

Hamiltonian graphs with minimum number of edges for fault-tolerant topologies

Krishnendu Mukhopadhyaya and Bhabani P. Sinha

Electronics Unit, Indian Statistical Institute, 203 Barrackpore Trunk Road, Calcutta 700 035, India

Communicated by M.J. Atallah

Abstract

Mukhopadhyaya, K. and B.P. Sinha, Hamiltonian graphs with minimum number of edges for fault-tolerant topologies, Information Processing Letters 44 (1992) 95–99.

For some applications it may be necessary that a hamiltonian cycle be present in the connection topology of the processing elements. We propose a family of network topologies, each of which has a hamiltonian cycle in the fault-free situations as well as when there is a single node or link failure. The proposed topologies require minimum number of edges among all possible interconnection structures having the above properties.

Keywords: Distributed systems; fault-tolerance; hamiltonian cycle; node failure; link failure

1. Introduction

Design of the topology of a computer network is guided by many mutually conflicting requirements. It is not always possible to get a network design, which is optimum from all angles. Depending on the requirements and their priorities, one has to design a suitable network structure. Certain applications need some particular properties to be present in the design. We consider the application areas requiring the presence of a hamiltonian cycle in the structure. An example of such an application may be a distributed operating system where mutual exclusion of certain shared resources is implemented by a “Token Passing” approach [2]. In such cases, the token passes along a logical ring. As long as the struc-

ture is connected, it is possible to construct a logical ring. But it is desirable that a hamiltonian cycle be present in the network so that all the nodes in the network get equal chance to grab the token.

Let $G = (V, E)$ be a graph, where V is the set of nodes and E is the set of links. Let $|V| = N$. Two nodes are said to be *adjacent* if there is a link joining them. A *cycle* is a sequence of three or more nodes such that (i) two consecutive nodes are adjacent and (ii) the first and the last nodes are the same. A cycle is called a *hamiltonian cycle*, if its nodes are distinct and they span V . A graph having a hamiltonian cycle is called a *hamiltonian graph*. A graph will be called *1-node-deleted hamiltonian* if the graph is hamiltonian when any one of its nodes along with all the links incident on that node are deleted. Similarly a graph will be called *1-link-deleted hamiltonian* if it is hamiltonian after the deletion of any one of its links.

Correspondence to: K. Mukhopadhyaya, Electronics Unit, Indian Statistical Institute, 203 Barrackpore Trunk Road, Calcutta 700 035, India.

Many network structures have hamiltonian cycles embedded in them. In this paper we try to construct a graph which is hamiltonian as well as 1-link-deleted hamiltonian and 1-node-deleted hamiltonian. After a node or a link fails, the degree of some of the nodes may decrease by one. But the presence of a hamiltonian cycle needs that the degree of every node should be at least 2. Hence, for a graph to be 1-link-deleted hamiltonian or 1-node-deleted hamiltonian, all nodes must have degree at least 3. Here we propose a design such that (i) when the total number of nodes, N , is even, all nodes are of degree 3 and (ii) when N is odd, all but one of the nodes are of degree 3 and the remaining node is of degree 4. The proposed graph has minimum number of links among all the graphs which are 1-link-deleted hamiltonian or 1-node-deleted hamiltonian.

2. Design of the graph

First we consider the construction of the graph for $N = 6n + 4$ nodes for $n \geq 0$.

1. Connect $6n + 3$ nodes to form a ring.
2. Place the remaining node, labelled as $V(0)$, at the centre of the ring and connect it to three equidistant nodes on the ring. Label these nodes on the ring as $V(1)$, $V(2)$ and $V(3)$.
3. Starting from each $V(i)$, $i = 1, 2, 3$, label the n nodes encountered in traversing the ring counterclockwise as $V(i, 1)$, $V(i, 3)$, ..., $V(i, 2n - 1)$. Similarly, the n nodes encountered in the clockwise traversal from $V(i)$, $i = 1, 2, 3$, are labelled as $V(i, 2)$, $V(i, 4)$, ..., $V(i, 2n)$. As the distance along the ring between $V(i)$ and $V(j)$, $i, j \in \{1, 2, 3\}$, $i \neq j$, is $2n$, there will not be any conflict in such labelling of the nodes.
4. Join $V(i, 2j - 1)$ to $V(i, 2j)$ by a chord-link for all i, j , $i = 1, 2, 3$ and $1 \leq j \leq n$.

Let us call the resultant graph G . Figure 1 shows an example of such a graph with $N = 16$ nodes.

G can easily be generalized for any even number of nodes. Starting from $6n + 4$ nodes we can put two more nodes in G . We introduce the two

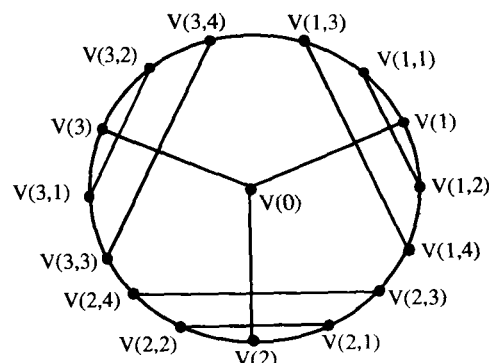


Fig. 1. An example of the proposed graph for $N = 16$ nodes.

new nodes on the ring as follows: (i) $V(1, 2n + 1)$ is placed between $V(1, 2n)$ and $V(3, 2n)$, (ii) $V(1, 2n + 2)$ is put between $V(1, 2n)$ and $V(2, 2n - 1)$. Then we join $V(1, 2n - 1)$ and $V(1, 2n)$ by a link. This gives us the graph for $N = 6(n + 1)$ nodes. Figure 2 shows an example for $N = 18$ nodes. Similarly, we can put two more nodes, $V(2, 2n + 1)$ (in between $V(2, 2n - 1)$ and $V(1, 2n + 2)$ on the ring) and $V(2, 2n + 2)$ (in between $V(2, 2n)$ and $V(3, 2n - 1)$ on the ring). An example for $N = 20$ nodes is shown in Fig. 3. Finally, two more nodes, $V(3, 2n + 1)$ (in between $V(3, 2n - 1)$ and $V(2, 2n + 2)$ on the ring) and $V(3, 2n + 2)$ (in between $V(3, 2n)$ and $V(1, 2n + 1)$ on the ring) can be added to generate the graph for $N = 6(n + 1) + 4$ nodes.

We now extend the design technique to the case where the total number of nodes N is odd. For odd N , it follows from the degree-sum crite-

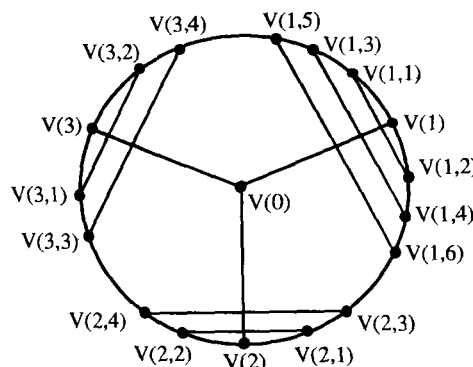


Fig. 2. The proposed graph with $N = 18$ nodes.

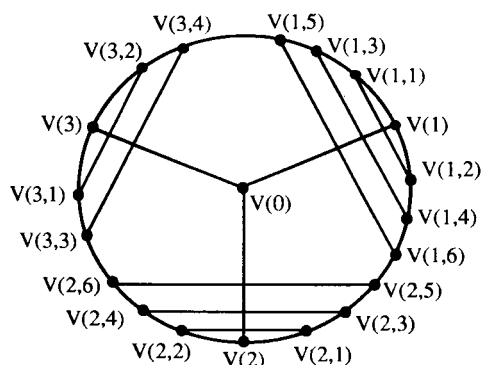


Fig. 3. The proposed graph for $N = 20$ nodes.

tion that we cannot make the graph 3-regular. However, allowing one node of degree 4 and the rest of degree 3, we can easily generalize the previous design algorithm to get a graph G' . We take the case of $8n + 5$ nodes. First we place $8n + 4$ nodes on a ring. Then we place a central node and join it to 4 nodes on the ring such that the spokes divide the ring into 4 equal parts (as opposed to 3 in the previous case). Then we similarly name the nodes around each spoke and join them. Again, as in the case of an even number of nodes, we can add to it two nodes at a time, thus designing the network for any odd N .

3. Properties

Theorem 1. *The graph G with an even number of nodes has the following properties:*

- (a) *It is trivalent.*
- (b) *It is planar.*
- (c) *It is hamiltonian.*
- (d) *It is 1-link-deleted hamiltonian.*
- (e) *It is 1-node-deleted hamiltonian.*
- (f) *It has diameter $\lfloor N/6 \rfloor + 2$.*
- (g) *It is incrementally extensible by two nodes.*

Proof. We prove the results for $N = 6n + 4$. For other even values of N , the proof is similar.

(a) Obvious.

(b) Planarity follows if we draw the chord-links between $V(i, 2j - 1)$ and $V(i, 2j)$, $i = 1, 2, 3$, $1 \leq j \leq n$, outside the ring.

Figure 4 shows a planar embedding of the graph for $N = 16$ nodes.

(c) First we consider the case when n is odd. The following sequence of nodes gives one hamiltonian cycle:

$$\begin{aligned}
 &V(0), V(1), V(1, 1), V(1, 2), V(1, 4), V(1, 3), \\
 &V(1, 5), V(1, 6), \dots, V(1, 2n - 1), V(1, 2n), \\
 &V(2, 2n - 1), V(2, 2n - 3), V(2, 2n - 5), \dots, \\
 &V(2, 1), V(2), V(2, 2), V(2, 4), V(2, 6), \dots, \\
 &V(2, 2n), V(3, 2n - 1), V(3, 2n), V(3, 2n - 2), \\
 &V(3, 2n - 3), V(3, 2n - 5), \dots, V(3, 1), \\
 &V(3, 2), V(3), V(0)
 \end{aligned}
 \tag{1}$$

The proof is similar when n is even.

(d) We first note that we can get hamiltonian cycles, other than the one described in sequence (1), by interchanging the roles of $V(1)$, $V(2)$ and $V(3)$.

Case I: Let the faulty link be one between $V(k, 2j - 1)$ and $V(k, 2j)$, for some k and j , $k \in \{1, 2, 3\}$, $1 \leq j \leq n$. If $k = 2$, the link is anyway not included in the hamiltonian cycle described in part (c). If $k = 1$ or 3, we can interchange the roles of $V(2)$ and $V(k)$ in sequence (1) to get a hamiltonian cycle without using the faulty link.

Case II: Let the faulty link be between $V(0)$ and $V(k)$, $k \in \{1, 2, 3\}$. If $k = 2$, the link is anyway not included in the hamiltonian cycle described in part (c). If $k = 1$ or 3, we can interchange the roles of $V(2)$ and $V(k)$ in sequence (1) to get a hamiltonian cycle without using the faulty link.

Case III: Let the faulty link be between two nodes $V(k, j)$ and $V(k, j + 2)$, for some k and j ,

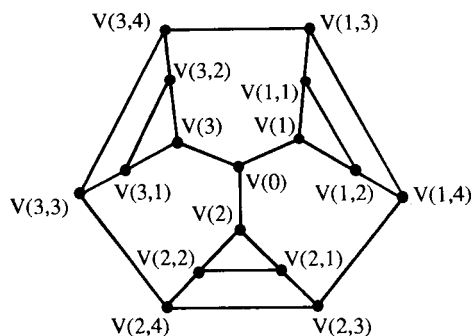


Fig. 4. A planar embedding of the graph in Fig. 1.

$k \in \{1, 2, 3\}$, $1 \leq j \leq n-2$. For the hamiltonian cycle, we start from $V(0)$ and go to $V(k)$ first. The choice of the next node is guided by the value of j so as to avoid the faulty link. If $j = 4p$ or $4p + 1$, then we go to $V(k, 1)$ else we go to $V(k, 2)$ from $V(k)$. The rest of the sequence can be computed in the manner similar to that discussed in part (c).

(e) First we take the case when n is odd.

Case I: $V(0)$ is faulty. The hamiltonian cycle exists along the ring.

Case II: $V(k)$, $k \in \{1, 2, 3\}$ is faulty. Without loss of generality, let $V(2)$ be faulty. We can modify the hamiltonian cycle as described in part (c), by replacing the subsequence " $V(2, 1), V(2), V(2, 2)$ " by " $V(2, 1), V(2, 2)$ ".

Case III: $V(1, i)$, $1 \leq i \leq 2n$, is faulty. Let $V(1, i)$ be connected to $V(1, i')$ by a chord-link, where $i' = i + 1$ or $i - 1$. To get a hamiltonian cycle, we start with the sequence $V(0), V(1)$. Next we move either to $V(1, 1)$ or to $V(1, 2)$ depending on the value of i so that $V(1, i)$ can be bypassed by the subsequence given as follows:

For $n = 1$,

(i) $V(1), V(1, i'), V(2, 1)$ for $i' = 2$,

(ii) $V(1), V(1, i'), V(3, 2)$ for $i' = 1$.

For $n > 1$,

(i) $V(1), V(1, i'), V(1, i' + 2)$ for $i' \leq 2$,

(ii) $V(1, i' - 2), V(1, i'), V(1, i' + 2)$ for $3 \leq$

$i' \leq 2n - 2$,

(iii) $V(1, i' - 2), V(1, i'), V(3, 2n)$ for $i' =$

$2n - 1$,

(iv) $V(1, i' - 2), V(1, i'), V(2, 2n - 1)$ for $i' =$

$2n$.

The rest of the sequence can be computed in the manner similar to that described in part (c).

(f) From each node we first find the maximum number of steps needed.

(i) From $V(0)$ we can reach any other node within $n + 1$ steps.

(ii) From $V(1), V(2)$ or $V(3)$ we can reach any other node (via $V(0)$) within $n + 2$ steps.

(iii) From $V(1, i)$ (without loss of generality let $i = 2m$), $V(0)$ can be reached within $m + 1$ steps. So, $V(k, j)$, $k = 2, 3$, can be reached via $V(0)$, within $n + 2$ steps for $1 \leq j \leq 2(n - m)$. Now, $V(2, 2n - 1)$ can be reached via $V(1, 2n)$ in $n - m + 1$ steps. So, $V(2, 2j - 1)$ for $n - m < j \leq n$

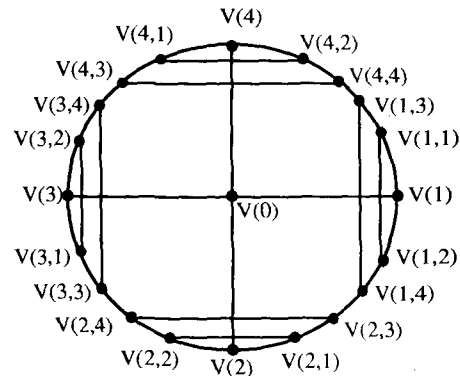


Fig. 5. The proposed graph for $N = 21$ nodes.

can be reached within n steps. Hence $V(2, 2j)$ for $n - m < j \leq n$ can be reached within $n + 1$ steps. Similarly, we can go to $V(1, 2m - 1)$ in one step. From there we can go to $V(3, j)$ for $2(n - m) \leq j \leq 2n$ in maximum of $n + 1$ more, i.e., $n + 2$ total steps.

We also note that the distance between some pairs of nodes, for example $V(2)$ and $V(3, 2n)$ is exactly $n + 2$. Hence the diameter is exactly $n + 2$.

(g) Follows from the design algorithm. \square

Theorem 2. The graph G' with an odd number of nodes has the following properties:

(a) All nodes except $V(0)$ have degree 3 and $V(0)$ has degree 4.

(b) It is planar.

(c) It is hamiltonian.

(d) It is 1-link-deleted hamiltonian.

(e) It is 1-node-deleted hamiltonian.

(f) It has diameter $\lfloor N/8 \rfloor + 3$.

(g) It is incrementally extensible by two nodes.

Proof. Similar to that of Theorem 1. \square

Figure 5 shows an example of the graph with $N = 21$ nodes and Fig. 6 shows a planar embedding of that graph.

4. Conclusion

Although the graphs we have proposed in this paper are hamiltonian, 1-node-deleted hamiltonian

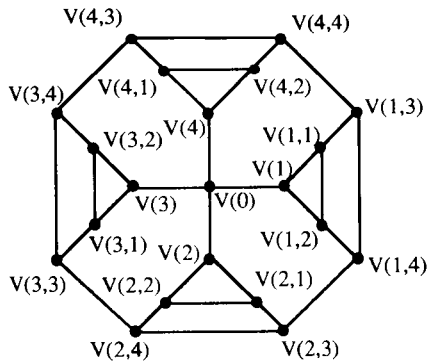


Fig. 6. A planar embedding of the graph in Fig. 5.

nian and 1-link-deleted hamiltonian, they are not optimal with regard to diameter. There are many denser trivalent and tetravalent graphs reported in the literature [1,4]. As an example, if we take the case of the trivalent Möbius graph, it has diameter $O(\log N)$ and it also has a hamiltonian cycle. However, existence of a hamiltonian cycle with a single node or link failure in a Möbius graph remains yet to be established. Also the problem with the Möbius graph is that it is defined only for $N = 2^n$ nodes. In other words, the Möbius graph is not incrementally extensible.

Further, for odd n , two of its links overlap, thus making it not exactly trivalent. Similar problems regarding incremental extensibility also appear in case of the dense tetravalent de Bruijn graph [3]. The graph structure that we have proposed here is, on the other hand, incrementally extensible.

Acknowledgment

The authors would like to thank the anonymous referee for his careful review and many helpful comments.

References

- [1] W.E. Leland and M.H. Solomon, Dense trivalent graphs for processor interconnection, *IEEE Trans. Comput.* **31** (1982) 219–222.
- [2] J.L. Peterson and A. Silberschatz, *Operating System Concepts* (Addison-Wesley, Reading, MA, 1985).
- [3] D.K. Pradhan and S.M. Reddy, A fault-tolerant communication architecture for distributed systems, *IEEE Trans. Comput.* **31** (1982) 863–870.
- [4] P.K. Srimani, B.P. Sinha, B.B. Bhattacharya and S. Ghosh, Properties of a class of trivalent network graphs and optimal routing, *Comput. Math. Appl.* **22** (2) (1991) 39–47.