BOUND STATES FOR MOMENTUM AND ASYMPTOTIC COMPLETENESS IN $L^2(R^n)$:

I. TRACE CLASS COMMUTATORS FOR $n = 1$

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In $L^2(R)$, asymptotic completeness is proved for $h(P) + W(Q)$ where $h(P)$ is a complete operator and W is a long range potential. In $L^2(R^n)$ for $n \geqslant 2$, asymptotic completeness is proved if h is a polynomial such that $\sum_j |D_j h(\xi)|$ grows very fast with $|\xi|$ and W is a smooth short range potential decaying faster than $|x|^{-(n+1)}$ at ∞.

§ 1 INTRODUCTION AND STATEMENT OF THE RESULT

Let $\mathscr{H} = L^2(R^n)$ be the Hilbert space of complex valued square integrable functions on R^n w.r.t the Lebesgue measure. Let $Q = (Q_1, \ldots, Q_n)$, $P = (P_1, \ldots, P_n)$ be the position and momentum operators on $L^2(R^n)$ given by $(Q_j f)(x) = x_j f(x)$, $(P_j f)(x) = -\mathrm{i}(D_j f)(x)$; $D_j = \partial / \partial x_j$.

Let $h : R^n \to R$ be any smooth function such that h and all its derivatives have at most polynomial growth. Further let $\{\xi \text{ in } R^n : \nabla h(\xi) = 0\}$ have zero Lebesgue measure. Let $W(x) = W_S(x) + W_L(x)$ where $W_S, W_L : R^n \to R$ are bounded short range and long range potentials respectively. Put $H = h(P) + W(Q) = H_0 + W(Q)$. In [3], as early as 1976, the following theorem was proved.

THEOREM 1.1 (i) *There exists a smooth function $X : R \times R^n \to R$ such that the wave operators $\Omega_\pm = \Omega_\pm(H_0, H)$ given by*

$$\Omega_\pm f = s\text{-}\lim_{t \to \pm\infty} \exp[\mathrm{i}tH] \exp[-\mathrm{i}X(t, P)] f$$

exist for each f in $L^2(R^n)$.

(ii) *Range $\Omega_\pm \subseteq \mathscr{H}_{ac}(H)$, the absolutely continuous space for the self adjoint operator H.*

Note 1.2. If $W_L = 0$ then we can take $X(t, \xi) = th(\xi)$

Definition 1.3. Asymptotic completeness [abbreviated: AC] is said to hold for the pair (H_0, H) if equality holds in Theorem 1.1 (ii) i.e. Range $\Omega_\pm = \mathscr{H}_{ac}(H)$.

AC for (H_0, H) is known only for a limited cases of h viz (i) h is an elliptic polynomial [5] (ii) h is vaguely elliptic i.e. $\lim_{|\xi| \to \infty} |h(\xi)| = \infty$ [7, 15] (iii) h is simply characteristic i.e.

(a) $\lim_{|\xi| \to \infty} 1 + |h(\xi)| + |\nabla h(\xi)| = \infty$ and

(b) $|D^\alpha h(\xi)| \leqslant K_\alpha(1 + |h(\xi)| + |\nabla h(\xi)|)$, where $|\alpha| \geqslant 2$, $K(\alpha)$ are suitable constants, [4, 11] and (iv) $H = H_0 + W_S$, where

(a) h is a monomial with $\lim \sum |D^\alpha h(\xi)| = \infty$ or

(b) h is a non negative finite linear combination of non-negative monomials, with $\lim_{|\xi|\to\infty} \sum_{\alpha} |D^{\alpha}h(\xi)| = \infty$ [8].

It was conjectured in [8] that A' holds for $(h(P), h(P) + (1 + |Q|)^{-1-\varepsilon})$, where $\varepsilon > 0$, when h is any polynomial with $\lim_{|\xi|\to\infty} \sum_{\alpha} |D^{\alpha}h(\xi)| = \infty$. In this article we capture some class of $(h(P), h(P) + W(Q))$ for which AC holds.

Definition 1.4. [14, p. 255] Let $h : R^n \to R$ be any polynomial; h is said to be a complete polynomial if

$$\{0\} = \{\eta \text{ in } R^n : h(\xi + t\eta) = h(\xi) \quad \text{for all} \quad \xi \text{ in } R^n,\ t \text{ in } R\}.$$

While $h : R^2 \to R$ given by $h(\xi_1, \xi_2) = \xi_1$ or $h(\xi_1, \xi_2) = \xi_1 - \xi_2$ or $h(\xi_1, \xi_2) = (\xi_1 + 100\,\xi_2)^{10} + 2000(\xi_1 + 100\,\xi_2)^2$ is not a complete polynomial, the polynomial $h(\xi_1, \xi_2) = \xi_1 - \xi_2^2$ or $h(\xi_1, \xi_2) = \xi_1\xi_2$ is complete.

THEOREM 1.5 : *Let $h : R^n \to R$ be any polynomial. Then the following four conditions are equivalent :*

(i) *h is a complete polynomial*

(ii) $\lim_{|\xi|\to\infty} \sum |D^{\alpha}h(\xi)| = \infty$

(iii) $\lim_{|\xi|\to\infty} \int_{|\eta|<1} d\eta(1 + |h(\xi + \eta)|)^{-2} = 0$

(iv) $\chi(|Q| \leqslant r)\,[h(P) + \mathrm{i}]^{-1}$ *is a compact operator for each* $r > 0$. *Here χ stands for the indicator function.*

Proof : We refer to the appendix. Q.E.D.

The condition (iv) of the previous Theorem 1.5 justifies the following Definition 1.6.

Definition 1.6. Let $h : R^n \to R$ be any continuous function. Then h is said to be a complete function if $\chi(|Q| \leqslant r)\ [h(P) + \mathrm{i}]^{-1}$ is compact for each $r > 0$.

Sufficient conditions for h to be a complete function are given in Theorem 1.7.

THEOREM 1.7 : a) *Let $h : R^n \to R$ be any continuous function such that $\lim_{|\xi|\to\infty} |h(\xi)| = \infty$. Then $\chi(|Q| \leqslant r)[h(P) + \mathrm{i}]^{-1}$ is compact for each* $r > 0$.

b) *Let $h : R^n \to R$ be a C^{k+1} function for some $k \geqslant 0$ such that*

(i) $\lim_{|\xi|\to\infty} \sum_{|\alpha|\leqslant k} |D^{\alpha}h(\xi)| = \infty$ *and*

(ii) $|D^{\beta}h(\xi)| \leqslant K \sum_{|\alpha|\leqslant k} |D^{\alpha}h(\xi)|$ *for all* β with $|\beta| = k + 1$. *Here K is a suitable constant. Then $\chi(|Q| \leqslant r)\ [h(P) + \mathrm{i}]^{-1}$ is compact for each* $r > 0$.

Proof. Part (a) is clear. For (b) we refer to Theorem 9 and Theorem A1 of [2]. Q.E.D.

We are ready to state our assumptions A1, A2,...A7 and our results Theorems 1.8, and 1.9.

A1 : $h : R^n \to R$ is a C^∞ function such that h and all its derivatives have at most polynomial growth.

A2 : $\{\xi$ in $R^n : \nabla h(\xi) = 0\}$ has zero Lebesgue measure.

A3 : (On critical values) If $C_v = \{h(\xi) : \Delta h(\xi) = 0\}$ is the set of critical values of h, then $\bar{C}_v$, the closure of C_v is a countable set.

A4 : (Local compactness = complete h) For each $r > 0$, the operator $\chi(|Q| \leqslant r)\ [h(P) + i]^{-1}$ is compact.

A5 : (Long range potential) $W_L : R^n \to R$ in a C^∞ function and there exists some ε_0 in $(0,1]$ such that

$$|D^\alpha W_L(x)| \leqslant K(\alpha)\ (1 + |x|)^{-|\alpha|-\varepsilon_0}$$

for all multi indices. Here $K(\alpha)$ is a suitable constant.

A6 : (Short range potential) $W_S : R^n \to R$ is a bounded continuous function and for some ε_1 in $(0, 1]$ we have

$$|W_S(x)| \leqslant K(1 + |x|)^{-1-\varepsilon_1}$$

Here K is a constant.

A7 : (Smoothness of the short range potential) W_S of A6 is of class C^2 and for some $m > n = (\dim R^n)$

$$|D^\alpha W_S(x)| \leqslant K(1 + |x|)^{-m}$$

for all multi indices α with $1 \leqslant |\alpha| \leqslant 2$. Here K is a constant.

Unless otherwise specified the letter K with or without suffix will stand for a generic constant.

Note that we shall assume $n = 1$ in A5.

THEOREM 1.8. *Let $n = 1$. Let h, W_S, W_L satisfy* A1, ..., A7. *Put* $H_0 = h(P)$, $H = H_0 + W_S(Q) + W_L(Q)$, $U_t = \exp[-itH_0]$ *and* $V_t = \exp[-itH]$. *Then there exists a C^∞ function $X : R \times R^n \to R$ such that*

(i) $\Omega_\pm = s\text{-}\lim_{t \to \pm\infty} V_t^* \exp[-iX(t, P)]$ *exists*

(ii) $\Omega_\pm^* \Omega_\pm =$ *identity* ie $\Omega_\pm$ *is an isometry*

(iii) (*Intertwining relations*) $V_t \Omega_\pm = \Omega_\pm U_t$ *for all real* t

(iv) Range $\Omega_\pm \subseteq \mathscr{H}_{ac}(H)$ *and*

(v) Range $\Omega_\pm = \mathscr{H}_{ac}(H)$.

THEOREM 1.9. *Let $n \geqslant 2$. Let h satisfy* A1, A2, A3 *and the conditions* b(i), (ii) *of Theorem* 1.7. *for some $k \geqslant 0$ so that* A4 *is also satisfied. Let*

$$\tilde{h}(\xi) = 1 + \sum \{ |D^\alpha h(\xi)| : |\alpha| \leqslant k\} \geqslant K(1 + |\xi|)^N$$

Since $\{(H+i)^{-2}f : f\in\mathscr{H}_{ac}(H)\}$ is dense in $\mathscr{H}_a(H)$ we deduce

$$\limsup_{r\to\infty}\ \sup_t \|P^2(P^2+r^2)^{-1}V_tf\| = 0 \quad \text{for each } f \text{ in } \mathscr{H}_{ac}(H)$$

Now the result is clear, because, for all g in $L^2(R^n)$ we have

$$\|\chi(|P|\geqslant r)g\| \leqslant 2\,\|P^2(P^2+r^2)^{-1}g\|. \quad \text{Q.E.D.}$$

Now we state the ingredient from [7].

THEOREM 2.3. *Let h, W_S, W_L satisfy the assumptions* A1, A2,...,A6. *Put* $H_0 = h(P)$, $H = H_0 + W_S(Q) + W_L(Q)$, $U_t = \exp[-it\,H_0]$, *and* $V_t = \exp[-itH]$. *Then there exists a C^∞ function $X : R\times R^n \to R$ such that*

(i) $\Omega_\pm = s\text{-}\lim_{t\to\pm\infty} V_t^*\exp[-iX(t, P)]$ *exists*

(ii) $\Omega_\pm$ *is an isometry*

(iii) $V_t\Omega_\pm = \Omega_\pm U_t$ *for all real* t

(iv) Range $\Omega_\pm \subseteq \mathscr{H}_{ac}(H)$ *and*

(v) Put $G = \{\xi$ in $R^n : \nabla h(\xi)\neq 0\}$. *Then* $\mathscr{H}_{ac}(H)\ominus$ Range $\Omega_\pm = \{f \text{ in } \mathscr{H}_{ac}(H) : \lim_{t\to\pm\infty}\|r(P)V_tf\| = 0$ *for each r in* $C_0^\infty(G)\}$.

Note that if $W_L = 0$, then we can, and do, take $X(t,\xi) = th(\xi)$.

Proof. Let φ be in $C_0^\infty(R^n)$. Then $(1+Q_1^2+\ldots+Q_n^2)^{-1}\varphi(P)(1+Q_1^2+\ldots+Q_n^2)$ is a bounded operator by the commutation relation between P and Q. So by the interpolation techniques [12], $(1+|Q|)^{-1-\varepsilon}\varphi(P)(1+|Q|)^{1+\varepsilon}$ is bounded for each ε in $[0,1]$. So $W_S(Q)\varphi(P)(1+|Q|)^{1+\varepsilon_1}$ is a bounded operator for each φ in $C_0^\infty(R^n)$.

Let $W(x) = W_S(x) + W_L(x)$. Since W is bounded and $\lim_{|x|\to\infty} W(x) = 0$, by the assumption A4, the operator $W(Q)(H_0+i)^{-1}$ is compact. Since

$$(H+i)^{-1} - (H_0+i)^{-1} = -(H+i)^{-1}W(Q)(H_0+i)^{-1}$$

the operator $(H+i)^{-1} - (H_0+i)^{-1}$ is compact. Again by A4, the operator $\chi(|Q|\leqslant r)\,(H+i)^{-1}$ is compact for each $r>0$. Now the result follows by Theorem 2.3 and the proof of Theorem 2.2 (ii) of [7]. Q.E.D

THEOMEM 2.4. *Let $h, W_S, W_L, W = W_S + W_L$ satisfy the assumptions* A1,...A6 *and let*

$$\lim_{r\to\infty}\|(H_0+i)^{-1}[P^2(P^2+r^2)^{-1}, W(Q)]\,(H_0+i)^{-1}\|_1 = 0$$

Then, Range $\Omega_\pm = \mathscr{H}_{ac}(H)$.

Proof. This theorem is a consequence of Theorems 2.2 and 2.3.

Step 1 : Let $\mathscr{A}$ be the Banach algebra given by

$$\mathscr{A} = \{\varphi : R\to\mathbb{C},\ \varphi \text{ is continuous, } \lim_{|t|\to\infty}\varphi(t) = 0\}$$

with the sup norm. Let $\mathscr{B}$ be given by

$$\mathscr{B} = \{\varphi \text{ in } \mathscr{A} : \varphi(H) - \varphi(H_0) \text{ is compact}\}$$

$\mathscr{B}$ is easily seen to be a closed * sub algebra of $\mathscr{A}$. The proof of Theorem 2.3 shows that, if $\alpha(x) = (x + i)^{-1}$, then α is in $\mathscr{B}$. By Stone-Weierstrass theorem we have $\mathscr{A} = \mathscr{B}$. Thus for each continuous $\varphi : R \to \mathbb{C}$ with $\lim_{|t|\to\infty} \varphi(t) = 0$ the operator $\varphi(H) - \varphi(H_0)$ is compact.

Step 2. Let φ be in $C_0^\infty(R \setminus \bar{C}_v)$. Choose ψ in $C_0^\infty(R^n)$ such that $\psi(\xi) = 1$ for $|\xi| \leqslant 1$ and 0 for $|\xi| \geqslant 2$. Let f be in $\mathscr{H}_{ac}(H) \ominus$ Range Ω_+. Since the function $\varphi(h(\xi))\,\psi(\xi/r)$ is in $C_0^\infty(G)$ for each $r > 0$, by Theorem 2.3 (v)

(3) $$\lim_{t\to\infty} \|\varphi(H_0)\ \psi(P/r)V_t f\| = 0$$

By Theorem 2.2,

(4) $$\lim_{r\to\infty} \sup_{t>0} \|[1 - \psi(P/r)]V_t f\| = 0$$

By (3) and (4) we have

(5) $$\lim_{t\to\infty} \|\varphi(H_0)V_t f\| = 0$$

Since $\varphi(H) - \varphi(H_0)$ is compact and weak $\lim_{t\to\infty} V_t f = 0$ one gets

(6) $$\lim_{t\to\infty} \|\{\varphi(H) - \varphi(H_0)\}\,V_t f\| = 0$$

By (5) and (6) we have $\varphi(H)f = 0$ for each φ in $C_0^\infty(R \setminus \bar{C}_v)$. Since $\bar{C}_v$ is a countable set and f is in $\mathscr{H}_{ac}(H)$ we conclude that $f = 0$. Thus $\mathscr{H}_{ac}(H) \ominus$ Range $\Omega_+ = 0$ i.e. $\mathscr{H}_{ac}(H) =$ Range Ω_+. Similarly $\mathscr{H}_{ac}(H) =$ Range Ω_-.
Q.E.D

§ 3. PROOF OF THEOREM 1.8.

LEMMA 3.1. *Let $h : R \to R$ be a continuous function such that $\chi(|Q| \leqslant r)\ [h(P) + i]^{-1}$ is a compact operator for each $r > 0$. Then*

$$\lim_{|n|\to\infty} \int_n^{n+1} (|h(\xi)|^2 + 1)^{-1}\, d\xi = 0$$

Proof. It is obvious that $(1 + Q^2)^{-1}\ [h(P) + i]^{-1}$ is compact. By interchanging P and Q we easily conclude that $[h(Q) + i]^{-1}\ (1 + P^2)^{-1}$ is compact. Let f in $C_0^\infty(R)$ be such that $f(x) = 1$ for $|x| \leqslant 1$ and 0 for $|x| \geqslant 2$. Define $f_j(x) = f(x - j)$ for any integer j. It is clear that weak $\lim_{|j|\to\infty} (1 + P^2)f_j = 0$. So $\lim_{|j|\to\infty} \|(h(Q) + i)^{-1}f_j\| = 0$ proving the Lemma. Q.E.D.

Now we prove Theorem 1.8. By Theorem 2.4 we only need $0 = \lim_{r\to\infty} \|(H_0 + i)^{-1}[P^2(P^2 + r^2)^{-1},\ W(Q)](H_0 + i)^{-1}\|_1$. For this, note that

$$P^2(P^2 + r^2)^{-1} = 1 + ir2^{-1}\{(P - ir)^{-1} - (P + ir)^{-1}\}$$

and

$$[(P \pm ir)^{-1}, W(Q)] = i(P \pm ir)^{-1}W'(Q)(P \pm ir)^{-1}.$$

Since $|W'(x)| \leq K(1 + |x|)^{-2\delta}$ with $\delta > 1/2$ we easily have

$$\|(H_0 + i)^{-1}[P^2(P^2 + r^2)^{-1}, W(Q)](H_0 + i)^{-1}\|_1 \leq$$

$$\leq Kr\|(1 + |Q|)^{-\delta}(P \pm ir)^{-1}[h(P)+i]^{-1}\|_2^2 \leq Kr \int d\xi\, |\xi + ir|^{-2}|h(\xi) + i|^{-2}.$$

The proof is complete by the following Lemma 3.2.

LEMMA 3.2. $\lim_{r\to\infty} r \int (|x| + r)^{-2}(1 + |h(x)|)^{-2}\, dx = 0.$

Proof. We show that $\lim_{r\to\infty} r\int_0^\infty (|x| + r)^{-2}(1 + |h(x)|)^{-2}\, dx = 0.$

$\left(\text{Similarly we can prove that } \lim_{r\to\infty} \int_\infty^0 (|x| + r)^{-2}(1 + |h(x)|)^{-2}dx = 0.\right)$

Let $a_n = \int_n^{n+1} (1 + |h(x)|)^{-2}\, dx$. Then by Lemma 3.1, $\lim_{n\to\infty} a_n = 0$. Now, clearly we can assume $r \geq 1$. We have

$$r \int_0^\infty dx(|x| + r)^{-2}(1 + |h(x)|)^{-2} \leq$$

$$\leq r \sum_0^\infty a_n \sup\{(|x| + r)^{-2} : n \leq x \leq n + 1\} \leq$$

$$\leq r \sum_0^\infty a_n(n + r)^{-2} \leq$$

$$\leq r \sum_0^N a_n(n + r)^{-2} + r \sup\{a_n : n \geq N + 1\} \cdot 4 \cdot \int_N^\infty (x + r)^{-2}\, dx \leq$$

$$\leq r \sum_0^N a_n(n + r)^{-2} + 4 \sup\{a_n : n \geq N + 1\}$$

Now the result is obvious since $\lim_{n\to\infty} a_n = 0$. Q.E.D.

Remark 3.3. A careful look at the proof of Theorem 1.8 shows that it suffices to assume in A7 that W_S is in $C^1(R)$ and $|W_S'(x)| \leq K(1 + |x|)^{-m}$ for some $m > 1$.

Remark 3.4. If h is a non constant polynomial on R, then it is vaguely elliptic and hence satisfies the assumption A4.

Example 3.5. Let $h : R \to R$ be given by $h(\xi) = \xi \sin \xi$. Then h is simply characteristic.

§ 4. PROOF OF THEOREM 1.9

LEMMA 4.1. *Let for x in R^n, $\langle x\rangle = (1 + x^2)^{1/2}$. Then for any two real numbers a, b the operators $\langle P\rangle^a \langle Q\rangle^b \langle P\rangle^{-a} \langle Q\rangle^{-b}$ and $\langle Q\rangle^a \langle P\rangle^b \langle Q\rangle^{-a} \langle P\rangle^{-b}$ are bounded.*

Proof. Refer to p 284 of [5]. Q.E.D.

THEOREM 4.2. *Let h, W_S satisfy the assumptions A1, . . ., A4, A6 and A7. Put $H_0 = h(P)$, $H = H_0 + W_S(Q)$. If further*

$$\lim_{r\to\infty} r^2 \int (1 + |x|)(|x| + r)^{-4}(1 + |h(x)|)^{-2} = 0,$$

then Range $\Omega_\pm = \mathscr{H}_{ac}(H)$.

Proof. We apply Theorem 2.4. Put $W(x) = W_S(x)$. Now,

$$\|(H_0 + i)^{-1} [P^2(P^2 + r^2)^{-1}, W(Q)](H_0 + i)^{-1}\|_1 =$$

$$(7) \qquad = r^2\|(H_0 + i)^{-1}[(P^2 + r^2)^{-1}, W(Q)](H_0 + i)^{-1}\|_1 =$$

$$= r^2\|(H_0 + i)^{-1}(P^2 + r^2)^{-1} \sum_j [P_j^2, W(Q)](P^2 + r^2)^{-1}(H_0 + i)^{-1}\|_1$$

Note that

$$(8) \qquad [P_j^2, W(Q)] = -i\{P_j(D_jW)(Q) + (D_jW)(Q)P_j\}$$

$$P_j(D_jW)(Q) = \langle P\rangle^{\frac{1}{2}}\langle Q\rangle^{\frac{-m}{2}}\{\langle Q\rangle^{\frac{m}{2}}\langle P\rangle^{\frac{-1}{2}} P_j(D_jW)(Q)\langle P\rangle^{\frac{-1}{2}}\langle Q\rangle^{\frac{m}{2}}\} \langle Q\rangle^{\frac{-m}{2}} \langle P\rangle^{\frac{1}{2}}$$
(9)

$$(D_jW)(Q)P_j = \langle P\rangle^{\frac{1}{2}}\langle Q\rangle^{\frac{-m}{2}}\{\langle Q\rangle^{\frac{m}{2}}\langle P\rangle^{\frac{-1}{2}}(D_jW)(Q)P_j\langle P\rangle^{\frac{-1}{2}}\langle Q\rangle^{\frac{m}{2}}\}\langle Q\rangle^{\frac{-m}{2}}\langle P\rangle^{\frac{1}{2}}$$
(10)

CLAIM. $A_j = \langle Q\rangle^{\frac{m}{2}}\langle P\rangle^{\frac{-1}{2}} P_j(D_jW)(Q)\langle P\rangle^{\frac{-1}{2}}\langle Q\rangle^{\frac{m}{2}}$ and

$B_j = \langle Q\rangle^{\frac{m}{2}}\langle P\rangle^{\frac{-1}{2}}(D_jW)(Q)P_j\langle P\rangle^{\frac{-1}{2}}\langle Q\rangle^{\frac{m}{2}}$ are bounded operators.

We assume the CLAIM and proceed. In the end, we prove the CLAIM. By (7), (8), (9), (10) and the CLAIM we get

$$\|(H_0 + i)^{-1} [P^2(P^2 + r^2)^{-1}, W(Q)] (H_0 + i)^{-1}\|_1 \leqslant$$

$$\leqslant Kr^2 \|(H_0 + i)^{-1} \langle P\rangle^{1/2} (P^2 + r^2)^{-1} \langle Q\rangle^{-m/2}\|_2^2.$$

Now Theorem 4.2 follows from Theorem 2.4 since $m > n$ and $\|f(P)g(Q)\|_2 = \|f\|_2 \|g\|_2$.

Now we prove the CLAIM. We show that A_j is bounded; similarly one can prove that B_j is bounded. In what follows the letter B will stand for a generic bounded operator. Now

$$A_j = \langle Q\rangle^{\frac{m}{2}}\langle P\rangle^{\frac{-1}{2}} P_j(D_jW)(Q)\langle P\rangle^{\frac{-1}{2}}\langle Q\rangle^{\frac{m}{2}} =$$

$$= \{\langle Q\rangle^{\frac{m}{2}}\langle P\rangle^{\frac{-1}{2}}\langle Q\rangle^{\frac{-m}{2}}\langle P\rangle^{\frac{1}{2}}\}\langle P\rangle^{\frac{-1}{2}}\langle Q\rangle^{\frac{m}{2}} P_j(D_jW)(Q)\langle Q\rangle^{\frac{m}{2}}\langle P\rangle^{\frac{-1}{2}}\cdot$$

$$\cdot \{\langle P\rangle^{\frac{1}{2}}\langle Q\rangle^{\frac{-m}{2}}\langle P\rangle^{-\frac{1}{2}}\langle Q\rangle^{\frac{m}{2}}\}$$

Now use Lemma 4.1 to get

$$A_j = B\langle P\rangle^{\frac{-1}{2}}[\langle Q\rangle^{\frac{m}{2}}, P_j](D_jW)(Q)\langle Q\rangle^{\frac{m}{2}}\langle P\rangle^{\frac{-1}{2}} B +$$

$$+ B\langle P\rangle^{\frac{-1}{2}} P_j\langle Q\rangle^{\frac{m}{2}}(D_jW)(Q)\langle Q\rangle^{\frac{m}{2}}\langle P\rangle^{\frac{-1}{2}} B$$

Note that by the assumption A7 the first summand for A_j is bounded; in the second summand write $P_j = P_j\langle P\rangle^{-1}\langle P\rangle$ to get

(11) $$A_j = B + B\langle P\rangle^{\frac{1}{2}}\langle Q\rangle^m(D_jW)(Q)\langle P\rangle^{-\frac{1}{2}} B$$

The operator $P_k\langle Q\rangle^m(D_jW)(Q)(P_k + \mathrm{i})^{-1}$ is, using A7, seen to be bounded by commuting the P_k across. So we get

(12) $$\langle P\rangle\langle Q\rangle^m(D_jW)(Q)\langle P\rangle^{-1} = B$$

Clearly by A7

(13) $$\langle P\rangle^0\langle Q\rangle^m(D_jW)(Q)\langle P\rangle^{-0} = B$$

By (12), (13) and interpolation techniques [12] we get the boundedness of $\langle P\rangle^{1/2}\langle Q\rangle^m(D_jW)(Q)\langle P\rangle^{-1/2}$. So by (11) the operator A_j is bounded. Q.E.D.

The next Lemma 4.3, though simple, is crucial to connect h and $\tilde{h}$ through integrals in Lemma 4.4.

LEMMA 4.3. *Let $f: [0, b] \to R$ be in $C^j\,[0, b]$ for some j. Let $|f^{(j)}(t)| \geqslant \beta > 0$ for all t in $[0, b]$. Then for $r \geqslant 0$.*

$$\int_0^b [1 + |f(t)|]^{-r}\,\mathrm{d}t \leqslant 2(K_j + b)\beta^{-r/(1+jr)}$$

where K_j depends only on j; K_j is independent of b, r, β.

Proof. Let $I_j = [0, b]$. Fix $\delta > 0$. Define

$$I_{j-1} = \{t \text{ in } I_j : |f^{(j-1)}(t)| \leqslant \delta\beta\}$$

Since $f^{(j)}$ never vanishes in I_j, the function $f^{(j-1)}$ is monotone on I_j. So I_{j-1} is an interval and the complement $\{t \text{ in } I_j : |f^{(j-1)}(t)| \geqslant \delta\beta\}$ is a union of at most two intervals. Now we calculate $|I_{j-1}|$, the length of I_{j-1}. Let $I_{j-1} = [t_1, t_2]$. An easy calculation using mean value theorem gives

$$2\delta\beta \geqslant |f^{(j-1)}(t_1) - f^{(j-1)}(t_2)| \geqslant (t_2 - t_1)\beta$$

Thus $|I_{j-1}| \leqslant 2\delta$. We easily conclude

$$\int_{I_j} [1 + |f(t)|]^{-r}\, dt \leqslant 2\delta + \int_{\{|f^{(j-1)}(t)| \geqslant \delta\beta\}} [1 + |f(t)|]^{-r}\, dt$$

Now the set $\{t : |f^{(j-1)}(t)| \geqslant \delta\beta\}$ is a union of at most two intervals and we keep on repeating the procedure to get.

$$\int_{[0,b]} [1 + |f(t)|]^{-r}\, dt \leqslant$$

$$\leqslant 2\delta + 2.2\delta + \int_{\{|f^{(j-2)}(t)| \geqslant \delta^2\beta\}} [1+|f(t)|]^{-r} dt \leqslant$$

$$\leqslant 2\delta + 2.2\delta + \ldots + 2^{j-1}\cdot 2\delta + \int_{\{|f(t)| \geqslant \delta^j\beta\}} [1+|f(t)|]^{-r} dt \leqslant$$

$$\leqslant K_j\delta + b\,[1 + \delta^j\beta]^{-r}$$

where $K_j = 2 + 2^2 + \ldots, +2^j$. So

$$\int_0^b [1 + |f(t)|]^{-r}\, dt \leqslant (K_j + b)(\delta + \delta^{-jr}\beta^{-r})$$

Taking $\delta = \beta^{-r/(1+jr)}$ we get the result. Q.E.D.

LEMMA 4.4. *Let $h : R^n \to R$ satisfy the conditions of Theorem* 1.7b (i), (ii). *Then there exists a partition of R^n into congruent parallel cubes C_j such that*

$$\int_{C_j} [1 + |h(y)|]^{-2}\, dy \leqslant K_1 \int_{C_j} \Big[1 + \sum_{|\alpha| \leqslant k} |D^\alpha h(y)| \Big]^{-2/(1+2k)} dy$$

for each j. Here K_1 depends on k, K of Theorem 1.7 (b) (i) *and a bound on the length of diagonal of C_j.*

Proof. We apply the results of [2] and Lemma 4.3. Define $F : R^n \times \times R^n \to R$ and $g : R^n \to R$ by

$$F(x, y) = \sum_{p \leqslant k} \Big\{ \sum_{|\alpha| = p} (D^\alpha h)(x) y^\alpha \Big\}^2$$

$$g(x) = \sum_{|\alpha| \leqslant \alpha} [D^\alpha h)(x)]^2$$

Then the proof of Lemma A3 of [2] shows that we can partition R^n into congruent parallel cubes C_j with length of diagonal of $C_j \leq 1$ and there exist unit vectors y_j of R^n such that

(14) $$S_j = \sup\{g(x) : x \text{ in } C_j\} \leq K_1 \inf \{g(x) : x \text{ in } C_j\}$$

(15) $$F(x, y_j) \geq K_2\, g(x) \text{ for all } x \text{ in } C_j,\ K_2 > 0$$

(16) $$(D^\beta h)(x) y_j^\beta]^2 \leq K_3 F(x, y_j) \text{ for all } x \text{ in } C_j, \text{ all } \beta \text{ with } |\beta| = k + 1.$$

Now let us fibrate C_j along y_j. For x in C_j, let

$$I(x) = \{t \text{ in } R : x + ty_j \text{ is in } C_j\}$$

Then clearly $I(x)$ is an interval of length ≤ 1 since length of diagonal of $C_j \leq 1$. Clearly one has

(17) $$\int_{C_j} [1 + |h(y)|^2]^{-1}\, dy \leq \sup_{x \in C_j} \int_{I(x)} [1 + |h(x + ty_j)|^2]^{-1}\, dt$$

Now fix x in C_j and define $f : I(x) \to R$ by $f(t) = h(x + ty_j)$. The proof of Lemma A4 of [2] shows that $I(x)$ can be divided into a finite number of intervals $I_1, \ldots, I_N$ with N depending only on k and independent of x such that on each of these intervals

(18) $$|f^{(i)}(t)| \geq K_4\, S_j^{1/2}$$

for some $0 \leq i \leq k$. Here K_4 depends only on K_1, K_2, K_3 and an upper bound on the length of $I(x)$. Now by (17), (18) and Lemma 4.3 we have

(19) $$\int_{C_j} [1 + |h(y)|]^{-2}\, dy \leq K S_j^{-1/(1+2k)}$$

Now by (14) and (19) we have

$$\int_{C_j} [1 + |h(y)|]^{-2}\, dy \leq K \int_{C_j} \left[\sum_{|\alpha|=0}^{k} |D^\alpha h(x)| \right]^{-2/(1+2k)} dx \leq$$

(20)

$$\leq K_1 \int_{C_j} \left[1 + \sum_{|\alpha|=0}^{k} |D^\alpha h(x)| \right]^{-2/(1+2k)} dx$$

Q.E.D.

Lemma 4.5. *Let h be as in Lemma 4.4. Put*

$$\tilde{h}(x) = 1 + \sum_{|\alpha| \leq k} |D^\alpha h(x)|$$

a) *If* $\lim_{r\to\infty} r^2 \int [(1+|x|)(|x|+r)^{-4}[\tilde{h}(x)]^{-2/(1+2k)}dx = 0$, *then*

$$\lim_{r\to\infty} r^2 \int (1+|x|)(|x|+r)^{-4}[|1+h(x)|]^{-2}\,dx = 0.$$

b) *Consequently if* $\tilde{h}(x) \geq K_0(1+|x|)^N$ *for some* $K_0 > 0$ *and* $N > (n-1)[k+(1/2)]$, *then*

$$\lim_{r\to\infty} r^2 \int (1+|x|)(|x|+r)^{-4}[1+|h(x)|]^{-2}\,dx = 0$$

Proof. a) Let the cubes C_j be as in Lemma 4.4. Clearly we can assume $r \geq 1$. Now

$$\int (1+|x|)(|x|+r)^{-4}[1+|h(x)|]^{-2}\,dx =$$

$$(21)\qquad = \sum_j \int_{C_j} [(1+|x|)(|x|+r)^{-4}[1+|h(x)|]^{-2}\,dx \leq$$

$$\leq \sum_j \sup\{(1+|x|)(|x|+r)^{-4} : x \text{ in } C_j\}\, K \int_{C_j} [\tilde{h}(x)]^{-2/(1+2k)}dx.$$

In the last step we have used Lemma 4.4. Clearly for some K_0 independent of j and r we have

$$(22)\ \sup\{(1+|x|)(|x|+r)^{-4} : x \text{ in } C_j\} \leq K_0 \inf\{(1+|x|)(|x|+r)^{-4} : x \text{ in } C_j\}.$$

Substituting (22) in (21) we get

$$\int (1+|x|)(|x|+r)^{-4}[1+|h(x)|]^{-2}\,dx \leq$$

$$(23)\qquad \leq KK_0 \int (1+|x|)(|x|+r)^{-4}[\tilde{h}(x)]^{-2/(1+2k)}\,dx.$$

Now the result is obvious.

b) Changing to polar coordinates, we have,

$$\lim_{r\to\infty} r^2 \int (1+|x|)(|x|+r)^{-4}[1+|x|]^{-2N/(1+2k)}\,dx=$$

$$= K \lim_{r\to\infty} r^2 \int_0^\infty (1+y)(y+r)^{-4}\,(1+y)^{-2N/(1+2k)}y^{n-1}\,dy \leq$$

$$\leqslant K \lim_{r\to\infty} r^2 \int_1^\infty (1+y)(y+r)^{-4}(1+y)^{-2N/(1+2k)} y^{n-1}\, dy \leqslant$$

$$\leqslant K \lim_{r\to\infty} \int_1^\infty y^{1+n-1-2-2N/(1+2k)} y^2 r^2 (y+r)^{-4}\, dy \tag{24}$$

Now $y^2r^2(y+r)^{-4} \leqslant 1$ and $\lim_{r\to\infty} y^2r^2(y+r)^{-4} = 0$. So by Lebesgue dominated convergence theorem right hand side of (24) is 0 if $n-2-2N/(1+2k) < -1$ i.e. if $N > (n-1)[(1/2)+k]$. Q.E.D.

Proof. of Theorem 1.9. It is clear that Theorem 1.9 is a consequence of Theorem 4.2 and Lemma 4.5 (b). Q.E.D.

Example 4.6. Let $h: R^2 \to R$ be given by $h(\xi_1, \xi_2) = \xi_1\xi_2$ and $W(x) = \langle x\rangle^{-m}$ where $m > 1$. then for the pair $(P_1P_2, P_1P_2 + (1+Q^2)^{-m/2})$ AC holds. We prove this by Lemma 4.5 (a) and Theorem 4.2. In this example $k=1$. So

$$[\tilde{h}(x,y)]^{-2/3} = (1+|x|)^{-2/3}(1+|y|)^{-2/3}.$$

It is easily seen that

$$0 = \lim_{r\to\infty} r^2 \int (1+|x|+|y|)(|x|+|y|+r)^{-4}(1+|x|)^{-2/3}(1+|y|)^{-2/3}\, dx\, dy.$$

Remark 4.7. Example 4.6 has been treated even with long range potential in [11] since h is simply characteristic. We treated the above example with the only aim of showing that the new method "Bound states for momentum" is useful to treat partial differential operators on $L^2(R^n)$ for $n \geqslant 2$.

APPENDIX

We prove Theorem 1.5. We show (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

(i) $\Rightarrow$ (ii) We refer to Proposition 10.2.9 of [4].

(ii) $\Rightarrow$ (i) Obvious

(ii) $\Leftrightarrow$ (iii) We refer to Lemma 7.8 Chapter 5 of [14].

(iii) $\Rightarrow$ (iv) We refer to Lemma 8 and Theorem 9 of [2]

(iv) $\Rightarrow$ (iii) The proof is similar to the proof of Lemma 3.1. Q.E.D.

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