HYPOTHESES CONCERNING THE MEANS OF OBSERVATIONS IN NORMAL SAMPLES

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1. Introduction

It is usual, as in Student's hypothesis, to regard all the observations in a normal sample to have the same mean and standard deviation. That is, they are taken as independent and identically distributed and the hypothesis concerning the normal population to which the sample belongs concerns the mean or the standard deviation or the ratio of the two. If, however, the observations are normal but not identically distributed, we can postulate several hypotheses regarding the mean and s.d. of each of them. Some of these hypotheses could be tested by the common t- or χ^2-tests.

Suppose $x_1, x_2, \ldots, x_n$ are independent normal variates and we consider the hypothesis that $E(x_i) = \mu_i$, $i = 1$ to n, given that var $(x_i) = \sigma^2$ (unknown) for all i. Let $\bar{x} = \Sigma x_i/n$ and $\bar{\mu} = \Sigma \mu_i/n$. Then $(\bar{x}-\bar{\mu})\sqrt{n}/\sigma$ is distributed as $N(0, 1)$. An unbiased estimate of σ^2 is found in $s^2 = [\Sigma(x_i-\mu_i)^2 - n(\bar{x}-\bar{\mu})^2]/(n-1)$. It can be easily shown by the method of transformation of the variates $(x_i-\mu_i)$, that $[\Sigma(x_i-\mu_i)^2 - n(\bar{x}-\bar{\mu})^2]$ is distributed as $\chi^2\sigma^2$ with $(n-1)$ degrees of freedom independently of $n(\bar{x}-\bar{\mu})^2$. Hence the ratio

$$\frac{(\bar{x}-\bar{\mu})\sqrt{n}}{s} \qquad \ldots \quad (1)$$

is distributed as a t with $(n-1)$ degrees of freedom. If we have an alternative hypothesis specifying that $E(x_i) = \mu_i'$ such that $\mu_i'/\mu_i =$ constant for all i, then the new centre of the distribution will lie on the equi-angular line through (μ_i), namely, $x_1-\mu_1 = x_2-\mu_2 = \ldots = x_n-\mu_n$; and as in the case of equal μ's, the best critical region for the test will be a right-circular cone with vertex at (μ_i) and axis along this line. Thus the usual t-test using the statistic (1) provides the unbiased most powerful test of the hypothesis $E(x_i) = \mu_i$, with respect to alternatives specifying a set of means proportional to the null set. It is obvious that with more general alternative hypotheses concerning the means, the above test is not likely to be the best.

Next suppose we have the hypothesis that var $(x_i) = \sigma_i^2$, $i = 1$ to n, given that the standardised means of the x's are equal, that is, that $E(x_i/\sigma_i) = \rho$(unknown). If we write $y_i = x_i/\sigma_i$, the hypothesis becomes var $(y_i) = 1$, given that the means of y_i are the same. Thus the criterion $\chi^2 = \Sigma(y_i-\bar{y})^2$ can be used to test the above hypothesis. If we consider alternatives specifying var $(x_i) = \sigma_i'^2$ such that $\sigma_i'/\sigma_i =$ a constant for all i, then the χ^2-test is the best test with respect to such alternatives but this may not be so for a more general class of alternatives.

We shall now consider another type of hypothesis on the x's which specifies their standardised means, that is, the ratios of the means to the corresponding s.d.'s, viz., $\frac{E(x_i)}{\sqrt{\text{var}(x_i)}} = \rho_i$. The hypothesis is now most general and cannot be tested by any of the current tests. If, however, it is given that the variance of x_i is the same for all i, then similar regions with respect to the unspecified σ exist. We shall examine this problem in this paper and show how to construct a certain class of critical regions for testing the hypothesis regarding the standardised means.

A special case of the above hypothesis arises when all the x's in the sample are identically normally distributed, so that the hypothesis concerns a single normal population with unknown s.d. but whose standardised mean is ρ. A test of this hypothesis was considered by the present writer (Patnaik, 1952). It was shown that a region between hypercones with vertices at the origin ($x_i = 0$) and axes along the equiangular line is similar with regard to σ and that the best critical region from this class for testing $\rho = \rho_0$ is the interior of a single cone when one-sided alternatives, either $\rho > \rho_0$ or $\rho < \rho_0$ are admissible. The problem of a general class of similar regions is studied in this paper and it is shown that the above region is the best.

The formulation of the test for ρ would involve the non-central t distribution. Johnson and Welch (1939) employed this distribution for obtaining a test of the hypothesis specifying the coefficient of variation $V(= 100/\rho)$ in a normal population. They have provided tables which, however, are not suitable for applying the test. Their tables give values of V for which $P(v \geqslant v_0 | V) = \epsilon$, v being the sample coefficient of variation. We have derived from these tables a table of upper and lower 5% points of t' (non-central t). Also approximate values can be found from a fairly good approximation to the t'-distribution considered in this paper.

When testing for $\rho = \rho_0$, both-sided alternatives $\rho > \rho_0$ and $\rho < \rho_0$ may be admissible. The method of constructing unbiased critical regions is discussed and an unbiased locally most powerful test obtained. The critical values of t' for this test may be obtained by using the t approximation to the t'-distribution.

Suppose we have a slightly more general hypothesis than the one just considered, namely, that m of the n observations have standardised means equal to ρ and the rest have $\rho + \gamma$, given as before, that the s.d. is the same for all. This means that the sample is not homogeneous, a part of it having one mean and the rest a different mean. If the presence of such 'rogue' observations is not recognised and the usual t-test for the mean is employed, we may get quite misleading conclusions. Even a few extraneous elements in a sample are likely to vitiate the test. As it is not possible to know which are the rogue observations, we cannot throw them out and make the sample homogeneous. We shall obtain in this paper a test for the standardised means in such mixed samples.

In practical situations it is likely that we suspect a few of the observations in a sample to have a known bias in the mean, as for instance, when the observations are taken by two experimenters one of whom has a subjective bias known from previous experience but which cannot be allowed for, since the observations are mixed up. In such a case we shall consider a two-sample procedure to test the hypothesis concerning the means.

The usual t-test for the mean is known to be the best when an alternative hypothesis specifies a different mean. But clearly, it will not be the best with respect to generalised alternatives specifying a different mean for each observation. This problem is also examined in this paper.

2. Test for the standardised mean when observations are normal and identically distributed

2.1. *Construction of similar regions*

Neyman's mechanism of constructing similar regions by combining sections from suitable ϕ-surfaces in the sample space does not work in the present case as there is no statistic ϕ sufficient for σ. The probability density function

$$p(x_1, x_2, \ldots, x_n; \sigma) = \frac{1}{(\sqrt{2\pi}\sigma)^n} \cdot e^{-\frac{1}{2\sigma^2}\Sigma(x_i - \rho_0)^2}$$

does not satisfy the condition for the existence of a sufficient statistic, namely,

$$p(x_1, x_2, \ldots, x_n; \sigma) = p_1\, e^{\Theta_1 g(\phi) + \Theta_2}$$

in which p_1 and $g(\phi)$ should be functions of $x_1, x_2, \ldots x_n$ only and Θ_1, Θ_2 functions of σ.

We shall show, following a different method, how to construct a general class of similar regions with respect to σ. Suppose D is a similar region of mass α under the hypothesis $\rho = \rho_0$. Then from (2), we have

$$\frac{1}{(\sqrt{2\pi}\sigma)^n} \int \underset{D}{\cdots} \int e^{-\frac{1}{2\sigma^2}\Sigma(x_i - \rho_0\sigma)^2} dx_1 \ldots dx_n = \alpha. \qquad \ldots (3)$$

First, we shall consider the case when D is a single connected region which is met by any straight line through the origin in not more than two points. The surface bounding D must be independent of σ.

Make the following transformation:

$$\left.\begin{aligned} x_1 &= r\cos\theta_1 \cos\theta_2 \;\ldots\; \cos\theta_{n-1}, \\ x_2 &= r\cos\theta_1 \ldots \cos\theta_{n-2}\sin\theta_{n-1}, \\ &\ldots \quad \ldots \quad \ldots \\ x_n &= r\sin\theta_1. \end{aligned}\right\} \qquad \ldots (4)$$

Then $r^2 = \Sigma x_i^2$, i.e. r is the distance of the point (x_i) from the origin. The Jacobian of the transformation simplifies into

$$r^{n-1} \cos^{n-2}\theta_1 \cos^{n-3}\theta_2 \ldots \cos\theta_{n-2}.$$

Consider the cone C with vertex at the origin enveloping the region D. In terms of the new coordinates let the region C be defined by $0 < r < \infty$ and $\phi(\theta_1, \ldots, \theta_{n-1}) < 0$. Let the line of contact of this enveloping cone with the surface of D divide it into two parts represented by the equations $r = r_1(\theta_1, \ldots, \theta_{n-1})$ and $r = r_2(\theta_1, \ldots, \theta_{n-1})$. Hence D will be defined by

$$r_1(\theta_1, \ldots, \theta_{n-1}) < r < r_2(\theta_1, \ldots, \theta_{n-1}) \text{ and } \phi(\theta_1, \ldots, \theta_{n-1}) \leqslant 0.$$

Thus under the transformation (4), (3) becomes

$$\frac{1}{(\sqrt{2\pi}\sigma)^n} \int \cdots \int_{\phi(\theta_1, \ldots, \theta_{n-1}) < 0} \int_{r_1(\theta_1, \ldots, \theta_{n-1})}^{r_2(\theta_1, \ldots, \theta_{n-1})} \text{Exp}\left\{-\tfrac{1}{2}\left[r^2 + n\rho_0^2\sigma^2 - 2\rho_0\sigma r(\cos\theta_1 \ldots \cos\theta_{n-1} + \ldots + \sin\theta_{n-1})\right]\right\}$$

$$\times r^{n-1} \cos^{n-2}\theta_1 \ldots \cos\theta_{n-2}\, dr d\theta_1 \ldots d\theta_{n-1} = \alpha. \quad \ldots \quad (5)$$

From the fact that the above integral is a constant for all values of σ, we shall show that $r_1 = 0$ and $r_2 = \infty$ and derive the result that D is identical with the conical region C.

Putting $r/\sigma = R$, (5) becomes

$$\frac{1}{(\sqrt{2\pi})^n} \int \cdots \int_{\phi(\theta_1, \ldots, \theta_{n-1}) < 0} \int_{\frac{r_1(\theta_1, \ldots, \theta_{n-1})}{\sigma}}^{\frac{r_2(\theta_1, \ldots, \theta_{n-1})}{\sigma}} \text{Exp}\left\{-\tfrac{1}{2}\left[R^2 + n\rho_0^2 - 2R\rho_0(\cos\theta_1 \ldots \cos\theta_{n-1} + \ldots + \sin\theta_{n-1})\right]\right\}$$

$$\times R^{n-1} \cos^{n-2}\theta_1 \ldots \cos\theta_{n-2}\, dR d\theta_1 \ldots d\theta_{n-1} = \alpha$$

which we shall write for convenience as

$$\int \cdots \int_{\phi < 0} \int_{r_1/\sigma}^{r_2/\sigma} f(R, \theta) dR d\theta = \alpha. \quad \ldots \quad (6)$$

In this integral both r_1 and r_2 cannot be finite. For, let σ have an indefinitely large value. Since r_1 and r_2 are independent of σ (the surface of D being independent of σ), the limits of R will then become indefinitely small and therefore the integral in (6) will tend to zero. This is impossible as it violates the condition (6). Hence one of the limits must be infinite and this will be the upper limit, as the integrand is positive.

Now putting $r_2 = \infty$ in the integral, we see that r_1 cannot be greater than zero; for, by making σ indefinitely small, the lower limit can be made indefinitely large and then the integral will tend to zero. Hence $r_1 = 0$. This means that any straight line through the origin meets the surface of D at distances zero and infinity. Hence the similar region is the cone C.

Next we shall consider the case when a similar region consists of two disconnected domains D_1 and D_2 each of which is met by any line through the origin in not more than two points. Corresponding to (6), we now have the relation

$$\int_{\phi_1<0}\cdots\int \int_{r_1^{(1)}/\sigma}^{r_2^{(1)}/\sigma} f(R,\theta)dR\,d\theta + \int_{\phi_2<0}\cdots\int \int_{r_1^{(2)}/\sigma}^{r_2^{(2)}/\sigma} f(R,\theta)dR d\theta = \alpha. \qquad \cdots \quad (7)$$

In the above integrals the r's cannot all be finite. For, by taking σ indefinitely large, the left side of (7) tends to zero. Hence at least one of the integrals should not vanish. Suppose this is the first one; as before, we see that its upper limit should be infinity, i.e., $r_2^{(1)} = \infty$. Now as σ tends to zero, both integrals would tend to vanish unless $r_1^{(1)}$ or $r_1^{(2)}$ is zero. If $r_1^{(1)} = 0$, the domain of the first integral, D_1 becomes a cone and so has a fixed mass. This can be seen to be α from the fact that the second integral vanishes as σ tends to infinity. Hence the similar region is the single cone D_1, the other domain having zero mass for all σ.

Instead of $r_1^{(1)}$, $r_1^{(2)}$ may be zero. Then (7) can be written as

$$\int_{\phi_1<0}\cdots\int \int_{r_1^{(1)}/\sigma}^{\infty} f(R,\theta)dR\,d\theta + \int_{\phi_2<0}\cdots\int \int_{0}^{r_2^{(2)}/\sigma} f(R,\theta)dR d\theta = \alpha. \qquad \cdots \quad (8)$$

On differentiating both sides of (8) with respect to σ, we get

$$-\int_{\phi_1<0}\cdots\int f\left(\frac{r_1^{(1)}}{\sigma},\theta\right)d\theta + \int_{\phi_2<0}\cdots\int f\left(\frac{r_2^{(2)}}{\sigma},\theta\right)d\theta = 0, \qquad \cdots \quad (9)$$

which must be true for all values of σ. Looking at the form of the expression $f(R,\theta)$ in (6), we see that there are only two possibilities, namely,

(a) $r_1^{(1)} = 0$ and $r_2^{(2)} = \infty$, for then each of the integrals on the left of (9) vanishes. Then the similar region consists of the two cones D_1 and D_2.

(b) $r_1^{(1)} = r_2^{(2)}$ and the domains of integration $\phi_1(\theta_1,\ldots,\theta_{n-1}) \leqslant 0$ and $\phi_2(\theta_1,\ldots,\theta_{n-1}) \leqslant 0$ are equivalent. This does not mean that the cones $\phi_1 \leqslant 0$, $0 \leqslant r \leqslant \infty$ and $\phi_2 \leqslant 0$, $0 \leqslant r \leqslant \infty$ should be identical. They can have different positions in the sample space, but their 'angles' are such that the masses are equal. For if we put $r_1^{(1)} = r_2^{(2)}$ in (8), the first integral becomes α and similarly if $r_1^{(1)} = r_2^{(2)} = \infty$, the second integral is α. Thus D_1 and D_2 are parts of the above cones and are defined by $\phi_1 \leqslant 0$, $r_1(\theta_1,\ldots,\theta_{n-1}) \leqslant r \leqslant \infty$ and $\phi_2 \leqslant 0$, $0 \leqslant r \leqslant r_1(\theta_1,\ldots,\theta_{n-1})$. Since $\phi_2 \leqslant 0$ is equivalent to $\phi_1 \leqslant 0$, the two integrals giving their masses can be combined into one integral, and in the place of (8) we have

$$\int_{\phi_i<0} \cdots \int \int_0^\infty f(R, \theta) dR d\theta = \alpha.$$

In the most general case when several domains D_1, D_2, ..., finite or infinite in number, constitute a similar region, we have, with a notation similar to that in (8), the condition

$$\sum_j \int_{\phi_j<0} \cdots \int \int_{r_1^{(j)}/\sigma}^{r_2^{(j)}/\sigma} f(R, \theta) dR d\theta = \alpha. \qquad \dots \quad (10)$$

Suppose D is a single domain which contains all the domains D_1, D_2, ... and is defined by $\phi < 0$ and $r_1 < r < r_2$. Then the left side of (10) is less than or equal to

$$\int_{\phi<0} \cdots \int \int_{r_1/\sigma}^{r_2/\sigma} f(R, \theta) dR d\theta.$$

Now as σ tends to zero or infinity, this integral tends to zero and therefore, the left side of (10) tends to zero. Following the same line of argument as in the case of two domains, we can show that some of the domains D_1, D_2,... are complete cones and the others sections of cones whose total mass, however, will be equal to that of a single complete cone for every σ.

2.2. *Best Critical regions*

We shall now show that amongst the regions similar with respect to σ discussed in the last subsection, the best critical region for testing the hypothesis H_0: $\rho = \rho_0$ against an alternative hypothesis H_1: $\rho = \rho_1$ is the interior of a right-circular cone with vertex at the origin and axis along the equi-angular line $x_1 = x_2 = \dots = x_n$.

Let C be a similar region consisting of complete cones and sections of cones. We have shown that its mass can be expressed as a single multiple integral. Thus under H_0, we have from (5),

$$\frac{1}{(\sqrt{2\pi}\sigma)^n} \int_{\phi<0} \cdots \int \int_0^\infty \text{Exp}\left\{-\frac{1}{2\sigma^2}\left[r^2 + n\rho_0^2\sigma^2 - 2\sqrt{n}\rho_0\sigma \cos\theta\right]\right\}$$

$$\times r^{n-1} \cos^{n-2}\theta_1 \dots \cos\theta_{n-2} dr d\underset{\sim}{\theta_i} = \alpha. \qquad \dots \quad (11)$$

Here $\sqrt{n}\cos\theta$ is written for $(\cos\theta_1\ldots\cos\theta_{n-1}+\ldots+\sin\theta_1)$ in the integrand so that θ is the angle between the line joining the origin to the point $(r, \theta_1,\ldots,\theta_{n-1})$ and the equi-angular line as seen from the transformation (4).

Under H_1 $(\rho=\rho_1, \sigma=\sigma_1)$, we have

$$\frac{1}{(\sqrt{2\pi}\sigma_1)^n}\int_{\phi<0}\!\!\ldots\!\int\int_0^\infty \text{Exp}\left\{-\frac{1}{2\sigma_1^2}\left[r^2+n\rho_1^2\sigma_1^2-2\sqrt{n}\rho_1\sigma_1\cos\theta\right]\right\}$$

$$\times r^{n-1}\cos^{n-2}\theta_1\ldots\cos\theta_{n-2}\,dr\,d\underset{\sim}{\theta_i}=\beta.$$

Now the best region is that for which β is a maximum subject to the condition (11). Following the usual reasoning, we find that the best region defined by $\phi<0$ is that for which

$$\frac{1}{(\sqrt{2\pi}\sigma_1)^n}\int_0^\infty \text{Exp}\left\{-\frac{1}{2\sigma_1^2}\left[r^2+n\rho_1^2\sigma_1^2-2\sqrt{n}\rho_1\sigma_1\cos\theta\right]\right\}r^{n-1}dr$$

$$>k\,\frac{1}{(\sqrt{2\pi}\sigma)^n}\int_0^\infty \text{Exp}\left\{-\frac{1}{2\sigma^2}\left[\left[r^2+n\rho_0^2\sigma^2-2\sqrt{n}\rho_0\sigma\cos\theta\right]\right]\right\}r^{n-1}dr,$$

where k is a constant depending on (11). Since the region is similar with respect to σ, we can take the same value on both sides of the inequality and put this equal to unity. Then the inequality becomes

$$\int_0^\infty e^{-\frac{1}{2}[(r^2+n\rho_1^2-2\sqrt{n}r\rho_1\cos\theta)]}r^{n-1}dr>k\int_0^\infty e^{-\frac{1}{2}[r^2+n\rho_0^2-2\sqrt{n}r\rho_0\cos\theta]}r^{n-1}dr.$$

On performing the integration, we get

$$\sum_{j=0}^{\infty}\frac{e^{-\frac{n\rho_1^2}{2}}(\sqrt{2n}\rho_1\cos\theta)^j\Gamma\left(\frac{n+j}{2}\right)}{j!}>k\sum_{j=0}^{\infty}\frac{e^{-\frac{n\rho_0^2}{2}}(\sqrt{2n}\rho_0\cos\theta)^j\Gamma\left(\frac{n+j}{2}\right)}{j!}.\qquad\ldots\ (12)$$

Denoting the sum of the series on the left side as $f(\rho_1\cos\theta)$ and therefore the right side as $kf(\rho_0\cos\theta)$, it can be shown that $f(\rho_1\cos\theta)/f(\rho_0\cos\theta)$ is a monotonic function of $\cos\theta$.

Consider $f(\rho_1 x)/f(\rho_0 x)$ and differentiate with respect to x. The numerator $\rho_1 f'(\rho_1 x)f(\rho_0 x)-\rho_0 f'(\rho_0 x)f(\rho_1 x)$ is equal to

$$\sum_i \frac{(\sqrt{2n}\rho_1)^i x^{i-1}\Gamma\left(\frac{n+i}{2}\right)i}{i!}\sum_j \frac{(\sqrt{2n}\rho_0)^j x^j\Gamma\left(\frac{n+j}{2}\right)}{j!}-$$

$$-\sum_i \frac{(\sqrt{2n}\rho_1)^i x^i\Gamma\left(\frac{n+i}{2}\right)}{i!}\sum_j \frac{(\sqrt{2n}\rho_0)^j x^{j-1}\Gamma\left(\frac{n+j}{2}\right)j}{j!}$$

which simplifies to

$$\sum_i\sum_j \frac{\Gamma\left(\frac{n+i}{2}\right)\Gamma\left(\frac{n+j}{2}\right)}{i!\,j!}x^{i+j-1}(\sqrt{2n}\rho_0)^{i+j}\left[\left(\frac{\rho_1}{\rho_0}\right)^i-\left(\frac{\rho_1}{\rho_0}\right)^j\right](i-j).$$

This is greater than or equal to zero if $\rho_1 > \rho_0$ and less than or equal to zero if $\rho_1 < \rho_0$. Thus $f(\rho_1 \cos\theta)/f(\rho_0 \cos\theta)$ is monotonic increasing in $\cos\theta$ if $\rho_1>\rho_0$ and decreasing if $\rho_1<\rho_0$. Hence the relation (12) which is

$$\frac{f(\rho_1\cos\theta)}{f(\rho_0\cos\theta)} > k$$

is equivalent to

$$\cos\theta > \cos\theta_0, \text{ if } \rho > \rho_0$$

and

$$\cos\theta < \cos\theta'_0, \text{ if } \rho < \rho_0.$$

where θ_0 and θ'_0 depend on (11).

Since θ is the angle between the direction of the sample point and the line $x_1 = x_2 = \ldots = x_n$, it follows that the best critical region is a right circular cone with vertex at the origin and axis along this equi-angular line and semi-angle θ_0 when the alternative $\rho=\rho_1>\rho_0$ is considered, and is a similar cone with axis in the opposite direction and of semi-angle $(\pi-\theta'_0)$ when the alternative specifies $\rho=\rho_1<\rho_0$. Clearly, each of these regions is uniformly most powerful with respect to either one or the other of the families of alternative hypotheses, $\rho>\rho_0$ and $\rho<\rho_0$.

The values of θ_0 and θ'_0 are determined from the distribution of θ. This can be easily obtained from the following distribution of the sample mean and standard deviation:

$$p(\bar{x}, s)d\bar{x}ds = \frac{n^{n/2}}{2^{\frac{n}{2}-1}\Gamma\left(\frac{n-1}{2}\right)(\sqrt{2\pi}\sigma)^n}\exp\left\{-\frac{n}{2\sigma^2}\left[(\bar{x}-\rho\sigma)^2+s^2\right]\right\}s^{n-2}d\bar{x}ds.$$

Now $\sqrt{n}\bar{x} = r\cos\theta$ and $\sqrt{n}s = r\sin\theta$. So we get

$$p(r,\theta)drd\theta = \frac{1}{2^{\frac{n}{2}-1}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}\exp\left\{-\frac{1}{2}\left[\frac{r^2}{\sigma^2}-\frac{2\sqrt{n}r\rho}{\sigma}\cos\theta+n\rho^2\right]\right\}\times r^{n-1}\sin^{n-2}\theta drd\theta$$

which on integration with respect to r gives, as in (12), the desired distribution

$$p(\theta)d\theta = \sum_{j=0}^{\infty}\frac{e^{-\frac{n\rho^2}{2}}(\sqrt{2n}\rho)^j\Gamma\left(\frac{n+j}{2}\right)}{j!\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}\sin^{n-2}\theta\cos^j\theta\, d\theta.$$

The distribution of $v = \cos\theta\ (= \bar{x}/s)$ is therefore

$$p(v)dv = \sum_{j=0}^{\infty}\frac{e^{-\frac{n\rho^2}{2}}(\sqrt{2n}\rho)^j\Gamma\left(\frac{n+j}{2}\right)}{j!\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}\frac{v^j}{(1+v^2)^{\frac{n+j}{2}}}dv. \quad \ldots (13)$$

The two critical cones correspond to $v \geqslant v_0$ and $v \leqslant v_0'$ so that v_0 and v_0' are the lower and upper $100\alpha\%$ points of the above distribution.

In practice it is desirable to use the statistic $\frac{\bar{x}\sqrt{n-1}}{s}$ for testing the hypothesis $\rho = \rho_0$. This is a non-central t ratio, denoted as t', with $(n-1)$ degrees of freedom and parameter $\sqrt{n}\rho$. Its distribution follows (13) since $t' = v\sqrt{n-1}$. Thus the test procedure consists in comparing the observed value of $\bar{x}\sqrt{n-1}\Big/\sqrt{\left(\frac{\Sigma(x_i-\bar{x})^2}{n}\right)}$ with the appropriate percentage points of t'. We shall in the next subsection study some of the properties of this distribution and methods of finding its percentage points.

2.3. *Percentage points of t'*

Suppose t_α' is the upper $100\alpha\%$ point of the t'-distribution

$$p(t')dt' = \sum_{j=0}^{\infty}\frac{\Gamma\left(\frac{\nu+j+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\nu}{2}\right)j!}\frac{e^{-\delta^2/2}(\sqrt{2}\delta)^j}{\nu^{\frac{j+1}{2}}}\frac{t'^j}{\left(1+\frac{t'^2}{\nu}\right)^{\frac{\nu+j+1}{2}}}dt' \quad \ldots (14)$$

with non-central parameter $\delta = \sqrt{\nu+1}\,\rho$ and degrees of freedom $\nu = n-1$. Then

$$P\left\{\frac{\bar{x}\sqrt{\nu}}{s} > t_\alpha'\right\} = \alpha,$$

which can be written as

$$P\left\{\frac{(\bar{x}-\mu)\sqrt{\nu}}{\sigma}+\rho\sqrt{\nu} > t_\alpha'\frac{s}{\sigma}\right\} = \alpha.$$

This is the same as $$P\left\{-\frac{(\bar{x}-\mu)\sqrt{\nu+1}}{\sigma}+t'_\alpha\frac{s}{\sigma}\frac{\sqrt{\nu+1}}{\sqrt{\nu}}<\delta\right\}=\alpha.$$

Denoting by u the expression on the left side of the inequality in the above, we see that u is distributed approximately normal since $\sqrt{\Sigma(x_i-\bar{x})^2}/\sigma$ is like a normal deviate even for moderately large n. Using the Cornish-Fisher method (1937), we can obtain accurate values of the percentage points of

$$\xi=\frac{u-t'_\alpha}{\sqrt{(1+t'^2_\alpha/2\nu)}}\,.$$

If ξ_α is the lower 100α % point of ξ, then on solving this quadratic

$$\xi_\alpha\sqrt{(1+t'^2_\alpha/2\nu)}+t'_\alpha=\delta$$

we have $$t'_\alpha=\frac{\delta+\xi_\alpha\left(1+\frac{\delta^2}{2\nu}-\frac{\xi^2_\alpha}{2\nu}\right)^{1/2}}{\left(1-\frac{\xi^2_\alpha}{2\nu}\right)} \quad \ldots \quad (15)$$

The lower 100α% point of t' is given by an expression similar to (15).

Johnson and Welch (1939) have given a table of values of ξ_α for $\alpha = 0.05$. Using these values, the upper and lower 5% points are calculated for $\nu = 4$ (1) 15, 20 and $\delta = 0(0.5)$ 4.0 and are given in Tables I and II in the Appendix. From the degree of accuracy of the values of ξ, it is expected that the figures in these tables are correct to the number of decimal places shown. For intermediate values of δ, linear interpolation seems to be adequate. If ν exceeds 20, by taking a normal deviate in place of ξ_α in (15), sufficiently accurate values of t'_α may be obtained.

Dudding and Jennett (1942) have given charts for reading off the upper and lower 2.5% and 0.1% limits of $u=\frac{\bar{x}-L}{s}$ for particular values of $U=\frac{\mu-L}{\sigma}$ lying between 0 and 3 and for $n = 5, 7, 10, 15$ and 20. These charts were provided by Johnson and Welch on the basis of the approximation given in (15) in which ξ_α is a normal percentage point.

We shall now consider an approximation to the distribution of t' from which any percentage limit of t' can be easily derived. The first five moments about the origin of the t'-distribution in (14) are

$$\left.\begin{aligned}\mu'_1&=\frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\frac{\sqrt{\nu}\delta}{\sqrt{2}};\quad \mu'_2=\frac{1+\delta^2}{\nu-2}\cdot\nu;\quad \mu'_3=\frac{3+\delta^2}{\nu-3}\nu.\mu'_1;\\ \mu'_4&=\frac{3+6\delta^2+\delta^4}{(\nu-2)(\nu-4)}\nu^2;\quad \mu'_5=\frac{15+10\delta^2+\delta^4}{(\nu-3)(\nu-5)}\nu^2.\mu'_1.\end{aligned}\right\} \quad \ldots \quad (16)$$

It is known that the distribution tends to the normal as $\nu \to \infty$. If δ^2 is small compared with ν, it can be seen from the expressions for the moments that the distribution of $(t'-E(t'))$ is very close to a t-distribution. We shall therefore approximate the distribution of $(t'-E(t'))$ by that of ct where c is a constant and m is the degrees of freedom of t. To find c and m we equate the second and fourth moments, thus

$$\left.\begin{aligned} \frac{c^2 m}{m-2} &= \frac{1+\delta^2}{\nu-2}\,\nu-\mu_1'^2, \\ \frac{3c^4 m^2}{(m-2)(m-4)} &= \frac{3+6\delta^2+\delta^4}{(\nu-2)(\nu-4)}\,\nu^2-\mu_1'^2\left(\frac{12+4\delta^2}{\nu-3}\,\nu-\frac{6+6\delta^2}{\nu-2}\,\nu+3\mu_1'^2\right) \end{aligned}\right\} \quad \ldots \quad (17)$$

where μ_1' is given in (16). If further we equate the third moments, we get an equation which is inconsistent with the two above.

Substituting the values of ν and δ on the right side expressions in (17) and solving the equations, we get unique values of m and c. The probability integral of t' can be obtained approximately from that of t, for,

$$\int^{t'_\alpha} p_\nu(t'\,|\,\delta)\,dt' = \int^{\frac{t'_\alpha-\mu_1'}{c}} p_m(t)\,dt.$$

and so the upper $100\alpha\%$ point of t' is given by $(ct_\alpha+\mu_1')$, where t_α is $100\times 2\alpha\%$ point of t with m degrees of freedom. Similarly, the lower percentage point is given by $(ct_\alpha+\mu_1')$.

These approximations for the probability integral and percentage points could be improved by using the third, fifth and higher cumulants of t'. Let $k_1, k_2, k_3, k_4, \ldots$ stand for the cumulants of t'. Then the cumulants of $\xi = (t'-k_1)/\sqrt{k_2}$ are 0, 1, $k_3/k_2^{3/2}$, k_4/k_2^2, $k_5/k_2^{5/2}, \ldots$ Let $p(\xi)$ be the p.d.f. of ξ. Taking the fitted distribution of ct, the cumulants of $\xi = ct/\sqrt{k_2}$ are 0, 1, 0, k_4/k_2^2, 0, ... The fourth cumulant here is k_4, since in the fitting, both second and fourth cumulants are identified. Let the fitted density function of ξ be $f(\xi)$ and let $\alpha(\xi)$ be the normal density function $\frac{1}{\sqrt{2\pi}}e^{-\xi^2/2}$

The Gram-Charlier Type A series for $p(\xi)$ and $f(\xi)$ are the following (see Kendall I § 6.32)—

$$\begin{aligned} p(\xi) = \alpha(\xi)\Big[1+\frac{1}{6}\frac{k_3}{k_2^{3/2}}\xi_3+\left(\frac{1}{24}\frac{k_4}{k_2^2}\xi_4+\frac{1}{72}\frac{k_3^2}{k_2^3}\xi_6\right)- \\ -\left(\frac{1}{120}\frac{k_5}{k_2^{5/2}}\xi_5+\frac{1}{144}\frac{k_3k_4}{k_2^{7/2}}\xi_7+\frac{1}{1296}\frac{k_3^3}{k_2^{9/2}}\xi_9\right)+\ldots\Big] \end{aligned} \quad \ldots \quad (18)$$

$$f(\xi) = \alpha(\xi)\left[1+\frac{1}{24}\frac{k_4}{k_2^2}\xi_4-\ldots\right] \quad \ldots \quad (19)$$

where $\xi_3, \xi_4, \ldots$ are Hermite polynomials.

Subtracting (10) from (18), we get

$$p(\xi) = f(\xi)+\alpha(\xi)\Big[\frac{1}{6}\frac{k_3}{k_2^{3/2}}\xi_3+\frac{1}{72}\frac{k_3^2}{k_2^3}\xi_6+$$

$$+\Big(\frac{1}{120}\frac{k_5}{k_2^{5/2}}\xi_5+\frac{1}{144}\frac{k_3k_4}{k_2^{7/2}}\xi_7+\frac{1}{1296}\frac{k_3^3}{k_2^{9/2}}\xi_9\Big)+\ldots\Big].$$

Even though this series is not uniformly convergent, we shall integrate it term by term and use the first few terms. Thus, retaining terms up to $O(n^{-3/2})$, we have

$$\int_0^\xi p(\xi)d\xi = \int_0^\xi f(\xi)d\xi+\alpha(\xi)\Big[-\frac{1}{6}\frac{k_3}{k_2^{3/2}}\xi_2-\frac{1}{72}\frac{k_3^2}{k_2^3}\xi_5-$$

$$-\Big(\frac{1}{120}\frac{k_5}{k_2^{5/2}}\xi_4+\frac{1}{144}\frac{k_3k_4}{k_2^{7/2}}\xi_6+\frac{1}{1296}\frac{k_3^3}{k_2^{9/2}}\xi_8\Big)\Big].$$

This will give a good approximation for the probability integral of t'.

Similarly an improved approximation for the percentage points of t' is obtained by using the Cornish-Fisher inversion of the Gram-Charlier Series (1937). If t', ct, ξ are corresponding percentage points of the t', ct and normal distributions, we have

$$\frac{t'-k_1}{\sqrt{k_2}} = \xi+\frac{1}{6}\frac{k_3}{k_2^{3/2}}(\xi^2-1)+\Big\{\frac{1}{24}\frac{k_4}{k_2^2}(\xi^3-2\xi)-\frac{1}{36}\frac{k_3^2}{k_2^3}(2\xi^3-5\xi)\Big\}+\ldots$$

$$\frac{ct}{\sqrt{k_2}} = \xi+\frac{1}{24}\frac{k_4}{k_2^2}(\xi^3-2\xi)+\ldots$$

Hence differencing and retaining terms up to $O(n^{-3/2})$, we get on rearrangement

$$t' = k_1+ct+\frac{1}{6}\frac{k_3}{k_2}(\xi^2-1)-\frac{1}{36}\frac{k_3^2}{k_2^{5/2}}(2\xi^3-5\xi)+$$

$$+\Big\{\frac{1}{120}\frac{k_5}{k_2^2}(\xi^4-6\xi^2+3)-\frac{1}{24}\frac{k_3k_4}{k_2^3}(\xi^4-5\xi^2+2)+$$

$$+\frac{1}{324}\frac{k_3^3}{k_2^4}(12\xi^4-53\xi^2+17)\Big\}. \quad \ldots \quad (20)$$

This expression gives the upper percentage point. The lower point is got by reversing the signs of ct and ξ in the above.

A fairly good approximation for t' can be obtained by taking only the first four terms on the right of (20). Using this method, approximate 5 per cent points have been calculated in a few cases and shown in the following table along with exact values from the tables in the Appendix.

TABLE OF FIVE PER CENT POINTS OF t'

		upper points		lower points	
ν	δ	approx.	exact	approx.	exact
9	1	3.13	3.09	−0.63	−0.68
	2	4.51	4.45	0.46	0.36
	3	5.98	5.88	1.47	1.31
15	1	2.91	2.90	−0.65	−0.67
	2	4.12	4.04	0.39	0.36
	3	5.35	5.33	1.40	1.33
20	1	2.83	2.83	−0.63	−0.66
	2	3.99	3.98	0.37	0.36
	3	5.18	5.17	1.36	1.33

From the above table it is clear that the approximation using the first four cumulants of t' is correct to the first decimal place and the error in the second decimal place is likely to be very small for large ν. The expression (20) can be expected to give a much closer approximation.

2.4. *Unbiased tests for composite hypotheses*

The critical regions obtained in sec. 2.2 are the best for testing $\rho = \rho_0$ against alternatives $\rho > \rho_0$ or $\rho < \rho_0$. But each of these regions is biased when both-sided alternatives are relevant. This kind of situation is quite general with tests of composite hypotheses and we shall consider here the problem of constructing unbiased tests in such cases.

Suppose $p(x|\theta, \psi_1,...,\psi_k)$ is a probability density function admitting a sufficient statistic for ϕ for the parameters $\psi : \psi_1,...,\psi_k$. It can be shown that the Type A region for testing the hypothesis $\theta = \theta_0$ could be obtained by combining regions having the same Type A property on the surfaces ϕ = constant.

Now we have

$$p(x_i|\theta, \psi) = p(\phi|\theta, \psi).p(x_i|\theta, \phi).$$

Let the sample space be divided by a system of ϕ-surfaces. From the shell $W(\phi)$ between two such surfaces close to one another let $w(\phi)$ be a section whose mass is

$$\int_{w(\phi)} p(x_i|\theta, \phi)dx_i = \beta(\theta|\phi).$$

We shall define the region $w(\phi)$ as of conditional Type A if

$$\beta(\theta_0|\phi) = \alpha, \qquad \dots \quad (21)$$

$$\frac{\partial\beta(\theta_0|\phi)}{\partial\theta} = 0 \qquad \dots \quad (22)$$

and $\dfrac{\partial^2\beta(\theta_0|\phi)}{\partial\theta^2}$ is a maximum for all $w(\phi)$. ... (23)

Combining all such sections $w(\phi)$ on the ϕ-surfaces, we get the Type A region w. For, the over-all power function is given by

$$\beta(\theta) = \int_R \beta(\theta|\phi) p(\phi|\theta, \psi) d\phi,$$

R being the range of values of ϕ and differentiating with respect to θ, we have

$$\frac{\partial \beta}{\partial \theta} = \int_R \left\{ \frac{\partial \beta(\theta|\phi)}{\partial \theta} p(\phi|\theta, \psi) + \beta(\theta|\phi) \frac{\partial p(\phi|\theta, \psi)}{\partial \theta} \right\} d\phi,$$

assuming that the range R is independent of θ. By using (21) and (22) we see that

$$\beta(\theta_0) = \alpha; \quad \left[\frac{\partial \beta(\theta)}{\partial \theta} \right]_{\theta_0} = 0.$$

Now $$\frac{\partial^2 \beta}{\partial \theta^2} = \int_R \left\{ \frac{\partial^2 \beta(\theta|\phi)}{\partial \theta^2} p(\phi) + 2 \frac{\partial \beta(\theta|\phi)}{\partial \theta} \frac{\partial p(\phi)}{\partial \theta} + \beta(\theta|\phi) \frac{\partial^2 p(\phi)}{\partial \theta^2} \right\} d\phi$$

$$= \int_R \frac{\partial^2 \beta(\theta|\phi)}{\partial \theta^2} p(\phi) d\phi.$$

Therefore, from (23), $\left[\frac{\partial^2 \beta(\theta)}{\partial \theta^2} \right]_{\theta_0}$ is a maximum for all w.

The construction of conditional Type A regions on each ϕ-surface is based on Neyman and Pearson's Lemma (1936). Thus $w(\phi)$ is such that within it

$$p''(\underline{x}|\phi, \theta_0) \geqslant k_1(\phi) p'(\underline{x}|\phi, \theta_0) + k_2(\phi) p(\underline{x}|\phi, \theta_0) \quad \dots \quad (24)$$

and the reverse outside; the functions $k_1(\phi)$ and $k_2(\phi)$ should satisfy the conditions (21) and (22).

We shall illustrate the above method of constructing an unbiased locally most powerful critical region for testing $\sigma = \sigma_0$ in a normal population. It is easily seen that the ϕ-surfaces in this case are planes $\bar{x}$ = constant. The conditional distribution of $x_1, x_2, \dots, x_n$ is given by

$$p(\underline{x}|\bar{x}_i, \sigma_0) = C\, e^{-ns^2/\sigma_0^2}$$

Hence the section $w(\bar{x})$ is that within which

$$C e^{-ns^2/\sigma_0^2} \left\{ \frac{4n^2 s^4}{\sigma_0^6} - \frac{6ns^2}{\sigma_0^4} - k_1 \frac{2ns^2}{\sigma_0^3} - k_2 \right\} > 0$$

that is, $s^4 + l_1 s^2 + l_2 > 0$, where l_1 and l_2 are functions of $\bar{x}$ and σ. This inequality yields $s^2 < a_1^2$ and $s^2 > a_2^2$ which define respectively regions on the plane $\bar{x}$ = const. outside and inside two circles, centre $(\bar{x})$ and radii a_1 and a_2. Since s^2 is independent of $\bar{x}$, a_1 and a_2 are constant for every $\bar{x}$ plane and thus combining these conditional type A regions we get the familiar cylindrical regions which correspond to suitable tail areas of the distribution of ns^2/σ_0^2.

2.5. *An unbiased test for* $\rho = \rho_0$

It was seen in sec. 2.1 that in the case of the normal distribution with parameters ρ and σ, there is no statistic sufficient for σ. Hence the method discussed above of constructing an unbiased critical region by combining sections of the ϕ-surfaces which themselves are unbiased is not applicable in the present case. We shall, however, show that an unbiased locally most powerful critical region exists and that this consists of the interior of two right circular cones with axes in opposite directions along the equiangular line.

It was shown that for testing $\rho = \rho_0$, the best region is the cone defined by $\theta < \theta_0$, when the alternatives are $\rho > \rho_0$ and is the cone $\theta > \theta_0'$ for alternatives $\rho < \rho_0$. Now when both-sided alternatives are relevant, the best critical region should be an envelope of these regions, that is, should consist of two such cones, the angles of the cones being adjusted for making the region unbiased.

We can obtain this result directly by considering a similar region (consisting of one or more domains) satisfying (11). The conditions for an unbiased locally most powerful region are given by (11) and

$$\left[\frac{\partial\beta}{\partial\rho}\right]_{\rho_0} = 0; \quad \left[\frac{\partial^2\beta}{\partial\rho^2}\right]_{\rho_0} \text{ is maximum,}$$

where β is the integral in (11) with ρ put for ρ_0. Now suppose $p(\underline{\theta}_i)$ is the probability function of $\theta_1, \ldots, \theta_{n-1}$ obtained by integrating out r in $p(r, \theta_1, \ldots, \theta_{n-1})$ which is the integrand in (11), that is,

$$p(\underline{\theta}_i) = \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{n+j}{2}\right) e^{-n\rho^2/2} (\sqrt{2n}\ \rho \cos\theta)^j}{(\sqrt{2\pi})^n\ j!} \cos^{n-1}\theta_1 \ldots \cos\theta_{n-2}. \quad \ldots \quad (25)$$

Then the required region $\phi(\theta_i) < 0$ should satisfy the following

$$\int_{\phi(\underline{\theta}_i)<0} \cdots \int p_0(\underline{\theta}_i) d\underline{\theta}_i = \alpha \quad \ldots \quad (26)$$

$$\left[\frac{\partial}{\partial\rho} \int_{\phi(\underline{\theta}_i)<0} \cdots \int p(\underline{\theta}_i) d\underline{\theta}_i\right]_{\rho_0} = 0 \quad \ldots \quad (27)$$

and

$$\left[\frac{\partial^2}{\partial\rho^2} \int_{\phi(\underline{\theta}_i)<0} \cdots \int p(\underline{\theta}_i) d\underline{\theta}_i\right]_{\rho_0} \text{ is maximum.} \quad \ldots \quad (28)$$

From Neyman and Pearson's lemma, $w(\theta)$ will be that region in the θ-space within which

$$p_0''+k_1p_0'+k_2p_0 > 0, \quad \dots \quad (29)$$

k_1 and k_2 being constants depending on the conditions (25) and (26).

Writing
$$p_0(\theta) = \sum_{j=0}^{\infty} u_j(\theta),$$

it is seen that

$$p_0'(\theta) = \sum_{j=0}^{\infty} u_j(\theta) \left[\frac{\sqrt{2n}\,\Gamma\left(\frac{n+j+1}{2}\right)}{\Gamma\left(\frac{n+j}{2}\right)} \cos\theta - n\rho_0 \right].$$

$$p_0''(\theta) = \sum_{j=0}^{\infty} u_j(\theta) \left[\frac{2n\,\Gamma\left(\frac{n+j+2}{2}\right)}{\Gamma\left(\frac{n+j}{2}\right)} \cos^2\theta - 2n\rho_0\sqrt{2n}\, \frac{\Gamma\left(\frac{n+j+1}{2}\right)}{\Gamma\left(\frac{n+j}{2}\right)} \cos\theta + n^2\rho_0^2 \right].$$

The condition (29) will be equivalent to the inequality

$$A \cos^2\theta + B\cos\theta + C > 0$$

in which A, B and C are constants depending on n, k_1, k_2 and ρ_0. This will give the conditions $\cos\theta > \cos\theta_0$ and $\cos\theta < \cos\theta_0'$. Thus $w(\theta)$ is the region $\theta < \theta_0$ and $\theta > \theta_0'$ at the two tails of the θ-distribution. In the x-space, this corresponds to the interior of two right circular cones of semi-angles θ_0 and θ_0'. The values of θ_0 and θ_0' will be such that the sum of the tails is α according to (26) and that the region is unbiased (condition 27).

Since we use $t' = \frac{\bar{x}\sqrt{n-1}}{s}$ as our criterion for testing $\rho = \rho_0$, we shall obtain the locally most powerful unbiased critical region in terms of t'. From the above discussion, we see that it consists of the two tails $t' < t_1'$ and $t' > t_2'$, the values of t_1' and t_2' satisfying conditions corresponding to (26) and (27). Writing for convenience

$$v = \frac{t'}{\sqrt{n-1}} \text{ and } \delta = \sqrt{n}\rho_0,$$

these conditions are

$$\left[\int_{-\infty}^{v_1} p(v)dv + \int_{v_2}^{\infty} p(v)dv \right]_{\delta_0} = \alpha \quad \dots \quad (30)$$

and

$$\left[\frac{\partial}{\partial\delta} \left\{ \int_{-\infty}^{v_1} + \int_{v_2}^{\infty} p(v)dv \right\} \right]_{\delta_0} = 0, \quad \dots \quad (31)$$

where $p(v)$ is that given in (13) with δ for $\sqrt{n}\rho$.

Performing the differentiation under the sign of integration, (31) becomes

$$-\sum_{j=0}^{\infty}\frac{\Gamma\left(\frac{n+j}{2}\right)(\sqrt{2})^{j}e^{-\delta^2/2}\,\delta^{j+1}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)j!}\left[\int_{-\infty}^{v_1}+\int_{v_2}^{\infty}\frac{v^j dv}{(1+v^2)^{\frac{n+j}{2}}}\right]+$$

$$+\sum_{j=0}^{\infty}\frac{\Gamma\left(\frac{n+j}{2}\right)(\sqrt{2})^{j}e^{-\delta^2/2}\,\delta^{j-1}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)(j-1)!}\left[\int_{-\infty}^{v_1}+\int_{v_2}^{\infty}\frac{v^j dv}{(1+v^2)^{\frac{n+j}{2}}}\right]=0 \quad \dots \quad (32)$$

Now

$$\int\frac{v^j dv}{(1+v^2)^{\frac{n+j}{2}}}=-\frac{v^{j-1}}{(n+j-2)(1+v^2)^{\frac{n+j-2}{2}}}+\frac{j-1}{n+j-2}\int\frac{v^{j-2}dv}{(1+v^2)^{\frac{n+j-2}{2}}}.$$

Hence the second series in (32) can be written as

$$\sum_{j=2}^{\infty}\frac{\Gamma\left(\frac{n+j-2}{2}\right)(\sqrt{2})^{j-2}e^{-\delta^2/2}\,\delta^{j-1}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma(\frac{1}{2})(j-2)!}\left[\int_{-\infty}^{v_1}+\int_{v_2}^{\infty}\frac{v^{j-2}dv}{(1+v^2)^{\frac{n+j-2}{2}}}\right]-$$

$$-\sum_{j=1}^{\infty}\frac{\Gamma\left(\frac{n+j-2}{2}\right)(\sqrt{2})^{j-1}e^{-\delta^2/2}\,\delta^{j-1}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma(\frac{1}{2})(j-1)!}\left[\frac{v_1^{j-1}}{(1+v_1^2)^{\frac{n+j-2}{2}}}-\frac{v_2^{j-1}}{(1+v_2^2)^{\frac{n+j-2}{2}}}\right].$$

Putting j for $(j-2)$ in the first expression and j for $(j-1)$ in the second expression above, the equation (32) simplifies to

$$\sum_{j=0}^{\infty}\frac{\Gamma\left(\frac{n+j-1}{2}\right)(\sqrt{2})^{j}e^{-\delta^2/2}\,\delta^{j}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)j!}\left[\frac{v_1^j}{(1+v_1^2)^{\frac{n+j-1}{2}}}-\frac{v_2^j}{(1+v_2^2)^{\frac{n+j-1}{2}}}\right]=0 \quad \dots \quad (33)$$

This together with (30) will determine v_1 and v_2 and therefore t_1' and t_2'.

If we multiply (33) by $\Gamma\left(\frac{n-1}{2}\right)\Big/\Gamma\left(\frac{n-2}{2}\right)$, it will be seen that the ordinates of the curve of the distribution of v with $(n-1)$ degrees of freedom and the same parameter δ are equal at $v=v_1$ and v_2, i.e., $p_{n-1}(v_1)=p_{n-1}(v_2)$.

Using the approximation to the distribution of $\xi=\frac{t'-k_1}{\sqrt{k_2}}$ given in (20), we can find by a process of iteration, fairly accurate values of t_1' and t_2'.

2.6. The Test for the Standardised Mean

There are several situations in Biometry, Psychology and Industry where the ratio of mean to standard deviation is of greater practical significance than the mean alone. In such cases, the test procedure will be based on the t'-distribution, not the t-distribution. Suppose it is desired to test for consistency of manufactured articles. Taking a certain quality characteristic we may consider its standardised mean as a measure of consistency and we may set an acceptance value ρ. To test whether the manufacturing process is under statistical control, we evaluate $t' = \frac{\bar{x}\sqrt{(n-1)}}{s}$ and see if it exceeds its probability limit . A control chart could be constructed in which values of t' or $v\left(=\frac{\bar{x}}{s}\right)$ calculated for samples of a constant size n are plotted.

We may be sometimes interested in detecting departures of a process mean relative to the standard deviation in either direction. Then the chart will have two control lines, the control limits being the upper $100\alpha/2$ and lower $100\alpha/2$ percentage points of t' with $(n-1)$ d.f. and non-centrality $\sqrt{n}\rho$ where ρ is the specified standardised mean. By taking unbiased points of t' as described in the last section, we can have unbiased control lines. Very often, this ρ is determined from the observations themselves. Thus taking, say, k pairs of values of $\bar{x}$ and s over a period during which production was under control, we find $\bar{X} = \Sigma\bar{x}/k$ and $S = \sqrt{\Sigma s^2/k}$ and take $\frac{\bar{X}}{S}$ for ρ.

If the samples taken are not of the same size, the control limits will naturally vary and the control lines will not be straight lines. The estimate of ρ is then obtained by taking weighted $\bar{x}$'s and s^2's.

In this type of application we take the statistic $t' = \frac{(\bar{x}-L)\sqrt{n-1}}{s}$, where L is some fixed number, rather than $\frac{\bar{x}\sqrt{n-1}}{s}$. The population ρ will be $\frac{\mu-L}{\sigma}$. If $L=\mu$, then $\rho = 0$ and the statistic will be a t. The number L may be chosen beforehand in such a way that $\frac{\mu-L}{\sigma}\sqrt{n}$, the non-central parameter of the distribution of t', is not too high or too low, and lies say, between -4 and $+4$. Then we do not need tables of percentage points of t' beyond 4.

In problems of two samples, corresponding to the hypothesis that the means of two normal populations differ by a specified quantity, we can consider the hypothesis regarding their standardised means. If we assume that the standard deviations of the populations are equal, the statistic

$$\frac{\bar{x}_1-\bar{x}_2}{s\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} \qquad \dots \quad (34)$$

where $$s^2 = \{\sum_1^{n_1}(x_{1i}-\bar{x}_1)^2+\sum_1^{n_2}(x_{2i}-\bar{x}_2)^2\}/(n_1+n_2-2)$$

is distributed as a t' with $\nu = (n_1+n_2-2)$ degrees of freedom and parameter $\delta = k\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}$, k being the specified difference between the two population standardised means. The hypothesis here is that the means differ by a multiple k of the common s.d. The test consists in comparing the value of (34) with the appropriate t'-points. The test is two-sided if the alternatives are that the difference $(\rho_1-\rho_2) > k$ or $< k$ and is one-sided otherwise. As in the case of the two-sample t-test, we may first test for equality of the standard deviations and then apply the t'-test.

3. Test for standardised means of individual observations in a sample

3.1. *Test of* $H_0 : \rho_i = \rho_{i0}$, $i = 1$ *to* n, *when* $\sigma_i = \sigma$ *(unknown)*

In a sample of n observations $x_1, x_2, \ldots, x_n$, suppose we are given that the x_i's are normal variates having the same but unknown s.d. Consider the null hypothesis H_0 which specifies the standardised means $\rho_i = \dfrac{E(x_i)}{\sqrt{\text{var}(x_i)}}$ of each observation thus, $\rho_i = \rho_{i0}$. In the n-dimensional sample space, the centre of the normal distribution is at $(\rho_{10}\sigma, \ldots, \rho_{n0}\sigma)$ and since σ is unknown, this will lie at some point on the line

$$\frac{x_1}{\rho_{10}} = \frac{x_2}{\rho_{20}} = \ldots = \frac{x_n}{\rho_{n0}}. \qquad \ldots \quad (35)$$

As in sec. 2.1, it can be shown that any conical region with vertex at the origin 0 is similar with respect to σ.

Suppose an admissible alternative hypothesis H_1, specifies a set of ρ_i which have a constant ratio to the corresponding ρ_{i0}. Thus $\rho_{i1}/\rho_{i0} = C$ for all i. The new centre of the distribution $(\rho_{i1}\sigma)$ is on the line (35). Then it can be shown, as in section 2.2 that the best critical region for testing H_0 against H_1 is a cone, vertex origin and axis along the line defined by (35). A cone with axis in the direction of $\Sigma(x_i\rho_i)$ increasing is the most powerful with respect to all alternative hypotheses of the type H_1 for which $C > 1$; and a cone with axis in the opposite direction is best when $C < 1$.

The distance of any point (x_i) from the origin is Σx_i^2 and that from the axis is $\Sigma x_i\rho_{i0}/\sqrt{\Sigma\rho_{i0}^2}$. Hence the interiors of these two cones with semi-angles θ_0 and θ_0' are defined by

$$l = \frac{\Sigma x_i\rho_{i0}/\sqrt{\Sigma\rho_{i0}^2}}{\sqrt{\left\{\Sigma x_i^2-\dfrac{(\Sigma x_i\rho_{i0})^2}{\Sigma\rho_{i0}^2}\right\}}} \geqslant \cot\theta_0 \text{ and } \leqslant \cot\theta_0' \qquad \ldots \quad (36)$$

respectively. The values of θ_0, θ_0' can be determined from the distribution of l.

Introduce the orthogonal linear transformation

$$\left.\begin{aligned} y_1 &= \sum_{i=1}^{n} \frac{\rho_{i0}}{\sqrt{\Sigma\rho_{i0}^2}} x_i , \\ y_2 &= \sum_{i=1}^{n} c_{2i}\, x_i , \\ &\cdots \quad \vdots \quad \cdots \quad \cdots \\ y_n &= \sum_{i=1}^{n} c_{ni}\, x_i . \end{aligned}\right\} \quad \ldots \quad (37)$$

Then l is transformed into $y_1/\sqrt{(y_2^2+\ldots+y_n^2)}$, where the y's are mutually independent. From (37) we find that y_1/σ is a normal variate with mean $\sqrt{\Sigma\rho_{i0}^2}$ and s.d. unity and that y_j/σ ($j = 2$ to n) is also a normal variate with mean $\sum_{i=1}^{n} c_{ji}\rho_{i0}$ and s.d. unity. The sum

$$\sum_{j=2}^{n} E(y_j/\sigma)^2 = \sum_{j=2}^{n} \left(\sum_{i=1}^{n} c_{ji}\,\rho_{i0}\right)^2$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} c_{ji}\,\rho_{i0}\right)^2 - \left(\sum_{i=1}^{n} c_{1i}\,\rho_{i0}\right)^2$$

$$= \sum_{i=1}^{n} \rho_{i0}^2 - \left(\frac{\Sigma\rho_{i0}^2}{\sqrt{\Sigma\rho_{i0}^2}}\right)^2 = 0$$

in virtue of the properties of the constants in the transformation (31). Hence l in (36) is distributed as the ratio of a non-central normal deviate to an independent χ, that is, as $t'/\sqrt{(n-1)}$ with $(n-1)$ degrees of freedom and parameter $\sqrt{\Sigma\rho_{i0}^2}$.

The test for the hypothesis H_0 specifying the set (ρ_0) when the alternative hypothesis specifies a set (ρ_i) in a fixed proportion to the null set consists in evaluating

$$t' = \frac{\sqrt{(n-1)}\Sigma(x_i\rho_{i0})}{\sqrt{\{(\rho\Sigma^2_{i0})(\Sigma x_i^2)-(\Sigma\rho_{i0}x_i)^2\}}}$$

and comparing with the upper or lower $100\alpha\%$ point of the appropriate t'-distribution.

If $\rho_{10} = \rho_{20} = \ldots = \rho_{n0} = \rho_0$, this statistic reduces to $\bar{x}\sqrt{(n-1)}/s$ and the test is identical with that discussed in sec. 2.2.

We shall next consider the test of $H_0(\rho_i = \rho_{i0})$ when the alternative H_1 is more general than the previous one. Let the set of standardised means be ρ_{i1}, $i = 1, 2, \ldots, n$.

As already proved in sec. 2.1, any cone with vertex at the origin or any region consisting of sections of cones which are together equivalent to a single cone for all values of σ is similar with respect to this unknown parameter. To find the best amongst all such regions, let A_1 be the point $(\rho_{i0}\sigma)$ and A_2 the point $(\rho_{i1}\sigma)$ in the x-space. We shall show that the most powerful region for testing H_0 against H_1 is the interior of a right circular cone with axis along the line OK through the origin parallel to A_1A_2. We shall follow a method slightly different from that used in sec. 2.2 for the case of equal standardised means.

Let C be a similar region of size α. Consider the class S of all possible regions of size α which lie between two or more right circular cones with vertices at the origin O and the line OK as axes. Under the alternative hypothesis $H_1(\rho_i = \rho_{i1})$, the mass of C will be equal to that of a region D_1 belonging to S when $\sigma = \sigma_1$, to a region D_2 when $\sigma = \sigma_2$, and so on. That is, there will be some members of S having the same mass as C under both H_0 and H_1. Now if we show that there is a single region D in S which has greater mass than every member of S when H_1 is true, then it would follow that D is better than D_1 if $\sigma = \sigma_1$, better than D_2 if $\sigma = \sigma_2$, and so on, and hence is better than C for every σ.

We shall, therefore, proceed to prove that amongst all regions of the class S, the one bounded by a single cone is the most powerful. Since we are dealing with similar regions we shall take $\sigma = 1$ for convenience. Consider the following orthogonal transformation similar to (37):

$$z_1 = \sum_{i=1}^{n} \frac{(\rho_{i1}-\rho_{i0})}{\sqrt{\Sigma(\rho_{i1}-\rho_{i0})^2}} \cdot x_i,$$

$$z_j = \sum_{i=1}^{n} c_{ji} x_i \,;\, j = 2 \ldots n.$$

We shall write $S^2 = \sum_{j=2}^{n} z_j^2 = \sum_{i=1}^{n} x_i^2 - z_1^2$. It is easily seen that under H_0, z_1 is distributed as a normal deviate with s.d. unity and mean

$$\delta = \frac{\sum_{i=1}^{n} (\rho_{i1}-\rho_{i0})\rho_{i0}}{\sqrt{\sum_{i=1}^{n} (\rho_{i1}-\rho_{i0})^2}} \qquad \ldots \quad (38)$$

and S^2 as a non-central χ^2 i.e. χ'^2 with $(n-1)$ d.f. and parameter

$$\lambda = \sum_{j=2}^{n} \left(\sum_{i=1}^{n} c_{ji}\rho_{i0}\right)^2 = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} c_{ji}\rho_{i0}\right)^2 - \left(\sum_{i=1}^{n} c_{1i}\rho_{i0}\right)^2 \qquad \ldots$$

$$= \sum_{i=1}^{n} \rho_{i0}^2 - \frac{\left(\sum_{i=1}^{n} (\rho_{i1}-\rho_{i0})\rho_{i0}\right)^2}{\Sigma(\rho_{i1}-\rho_{i0})^2} \qquad \ldots \quad (39)$$

The distribution of χ' with ν d.f. is the following (see Patnaik, 1949):

$$p(\chi')d\chi' = \sum_{i=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^i\sqrt{2}}{i\,!\,\Gamma\left(\frac{\nu+2i}{2}\right)} e^{-\chi'^2/2}(\chi'^2/2)^{\frac{\nu+2i-1}{2}} d\chi'.$$

Dropping the suffix of z_1, the joint distribution of z and S which are independent can be written down; thus,

$$p(z, S)dzdS = \text{Exp}\,\{-\tfrac{1}{2}[(z-\delta)^2+S^2+\lambda]\} \sum_{i=0}^{\infty} \frac{\left(\frac{\lambda}{2}\right)^i\left(\frac{S^2}{2}\right)^{\frac{\nu+2i-1}{2}}}{i!\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\nu+2i}{2}\right)} dzdS, \quad \ldots \quad (40)$$

in which δ and λ are given by (38) and (39).

Now let $z = r\cos\theta$, $S = r\sin\theta$. Then (40) transforms into

$$p(r, \theta)drd\theta = \text{Exp}\,\{-\tfrac{1}{2}(r^2-2r\delta\cos\theta+\delta^2+\lambda)\} \sum_{i=0}^{\infty} \frac{(\lambda/2)^i r^{\nu+2i}(\sin\theta)^{\nu+2i-1}}{i!\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\nu+2i}{2}\right)(\sqrt{2})^{\nu+2i-1}}\; drd\theta.$$

$$\ldots \quad (41)$$

Any region of the class D (i.e. lying between right circular cones with axes along OK parallel to the line joining (ρ_{i0}) and (ρ_{i1})) will correspond to one or more parallel strips in the (r, θ)-plane. Thus when H_0 is true,

$$\int_{w(\theta)}\int_0^{\infty} p_0(r, \theta)drd\theta = \alpha, \quad \ldots \quad (42)$$

$p_0(r,\theta)$ being the expression in (41) with δ and λ given by (38) and (39). Under H_1 we have

$$\int_{w(\theta)}\int_0^{\infty} p_1(r,\theta)drd\theta = \beta$$

in which $p_1(r, \theta)$ has the expression in (41) with

$$\delta = \frac{\sum_{i=1}^{n}(\rho_{i1}-\rho_{i0})\rho_{i1}}{\sqrt{\Sigma(\rho_{i1}-\rho_{i0})^2}}\;,\quad \lambda = \sum_{i=1}^{n}\rho_{i0}^2 - \frac{\left(\sum_{i=1}^{n}(\rho_{i1}-\rho_{i0})\rho_{i1}\right)^2}{\sum_{i=1}^{n}(\rho_{i1}-\rho_{i0})^2}.$$

The best interval of θ will be that within which

$$\int_0^{\infty} p_1(r, \theta)dr > k\int_0^{\infty} p_0(r, \theta)dr,$$

that is,

$$p_1(\theta) > kp_0(\theta), \quad \ldots \quad (43)$$

k depending on the relation (42). Integrating $p(r, \theta)$ with respect to r, we find that

$$p(\theta)d\theta = \sum_{i=0}^{\infty} \frac{e^{-\frac{\delta^2+\lambda}{2}}(\lambda/2)^i(\sin\theta)^{\nu+2i-1}d\theta}{i!\,\Gamma(\tfrac{1}{2})\,\Gamma\left(\frac{\nu+2i}{2}\right)(\sqrt{2})^{\nu+2i-1}} \cdot \int_0^{\infty} e^{-\frac{r^2}{2}} r^{\nu+2i}(1+r\cos\theta+\ldots)dr$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-\frac{\delta^2+\lambda}{2}}(\lambda/2)^i(\sqrt{2}\delta)^j\Gamma\left(\frac{\nu+2i+j+1}{2}\right)}{i!j!\Gamma(\tfrac{1}{2})\Gamma\left(\frac{\nu+2i}{2}\right)} (\sin\theta)^{\nu+2i-1}\cos^j\theta\, d\theta. \quad \ldots \quad (44)$$

On substituting the double infinite series for $p(\theta)$ in (43), with the appropriate expressions for δ and λ, the resulting inequality will be satisfied by

$$\cos\theta \geqslant \cos\theta_0, \qquad i.e. \quad \theta \leqslant \theta_0,$$

where θ_0 is a constant depending on (42). This follows from the fact that $(p_1(\theta)/p_0(\theta))$ is monotonic increasing in $\cos\theta$. Thus we get the result that a right circular cone, vertex origin and axis along a line with direction cosines proportional to $(\rho_{i1}-\rho_{i0})$ is the best critical region for testing the hypothesis $H_0(\rho_{i0})$ against an alternative $H_1(\rho_{i1})$. Clearly, if a different alternative hypothesis is considered, the best critical region will have a different axis.

Since $\cot\theta = z/S$, the test criterion is easily seen to be

$$\frac{z\sqrt{n-1}}{S} = \frac{\sqrt{n-1}\,\Sigma(\rho_{i1}-\rho_{i0})x_i}{\sqrt{[(\Sigma x_i^2)\{\Sigma(\rho_{i1}-\rho_{i0})^2\}-\{\Sigma x_i(\rho_{i1}-\rho_{i0})\}^2]}}$$

whatever be the value of the unknown σ. Under H_0 this is the ratio of a non-central normal deviate and a non-central χ with $(n-1)$ d.f., the parameters δ and λ being given by (38) and (39). We shall denote it by t_1'. Thus the test procedure consists in comparing the observed value of this ratio with the $100\alpha\%$ point of the distribution of

3.2. *The t_1'-distribution*

The general distribution of t_1' can be written down from the distribution of θ in (44). Thus,

$$p(t_1')dt_1' = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-\frac{\delta^2+\lambda}{2}}(\lambda/2)^i(\sqrt{2}\delta)^j\Gamma\left(\frac{\nu+2i+j+1}{2}\right)}{i!\,j!\,\Gamma(1/2)\Gamma\left(\frac{\nu+2i}{2}\right)(\sqrt{\nu})^{j+1}} \times \frac{t_1'^j dt_1'}{\left(1+\frac{t_1'^2}{\nu}\right)^{\frac{\nu+2i+j+1}{2}}} \quad \ldots \quad (44a)$$

Robbins (1948) derived this distribution while considering the distribution of Student's when the population means are unequal. He was not dealing with the problem of testing the hypothesis concerning the means.

The following expressions for the first two moments of t'_1 are obtained by straightforward integration:

$$\left.\begin{aligned} \mu'_1 &= \left(\frac{\delta^2\nu}{2}\right)^{\frac{1}{2}} \sum_{i=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^i}{i!} \cdot \frac{\Gamma\left(\frac{\nu+2i-1}{2}\right)}{\Gamma\left(\frac{\nu+2i}{2}\right)}, \\ \mu'_2 &= (1+\delta^2)\nu \sum_{i=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^i}{i!} \cdot \frac{1}{\nu+2i-2}. \end{aligned}\right\} \quad \dots \quad (45)$$

Each term in either of the above series is the product of a term of a Poisson series and the ratio of two Gamma functions. These moments can be computed by using existing tables. If λ is small, the Poisson terms after the first few become so small that a limited number of terms in (45) should be adequate for evaluating μ'_1 and μ'_2.

We can approximate the distribution of t'_1 by that of a t' (non-central t) by identifying the first two moments. This is the same as approximating the χ' in the denominator of t'_1 by a χ with modified degrees of freedom and a scale factor. Patnaik (1949) has shown that χ'^2 can be well approximated by a χ^2 and has used this approximation for representing an F', i.e. $\left(\frac{\chi'^2_1/\nu_1}{\chi^2_2/\nu_2}\right)$ by $\left(k\,\frac{\chi^2_1/\nu}{\chi^2_2/\nu_2}\right)$. This is found to be quite satisfactory (see Pearson and Hartley, 1951). We may similarly expect that the proposed approximation to t'_1 would be good.

Taking the first two moments of ct' with f d.f. and parameter δ, and equating them to the corresponding moments of t'_1 given in (45), we have

$$c\left(\frac{\delta^2}{2}\right)^{\frac{1}{2}}\sqrt{f}\,\frac{\Gamma\left(\frac{f-1}{2}\right)}{\Gamma(f/2)} = \mu'_1 = \left(\frac{\delta^2}{2}\right)^{\frac{1}{2}} g, \text{ say,}$$

$$c^2(1+\delta^2)\,\frac{f}{f-2} = \mu'_2 = (1+\delta^2)h, \quad \text{say.}$$

Employing Stirling's approximation to the Gamma function, we get to $0(f^{-3})$, the following relations

$$c\sqrt{2}\left(1+\frac{3}{4f}+\frac{25}{32f^2}\right) = g,$$

$$c^2\left(\frac{f}{f-2}\right) = h,$$

which yield expressions for f and c.

$$\left.\begin{aligned} \frac{1}{f} &= \frac{2}{7}\left\{-1+\sqrt{\left(15-\frac{7g^2}{h}\right)}\right\} \\ c &= \sqrt{\left\{h\left(1-\frac{2}{f}\right)\right\}}. \end{aligned}\right\} \quad \dots \quad (46)$$

The above approximation can be used to find the percentage points of t'_1. The upper and lower 5% points may be derived from the t' table of percentage points given in the Appendix.

4. Test with non-homogeneous samples

4.1. *Test for the standardised mean*

Suppose $(x_1, x_2, \ldots, x_n)$ is a sample and it is known that it contains observations from two normal populations with the same variance but with different means. It may be that the sample has a few rogue observations which have got mixed up with the others. For instance, in a biometric study of a certain species of organisms, measurements of a different, though closely allied, species having the same variability might get included; or again, the observations in the sample might have come from two analysts one of whom has a definite subjective bias. It will be shown that if we know their number and their divergence in their standardised means from the rest, we can develop a test corresponding to the test for ρ in a homogeneous sample, discussed in section 2.

Suppose we have the hypothesis that in a sample of n, $(n-m)$ observations have standardised mean ρ and the rest $\rho+\gamma$, the s.d. being the same for all. Here γ is fixed. We have to test the hypothesis that $\rho = \rho_0$ against alternatives $\rho > \rho_0$ or $\rho < \rho_0$. It will be noted that since we are not testing for γ also this is not a special case of the hypothesis considered in sec. 3.1.

In the n-dimensional sample space, the centre of the normal distribution is at a point C whose co-ordinates in some order are $(\rho_0\sigma, \ldots; (\rho_0+\gamma)\sigma, \ldots)$. Following the same line of proof as in sec. 2.1, it can be shown that any conical region with vertex at the origin 0 will be similar with respect to σ. Under the alternative hypothesis $\rho = \rho_1$, the centre of the distribution will be at D, $(\rho_1\sigma, \ldots; (\rho_1+\gamma)\sigma, \ldots)$. Now we may apply the general result of sec. 3.1 and find that the best critical region is a right-circular cone vertex 0 and axis parallel to CD, that is, parallel to the equi-angular line. There will be two possible regions with axes in opposite directions according as the alternatives $\rho_1 > \rho_0$ or $\rho_1 < \rho_0$. Each of these will be the most powerful for a family of alternatives, either $\rho > \rho_0$ or $\rho < \rho_0$.

The test criterion will be $\bar{x}\sqrt{(n-1)}/s$, in which $\bar{x}$ and s are the usual sample mean and s.d. Now

$$E\left(\frac{\sqrt{n}\bar{x}}{\sigma}\right) = \sqrt{n}\left(\rho + \frac{m\gamma}{n}\right),$$

$$\sum_{i=1}^{n} E(x_i-\bar{x})^2 = \left(m - \frac{m^2}{n}\right)\gamma^2.$$

So, $\bar{x}\sqrt{(n-1)}/s$ is distributed as a t_1' (defined in sec. 3.1) with

$$\left.\begin{aligned} \nu &= n-1, \\ \delta &= \sqrt{n}\left(\rho+\frac{m\gamma}{n}\right), \\ \lambda &= m\left(1-\frac{m}{n}\right)\gamma. \end{aligned}\right| \quad \ldots \quad (47)$$

Hence the test procedure consists in comparing the value of the familiar Student's ratio with the upper or lower $100\alpha\%$ point of the appropriate t_1' distribution. Approximate percentage points can be found as described in the last section.

When both-sided alternatives $\rho > \rho_0$ and $\rho < \rho_0$ are relevant, it can be shown that an unbiased locally most powerful critical region would consist of both the tails of the t_1' distribution. For if we write v for t_1'/ν and $n = \nu+1$ in (44a) and denote the density function by $p_n(v;\delta,\lambda)$, this can be expressed in terms of the density function $p_n(v;\delta)$ when $\lambda = 0$, (which is the same as in (13) with δ put for $\sqrt{n}\rho$). Thus

$$\left.\begin{aligned} p_n(v;\delta,\lambda) &= \sum_{i=0}^{\infty} a_i p_{n+2i}(v;\delta), \\ \text{where} \qquad a_i &= \frac{e^{-\lambda/2}(\lambda/2)^i}{i\,!}. \end{aligned}\right\} \quad \ldots \quad (48)$$

By an argument similar to that used in the case of $p(v;\delta)$, we see that the best region in our present case is defined by $v \geqslant v_1$ and $v \leqslant v_2$, where v_1 and v_2 satisfy

$$\int_{-\infty}^{v_1} + \int_{v_2}^{\infty} p_n(v;\delta_0,\lambda) = \alpha, \quad \ldots \quad (49)$$

$$\left[\int_{-\infty}^{v_1} + \int_{v_2}^{\infty} \frac{\partial}{\partial\delta} p_n(v;\delta,\lambda)dv\right]_{\delta_0} = 0.$$

The second condition becomes in virtue of (48),

$$\sum_{i=0}^{\infty} a_i \left[\int_{-\infty}^{v_1} + \int_{v_2}^{\infty} \frac{\partial}{\partial\delta} p_{n+2i}(v;\delta)dv\right]_{\delta_0} = 0.$$

It was shown in sec. 2.5 that the sum of the integrals within brackets at δ_0 is equal to

$$p_{n+2i-1}(v_1;\delta_0) - p_{n+2i-1}(v_2;\delta_0).$$

Hence we get the relation

$$\sum_{i=0}^{\infty} a_i p_{n+2i-1}(v_1;\delta_0) = \sum_{i=0}^{\infty} a_i p_{n+2i-1}(v_2;\delta_0)$$

which is the same as

$$p_{n-1}(v_1;\delta_0,\lambda) = p_{n-1}(v_2;\delta_0,\lambda)$$

in virtue of (48). This condition together with (49) determines the unbiased limits of v and hence of t_1'. In the above expressions, δ and λ take the appropriate values given in (47). It may be noticed that if n is not too small, the unbiased region is formed by the tail areas of the t_1' distribution cut off by equal ordinates.

4.2. *A two-sample test*

In the situation considered in the last sub-section, we assumed that the difference between the standardised mean of the rogue observations and of the rest i.e. γ is known. It is, however, more likely that the difference in the means is known; as for instance, when observations are taken at two places or on two occasions, there may be a difference between the expected values $(\mu+b)$ and μ which is known from previous experience. We may then employ a two-sample procedure (see Stein, 1946) to test the hypothesis that $\mu = \mu_0$. The power of this test will be independent of the unknown standard deviation.

Suppose we have a first sample of n_0 observations which are known to be identically distributed. Let

$$s^2 = \sum_{i=1}^{n_0} (x_i-\bar{x})^2/(n_0-1), \quad \text{where} \quad \bar{x} = \sum_{i=1}^{n_0} x_i/n_0 . \qquad \ldots \quad (50)$$

Choosing a number z, let

$$n = \text{Max}\left\{\left[\frac{s^2}{z}\right]+1,\ n_0+1\right\}$$

where $\left[\frac{s^2}{z}\right]$ is the largest integer less than s^2/z. Then we take a second sample of size $(n-n_0)$. Of these, suppose m observations happen to be rogues having the same s.d. as the rest but with mean exceeding the mean of the rest by b. Here m is a random variable.

Let $a_1, a_2, \ldots, a_n$ be a set of real numbers such that

(i) $a_1 = a_2 = \ldots = a_{n_0} = a'$, say;

$a_{n_0+1} = a_{n_0+2} = \ldots = a_n = a''$, say;

and (ii) $\sum_{i=1}^{n} a_i = n_0 a' + (n-n_0)a'' = 1$,

(iii) $\sum_{i=1}^{n} a_i^2 = n_0 a'^2 + (n-n_0)a''^2 = \frac{z}{s^2}$.

The hypothesis to be tested is that in the two samples together, the mean of $(n-m)$ observations is μ and that of the rest is $\mu+b$, b being known. Consider the statistic

$$\frac{\sum_{i=1}^{n} a_i x_i-(\mu+\eta)}{\sqrt{z}} \quad \dots \quad (52)$$

where $\eta = ma''b$. Now

$$E\left(\sum_{i=1}^{n} a_i x_i-(\mu+\eta)\right)= 0,$$

and
$$\text{Var}\left(\sum_{i=1}^{n} a_i x_i-(\mu+\eta)\right)= \sigma^2 \sum_{i=1}^{n} a_i^2, \text{ from (51).}$$

Since s^2 in (50) is an estimate of σ^2, the above variance is estimated by $s^2\Sigma a_i^2$ which is equal to z. Hence the ratio (52) is distributed as a t with (n_0-1) degrees of freedom, the numerator being independent of the denominator.

If there is an alternative hypothesis specifying the mean as μ_1, the statistic (52) is then distributed as

$$\frac{\Sigma a_i x_i-(\mu_1+\eta)}{\sqrt{z}}+\frac{\mu_1-\mu}{\sqrt{z}},$$

that is, as t+a constant. Thus the power of the above two-sample test for the mean is obtained from the distribution of t above and does not involve the unknown σ.

If we know that there are rogue observations in the first sample also, then n_0s^2/σ^2 will be a χ'^2 and its distribution will involve the standardised means and so the distribution of the statistic (52) will not be free of the unknown σ. Hence, in practice, the first sample should be carefully chosen so as to exclude any extraneous observations and so it might be kept small.

5. TEST FOR $\mu = \mu_0$ AGAINST ALL POSSIBLE ALTERNATIVES

In tests concerning the mean of a normal sample it is usual to assume that all the observations are drawn from the same normal population. Thus we test for $\mu = \mu_0$ against an alternative $\mu = \mu_1$, μ_0 and μ_1 being the expected values of every observation. We might, however, admit a more general class of alternative hypotheses in which the means of individual observations $E(x_i)$ are not all equal.

Suppose under H_0

$$E(x_i) = \mu_0, \quad i = 1 \text{ to } n,$$

given that var $(x_i) = \sigma^2$ (unknown) for all i. Let us consider H_1 specifying

$$\begin{aligned} E(x_i) &= \mu_1; \quad i = 1 \text{ to } m \\ &=\mu_0; \quad i = m+1 \text{ to } n, \end{aligned}$$

given that var $(x_i) = \sigma^2$ (unknown) for all i. This means that against the null hypothesis that the whole sample belongs to a normal population with mean μ_0, we admit the alternative hypothesis that a part of the sample belongs to a different normal population with mean μ_1.

The centre of the distribution is at a point C_0 $(\mu_0, \mu_0, \ldots)$ in the x-space under H_0 and at $C_1(\mu_1, \ldots; \mu_0, \ldots)$ under H_1. It can be easily shown that the best region for testing H_0 against H_1 is formed by combining sections on the ϕ-surfaces which are hyperspheres, centre C, cut off by planes parallel to the line C_0C_1. This will be a right circular cone with axis along C_0C_1. A point on its surface will satisfy the condition

$$\frac{\sum_{i=1}^{m}(x_i-\mu_0)(\mu_1-\mu_0)\Big/\sqrt{\sum_{i=1}^{m}(\mu_i-\mu_0)^2}}{\sqrt{\left[\sum_{i=1}^{n}(x_i-\mu_0)^2-\frac{\left\{\sum_{i=1}^{m}(x_i-\mu_0)(\mu_1-\mu_0)\right\}^2}{\sum_{i=1}^{m}(\mu_i-\mu_0)^2}\right]}} = \text{Constant.}$$

The above ratio simplifies to

$$\frac{\sqrt{m}(\bar{x}_{(m)}-\mu_0)}{\sqrt{\left[\sum_{i=1}^{m}(x_i-\bar{x})^2+\sum_{i=m+1}^{n}(x_i-\mu_0)^2\right]}},$$

in which $\bar{x}_{(m)}$ is written for $(x_1+\ldots+x_m)/m$. The denominator is distributed as $(\sigma^2\chi^2_{m-1}+\sigma^2\chi^2_{n-m})$ and is independent of the numerator which is a normal deviate with expectation zero. Hence under H_0, the above ratio follows a t-distribution with $(n-1)$ degrees of freedom and so provides the test criterion. It will be noted that this statistic is different from the usual t-ratio when H_1 specifies the same mean for every x_i.

Let us now consider a more general alternative $H_1 : E(x_i) = \mu_i$, variance being the same for all i. The best critical regions are right circular cones vertex $(\underline{\mu_0})$ and axes along the line joining (μ_0) and (μ_i). Then the statistic to be used turns out to be

$$\frac{\sqrt{(n-1)}\sum_{i=1}^{n}(x_i-\mu_0)(\mu_i-\mu_0)}{\sqrt{\left[\left(\sum_{i=1}^{n}(x_i-\mu_0)^2\right)\left(\sum_{i=1}^{n}(\mu_i-\mu_0)^2\right)-\left(\sum(x_i-\mu_0)(\mu_i-\mu_0)\right)^2\right]}}$$

which follows a t-distribution with $(n-1)$ degrees of freedom.

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APPENDIX

TABLE I. UPPER 5% POINTS OF THE t' DISTRIBUTION

$\nu \backslash \delta$	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
4	2.13	3.03	3.84	4.71	5.77	6.84	7.88	8.98	10.12
5	2.02	2.75	3.53	4.38	5.20	6.16	7.10	7.99	8.99
6	1.943	2.631	3.361	4.130	4.934	5.761	6.604	7.466	8.312
7	1.895	2.549	3.240	3.964	4.716	5.488	6.270	7.092	7.880
8	1.860	2.492	3.155	3.847	4.561	5.296	6.055	6.815	7.520
9	1.833	2.449	3.091	3.757	4.446	5.154	5.876	6.593	7.362
10	1.813	2.415	3.041	3.688	4.356	5.040	5.738	6.449	7.160
11	1.796	2.388	3.001	3.633	4.284	4.950	5.629	6.325	7.018
12	1.782	2.366	2.968	3.588	4.225	4.876	5.540	6.214	6.894
13	1.771	2.348	2.941	3.551	4.177	4.815	5.468	6.128	6.797
14	1.761	2.332	2.918	3.520	4.136	4.764	5.405	6.052	6.714
15	1.753	2.318	2.899	3.501	4.096	4.667	5.325	5.997	6.641
20	1.725	2.273	2.832	3.401	3.981	4.569	5.166	5.772	6.381

For the test for the standardised mean ρ with a sample of size n, take $\nu = n-1$ and $\delta = \sqrt{n}\rho$.

TABLE II. LOWER 5% POINTS OF THE t' DISTRIBUTION

$\nu \backslash \delta$	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
4	−2.13	−1.40	−0.74	−0.16	0.36	0.83	1.26	1.68	2.09
5	−2.02	−1.34	−0.72	−0.15	0.36	0.84	1.28	1.70	2.11
6	−1.943	−1.301	−0.705	−0.152	0.361	0.842	1.293	1.724	2.128
7	−1.895	−1.267	−0.695	−0.151	0.360	0.843	1.302	1.743	2.154
8	−1.860	−1.257	−0.689	−0.151	0.359	0.845	1.310	1.753	2.183
9	−1.833	−1.245	−0.684	−0.150	0.359	0.846	1.313	1.763	2.202
10	−1.813	−1.234	−0.680	−0.150	0.358	0.846	1.316	1.771	2.212
11	−1.796	−1.226	−0.677	−0.150	0.358	0.847	1.319	1.780	2.222
12	−1.782	−1.219	−0.674	−0.149	0.357	0.847	1.322	1.784	2.231
13	−1.771	−1.213	−0.673	−0.149	0.357	0.847	1.325	1.788	2.241
14	−1.761	−1.207	−0.671	−0.149	0.357	0.848	1.327	1.793	2.253
15	−1.753	−1.202	−0.668	−0.148	0.357	0.849	1.329	1.797	2.258
20	−1.725	−1.189	−0.656	−0.147	0.356	0.849	1.334	1.810	2.277

For the test for the standardised mean ρ with a sample of size n, take $\nu = n-1$ and $\delta = \sqrt{n}\rho$.

Paper received : July, 1954.