2,

(b) indicate a method of obtaining upper and lower bounds for the probability integral of the Von Neumann statistic δ^2 itself, and

(c) apply the principle underlying (a) to derive exact tests corresponding to the F- and t- tests, based on the new estimator proposed by us.

2. The proposed estimate

Let $x_1, x_2, ..., x_n$ be a sample sequence from the normal population with mean μ and variance σ^2. Then the estimate of σ^2 proposed by Von Neumann *et al* (1941), viz. the mean square successive difference, is

$$\delta^2 = \sum_{i=1}^{n-1} (x_i - x_{i+1})^2/(n-1) = X_{(n)}/(n-1). \quad \text{... (1)}$$

Let n be even, $= 2m$; then the estimate of σ^2 which we propose, is

$$\delta'^2 = \frac{1}{2(m-1)}\left\{\sum_{i=1}^{m-1}(x_i-x_{i+1})^2+\sum_{i=1}^{m-1}(x_{m+i}-x_{m+i+1})^2\right\}$$

$$= \frac{1}{2(m-1)}(X_{1(m)}+X_{2(m)}). \qquad \dots \quad (2)$$

δ'^2 is in fact half the sum of two independent δ^2 statistics. This means that we have omitted the middle term $(x_m-x_{m+1})^2$ from δ^2 for $2m$ observations with a consequent loss of information. It can be shown however that this loss is not considerable from the comparison of the efficiencies of δ^2 and δ'^2 given below:

$$\left.\begin{array}{ll}\text{Efficiency of} & \delta^2 = \frac{2}{3}\left(1+\frac{\cdot 1}{3n-4}\right).\\ \text{Efficiency of} & \delta'^2 = \frac{2}{3}\left(1-\frac{n-4}{(3n-8)(n-1)}\right).\end{array}\right\} \qquad \dots \quad (3)$$

In the following, we shall take $\sigma^2 = 1$ for the sake of simplicity of expressions.

3. The distribution of δ'^2

Von Neumann (1941) has shown that the characteristic roots of the matrix A where

$$X_{(m)} = \sum_{i=1}^{m-1}(x_i-x_{i+1})^2 \equiv x'Ax \qquad \dots \quad (4)$$

are
$$\lambda_j = 2-2\cos\frac{j\pi}{m} = 4\sin^2\frac{j\pi}{2m}, (j = 0, 1, 2, \dots, m-1) . \qquad \dots \quad (5)$$

The characteristic function of $X_{(m)}$ is therefore

$$\prod_{j=1}^{m-1}(1-2i\lambda_j t)^{-\frac{1}{2}}. \qquad \dots \quad (6)$$

Since $X_{1(m)}$ and $X_{2(m)}$ are independent it follows that the characteristic function of $X_{1(m)}+X_{2(m)}$ is given by

$$\phi(t) = \prod_{j=1}^{m-1}(1-2i\lambda_j t)^{-1} \qquad \dots \quad (7)$$

where λ_j are given by (5). Now $\phi(t)$ can be thrown into partial fractions with the help of the identity

$$\frac{\sin m\theta}{\sin\theta} \equiv 2^{m-1}\cos^{m-1}\theta-\frac{m-2}{1}2^{m-3}\cos^{m-3}\theta+\frac{(m-3)(m-4)}{2!}2^{m-5}\cos^{m-5}\theta - \dots \quad (8)$$

noting that the roots of the r.h.s. of (8) equated to zero are $c_j = \cos\frac{j\pi}{m}$ $(j = 1, 2, \ldots, m-1)$ and are therefore related to the characteristic roots λ_j by the relation $\lambda_j = 2-2c_j$. Let

$$\phi(t) = \prod_{j=1}^{m-1} (1-2i\lambda_j t)^{-1} = \sum_{j=1}^{m-1} A_j(1-2i\lambda_j t)^{-1} ;$$

then
$$A_j = \lambda_j . \prod_{k\neq j}^{m-2} (\lambda_j-\lambda_k)^{-1} = \left(-\frac{\lambda_j}{2}\right)^{m-2} \prod_{k\neq j} (c_j-c_k)^{-1}.$$

Now
$$\prod_{k\neq j} (c_j-c_k) = \left\{\frac{d}{d\cos\theta}\left(\frac{1}{2^{m-1}}\frac{\sin m\theta}{\sin\theta}\right)\right\}_{\theta=\frac{j\pi}{m}} = (-1)^{j+1}\frac{m}{2^{m-1}\sin^2\frac{j\pi}{m}}$$

and therefore
$$A_j = (-1)^{m-j-1}\frac{2}{m}\lambda_j^{m-2}\sin^2\frac{j\pi}{m} = (-1)^{m-j-1}\frac{1}{2m}\lambda_j^{m-1}\lambda_{m-j} .$$

Hence we have

$$\phi(t) = \frac{1}{2m}\sum_{j=1}^{m-1} (-1)^{m-j-1}\lambda_j^{m-1}\lambda_{m-j}(1-2i\lambda_j t)^{-1}. \quad \ldots \quad (9)$$

Inverting the characteristic function, the frequency function is obtained as

$$f(x) = \frac{1}{4m}\sum_{j=1}^{m-1}(-1)^{m-j-1}\lambda_j^{m-2}\lambda_{m-j}e^{-\frac{x}{2\lambda_j}}, \quad \ldots \quad (10)$$

and the probability integral for $\delta'^2 = \frac{1}{2(m-1)}(X_{1(m)}+X_{2(m)})$ is given by

$$Pr\{\delta'^2 < x\} = Pr\{X_{1(m)}+X_{2(m)} \leqslant 2(m-1)x\}$$

$$= \int_0^{2(m-1)x} f(x)dx$$

$$= 1-\frac{1}{2m}\sum_{j=1}^{m-1}(-1)^{m-j-1}\lambda_j^{m-1}\lambda_{m-j}e^{-\frac{(m-1)x}{\lambda_j}} . \ldots \quad (11)$$

The calculation of the probability points of δ'^2 becomes very troublesome for large $n = 2m$ because the coefficients of the exponential terms on the r.h.s. of (11) become very large in magnitude and have alternately positive and negative signs. We are giving here the upper 5% points for some values of n.

n	8	10	12	16	20	24
5% point	4.58	4.25	4.01	3.69	3.49	3.30

4. Upper and lower bounds for the probability points of the mean square successive difference δ^2

It will now be shown that the distribution function of the Von Neumann statistic $\delta^2 = \sum_{i=1}^{2m-1}(x_i-x_{i+1})^2/(2m-1)$ is bounded by two known distribution functions. To fix the ideas let $x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_{2m}$, be a sample from the normal population with zero mean and variance σ^2. Then it is clear that

$$\sum_{i=1}^{m-1}(x_i-x_{i+1})^2+\sum_{i=1}^{m-1}(x_{m+i}-x_{m+i+1})^2 \leqslant \sum_{i=1}^{2m-1}(x_i-x_{i+1})^2$$

$$\leqslant \{\sum_{i=1}^{m-1}(x_i-x_{i+1})^2+2x_m^2+2x_{m+1}^2+\sum_{i=m+1}^{2m-1}(x_i-x_{i+1})^2\} \quad \ldots \quad (12)$$

or using the notation $X_{(m)} = \sum_{i=1}^{m-1}(x_i-x_{i+1})^2$ and $Y_{(m)} = X_{(m)}+2x_m^2$, (12) can be written as

$$X_{1(m)}+X_{2(m)} \leqslant X_{(2m)} \leqslant Y_{1(m)}+Y_{2(m)},$$

that is
$$\frac{X_{1(m)}+X_{2(m)}}{2m-1} \leqslant \delta^2_{(2m)} \leqslant \frac{Y_{1(m)}+Y_{2(m)}}{2m-1}. \quad \ldots \quad (13)$$

From (13) it can be easily shown that, if for a fixed P

$$P = Pr\left\{\frac{X_{1(m)}+X_{2(m)}}{2m-1} \leqslant x_1\right\}$$

$$= Pr\left\{\delta^2_{(2m)} \leqslant x_2\right\} = Pr\left\{\frac{Y_{1(m)}+Y_{2(m)}}{2m-1} \leqslant x_3\right\},$$

then
$$x_1 \leqslant x_2 \leqslant x_3. \quad \ldots \quad (14)$$

But the distribution of $(X_{1(m)}+X_{2(m)})/(2m-1)$ can be simply written down from the analysis given above in (10) and (11) as

$$Pr\left\{\frac{X_{1(m)}+X_{2(m)}}{2m-1} \leqslant x\right\} = 1-\frac{1}{2m}\sum_{j=1}^{m-1}(-1)^{m-j-1}\lambda_j^{m-1}\lambda_{m-j}e^{-\frac{2m-1}{2\lambda_j}x}. \quad \ldots \quad (15)$$

It therefore remains to find the distribution of $(Y_{1(m)}+Y_{2(m)})/(2m-1)$.

5. The distribution of $(Y_{1(m)}+Y_{2(m)})/(2m-1)$

The quadratic form $Y_{(m)} = \sum_{i=1}^{m-1} (x_i - x_{i+1})^2 + 2x_m^2 = x'Bx$ has the matrix B given by

$$B = \begin{vmatrix} 1 & -1 & . & . & . & . & . & . & . & . \\ -1 & 2 & -1 & . & . & . & . & . & . & . \\ . & -1 & 2 & -1 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & -1 & 2 & -1 & . \\ . & . & . & . & . & . & -1 & 2 & -1 \\ . & . & . & . & . & . & . & -1 & 3 \end{vmatrix} \quad \dots \quad (16)$$

The characteristic roots of B can be found in the following manner. Let $(x_1, x_2, \dots, x_m)$ be a characteristic vector of B, then

$$(1-\lambda)x_1 - x_2 = 0,$$

$$x_{r-1}+x_{r+1} = (2-\lambda)x_r, \qquad (r = 2, 3, \dots, m-1)$$

$$-x_{m-1}+(3-\lambda)x_m = 0$$

where λ is the corresponding characteristic root of B. That is, x_r satisfies the difference equation

$$x_{r-1}+x_{r+1} = (2-\lambda)x_r, \qquad (r = 1, 2, \dots, m)$$

with the boundary conditions

$$x_0 = x_1, \qquad x_m = -x_{m+1}. \quad \dots \quad (17)$$

To solve (17), let
$$x_r = 2\cos(r-\tfrac{1}{2})\,\alpha\,; \quad \dots \quad (18)$$
then
$$x_{r-1}+x_{r+1} = (2\cos\alpha)x_r. \quad \dots \quad (19)$$

The condition $x_0 = x_1$ is automatically satisfied by (18) and in order that $x_m = -x_{m+1}$ we must have $\cos(m-\frac{1}{2})\alpha = -\cos(m+\frac{1}{2})\alpha$, which is satisfied for m distinct values of α given by $\alpha_j = \frac{(2j+1)\pi}{2m}$ for $j = 0, 1, 2, \dots, m-1$. Therefore the characteristic roots of B given by $2-\lambda = 2\cos\alpha$, are

$$\lambda = \nu_j = 2-2\cos\frac{(2j+1)\pi}{2m} = 4\sin^2\frac{(2j+1)\pi}{4m}, \qquad j = 0, 1, \dots, m-1. \quad \dots \quad (20)$$

The characteristic function of $Y_{(m)}$ is, therefore,

$$\prod_{j=0}^{m-1} (1-2i\nu_j t)^{-\frac{1}{2}}. \quad \dots \quad (21)$$

Since $Y_{1(m)}$ and $Y_{2(m)}$ are independent the characteristic function of $Y_{1(m)}+Y_{2(m)}$ is

$$\phi(t) = \prod_{j=0}^{m-1} (1-2i\nu_j t)^{-1}. \quad \dots \quad (22)$$

This expression can be thrown into partial fractions with the help of the trigonometric identity

$$\cos m\theta = 2^{m-1}\cos^m\theta - \frac{m}{1}\,2^{m-3}\cos^{m-2}\theta + \frac{m(m-3)}{2!}\,2^{m-5}\cos^{m-4}\theta - \dots \quad \dots \quad (23)$$

noting that the r.h.s. of (23) equated to zero has m distinct roots $\cos\frac{(2j+1)\pi}{2m}$ $(j = 0, 1, 2, ..., m-1)$ which are related to the characteristic roots ν_j by the relation (20). Following the procedure similar to that adopted in section 3 above, it can be shown that

$$\phi(t) = \frac{1}{m}\sum_{j=0}^{m-1} (-1)^{m-j-1} \nu_j^{m-1} \sin\frac{(2j+1)\pi}{2m} (1-2i\nu_j t)^{-1} \quad ... \quad (24)$$

The frequency function of $Y_{1(m)}+Y_{2(m)}$ is therefore

$$f(x) = \frac{1}{2m}\sum_{j=0}^{m-1}(-1)^{m-j-1}\nu_j^{m-2} \sin\frac{(2j+1)\pi}{2m} e^{-\frac{x}{2\nu_j}}, \quad ... \quad (25)$$

and the probability integral of $(Y_{1(m)}+Y_{2(m)})/(2m-1)$ is given by

$$Pr\left\{\frac{Y_{1(m)}+Y_{2(m)}}{2m-1} < x\right\} = Pr\{Y_{1(m)}+Y_{2(m)} < (2m-1)x\} = \int_0^{(2m-1)x} f(x)dx$$

$$= 1-\frac{1}{m}\sum_{j=0}^{m-1} (-1)^{m-j-1}\nu_j^{m-1} \sin\frac{(2j+1)\pi}{2m} e^{-\frac{2m-1}{2\nu_j}x}. \quad ... \quad (26)$$

Using results (14), (15) and (26) we can obtain the limits between which the probability points of δ^2 lie. It can be established for instance that for $n = 20$ the upper 5% point of δ^2 lies between 3.29 and 3.61. Although these limits are not very close it is clear that they will become closer when n is larger. The calculations, unfortunately, become rather laborious for large n for the reason stated in section 3 above.

6. Derived statistics : A test corresponding to F-test

Using δ'^2 it is possible to derive the *exact* distributions of (*i*) a statistic corresponding to the F-ratio, and (*ii*) a statistic corresponding to Student's t, both based on the estimator δ'^2 of the variance.

Let $\delta'^2_{(m)}$ and $\delta'^2_{(n)}$ be two independent estimates of σ^2 based on $2m$ and $2n$ observations respectively. Then the frequency functions of $\delta'^2_{(m)}$ and $\delta'^2_{(n)}$ are, from (11),

$$\delta'^2_{(m)} : \quad f(x) = \sum_{j=1}^{m-1} A_j \exp[-(m-1)x/\lambda_j]$$

$$A_j = (-1)^{m-j-1}\lambda_j^{m-2}\lambda_{m-j}\left(\frac{m-1}{m}\right), \; \lambda_j = 4\sin^2\frac{j\pi}{2m}, \quad (j = 1, 2, ..., m-1). \quad ... \quad (27)$$

$$\delta'^2_{(n)} : \quad f(y) = \sum_{k=1}^{n-1} A'_k \exp[-(n-1)y/\lambda'_k]$$

$$A'_k = (-1)^{n-k-1}\lambda'^{n-2}_k\lambda'_{n-k}\left(\frac{n-1}{n}\right), \; \lambda'_k = 4\sin^2\frac{k\pi}{2n}, \quad (k = 1, 2, ..., m-1). \quad ... \quad (28)$$

Hence the joint frequency function of $\delta'^2_{(m)}$ and $\delta'^2_{(n)}$ is given by

$$f(x, y) = \sum_j \sum_k A_j A'_k \exp\left[-\left(\frac{m-1}{\lambda_j}x + \frac{n-1}{\lambda'_k}y\right)\right],$$

$$0<x, \quad y<\infty. \qquad \ldots \quad (29)$$

Substituting $x = yv$ and integrating out y we have the frequency function of $\phi_{m,n} = \dfrac{\delta'^2_{(m)}}{\delta'^2_{(n)}}$ as

$$\phi_{m,n}: \qquad f(v) = \sum_j \sum_k A_j A'_k \left(\frac{m-1}{\lambda_j}v + \frac{n-1}{\lambda'_k}\right)^{-2} \qquad \ldots \quad (30)$$

and its probability integral is given by

$$Pr\left\{\phi_{m,n} < v\right\} = \int_0^v f(v)dv$$

$$= 1 - \sum_j \sum_k \frac{\lambda_j A_j A'_k}{m-1}\left(\frac{m-1}{\lambda_j}v + \frac{n-1}{\lambda'_k}\right)^{-1}. \qquad \ldots \quad (31)$$

The ratio $\phi_{m,n}$ may be used to test the hypothesis that the two samples come from populations which have equal standard deviations even though their means may be having slow-moving continuous trends. We may also use it to see whether the dispersion of observations about their mean remains reasonably constant when the mean itself is known to have a slow shift.

7. A test corresponding to the t-test : Use in control charts

Let δ'^2 be an estimate of σ^2 based on $2n$ observations and let $\bar{x}$ be the mean of these observations; then it can be shown with the use the Helmert transformation that $\bar{x}-\mu$ and δ'^2 are independently distributed, μ being the constant mean of the parental normal population. Then it is possible to find the *exact* distribution of the ratio statistic

$$u = \frac{\bar{x}-\mu}{\delta'}. \qquad \ldots \quad (32)$$

Since $\bar{x}-\mu$ is $N\left(0, \dfrac{\sigma}{\sqrt{2n}}\right)$ we may write

$$u = \frac{\bar{x}-\mu}{\delta'} = \frac{1}{\sqrt{2n}}\frac{\xi}{\delta'} \qquad \ldots \quad (33)$$

where the frequency functions of ξ and δ' are

$$\xi: \qquad f(\xi) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\xi^2},$$

$$\delta': \qquad f(\delta') = 2\sum_{k=1}^{n-1} A_k e^{-\frac{(n-1)\delta'^2}{\lambda_k}}\delta' \qquad \ldots \quad (34)$$

A_k being the same as A'_k in (28). Taking the joint frequency function of ξ and δ', making the transformation $\xi = t\delta'$ and integrating out δ' we get the frequency function of ξ/δ' as

$$\frac{\xi}{\delta'}: \qquad f(t) = \sum_k A_k \left(t^2 + \frac{2n-2}{\lambda_k}\right)^{-3/2} \qquad \ldots \quad (35)$$

Hence the frequency function of $u = \dfrac{\bar{x}-\mu}{\delta'}$ is

$$f(u) = \frac{1}{2n} \Sigma A_k \left(u^2 + \frac{n-1}{n\lambda_k}\right)^{-3/2} \qquad \ldots \quad (36)$$

and the probability integral is given by

$$Pr\left\{\frac{\bar{x}-\mu}{\delta'} < u\right\} = \int_{-\infty}^{u} f(u)du$$

$$= \frac{1}{2} + \frac{1}{2(n-1)} \sum_{k=1}^{n-1} A_k \lambda_k u \left(u^2 + \frac{n-1}{n\lambda_k}\right)^{-1} \qquad \ldots \quad (37)$$

It is clear that the distribution of u is symmetrical about $u = 0$, and $F(0) = \frac{1}{2}$.

This distribution may be useful in control charts in quality control, when it is known that the mean is affected by a slow-moving trend and yet it is desired that the quality must not deteriorate beyond a certain limit.

CONCLUDING REMARKS

As observed in section 6 above δ'^2 is a weighted sum of χ^2 each with 2 d.f. In the same way $Y_{1(m)} + Y_{2(m)}$ is a weighted sum of χ^2 each with 2 d.f., while δ^2 and $Y_{(m)}$ are weighted sums of χ^2 each with 1 d.f. The distributions of δ'^2 and $Y_{1(m)} + Y_{2(m)}$ are obtainable in a finite analytic form but it seems difficult to do so for δ^2 or $Y_{(m)}$ (cf. Robbins and Pitman, 1949).

The fact that δ'^2 and $Y_{1(m)} + Y_{2(m)}$ are composed of χ^2 with 2 d.f., accounts for the resemblance of expressions (15) and (26) with the χ^2 distribution with 2 d.f. and of expressions (31) and (37) with the F-distribution with 2, 2 d.f. and the t-distribution with 2 d.f. respectively.

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