Journal of Statistical Planning and Inference 31 (1992) 263-271

Universal optimality and non-optimality of some row-column designs

Ashish Das

Stat-Math Division, Indian Statistical Institute, Calcutta 700035, India

A. Dey

Stat-Math Unit, Indian Statistical Institute, New Delhi 110 016, India

Received 21 February 1990; revised manuscript received 3 December 1990
Recommended by C.S. Cheng

Abstract: The optimality and non-optimality of some row-column designs is studied with respect to the universal optimality criterion. It is shown that, (i) in the non-regular setting, a generalized Youden design is never universally optimal, (ii) in the regular setting, if the number of treatments is three or more, a generalized Youden design (if it exists) is uniquely universally optimal, and (iii) a pseudo Youden design is never universally optimal in the non-regular setting.

AMS Subject Classification: Primary 62K05; secondary 62K10, 05B05.

Key words and phrases: Universal optimality; generalised Youden design; pseudo Youden design; Youden type design.

1. Introduction and preliminaries

Generalized Youden Designs (GYD) were introduced by Kiefer (1958) as a generalization of the usual Latin square and Youden square designs for eliminating heterogeneity in two directions. Under the usual additive and homoscedastic fixed effects model for comparing v treatments via $b_1 b_2$ experimental units arranged in a row-column design with b_1 rows and b_2 columns, the coefficient matrix of the reduced normal equations for estimating linear functions of treatment effects, using a design $d \in \mathscr{D}(v, b_1, b_2)$ is

$$C_d = R_d - b_2^{-1} N_{d1} N_{d1}' - b_1^{-1} N_{d2} N_{d2}' + (b_1 b_2)^{-1} r_d r_d' \qquad (1$$

where $R_d = \mathrm{diag}(r_{d1}, r_{d2}, \ldots, r_{dv})$, $r_d = (r_{d1}, r_{d2}, \ldots, r_{dv})'$, r_{di} is the replication of the i-th treatment, $i = 1, 2, \ldots, v$, $N_{d1} = ((n_{dij}^{(1)}))$, $N_{d2} = ((n_{dij}^{(2)}))$ are respectively the treatment-row and treatment-column incidence matrices and $\mathscr{D}(v, b_1, b_2)$ is the class of all *connected* row-column designs with v treatments, b_1 rows and b_2 columns.

A row-column setting is called *regular* if at least one of the following holds: (i) $v \mid b_1$, (ii) $v \mid b_2$, where $m \mid n$ means $n = 0 \bmod m$. Otherwise, the setting is called *non-regular*.

A design $d \in \mathscr{D}(v, b_1, b_2)$ is called a GYD if the following conditions hold for $s = 1, 2$:

$$\text{(i)} \quad \sum_{j=1}^{b_s} n_{dij}^{(s)} = r \quad \text{for } i = 1, 2, \ldots, v;$$

$$\text{(ii)} \quad \sum_{j=1}^{b_s} n_{dij}^{(s)} n_{dmj}^{(s)} = \lambda^{(s)} \quad \text{for } i \neq m, \ i, m = 1, 2, \ldots, v; \tag{1.2}$$

$$\text{(iii)} \quad |n_{dij}^{(3-s)} - b_s/v| < 1.$$

If in a GYD, the row (column) classification is ignored, the {columns} ({rows}) considered as blocks, form a Balanced Block Design (BBD) of Kiefer (1958). A GYD is regular if at least one of the following holds: (i) $v \mid b_1$, (ii) $v \mid b_2$; otherwise, the GYD is called *non-regular*.

Kiefer (1975) proved that a GYD (regular or non-regular) is A- and E-optimal, and is D-optimal if $v \neq 4$, in $\mathscr{D}(v, b_1, b_2)$. Further, it was shown by Kiefer (1975) that in the *regular setting*, a GYD is universally optimal. A design $d^* \in \mathscr{D}$, the class of competing designs, is called *universally optimal* in $\mathscr{D}$ if d^* minimizes $\phi(C_d)$ over $\mathscr{D}$ for all $\phi : \mathscr{B}_{v,0} \to (-\infty, \infty]$ satisfying

$$\phi \text{ is convex,} \tag{1.3a}$$

$$\phi(bC) \text{ is nonincreasing in the scalar } b \geq 0, \tag{1.3b}$$

$$\phi \text{ is invariant under each permutation of rows and (the same on) columns,} \tag{1.3c}$$

where $\mathscr{B}_{v,0}$ is the class of all $v \times v$ symmetric, non-negative definite matrices with zero row sums.

It is well known that if a design is universally optimal, then it is A-, D- and E-optimal as well. The universal optimality of *regular* GYD's can be established with the help of the following sufficient condition of Kiefer (1975).

Theorem 1.1 (Kiefer (1975)). *Suppose a class* $C = \{C_d, d \in \mathscr{D}\}$ *of matrices in* $\mathscr{B}_{v,0}$ *contains a* C_{d^*} *such that*

$$C_{d^*} \text{ is completely symmetric, i.e. } C_{d^*} \text{ is of the form } aI_v + bJ_v, \text{ where } I_v \text{ is the } v\text{-th order identity matrix and } J_v, \text{ a } v \times v \text{ matrix of all ones,} \tag{1.4a}$$

$$\operatorname{tr}(C_{d^*}) = \max_{d \in \mathscr{D}} \operatorname{tr}(C_d), \tag{1.4b}$$

then d^* *is universally optimal in* $\mathscr{D}$. *Here* $\operatorname{tr}(\cdot)$ *stands for the trace of a square matrix.*

Notice that since $-\mathrm{tr}\, C$ satisfies (1.3), condition (1.4b) is *necessary* for universal optimality. In subsequent sections, we utilize this fact to prove the non-universal optimality of certain designs.

The purpose of this presentation is to establish certain results regarding the universal optimality (or non-optimality) of designs within the class $\mathscr{D}(v, b_1, b_2)$. Specifically, we show that

(i) in *non-regular settings*, a GYD is never universally optimal.

Note that while in the regular setting, a GYD is universally optimal, the universal optimality of GYD's in all non-regular settings cannot be claimed. In fact, this prompted Kiefer (1975, p. 337) to make the following remark: "We do not know in which non-regular settings the GYD is still universally optimal." Result (i) above settles this question.

We also show that

(ii) in regular settings with $v>2$, a GYD (if it exists) is the only universally optimal design, and

(iii) a BBD with $v>2$ is the only universally optimal design in $\mathscr{D}_0(v, b, k)$, where $\mathscr{D}_0(v, b, k)$, is the class of all connected *block* designs with v treatments, b blocks and block size k.

The notion of GYD's has been extended to Pseudo Youden Designs (PYD) by Cheng (1981a). A design $d \in \mathscr{D}(v, b, b)$ is said to be PYD if

$$\bar{N}_d = [N_{d1} \mid N_{d2}]$$

is the incidence matrix of a BBD. Note that for a PYD, $b_1 = b_2 = b$. Clearly a GYD with $b_1 = b_2$ is a PYD, but the converse is not true unless the PYD is regular. It has been shown by Cheng (1981a) that a PYD is A- and E-optimal and is D-optimal if $v \neq 4$. We show that

(iv) in non-regular settings, a PYD is never universally optimal.

For the sake of completeness, we reproduce some results from Das and Dey (1989a,b).

Lemma 1.1 (Das and Dey, 1989a). *Let d be a block design with v treatments, b blocks and block size k such that its incidence matrix N_d has only two entries, x_d and $y_d = x_d + 1$, $x_d \geq 0$. Then $x_d = [k/v]$, where $[\cdot]$ is the greatest integer function.*

Theorem 1.2 (Das and Dey, 1989b). *Let d be a block design with v treatments, b blocks and block size k such that the i-th treatment is replicated r_i times for $i = 1, 2, \ldots, v$, $\sum r_i = bk$. Let us write the blocks of d as columns. Then the treatment symbols within columns can be so rearranged, that the i-th treatment symbol appears m_i times in each row, if and only if $r_i = km_i$ for $i = 1, 2, \ldots, v$.*

Suppose $d \in \mathscr{D}(v, b_1, b_2)$ is a row-column design. Associated with d is a block design d^{N}, obtained by treating the {columns} of d as blocks. Then from (1.1), it follows that

$$C_d = C_d^N - b_2^{-1} N_{d1}(I_{b_1} - b_1^{-1} J_{b_1}) N'_{d1}, \tag{1.5}$$

where

$$C_d^N = R_d - b_1^{-1} N_{d2} N'_{d2} \tag{1.6}$$

is the usual C-matrix of d^N.

We also need the following definition from Das and Dey (1989b).

Definition 1.1. A $b_1 \times b_2$ array containing entries from a finite set $\Omega = \{1, 2, \ldots, v\}$ of v treatment symbols will be called a Youden Type (YT) row–column design if the i-th treatment symbol occurs in each row m_i times for $i = 1, 2, \ldots, v$, where $m_i = r_i/k$ and r_i is the replication of the i-th treatment symbol, $i = 1, 2, \ldots, v$.

The class of YT designs contains all Youden square and *regular* GYD's.

The following result is due to Das and Dey (1989b).

Theorem 1.3 (Das and Dey, 1989b). *For $d \in \mathcal{D}(v, b_1, b_2)$, $C_d = C_d^N$ if and only if d is a YT design.*

2. The results

We first prove the following:

Lemma 2.1. *Given positive integers v, b_1, b_2 and $r_i, i = 1, 2, \ldots, v$, such that*

(i) $\sum_{i=1}^{v} r_i = b_1 b_2$

and

(ii) $b_2[b_1/v] \le r_i \le b_2([b_1/v] + 1), \quad i = 1, 2, \ldots, v, \qquad (2.1)$

there exists a block design d_0 with v treatments, b_2 blocks and block size b_1, such that the i-th treatment is replicated r_i times for $i = 1, 2, \ldots, v$ and the incidence matrix of d_0 contains only two entries, $x = [b_1/v]$ and $y = x + 1$.

Proof. Consider an array B of size $v \times b_2$, such that

(i) B contains only two integral entries, x and $y = x + 1$, $x \ge 0$, and

(ii) the i-th row sum of B is r_i for $i = 1, 2, \ldots, v$.

Clearly, such an array can always be formed for given positive integers $v, b_1, b_2, r_1, \ldots, r_v$ such that $\sum r_i = b_1 b_2$ and $b_2 x \le r_i \le b_2 y$ for $i = 1, 2, \ldots, v$. In fact, if f_i denotes the frequency of x in the i-th row of B, then

$$f_i = b_2 y - r_i, \quad i = 1, 2, \ldots, v. \tag{2.2}$$

Thus, if r_x denotes the number of times x appears in B, then

$$r_x = \sum_{i=1}^{v} f_i = b_2 v y - b_1 b_2 = b_2(vy - b_1). \tag{2.3}$$

It follows then that r_y, the frequency of y in $\boldsymbol{B}$ is

$$r_y = b_2(b_1 - vx). \tag{2.4}$$

Since r_x and r_y are both divisible by b_2, using Theorem 1.2, we can rearrange the symbols x and y in the rows of $\boldsymbol{B}$, to obtain an array A such that each column of A contains m_x x's and m_y y's, where $m_x = vy - b_1$ and $m_y = b_1 - vx$. Clearly, the column sums of A are all equal to b_1. Finally, using Lemma 1.1, we get $x = [b_1/v]$. The array A is the incidence matrix of the required block design d_0.

Corollary 2.1. *If $\boldsymbol{C}_{d_0}$ is the usual $\boldsymbol{C}$-matrix of d_0 of Lemma 2.1, then $\mathrm{tr}(C_{d_0})$ is maximum over $\mathscr{D}_0(v, b_2, b_1)$, where $\mathscr{D}_0(v, b_2, b_1)$ is the class of all connected block designs having v treatments, b_2 blocks and block size b_1.*

The following result can be proved on the lines of Theorem 3.2 of Agrawal (1966), using the results of Das and Dey (1989b).

Lemma 2.2. *Consider a block design having v treatments, b blocks and block size k, wherein the i-th treatment is replicated r_i times, $i = 1, 2, \ldots, v$, $\sum r_i = bk$. Let the blocks of the design be written as columns. The treatment symbols within columns can be so rearranged that the i-th treatment appears m_i times in $k - t_i$ rows and $m_i + 1$ times in t_i rows, if and only if $r_i = m_i k + t_i$, where $m_i \geq 0$ and $0 < t_i \leq k - 1$, for $i = 1, 2, \ldots, v$.*

We are now in a position to prove the main results of the paper.

Theorem 2.1. *In non-regular settings, a GYD is never universally optimal in $\mathscr{D}(v, b_1, b_2)$.*

Proof. Let d^* denote a $b_1 \times b_2$ GYD. Since the setting is non-regular, $b_i > v$ for $i = 1, 2$. Without loss of generality, let $b_2 \geq b_1$. Now, there exists a design $d_1 \neq d^*$ in $\mathscr{D}(v, b_1, b_2)$ such that

$$r_{d_1 i} = \begin{cases} r_{d^*} + 1 = mb_1 + t + 1 & \text{for } i = 1, 2, \ldots, [\tfrac{1}{2}v], \\ r_{d^*} - 1 = mb_1 + t - 1 & \text{for } i = [\tfrac{1}{2}v] + 1, \ldots, 2[\tfrac{1}{2}v], \end{cases} \tag{2.5}$$

and, if v is odd,

$$r_{d_1 v} = r_{d^*} = mb_1 + t,$$

where $r_{d^*} = b_1 b_2 / v$ is the replication of d^*, $m = [b_2/v]$, t is an integer, $1 < t \leq b_1 - 2$ and $r_{d_1 i}$ is the replication of the i-th treatment in d_1, $i = 1, 2, \ldots, v$.

It is not difficult to see that

$$b_2[b_1/v] \leq r_{d_1 i} \leq b_2([b_1/v] + 1) \quad \text{for } i = 1, 2, \ldots, v,$$

and

$$\sum_{i=1}^{v} r_{d_1 i} = b_1 b_2 .$$

Hence the replications in (2.5) satisfy the conditions of Lemma 2.1 and there exists a block design d_0 with v treatments, b_2 blocks and block size b_1, having replications given by (2.5). If we write the blocks of this design as columns, the resultant row-column design $d_1 \in \mathscr{D}(v, b_1, b_2)$, and the treatment-column incidence matrix of d_1 has only two entires, $x=[b_1/v]$ and $x+1$. In view of Lemma 2.2 and (2.5), we can rearrange the treatments in the columns of d_1 to get a row-column design $\bar{d}$ such that the treatments appear in each row of $\bar{d}$, m or $(m+1)$ times.

Now from (1.1), it is observed that the first three terms on the right-hand side of (1.1) contribute the same amount to $\operatorname{tr}(C_d)$ when $d=d^*$ or $\bar{d}$. Also,

$$\operatorname{tr}(r_{d^*} r_{d^*}') < \operatorname{tr}(r_{\bar{d}} r_{\bar{d}}')$$

and hence, $\operatorname{tr}(C_{\bar{d}}) > \operatorname{tr}(C_{d^*})$. Thus, d^* cannot be universally optimal in $\mathscr{D}(v, b_1, b_2)$, completing the proof.

Remark 1. Note that in (2.5), t cannot be equal to $b_1 - 1$. Since the setting is nonregular, $b_1 > v$ and with $t = b_1 - 1$,

$$r_{d^*} = b_1 b_2 / v = b_1 [b_2/v] + b_1 - 1,$$

giving

$$b_2 = v([b_2/v] + 1) - v/b_1 .$$

Thus, if $t = b_1 - 1$, b_2 cannot be integral. On similar lines, one can show that $t > 1$ and thus $1 < t \le b_1 - 2$.

We illustrate Theorem 2.1 by an example.

Example 2.1. Let d^* be the GYD with parameters $v=6$, $b_1=10$, $b_2=15$, reported by Ash (1981, p. 17). Corresponding to this GYD, we get another design $\bar{d}$, given below:

$$\bar{d} = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 1 & 2 & 3 & 4 & 5 & 6 & 3 & 4 & 2 \\ 5 & 3 & 4 & 5 & 6 & 1 & 2 & 3 & 4 & 5 & 6 & 1 & 2 & 3 & 1 \\ 3 & 4 & 5 & 6 & 1 & 2 & 3 & 4 & 5 & 6 & 1 & 2 & 4 & 1 & 5 \\ 4 & 5 & 6 & 1 & 2 & 3 & 4 & 5 & 6 & 2 & 1 & 3 & 1 & 6 & 4 \\ 5 & 6 & 1 & 2 & 3 & 4 & 5 & 6 & 2 & 1 & 3 & 4 & 6 & 5 & 3 \\ 6 & 1 & 2 & 3 & 4 & 5 & 2 & 1 & 3 & 3 & 4 & 5 & 5 & 2 & 6 \\ 3 & 5 & 1 & 6 & 5 & 4 & 3 & 2 & 1 & 6 & 4 & 2 & 4 & 5 & 3 \\ 6 & 3 & 2 & 1 & 2 & 5 & 1 & 4 & 5 & 4 & 6 & 3 & 5 & 4 & 2 \\ 2 & 1 & 5 & 4 & 6 & 1 & 4 & 3 & 6 & 3 & 2 & 5 & 6 & 2 & 1 \\ 1 & 2 & 4 & 2 & 1 & 3 & 6 & 5 & 4 & 5 & 3 & 6 & 3 & 6 & 4 \end{matrix}$$

Routine computations yield 150 $\mathrm{tr}(C_{d^*}) = 18300 < 18306 = 150\,\mathrm{tr}(C_{\bar{d}})$, showing that d^* is not universally optimal in $\mathscr{D}(6, 10, 15)$.

Remark 2. In (2.5), the replications can also be taken in the following manner:

$$r_{d_1 1} = r_{d^*} + 1 = mb_1 + t + 1,$$
$$r_{d_1 2} = r_{d^*} - 1 = mb_1 + t - 1,$$
$$r_{d_1 i} = r_{d^*} = mb_1 + t \quad \text{for } i = 3, 4, \ldots, v. \tag{2.6}$$

With replications as in (2.6) we can construct a design $\bar{\bar{d}}$ such that $\mathrm{tr}(C_{\bar{d}}) \geq \mathrm{tr}(C_{\bar{\bar{d}}}) > \mathrm{tr}(C_{d^*})$.

We next prove:

Theorem 2.2. *Under regular settings with $v > 2$, a GYD (if it exists) is the only universally optimal design in $\mathscr{D}(v, b_1, b_2)$.*

Proof. Let $d^* \in \mathscr{D}(v, b_1, b_2)$ be a regular GYD. Then, $C_{d^*} = C_{d^*}^{\mathrm{N}}$, $\mathrm{tr}(C_{d^*}) = \max_{d \in \mathscr{D}} \mathrm{tr}(C_d)$ and C_{d^*} is completely symmetric. It then follows that for any universally optimal design d, C_d must have maximum trace and constant nonzero eigenvalues. Then, $C_d^{\mathrm{N}} = C_d$ and C_d^{N} has maximum trace among all connected block designs and also has constant nonzero eigenvalues. It then follows from Das and Dey (1989a) that d^{N} is a BBD. Hence the result.

Remark 3. The condition $v > 2$ in Theorem 2.2 is necessary, since for $v = 2$, it is possible to have a design in the regular setting that is universally optimal but is *not* a GYD. For example, the following design $d \in \mathscr{D}(2, 3, 6)$ is universally optimal, but is clearly not a GYD:

$$d = \begin{matrix} B & B & A & A & B & B \\ B & A & B & B & A & B \\ A & B & B & B & B & A \end{matrix}$$

A, B being the treatment symbols.

On lines similar to Theorem 2.2, one can prove the following:

Theorem 2.3. *For $v > 2$, a BBD, whenever existent is uniquely universally optimal in the class of all connected block designs.*

Finally, we prove the following result.

Theorem 2.4. *Under nonregular settings, a PYD is never universally optimal in $\mathscr{D}(v, b, b)$.*

Proof. For $b>v$, the result can be proved on the lines of that of Theorem 2.1 and is therefore, not repeated. We consider the case $b<v$ (the case $b=v$ does not arise because the setting is non-regular). It is not difficult to see that a PYD, d^*, with $b<v$ has necessarily the following parameters:

$$v=p^2/n, \quad b=p(p-n)/n, \qquad p>n, \tag{2.7}$$

where p, n are integers such that $n \mid p^2$. In fact, $p>n+1$ as the design with $p=n+1$ can be seen to be non existent.

Setting $r_{d^*}=b^2/v$, we have

$$r_{d^*}=(p-n)^2/n<p(p-n)/n-1=b-1. \tag{2.8}$$

Now, let $d_1 \in \mathscr{D}(v,b,b)$ be a design such that

$$r_{d_1 i}=\begin{cases} r_{d^*}+1 & \text{for } i=1,2,\ldots,[\frac{1}{2}v] \\ r_{d^*}-1 & \text{for } i=[\frac{1}{2}v]+1,\ldots,2[\frac{1}{2}v] \end{cases}$$

and, if v is odd,

$$r_{d_1 v}=r_{d^*}.$$

Then, proceeding on the lines of the proof of Theorem 2.1, we can show that corresponding to d_1, there exists a row-column $\bar{d}_1$ in $\mathscr{D}(v,b,b)$ such that $\mathrm{tr}(\boldsymbol{C}_{\bar{d}_1})>\mathrm{tr}(\boldsymbol{C}_{d^*})$. This completes the proof.

We illustrate this result by taking an example.

Example 2.2. Consider the following PYD, d^*, with $v=9$, $b=6$, reported by Cheng (1981a) and also by Kshirsagar (1957):

$$d^* = \begin{matrix} 4 & 7 & 8 & 6 & 9 & 5 \\ 3 & 1 & 2 & 8 & 7 & 9 \\ 2 & 5 & 1 & 3 & 6 & 4 \\ 9 & 3 & 6 & 2 & 5 & 8 \\ 7 & 6 & 9 & 4 & 1 & 3 \\ 5 & 8 & 4 & 7 & 2 & 1 \end{matrix}$$

Corresponding to this PYD, we can get a design $\bar{d}_1$ as follows:

$$\bar{d}_1 = \begin{matrix} 3 & 6 & 8 & 2 & 1 & 4 \\ 6 & 4 & 1 & 7 & 3 & 9 \\ 7 & 9 & 2 & 6 & 4 & 1 \\ 8 & 5 & 3 & 1 & 9 & 2 \\ 1 & 3 & 4 & 8 & 2 & 5 \\ 2 & 7 & 9 & 4 & 5 & 3 \end{matrix}$$

It is seen that $36\,\mathrm{tr}(C_{\bar{d}_1}) = 1016 > 1008 = 36\,\mathrm{tr}(C_{d^*})$ and thus, d^* is not universally optimal in $\mathscr{D}(9,6,6)$.

PYD's with $b<v$ are obviously of great practical utility. From the definition of a PYD, one can readily show that a necessary condition for the existence of a PYD, with $b<v$ is that $p \geq 2n$ and

$$2(p-n)^2\{(p-n)p-n\}/\{n(p^2-n)\}$$

is an integer, where p and n are as defined earlier. In view of these conditions, only two PYD's with $v \leq 100$ can possibly exist. These have parameters $v=9$, $b=6$ and $v=49$, $b=28$. Both these designs do exist, as they belong to the general series of PYD's reported by Cheng (1981b).

Acknowledgement

The authors wish to thank a referee for his careful reading and very useful suggestions on a previous version.

References

Agrawal, H. (1966). Some generalizations of distinct representatives with applications to statistical designs. *Ann. Math. Statist.* **37**, 525–528.

Ash, A.S. (1981). Generalized Youden designs: Construction and tables. *J. Statist. Plann. Inference* **5**, 1–25.

Cheng, C.S. (1981a). Optimality and construction of pseudo Youden designs. *Ann. Statist.* **9**, 201–205.

Cheng, C.S. (1981b). A family of pseudo Youden designs with row size less than the number of symbols. *J. Combin. Theory Ser. A* **31**, 219–221.

Das, A. and A. Dey (1989a). A note on balanced block designs. *J. Statist. Plann. Inference* **22**, 265–268.

Das, A. and A. Dey (1989b). A generalization of distinct representatives and its applications. *Calcutta Statist. Assoc. Bull.* **38**, 57–63.

Kiefer, J. (1958). On the nonrandomized optimality and randomized non-optimality of symmetrical designs. *Ann. Math. Statist.* **29**, 675–699.

Kiefer, J. (1975). Construction and optimality of generalized Youden designs. In: J.N. Srivastava, ed., *A Survey of Statistical Designs and Linear Models.* North-Holland, Amsterdam, 333–353.

Kshirsagar, A.M. (1957). On balancing in designs in which heterogeneity is eliminated in two directions. *Calcutta Statist. Assoc. Bull.* **7**, 469–476.