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QUANTUM STOCHASTIC FLOWS WITH INFINITE DEGREES OF FREEDOM AND COUNTABLE STATE MARKOV PROCESSES

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SUMMARY. Quantum stochastic flows are constructed for infinite degrees of freedom. The theory is then used to show that a classical countable state Markov process can be looked upon as one such commutative stochastic flow.

1. Introduction

The concept of quantum stochastic process was introduced by Accardi, Frigerio and Lewis (1982) and a construction of a quantum stochastic flow† satisfying a quantum stochastic differential equation was carried out by Evans and Hudson (1988) and Evans (1889). However this construction was achieved under two restrictive hypotheses—firstly that of finite degree of freedom for the noise space and secondly that of boundedness of the structure maps on the algebra of observables of the system. Here we build a theory of quantum diffusions removing both these restrictions i.e. with a countably infinite degree of freedom for the noise and replacing the boundedness of the structure maps with suitable strong summability hypotheses on them. In the last section we apply this theory to the case of countably infinite state Markov chain and show that they can be understood as commutative quantum (classical) stochastic flows over the commutative algebra of functions on the state space. This extends the previous studies by Meyer (1989) and Parthasarathy and Sinha (1990).

2. Notations and preliminaries

All the Hilbert spaces that appear here are assumed to be complex and separable with scalar product $< \dots >$ linear in second variable. For any Hilbert space $\mathcal{H}$ we denote by $\Gamma(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$ respectively the boson Fock

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† Though the phrase "quantum diffusion" has often been used in the past, it seems that "quantum stochastic flow" is more appropriate in analogy with the terminology used in the theory of ordinary differential equations.

space over $\mathcal{H}$ and the C^*—algebra of all bounded linear operators in $\mathcal{H}$. Let $\mathcal{H}_0$ and $\mathcal{K}$ be two fixed Hilbert spaces and we write

$$\mathcal{H} = L_2(\mathcal{R}_+) \otimes \mathcal{K},$$
$$\tilde{\mathcal{H}} = \mathcal{H}_0 \otimes \Gamma(\mathcal{H}). \quad \ldots \quad (2.1)$$

For any $f \in \mathcal{H}$ we denote by $e(f)$ the exponential or coherent vector in $\Gamma(\mathcal{H})$ associated with f and by $\mathcal{E}$ the set of all vectors of the form $u \otimes e(f)$, $u \in \mathcal{H}_0$, $f \in \mathcal{H}$. Also we adopt the convention of writing $ue(f)$ in place of $u \otimes e(f)$. Note that $\mathcal{E}$ is total in $\tilde{\mathcal{H}}$.

We fix an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of $\mathcal{K}$ and set $E_j^i = |e_j><e_i|$ $(i, j \geqslant 1)$. The basic quantum stochastic processes of the theory are :

$$\Lambda_j^i = \begin{cases} \Lambda(\chi_{[0,t]} \otimes E_j^i) & , \quad i, j \geqslant 1 \\ a(\chi_{[0,t]} \otimes e_i) & , \quad i \geqslant 1, \ j = 0 \\ a^\dagger(\chi_{[0,t]} \otimes e_j) & , \quad i = 0, \ j \geqslant 1 \\ tI & , \quad i = j = 0. \end{cases} \quad \ldots \quad (2.2)$$

The quantum Ito's formula gives :

$$d\Lambda_j^i d\Lambda_l^k = \hat{\delta}_l^i d\Lambda_j^k, \quad i, j, k, l \geqslant 0 \quad \ldots \quad (2.3)$$

where
$$\hat{\delta}_l^i = 0 \quad \text{if } i = 0 \text{ or } l = 0 \quad \ldots \quad (2.4)$$
$$= \delta_l^i \quad \text{otherwise.}$$

For further details on these definitions and quantum Ito's formula the reader is referred to Evans (1989) and Hudson and Parthasarathy (1984).

Definition 2.1 : $L \equiv \{L_j^i(s)\}_{i,j \geqslant 0}$ is said to be an adapted square integrable family of processes (w.r.t. Λ_j^i) if they are adapted and satisfy for each $j \geqslant 0$, $t \geqslant 0$:

$$\sum_{i=0}^{\infty} \int_0^t \|L_j^i(s)\, ue(f)\|^2 \, d\nu_f(s) < \infty, \quad \ldots \quad (2.5)$$

where
$$\nu_f(t) = \int_0^t (1 + \|f(s)\|^2) ds, \; f \in \mathcal{M} \subseteq L^2(\mathcal{R}_+) \otimes \mathcal{K} \simeq L^2(\mathcal{R}_+, \mathcal{K})$$

being looked upon as $f \equiv \{f(s) \,|\, f(s) \in \mathcal{K}\}$, $\mathcal{M}$ a dense subset of $\mathcal{H}$.

We need to consider quantum stochastic integrals of the type $\sum_{i,j \geqslant 0} \int_0^t L_j^i(s) d\Lambda_i^j(s)$ and the next theorem sums up the result on their existence and their properties. We denote by $f^j(s) = <e_j, f(s)>$ and $f_j(s) \equiv f^j(s)$ for $j \geqslant 1$ and $f^0(s) = f_0(s) = 1$. We also choose $\mathcal{M} = \{f \in \mathcal{H} \,|\, f^j(.) = 0$ for all except finitely many j's} and for a given $f \in \mathcal{M}$, set $N(f) = \max\{j \, f^j(.) \neq 0\}$. Set $\mathcal{E}(\mathcal{M}) = \{ue(f) \mid u \in \mathcal{H}_0, \ f \in \mathcal{M}\}$.

Theorem 2.2 : *Suppose* $L \equiv \{L^i_j(s)\}$ *is an adapted square integrable operator family defined on* $\mathcal{E}$. *Then* $X(t) = \int_0^t \sum_{i,j \geq 0} L^i_j(s) d\Lambda^j_i(s)$ *exists in the strong sense on* $\mathcal{E}(\mathcal{M})$ *and defines a regular adapted process satisfying for* $u, v \in \mathcal{H}_0$, $f, g \in \mathcal{M}$

$$< ue(f), X(t)ve(g) > = \int_0^t ds \sum_{i,j} {}_0 f_i(s) g^j(s) < ue(f), L^i_j(s)ue(g) >, \quad \ldots \quad (2.6)$$

$$\|X(t)ue(f)\|^2 \leqslant 2 \exp(\nu_f(t)) \sum_{j=0}^{N(f)} \int_0^t \sum_{i \geqslant 0} \|L^i_j(s)ue(f)\|^2 d\nu_f(s). \quad \ldots \quad (2.7)$$

If $L' = \{L'^i_j(s)\}$ is another adapted square-integrable operator family and $X'(t) = \int_0^t \sum_{i,j \geqslant 0} L'^i_j(s) d\Lambda^j_i(s)$, then

$$\begin{aligned} <X'(t)\mathrm{ue}(f), X(t)\, ve\,(g)> = \int_0^t ds \sum_{i,j \geqslant 0} f_i(s) g^j(s) \{ &< X'(s)ue(f), L^i_j(s)ve(g) > \\ &+ < L'^j_i(s)ue(f), X(s)ve(g) > \\ &+ \sum_{k \geqslant 1} < L'^k_i(s)ue(f), L^k_j(s)ve(g) > \}. \end{aligned}$$

The proof is similar to that of Theorem 4.3 in Hudson and Parthasarathy (1984) and Theorem 2.1 in Parthasarathy and Sinha (1988). The second part is the quantum Ito formula.

Now suppose $L^i_j \in \mathcal{B}(\mathcal{H}_0)$, $i, j \geqslant 0$ and that for each $j \geqslant 0$, there exist constants $C_j \geqslant 0$ such that

$$\sum_{i \geqslant 0} \| L^i_j u \|^2 \leqslant C^2_j \| u \|^2 \text{ for all } u \in \mathcal{H}_0. \quad \ldots \quad (2.8)$$

Note that

$$\sum_i \| (L^i_j \otimes I)\psi \|^2 \leqslant C^2_j \| \psi \|^2 \text{ for all } \psi \in \mathcal{H}. \quad \ldots \quad (2.9)$$

Then we have

Theorem 2.3 : *Let* $L^i_j (i, j \geqslant 0)$ *satisfy* (2.8). *Then there exists a unique regular adapted process* $X \equiv \{X(t), \ 0 \leqslant t \leqslant T\}$ *satisfying :*

$$dX(t) = \left\{ \sum_{i,j \geqslant 0} L^i_j \, d\Lambda^j_i(t) \right\} X(t), \ X(0) = X_0 \in \mathcal{B}(\mathcal{H}). \quad \ldots \quad (2.10)$$

Proof : First we set up the iterative scheme :

$$X_0(t) = X_0$$

$$X_n(t) = x_0 + \int_0^t \sum_{i,j \geqslant 0} L^i_j \, X_{n-1}(s) d\Lambda^j_i(s), \; n \geqslant 1, \qquad \dots \quad (2.11)$$

and show that

$$\|X_n(t) - X_{n-1}(t)] ue(f)\|^2 \leqslant \frac{[\beta_f(T)\nu_f(t)]^n}{n!} \|X_0\|^2 \, \|u\|^2 \, \|e(f)\|^2, \quad \dots \quad (2.12)$$

where

$$\beta_f(T) = 2 \exp [\nu_f(T)] \sum_{j=0}^{N(f)} C_j^2. \qquad \dots \quad (2.13)$$

Note that X_1 is well defined by (2.8) and Theorem 2.2 and it is also easy to verify (2.12) for $n = 1$. Suppose that (2.12) is verified for $1 \leqslant n \leqslant k$. Then it follows that

$$\|X_k(t) ue(f)\| \leqslant \text{Const.} \, \|X_0\| \, \|u\| \, \|e(f)\|$$

so that X_{k+1} is well defined by (2.9) and we have by (2.7)

$$\|X_{k+1}(t) - X_k(t)] \, ue\,(f)\| \leqslant 2 \exp [\nu_f\,(T)]$$

$$\times \sum_{j=0}^{N(f)} \int_0^t \sum_{i \geqslant 0} \|L^i_j [X_k(s) - X_{k-1}(s)] \, ue\,(f)\| \, d\nu_f(s), \qquad \dots \quad (2.14)$$

which by (2.8), (2.9) and the induction hypothesis is

$$\leqslant 2 \exp [\nu_f(T)] \left[\sum_{j=0}^{N(f)} C_j^2 \right] \int_0^t \|[X_k(s) - X_{k-1}(s)] \, ue\,(f)\|^2 \, d\nu_f\,(s)$$

$$\leqslant \frac{\beta_f(T)^{k+1}}{K!} \left[\int_0^t \nu_f\,(s)^k \, d\nu_f\,(s) \right] \|X_0\|^2 \|u\|^2 \|e(f)\|^2$$

leading to (2.12). From (2.12) it easily follows that $X(t)ue(f) \equiv s\text{—}\lim_{n \to \infty} X_n(t)\, ue\,(f)$ exists for all $u \in \mathcal{H}_0$, $f \in \mathcal{M}$ and defines a regular adapted process. That $X\,(t)$ satisfies (2.10) follows easily from the above estimates.

Finally assume that there are two solutions X and X' satisfying $X(0) = X(0) = X_0$. Then by (2.7) and (2.9) we have

$$\|[X(t) - X'(t)] ue(f)\|^2 \leqslant \text{Const.} \int_0^t \|[X(s) - X'(s)] ue(f)\|^2 d\nu_f(s).$$

By iterating n times and observing that by virtue of (2.12), $\|X(t)ue(f)\|$, $\|X'(t)ue(f)\|$ both are uniformly bounded in $0 \leqslant t \leqslant T$, and letting $n \to \infty$ we conclude that $X = X'$.

Suppose $L_i (i \geqslant 1)$, S^i_j $(i, j \geqslant 1)$, $H \in \mathcal{B}(\mathcal{H}_0)$ with H self-adjoint satisfying :

$$\sum_{i \geqslant 1} \| L_i u \|^2 \leqslant C \|u\|^2 \text{ for all } u \in \mathcal{H}_0, \quad \dots \quad (2.15)$$

$$\sum_{k \geqslant 1} S^{k*}_i S^k_j = \sum_{k \geqslant 1} S^i_k S^{j*}_k = \delta^i_j.$$

Observe that the series involved in the second part of (2.15) converges in strong operator topology and that if we designate $S = \{S^i_j\}$ in $\mathcal{H}_0 \otimes \mathcal{K}$ in the matrix representation with respect to the canonical basis in $\mathcal{K}$, then this implies the unitarity of S in $\mathcal{H}_0 \otimes \mathcal{K}$. Now we make the following identification :

$$L^i_j = \begin{cases} S^i_j - \delta^i_j I & \text{if } 1 \leqslant i, j \\ L_i & \text{if } 1 \leqslant i, j = 0 \\ -\sum_{k \geqslant 1} L^*_k S^k_j & \text{if } 1 \leqslant j, i = 0 \\ iH - \frac{1}{2} \sum_{k \geqslant 1} L^*_k L_k & \text{if } i = j = 0. \end{cases} \quad \dots \quad (2.16)$$

That $\Sigma\, L^*_k L_k$ converges strongly is not difficult to see from (2.15) and the convergence of $\Sigma\, L^*_k S^k_j$ follows from (2.15) and Lemma 2.4. We also observe that the above L^i_j's satisfy the following identities :

$$L^i_j + L^{j*}_i + \sum_{k \geqslant 1} L^{k*}_i L^k_j = L^i_j + L^{j*}_i + \sum_{k \geqslant 1} L^i_k L^{j*}_k = 0. \quad \dots \quad (2.17)$$

The necessary convergences in (2.17) follow from (2.15) and Lemma 2.4.

Lemma 2.4 : *Suppose* $\{A_k\}$ *and* $\{B_k\}$, $k \geqslant 1$ *are two families of bounded operators in* $\mathcal{H}_0$ *such that* $\sum_{k \geqslant 1} A^*_k A_k$ *and* $\sum_{k \geqslant 1} B^*_k B_k$ *converge in strong operator topology. Then* $\sum_{k \geqslant 1} A^*_k B_k$ *also converges in strong topology.*

Proof : Let $u, v \in \mathcal{H}_0$. Then $\sum_{k \geqslant 1} \|A_k v\|^2 \leqslant C^2_1 \|v\|^2$ and $\sum_{k \geqslant 1} \|B_k u\|^2 \leqslant C^2_2 \|v\|^2$. Thus for $n > m$

$$\left| < v, \sum_{k=m}^{n} A^*_k B_k u > \right|^2 \leqslant \left(\sum_{m}^{n} \|A_k v\| \, \|B_k u\| \right)^2$$

$$\leqslant \sum_{k \geqslant 1} \|A_k v\|^2 \sum_{n=m}^{n} \|B_k u\|^2 \leqslant C^2_1 \|v\|^2 \sum_{k=m}^{n} \|B_k u\|^2$$

or

$$\left\| \sum_{k=m}^{n} A^*_k B_k u \right\| = \sup_v \frac{|< v \sum_{k=m}^{n} A^*_k B_k u >|}{\|v\|} \leqslant C_1 \left(\sum_{k=m}^{n} \|B_k u\|^2 \right)^{\frac{1}{2}} \to 0$$

as $m, n \to \infty$ and hence the result.

Theorem 2.5 : *Let* $\{L^i_j\}$, $i, j \geqslant 1$ *be as in* (2.16) *with* (2.15) *satisfied. Then the quantum stochastic differential equation* :

$$dU(t) = \Big(\sum_{i,j \geqslant 0} L^i_j \, d\Lambda^j_i(t)\Big)U(t), \; U(0) = I \qquad \ldots \quad (2.18)$$

has a unique unitary operator valued process as a solution.

Proof : From (2.15) and (2.16) it follows that L^i_j satisfies (2.8). Thus by Theorem 2.3 existence of a unique solution follows. The unitarity is an easy consequence of Ito's formula (2.17) and the proof is identical to that of Theorem 7.1 in Hudson and Parthasarathy (1984).

3. Quantum stochastic flow

Let $\mathscr{A}$ be a unital$*$-subalgebra of $\mathscr{B}(\mathscr{H}_0)$.

Definition 3.1 : As in Accardi *et al.* (1982) and Evans (1989), we define a quantum stochastic flow on $\mathscr{A}$ as a family $\{j_t, \; t \geqslant 0\}$ of identity—preserving $*$-homomorphisms from $\mathscr{A}$ into $\mathscr{B}(\tilde{\mathscr{H}})$ satisfying for $X \in \mathscr{A}$;

(1) $j_0(X) = X$

(2) $j_t\,(X)$ is an adapted process

(3) there exist structure maps $\mu^i_j : \mathscr{A} \to \mathscr{A}$; $i, \; j \geqslant 0$ such that $j_t(X)$ satisfy a quantum stochastic differential equation :

$$dj_t(X) = \sum_{i,j \geqslant 0} j_t(\mu^i_j(X))d\Lambda^j_i(t). \qquad \ldots \quad (3.1)$$

As in Evans (1989) it is easy to verify formally using Ito's formula that if such a $\{j_t\}$ exists then the structure maps have the following properties : for $x, \; y \in \mathscr{A}$,

(1) μ^i_j is linear on $\mathscr{A}$

(2) $\mu^i_j\,(I) = 0$ $\qquad \ldots \quad (3.2)$

(3) $\mu^i_j\,(X)^* = \mu^j_i\,(X^*)$

(4) $\mu^i_j\,(XY) = \mu^i_j\,(X)Y + X\mu^i_j\,(Y) + \sum_{k \geqslant 1} \mu^i_k(X)\mu^k_j\,(Y).$

Our aim is to construct a flow given the structure maps μ^i_j satisfying (3.2). Clearly we need some summability condition to make sense of the structure equation (4) in (3.2). We give one such condition which, for finite number of degrees of freedom i and j, reduces to that of Evans (1989).

Assumption 3.2 : In addition to (3.2), we suppose that for each $j \geqslant 0$ there exist constants $\alpha_j > 0$, a countable index set $\mathcal{I}_j$ and a family $\{D^i_j\}_{i \in I_j} \in \mathcal{B}(\mathcal{H}_0)$ such that for all $u \in \mathcal{H}_0$, $X \in \mathcal{A}$

$$\sum_{i \geqslant 0} \|\mu^i_j(X)u\|^2 \leqslant \sum_{i \in \mathcal{I}_j} \|XD^i_j u\|^2,$$

where ... (3.3)

$$\sum_{i \in \mathcal{I}_j} \|D^i_j u\|^2 \leqslant \alpha^2_j \|u\|^2.$$

Remark 3.4 : Note that with assumption 3.2, the structure equation (3.2) now makes rigorous sense by Lemma 2.4 since the sum on the right hand side of (4) in (3.2) is of the form $\sum_{k \geqslant 1} \mu^i_k(X)\mu^k_j(Y) = \sum_{k \geqslant 1} \mu^k_i(X^*)^* \mu^k_j(Y)$.

The construction of j_t in such a case is essentially along the lines of-the proof of Theorem 2.3. Before stating the theorem we need some notations.

Notations : We fix $f \in \mathcal{M}$. Then for $u \in \mathcal{H}_0$, $X \in \mathcal{A}$ we set

$K^{(0)}_f(X, u) = \|Xu\|^2$,

$$K^{(n)}_f(X, u) = \left[2e^{\nu_f(T)}\right]^n \sum_{\substack{i_k \in \mathcal{I}_{j_k},\ 0 \geqslant j_k \leqslant N(f) \\ 0 \leqslant k \leqslant n}} \left\|XD^{i_n}_{j_n} D^{i_{n-1}}_{j_{n-1}} \dots D^{i_2}_{j_2} D^{i_1}_{j_1} u\right\|^2, \quad \dots (3.4)$$

and

$$K_f(T) = \left[2e^{\nu_f(T)}\right] \sum_{j=0}^{N(f)} \alpha^2_j. \quad \dots (3.5)$$

By virtue of (3.3) we note that

$$\left[2e^{\nu_f(T)}\right] \sum_{i,\, 0 \leqslant j \leqslant N(f)} K^{(n)}_f(\mu^i_j(X), u) \leqslant K^{(n+1)}_f(X, u), \quad \dots (3.6)$$

and

$$K^{(n)}(X, u) \leqslant [K_f(T)]^n \|X\|^2 \|u\|^2. \quad \dots (3.7)$$

Also we set

$$S^{(n)}_f(X, u) = \sum_{k=0}^{n} \frac{K^{(k)}_f(X, n)}{(k\,!)^{\frac{1}{2}}} \sum_{k=0}^{n} \frac{\nu_f(T)^k}{(k!)^{\frac{1}{2}}}$$

and

$$S_f(X, u) = \lim_{n \to \infty} S^{(n)}_f(X, u) \leqslant \left[\sum_{k \geqslant 0} \frac{[K_f(T)]^k}{(k\,!)^{\frac{1}{2}}} \sum_{k \geqslant 0} \frac{\nu_f(T)^k}{(k!)^{\frac{1}{2}}}\right] \|X\|^2 \|u\|^2. \quad \dots (3.8)$$

Theorem 3.5 : *Let the structure maps* $\mu^i_j : \mathcal{A} \to \mathcal{A}$ *satisfy* (3.2) *and* (3.3). *Then there exists a constructive quantum flow* $\{j_t, t \geqslant 0\}$ *on* $\mathcal{A}$ *sftisfying* (3.1). *Furthermore the map* $(t, X) \to j_t(X)$ *is jointly continuous ιn strong topology with respect to the strong topology in* $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H}_0)$. *Also* j_t *satisfies the estimate*

$$\|j_t(X)ue(f)\|^2 \leqslant S_f(X, u)\|e(f)\|^2. \quad \dots (3.9)$$

We begin with a series of lemmas concerning structure maps satisfying (3.2) and Assumption 3.2.

Lemma 3.6 : *For $X \in \mathcal{A}$, there exist regular adapted processes $j_t^{(n)}(X)$ satisfying :*

$$j_t^{(0)}(X) = X$$

$$j_t^{(n)}(X) = X + \int_0^t \sum_{i,j \geqslant 0} j_s^{(n-1)}(\mu_j^i(X))d\Lambda_i^j, \qquad \ldots \quad (3.10)$$

such that for $u \in \mathcal{H}_0$, $f \in \mathcal{M}$, $0 \leqslant t \leqslant T$,

$$\|[j_t^{(n)}(X) - j_t^{(n-1)}(X)]ue(f)\|^2 \leqslant \frac{K_f^{(n)}X, u)\nu_f(t)^n}{n\,!} \|e(f)\|^2, \qquad \ldots \quad (3.11)$$

and

$$\|j_t^{(n)}(X)ue(f)\|^2 \leqslant S_f^{(n)}(X, u)]e(f)\|^2. \qquad \ldots \quad (3.12)$$

Proof : The proof runs along clines identical to that of Theorem 2.3 and so we avoid giving/details. The inequality (3.11) is obtained by the method of induction starting with $n = 1$ for which it is immediate. The inequality (3.12) follows from (3.11) and an application of Cauchy inequality :

$$\|j_t^{(n)}(X)ue(f)\|^2\| \leqslant \left(\left\| \sum_{k=1}^n [j_t^{(k)}(X) - j[_t^{(k-1)}(X)]ue(f) \right\| + \|Xue(f)\| \right)^2$$

$$\leqslant \left[\sum_{k=0}^n \frac{[K_f^{(k)}(X, u)]^{\frac{1}{2}}}{(k\,!)^{\frac{1}{2}}} \nu_f(T)^{k/2} \right]^2 \|e(f)\|^2$$

$$\leqslant S_f^{(n)}(X, u)\|e(f)\|^2.$$

Lemma 3.7 : *For each $X \in \mathcal{A}$, $j_t(X)ue(f) \equiv s\text{-}\lim_{n\to\infty} j_t^{(n)}(X)ue(f)$ exists and defines a regular adapted process on $\mathcal{H}_0 \otimes \mathcal{E}(\mathcal{M})$. Furthermore,*

(i) $$\|j_t(X)\, ue(f)\|^2 \leqslant S_f(X, u)\ \|e(f)\|^2$$
$$\leqslant \alpha(f, T)\, \|X\|^2 \, \|u\|^2 \, \|e(f)\|^2, \qquad \ldots \quad (3.13)$$

where

$$\alpha(f, T) = \left[\sum_{n=0}^\infty \frac{K_f(T)^n}{(n\,!)^{\frac{1}{2}}} \sum_{n=0}^\infty \frac{\nu_f(T)^n}{(n\,!)^{\frac{1}{2}}} \right],$$

and

(ii) $$\|[j_t(X) - j_t^{(n)}(X)]\, ue(f)\| \leqslant \sum_{n+1}^\infty \left[\frac{K_f^{(k)}(X, u)\, \nu_f(T)^k}{k\,!} \right]^{\frac{1}{2}} \|e(f)\| \ldots \quad (3.14)$$

(iii) *j_t satisfies (1), (2) and (3) of (3.1) ;*

(iv) *$(t, X) \to j_t(X)\, ue(f)$ is strongly continuous with respect to the strong operator topology of $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H}_0)$.*

(v) *j_t is $*$-preserving as a form on $\mathcal{H}_0 \otimes \mathcal{E}(\mathcal{M})$ i.e. $< ve(g), j_t(X^*)ue(f) > = < j_t(X)\, ve(g), ue(f) > \ \forall\ u, v \in \mathcal{H}_0$ and $f, g \in \mathcal{M}$.*

Proof: The existence of strong limit of $j_t^{(n)}(X)\, ue(f)$ follows from the summability of the square root of the right hand side of (3.11) by virtue of (3.7) while the estimate (3.13) is an easy consequence of (3.8) and (3.12). Similarly the inequality (3.14) follows from (3.11).

That $j_t(X)$ is linear in X and satisfies (1) and (2) of (3.1) is immediate from the construction of j_t. To show that $j_t(X)$ satisfies (3) of (3.1) we note that by (3.10) and (2.7)

$$\left\| \left[j_t(X) - X - \int_0^t \sum_{i,j \geq 0} j_s(\mu_j^i(X))\, d\Lambda_i^j \right] ue(f) \right\|^2$$

$$\leqslant 2 \, \| [j_t(X) - j_t^{(n)}(X)]\, ue(f) \|^2$$

$$+ 4 e^{\nu_f(t)} \sum_{j=0}^{N(f)} \int_0^t \| [j_s(\mu_j^i(X)) - j_s^{(n-1)}(\mu_j^i(X))]\, ue(f) \|^2 \, d\nu_f(s)$$

which converges to zero as $n \to \infty$ by (3.14).

From (3.11) and (3.4) we have that for $X, Y \in \mathcal{A}$

$$\| j_s(X-Y) ue(f) \|^2 \leqslant \| e(f) \|^2 \left(\sum_{n=0}^{\infty} \frac{\nu_f(T)^n}{\sqrt{n!}} \right)$$

$$\times \sum_{n=0}^{\infty} \frac{[2 \exp(\nu_f(T))]^n}{\sqrt{n!}} \sum_{\substack{0 \leqslant i_k,\, 0 \leqslant j_k \leqslant N(f) \\ 1 \leqslant k \leqslant n}} \| (X-Y) D_{j_n}^{i_n} \cdots D_{j_1}^{i_1} u \|^2$$

which proves the strong continuity of the map $X \to j_s(X) ue(f)$ with respect to the strong topology in $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H}_0)$ by virtue of the second part of (3.3) and an application of dominated convergence theorem. The continuity of the map $t \to j_t(X) ue(f)$ follows from the differential equation (3.1) satisfied by $j_t(X)$ and from (3.8), (3.13). These two observations together yield (iv).

Clearly $j_t^{(n)}(X^*) = j_t^{(n)}(X^*)$ for $n = 0$. Assume that $j_t^{(n-1)}(X^*) = j_t^{(n-1)}(X)^*$ for some n in the weak sense i.e. $< ve(g), j_t^{(n-1)}(X^*)\, ue(f) > = < j_t^{(n-1)}(X) ve(g), ue(f) >$ for $u, v \in \mathcal{H}_0$, $f, g \in \mathcal{M}$. Then by (3.10), (2.6), (3.2) $< ve(g), j_t^{(n)}(X^*) ue(f) > = < Xve(g), ue(f) > + < ve(g), \int_0^t \sum_{i,j>0} j_s^{(n-1)}(\mu_j^i(X^*)) d\Lambda_i^j ue(f) > = < Xve(g), ue(f) > + \int_0^t ds \sum_{i,j} g_i(s) f^j(s) < j_s^{(n-1)}(\mu_i^j(X))\, ve(g), ue(f) > = < j_s^{(n)}(X) ve(g), ue(f) >$ and passing to the limit we have (v).

Lemma 3.8 : *j_t is multiplicative on $\mathcal{A}$ as a form on $\mathcal{H}_0 \otimes \mathcal{E}\mathcal{M}$ i.e., for $X, Y \in \mathcal{A}$; $u, v \in \mathcal{H}_0$; $f, g \in \mathcal{M}$: $< ue(f), j_t(XY)ve(g) > = < j_t(X^*)ue(f), j_t(Y)ve(g) >$.*

Proof : The proof is by induction and is different from that in Evans (1989) For this we set for fixed $f, g \in \mathcal{M}$; $u, v \in \mathcal{H}_0$; $X, Y \in \mathcal{A}$.

$$R_f^{(0)}(X, u) = S_f(X, u)$$

$$R_f^{(n)}(X, u) = \sum_{i,\, 0 \leqslant j \leqslant N(f)} R_f^{(n-1)}(\mu_j^i(X), u).$$

First note that by (3.4)—(3.8), the above sum is well-defined and furthermore

$$R_f^{(n)}(X, u) \leqslant \alpha_f(T)\left(\sum_{j=0}^{N(f)} \alpha_j^2\right)^n \|X\|^2\, \|u\|^2, \qquad \dots \quad (3.15)$$

where $\alpha_f(T)$ is as in (3.13).

Set for $n \geqslant 0$,

$$B_t^{(n)}(X, Y) < j_t^{(n)}(X^*)ue(f), j_t^{(n)}(Y)ve(g) > - < ue(f), j_t^{(n)}(XY)ve(g) > \dots \quad (3.16)$$

with $j_t^{(n)}$ defined in (3.10). Then we claim that

$$|B_t^{(n)}(X, Y)| \leqslant \sum_{k=1}^{n} \frac{\nu(T)^{(n-k+1)/2}\, \nu_{f,g}(t)^k}{k!\, \sqrt{(n-k+1)!}} \sum \Big\{ G_{j_1 \dots j_k}^{i_1 \dots i_k}(X, u, f; Y, v, g)^{\frac{1}{2}}$$

$$+ G_{j_1 \dots j_k}^{i_1 \dots i_k}(Y^*, v, g; X^*, u, f)^{\frac{1}{2}} \Big\} \qquad \dots \quad (3.17)$$

where

$$\nu(T) = \max\{\nu_f(T), \nu_g(T)\},$$

$$\nu_{f,g}(t) = \int_0^t [(1+\|f(s)\|^2(1+\|g(s)\|^2)]^{\frac{1}{2}} ds,\ G_{j_1 \dots j_k}^{i_1 \dots i_k}(X, u, f); Y, v, g)$$

$$= \left[2e^{\nu_f(T)}\right]^{-\frac{1}{2}\sum^{k} i_r} K_f^{(n-k+\sum_1^k i_r)}(X^*, u) R_g^{(\Sigma j_r)}(Y, v) \|e(f)\|^2\, \|e(g)\|^2,$$

$$\dots \quad (3.18)$$

and the second summation in (3.17) is over i_r's and j_r's subject to the condition $i_1 = j_1 = 1$, $1 \leqslant i_r + j_r \leqslant 2$ for $2 \leqslant r \leqslant k$.

For $n = 0$, it is clear that $B_t^{(0)}(X, Y) = 0$ and hence satisfies (3.17). For n, we have by an application of Ito's formula, (3.3), Remark 3.4 and Lemma 3 7 (iv) that

$$B_t^{(n)}(X, Y) = \int_0^t \sum_{\substack{0 \leqslant i \leqslant N(j) \\ 0 \leqslant j \leqslant N(g)}} f_i(s) g^j(s) \{B_s^{(n-1)}(X, \mu_j^i(Y) + B_s^{(n-1)}(\mu_j^i(X), Y)$$

$$+ \sum_{k \geqslant 1} B_s^{(n-1)}(\mu_k^i(X), \mu_j^k(Y))\} ds + R_t^{(n)} X, Y)$$

where

$$R_t^{(n)} = \int_0^t ds \sum_{i,j} f_i(s) g^j(s) \{ < [j_s^{(n)}(X^* - j_s^{(n-1)}(X^*)] ue(f), j_s^{(n-1)}(\mu_j^i(Y)) ve(g) >$$

$$+ < j_s^{(n-1)}(\mu_j^i(X^*)) ue(f), [j_s^{(n-1)}(Y) - j_s^{(n-1)}(Y)] ve(g) > \}. \quad \ldots \quad (3.19)$$

By (3.11), (3.12) and application of Cauchy-Schwartz inequality it is easy to see that

$$| R_t^{(n)}(X, Y) | \leqslant \{[K_f^{(n)}(X^*, u) R_g^{(1)}(Y, \nu)]^{\frac{1}{2}} + [K_g^{(n)}(Y, v) R_f^{(1)}(X^*, u)]^{\frac{1}{2}}\}$$

$$\frac{\nu^{n/2}(T)\nu_{f,g}(t)}{\sqrt{n!}} \|e(f)\| \|e(g)\|. \quad \ldots \quad (3.20)$$

Thus for $n = 1$,

$B_t^{(1)}(X, Y) = R_t^{(1)}(X, Y)$ and easily can be seen to satisfy the estimate (3.17) with $n = 1$ by virtue of (3.20). It is easy to verify for any fixed $i_1 \ldots i_k$; $j_1 \ldots j_k$; $1 \leqslant k \leqslant n$ the followimg

$$\int_0^t ds \sum_{\substack{0 \leqslant i \leqslant N(f) \\ 0 \leqslant j \leqslant N(g)}} |f^i(s)| \, |g^j(s)| \, G_{j_1 \ldots j_k}^{i_1 \ldots i_k} (\mu_j^i(X), (u, f\,; Y, v, g)^{\frac{1}{2}} \leqslant G_{j_1 \ldots j_k, 1}^{i_1 \ldots i_k, 0}$$

$$(X, u, f\,; Y, v, g)\nu_{f, g}(t), \quad \ldots \quad (3.21)$$

$$\int_0^t ds \sum_{\substack{0 \leqslant i \leqslant N(f) \\ 0 \leqslant j < N(g)}} |f^i(s)| \, |g^j(s)| \, G_{j_1 \ldots j_k}^{i_1 \ldots i_k} (\mu_j^i(X), u, f\,; Y, v, g)^{\frac{1}{2}} \leqslant G_{j_1 \ldots j_k, 0}^{i_1 \ldots i_k, 1}$$

$$(X, u, f\,; Y, v, g)\nu_{f, g}(t), \quad \ldots \quad (3.22)$$

and

$$\int_0^t ds \sum_{\substack{0 \leqslant i \leqslant N(f) \\ 0 \leqslant j \leqslant N(g)}} |f^i(s)| \, |g^i(s)| \sum_{r > 1} G_{j_1 \ldots j_k}^{i_1 \ldots i_k} (\mu_r^i(X), u, f, \mu_j^r(Y), v, g)^{\frac{1}{2}}$$

$$\leqslant G_{j_1 \ldots j_k, 1}^{i_1 \ldots i_k, 1} (X, u, f\,; y, v, g)\nu_{f,g}(t). \quad \ldots \quad (3.23)$$

Next assume (3.17) for $(n-1)$. Using triangle inequality in (3.19) and the estimates (3.20)—(3.23ĭ, the estimate (3.17) for $B_t^{(n)}$)(X, Y) can be verified and then (3.17) is established for all n by induction.

By (3.5), (3.7), (3.15) and (3.18), we have

$$|G^{i_1\cdots i_k}_{j_1\cdots j_k}(X, u, f\,;\, Y, v, g)|$$

$$\leqslant \{\|X\|^2\,\|Y\|^2\,\|u\|^2\,\|v\|^2\,\|e(f)\|^2\,\|e(g)\|^2\}\,\alpha_g(T)\,(\Sigma\alpha_j^2)^{\Sigma j_r}\,[2e^{\nu f(T)}]^{-\Sigma i_r}\,[K_f(T)]^{n-k+\Sigma i_r}.$$

$$\leqslant \{\ldots\}\alpha_g(T)\left(\sum_1^{N(g)}\alpha_2^j\right)^{\Sigma j_r}\left[\sum_{j=1}^{N(f)}\alpha_j^2\right]^{\Sigma i_r}[K_f(T)]^{n-k}$$

$$\leqslant \{\ldots\}\alpha_g(T)\left(\max\left\{\sum_{j=1}^{N(g)}\alpha_j^2,\ \sum_{j=1}^{N(f)}\alpha_j^2,\ 1\right\}\right)^{2k} K_f(T)]^{n-k}.$$

Thus by (3.17)

$$|B_t^{(n)}(X, Y)| \leqslant \text{Constant}\sum_{k=1}^{n}\frac{a^{n-k}\,b^k}{k!\sqrt{(n-k+1)!}},$$

where

$$a = \sqrt{\nu(T)K_f(T)},\ b = 3\nu(T)\max\left\{\sum_{j=1}^{N(f)}\alpha_j^2,\ K_f(T),\ K_g(T),\ 1\right\}.$$

Since $\sum_{k=1}^{\infty}\frac{a^{n-k}b^k}{k!\sqrt{(n-k+1)!}} \leqslant \frac{1}{\sqrt{n!}}(a+b)^n \to 0$, we conclude that $B_t^{(n)}(X, Y) \to 0$ as $n \to \infty$ and we have the desired result by appealing to Lemma 3.7.

Proof of Theorem 3.5 : As in Evans (1989) we first claim that the weak multiplicatively of j_t as shown in Lemma 3.8 and (3.14) implies boundedness of $j_t(X)$ in $\tilde{\mathscr{H}}$ for every $X \in \mathscr{A}$ and in fact contractivity of j_t map. This result allows us to conclude (i) strong convergence of $j_t^{(n)}(X)$ in $\tilde{\mathscr{H}}$ as $n \to \infty$ (ii) strong continuity of $(t, X) \to j_t(X)$ (iii) $[j_t(X)]^* = j_t(X^*)$ for $X \in \mathscr{A}$ from Lemma 3.7 and that (iv) j_t is a homomorphism of $\mathscr{A}$ into $\mathscr{B}(\tilde{\mathscr{H}})$.

The next theorem shows that if we have a classical system of observables i.e., if $\mathscr{A}$ is a commutative $*$-subalgebra of $\mathscr{B}(\mathscr{H}_0)$, then the corresponding quantum flow is also commutative.

Theorem 3.9 : *Assume the hypotheses of Theorem* 3.5 *and suppose furthermore, $\mathscr{A}$ is commutative. Then for $X, Y \in \mathscr{A}$; $s, t \geqslant 0$*

$$[j_s(X), j_t(Y)] = 0.$$

Proof: It is identical to that of Theorem 2.2 of Parthasarathy and Sinha (1990) with $n = \max\{N(f), N(g)\}$.

4. Application to Markov chains

Consider a countably infinite state Markov chain. As in Parthasarathy, and Sinha (1990), let G be a infinite group acting on a separable σ-finite measure space $(\mathscr{X}, \mathscr{F}, \mu)$ so that μ is quasi-invariant under G action, and define the unitary representatiom S_g of G in $L^2(\mu)$ by

$$(S_g u)(X) = \sqrt{\frac{d\mu}{d\mu_g}}(g^{-1} x)v(g^{-1} x),\ u \in L^2(\mu). \qquad \ldots \quad (4.1)$$

where $\mu_g\ (E) = \mu(gE)$, $E \in \mathscr{F}$. Let m be a complex bounded measurable function on $G \times \mathscr{X}$ and let $\mathscr{A} = L_\infty(\mu)$, the commutative $*$-subalgebra of multiplication operators in $\mathscr{B}(L^2(\mu))$. For a countably infinite set $F \subseteq G$ indexed in any suitable fashion by $\mathcal{N}$ the set of natural numbers, we set $= S_i S_{g_i}$ and

$$L_i = S_i m_i, \qquad \ldots \quad (4.2)$$

where $m_i \equiv m_{g_i} \in L_\infty(\mu)$.

In $\widetilde{\mathscr{H}} = L_2(\mu) \otimes \Gamma(L_2(\mathscr{R}_+) \otimes L_2(F)) \simeq L^2(\mu) \otimes \Gamma(\bigoplus_{i \in N} L_2(\mathscr{R}_+))$, we write $d\Lambda_0^0 = dt$, $d\Lambda_i^0 = dA_i^+$, $d\Lambda_0^i = dA_i$, $d\Lambda_j^i = d\Lambda_i\ \delta_j^i$ $((i, j \geqslant 1)$, with respect to the standard basis of l_2.

The quantum stochastic differential equation :

$$dW = \left[\sum_i \left\{ \{L_i\ dA_i^\dagger + (S_i - 1)d\Lambda_i - L_i^* S_i dA_i - \frac{1}{2} L_i^* L_i\ dt \right\} \right] W \qquad \ldots \quad (4.3)$$

with initial value $W(O) = I$ has a unique unitary solution by Theorem 2.5 if we assume that $\sum_i L_i^* L_i = \sum_i |m_i|^2$ converges strongly. If we now define

$$j_t(X) = W(t)^* X \otimes I\ W(t). \qquad \ldots \quad (4.4)$$

so that $j(X_t)$ satisfies

$$d_{j_t}(X) = \Sigma\{j_t(S_i^{-1}[X, L_i])\ dA_i^+ + j_t(S_i^{-1}\ X\ S_i - X)d\Lambda_i$$
$$+ j_t([L_i^*, X]_{S_i}\ dA_i\} + j_t(\mathscr{L}(X))\ dt, \qquad \ldots \quad (4.5)$$

where

$$\mathscr{L}(X) = \sum_i \{L_i^* X L_i - \frac{1}{2} L_i^* L_i X - \frac{1}{2} X L_i^* L_i\}, \qquad \ldots \quad (4.6)$$

$X \in \mathscr{A}$. We observe that the series in (4.6) converges strongly by hypothesis stated above and by Lemma 2.4 and in fact shows that L is a bounded map on $\mathscr{A}$ with $\|\mathscr{L}(X)\| \leqslant 2\|X\|\ \|\sum_i L_i^*\ L_i\|$.

Next we apply Theorem 3.5 directly to (4.5) to get a quantum flow. For this we need only to verify (3.2) and (3.3) for structure maps. For $X \in \mathcal{B}(L_2(\mu))$ we note :

$$\mu_0^i(X) = S_i^{-1}[X,L_i],\ \mu_i^0(X) = \mu_0^i(X^*)^* = [L_i, X]S_i$$
$$\mu_j^i(X) = (S_i^{-1}XS_i - X)\delta_j^i (i, j \geqslant 1),\ \mu_0^0(X) = L(X). \quad \dots \quad (4.7)$$

Now, $\sum_i \|\mu_0^i(X)u\|^2 \leqslant \|L\|\ \|Xu\|^2 + \sum_i \|XL_iu\|^2$, where we have written

$$L = \sum_i L_i^* L_i.$$

For $j \geqslant 1$,

$$\sum_{i=1}^{\infty} \|\mu_j^i(X)u\|^2 = \|\mu_j^j(X)u\|^2 \leqslant 2\|Xu\|^2 + 2\|XS_ju\|^2.$$

Finally as we have already remarked, by Lemma 2.4

$$\|\mu_0^0(X)u\| \leqslant \frac{1}{2}\|L\|\ \|Xu\| + \frac{1}{2}\|XLu\| + \|L\|^{\frac{1}{2}} \left(\sum_i \|XL_iu\|^2 \right)^{\frac{1}{2}}.$$

Thus (3.2) and (3.3) are easily verified.

Theorem 4.1 : *Let $X \in \mathcal{A} = \mathcal{B}(L_2(\mu))$, and let μ_j^i be as given by* (4.7). *Then there exists a family of contractive adapted processes $\{j_t(X), t \geqslant 0\}$ satisfying* (4.5). *Furthermore, each j_t is a $*$-homomorphism of $\mathcal{A}$ and $(t, x) \to j_t(X)$ is strongly continuous with respect to the strong operator topology of $\mathcal{A}$.*

The proof of Theorem 4.1 follows from the verification of (3.2) and (3.3) and an application of Theorem 3.5.

Let $\mathcal{X}$ be the state space of a countably infinite state continuous time Markov chain and let $p_t(x, y)$ $(x, y \in \mathcal{X})$ be the stationary transition probabilities such that

$$l(x, y) = \frac{d}{dt} p_t(x, y)|_{t=0} \quad \dots \quad (4.8)$$

Then $l(x, y) \geqslant 0$ if $x \neq y$ and $\sum_{y \in \mathcal{X}} l(x, y) = 0$. We now realize this Markov chain as a flow. Put any group structure on $\mathcal{X}$ so that $G = \mathcal{X}$, μ is the counting measure and G acts on itself by left translation. Set

$$m_x(y) = \sqrt{l(y, xy)} \text{ if } x \neq e$$
$$= 0 \qquad \text{otherwise} \quad \dots \quad (4.9)$$

As in Parthasarathy and Sinha (1990), the structure maps can be computed for $\phi \in L_\infty(\mu)$,

$$(\mu_0^x(\phi)u)(y) = m_x(y)[\phi(xy) - \phi(y)]u(y),$$
$$(\mu_x^x(\phi)u)(y) = [\phi(xy) - \phi(y)]u(y)$$
$$(\mu_x^0(\phi)u)(y) = \overline{m_x(y)}[\phi xy) - \phi(y)]u(y),$$

and $$(\mu_0^0(\phi)u)(y) = \sum_{x \in \mathscr{X}} |m_x(y)|^2 [\phi(xy)-\phi(y)]u(y). \quad \ldots \quad (4.10)$$

The following theorem sums up the results in this case.

Theorem 4.2 : *Let the Markov chain be described as above with* $\sup_{x \in \mathscr{X}} |l(x, x)| \equiv \delta < \infty$, *and let* $\phi \in \mathscr{A} = L_\infty(\mu)$. *Then* $\{j_t(\phi),\ t \geqslant 0\}$ *is a classical (commutative) contractive, strongly continuous flow satisfying*

$$dj_t(\phi) = \sum_{x \in \mathscr{X}} \{j_t(\mu_0^x(\phi))dA_x^+ + j_t(\mu_x^x(\phi))d\Lambda_x^x + j_t(\mu_x^0(\phi))dA_x\} + j_t(\mu_0^0(\phi))dt, \quad \ldots \quad (4.11)$$

with $j_0(\phi) = \phi$.

Proof : We have only to verify the strong convergence of $\sum_x |m_x(.)|^2$ and appeal to Theorems 4.1 and 3.9. For $u \in L_2(\mu)$.

$$< u, \sum_{x \in \mathscr{X}} |m_x(.)|^2 u > = \sum_{x \neq e} \sum_{y \in \mathscr{X}} l(y, xy)|u(y)|^2$$

$$= \sum_{y \in \mathscr{X}} [\sum_{x \neq e} l(y, xy)] |u(y)|^2 = - \sum_{y \in \mathscr{X}} l(y, y)|u(y)\|^2$$

$$\leqslant \delta \|u\|^2.$$

Remark : Note that the condition of Theorem 4.2 is also sufficient to define the generator $\mathscr{L}$ as a bounded map where $\mathscr{L}(\phi)(x) = \sum_{y \in \mathscr{X}} l(x, y)\phi(y)$. In fact,

$$|\mathscr{L}(\phi)(x)| \leqslant \sum_{y \neq x} l(x, y)|\phi(y)| + |l(x, x)|\,|\phi(x)|$$

$$\leqslant 2 \sup_{x \in \mathscr{X}} |l(x, x)|\, \|\phi\|_\infty.$$

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