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FUZZY GOAL PROGRAMMING APPROACH FOR DESIGNING SINGLE SAMPLING ATTRIBUTE PLANS WHEN SAMPLE SIZE IS FIXED

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ABSTRACT: The problem of determining single sampling attribute plans when the sample size n is fixed and small is considered under the *fuzzy* environment of satisfying the producer's and consumer's risks *closely*. An exact and an approximate (for $n \geqslant 20$) solution procedures are described. Numerical examples are given to illustrate the methods.

1. Introduction

In many practical applications, it is required that the sample size n of the single sampling attribute plan (SSP) is fixed and small (less than 50). Chakraborty [2] modelled the problem as a Goal Programming and obtained the acceptance number c. Since the values of the producer's and the consumer's risks obtained from this class of plans are usually rather large compared to the accepted norms and also one of the risks is too high or too low, it is assumed that the decision maker (DM) wants the SSP satisfying the two risks 'closely around' some stated values. To deal with the imprecise requirements of the DM, we model the problem as a Fuzzy Goal Programming (FGP) and show that we can, if exists, obtain the 'close' solution.

For the Binomial case, the operating characteristic (OC) of the SSP as a function of p, the fraction defective is $P(p) = B(c, n, p) = \sum_{x=0}^{c} \binom{n}{x} p^x q^{n-x}$ where $q = 1 - p$. For satisfactory quality level p_1, $Q(p_1) = 1 - P(p_1) = \alpha$, called producer's risk and for unsatisfactory quality level p_2, $P(p_2) = \beta$, called consumer's risk.

The mathematical problem is : for a given fixed n, find a non-negative integer c such that the following fuzzy goals are satisfied as closely as possible

$$Q(p_1) = 1 - B(c, n, p_1) \fallingdotseq \alpha \quad \cdots \quad (1.1)$$

$$P(p_1) = B(c, n, p_2) \fallingdotseq \beta \quad \cdots \quad (1.2)$$

where the symbol $\fallingdotseq$ indicates the fuzzified version of the equality sign (see Chakraborty [1]).

2. PRELIMINARY RESULTS

Let $f_1 = 2(c+1)$, $f_2 = 2(n-c)$ and F_p is P fractile of F-distribution with degrees of freedom (f_1, f_2).

The following theorem 2.2 is a standard result and theorems 2.1 and 2.3 follow easily from standard results.

Theorem 2.1 : *The set of acceptance numbers* C^B *satisfying*

$$\alpha_L \leqslant Q(p_1) = 1 - B(c, n, p_1) \leqslant \alpha_U \quad \cdots \quad (2.1)$$

$$\text{ond } \beta_L \leqslant P(p_2) = \quad B(c, n, p_2) \leqslant \beta_U \quad \cdots \quad (2.2)$$

if exists is given by all c's satisfying

$$p_1 \leqslant \frac{f_1 F_{\alpha_U}}{f_2 + f_1 F_{\alpha_U}} \quad \cdots \quad (2.3)$$

$$p_2 \leqslant \frac{f_1 F_{1-\beta_U}}{f_2 + f_1 F_{1-\beta_U}} \quad \cdots \quad (2.4)$$

$$p_1 \geqslant \frac{f_1 F_{\alpha_L}}{f_2 + f_1 F_{\alpha_L}} \quad \cdots \quad (2.5)$$

$$p_2 > \frac{f_1 F_{1-\beta_L}}{f_2 + f_1 F_{1-\beta_L}} \quad \cdots \quad (2.6)$$

The Poisson OC is $G(c, np) = G(c, m) = \sum_{x=0}^{c} e^{-m} m^x / x!$

Theorem 2.2 : *For Poisson approximation to Binomial OC fractile, let* $B(c, n, p_\alpha) = G(c, m_\alpha) = \alpha$. For $(c+1)/(n+1) \leqslant 0.25$ and $n \geqslant 20$, *the approximation*

$$P_\alpha \simeq m_\alpha / n + \tfrac{1}{2}(m_\alpha - c) \quad \cdots \quad (2.7)$$

is normally sufficiently accurate.

Theorem 2.3 : *The set of acceptance numbers* C^Q *with Poisson* *approximation to Binomial OC fractile satisfying (3) and (4) if exists is given by*

$$m_{1-\alpha_U}(c)\left(\frac{1}{p_1}-\frac{1}{2}\right)+\frac{1}{2}c > n \quad \ldots (2.8)$$

$$m_{\beta_U}(c)\left(\frac{1}{p_2}-\frac{1}{2}\right)+\frac{1}{2}c \leqslant n \quad \ldots (2.9)$$

$$m_{1-\alpha_L}(c)\left(\frac{1}{p_1}-\frac{1}{2}\right)+\frac{1}{2}c \leqslant n \quad \ldots (2.10)$$

$$m_{\beta_L}(c)\left(\frac{1}{p_2}-\frac{1}{2}\right)+\frac{1}{2}c > n \quad \ldots (2.11)$$

Appendix I contains the values of Binomial OC fractile P_p for $n=3$, $c=0, 1$; $n=5, 8, 10, 12$; $c=0(1)2$; $n=15, 20$: $c=0(1)3$ *for twelve values of P. With this the set C^B can be found.*

The function $m_\alpha(c)$ is tabulated by Chakraborty [1] for $c=0(1)9$ for twenty eight values of α. With the help of m_α, one can easily find the set C^G.

3. Fuzzy goal programming problem and solution

FGP model (1) and (2) with the minimum operator to aggregate the membership function of fuzzy goals, is equivalent to four subproblems (SP) as shown in Chakrabarty [1]. These four subproblems SP 1 through SP 4 correspond to different combinations of membership functions of fuzzy goals and only the subproblem SP 1 is given below (for details, see Chakraborty [1]).

SP 1 Maximize λ ... (3.1)

subject to

$$\lambda \leqslant \frac{Q(p_1)-\alpha_L}{\alpha-\alpha_L} \quad \ldots (3.2)$$

$$\lambda \leqslant \frac{P(p_2)-\beta_L}{\beta-\beta_L} \quad \ldots (3.3)$$

$$\alpha_L \leqslant Q(p_1) \leqslant \alpha \quad \ldots (3.4)$$

$$\beta_L \leqslant P(p_2) \leqslant \beta \quad \ldots (3.5)$$

$$0 \leqslant \lambda \leqslant 1 \quad \ldots (3.6)$$

$$c \geqslant 0, \text{ integer} \quad \ldots (3.7)$$

The solution of that subproblem SP yielding the highest membership value λ will correspond to the solution of the FGP model (1) and (2).

Solution procedure : For all practical ranges of the parameters of the problem with fixed and small n, the set of feasible aceeptance number for each of the SP's would have (if exists) cardinality very small and can be obtained from the results of Theorem 2.2 with appropriate values of α's and β's. The problem can be solved easily by enumeration with the help of Appendix 1.

Example 1 : Let $n=20$, $p_1=0.02$, $p_2=0.21$, $\alpha=0.10$, $\alpha_L=0.00$, $\alpha_U=0.15$, $\beta=0.20$, $\beta_L=0.05$, $\beta_U=0.30$.

Solution : For case SP 1, with the help of Appendix 1, ($n=20$)

$$p_1=0.02 \leqslant 0.027=p_{1-0.10}\ (1) \rightarrow c \geqslant 1$$
$$p_1=0.02 \geqslant 0.000=p_{1-0.00}\ (n) \rightarrow c \leqslant 20$$
$$p_2=0.21 \geqslant 0.202=p_{0.20}\ \ (2) \rightarrow c \leqslant 2$$
$$p_2=0.21 \leqslant 0.216=p_{0.05}\ \ (1) \rightarrow c \leqslant 1$$

Therefore $C^B=\{1, 2\}$. To obtain the optimal plan for this case,

c	$Q(p_1)$	$\lambda_1=\dfrac{Q(p_1)-\alpha_L}{\alpha-\alpha_L}$	$P(p_2)$	$\lambda_2=\dfrac{P(p_2)-\beta_L}{\beta-\beta_L}$	$\lambda=\min\{\lambda_1, \lambda_2\}$
1	0.0599	0.599	0.0566	0.044	0.044
2	0.0071	0.071	0.1770	0.847	0.071

so that required $c=2$ with $\lambda=0.071$.

Similarly C^B's corresponding to SP 2, SP 3 and SP 4 are obtained and each C^B is empty and hence overall optimal c is 2.

4. Solution with Poisson approximation

For obtaining the acceptance number c for all given $n \geqslant 20$, we may use the approximation of Binomial fractile by Poisson fractile (Theorem 2.2) and obtain the set C^G's corresponding to each SP's. The optimal c can be found as in Section 3 applying Theorem 2.3 and with the help of m_α's given in Chakraborty [1].

Example 2 : We consider the problem given in Chakraborty [2], $n=40$, $p_1=0.01$, $p_2=0.08$. In addition we assume $\alpha_L=0.02$, $\alpha=0.05$, $\alpha_U=0.10$, $\beta_L=0.05$, $\beta=0.10$, $\beta_U=0.15$.

Solution : For all cases SP 1, SP 2, SP 3 and SP 4, $C^G=\phi$ indicating the non-existence of the required 'close' solution.

Example 3 : Same as example 2 but $\alpha_L=0.01$ and $\beta_U=0.20$. Here for all cases SP 1, SP 2 and SP 3, $C^G=\phi$ but for SP 4, $C^G=\{1\}$ hence required $c=1$ with $Q(p_1)=0.061$, $P(p_2)=0.159$ and $\lambda=0.78$.

Example 4 : Same as example 3, but $n=50$ and $\beta_U=0.25$. Here for SP 1, $C^G=\phi$, for SP 2, $C^G=\{2\}$ with $\lambda=0.095$. For SP 3, $C^G=\{1\}$ with $\lambda=0.212$. For SP 4, $C^G=\phi$. The overall optimal solution is obtained by comparing the λ's and we get $c=1$.

5. Concluding remarks

Remark 1 : Fuzzy set theoretic approach for designing SSP when the sample size is fixed provides plans 'close' to the goals specified or else indicates that the desired 'close to the goals' plans do not exist. In the latter case the DM could decide either to increase the sample size or relax the 'closeness' to the goals criteria for obtaining a satisfactory plan.

Remark 2 : For $n \geqslant 20$, the approximation for the exact Binomial solution is sufficiently accurate. However, if necessary, the exact binomial fractiles can also be obtained with the help of fractiles of F-distribution.

Remark 3 : The DM may also consider addition operator to aggregate the membership functions of the fuzzy goals and design optimal SSP by maximising the weighted fuzzy achievement function, (see Chakrabarty [1]).

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APPENDIX—1

Binomial OC fractiles, $P_p(c)$, $B(c, n, p_p) = P$.

		Probability of acceptance					
n	c	.99	.97	.95	.90	.85	.30
3	0	.00334	.0101	.0170	.0345	.0527	.331
	1	.0589	.103	.135	.196	.244	.637
	0	.00201	.00607	.0102	.0209	.0320	.214
5	1	.0327	.0581	.0764	.112	.142	.422
	2	.106	.157	.189	.247	.290	.610
	0	.00126	.00380	.00639	.0139	.0201	.140
8	1	.0197	.0351	.0464	.0686	.0873	.279
	2	.0601	.0913	.111	.147	.175	.407
	0	.00100	.00304	.00512	.0105	.0161	.113
10	1	.0155	.0278	.0368	.0543	.0695	.227
	2	.0475	.0715	.0873	.116	.138	.333
	0	.000837	.00253	.00426	.00874	.0135	.0955
12	1	.0128	.0230	.0305	.0452	.0578	.191
	2	.0390	.0588	.0719	.0956	.114	.281
	0	.000670	.00203	.00341	.00700	.0108	.0771
	1	.0102	.0183	.0242	.0360	.0461	.155
15	2	.0307	.0465	.0568	.0759	.0909	.228
	3	.0594	.0823	.0967	.122	.141	.299
	0	.000502	.00152	.00256	.00525	.00809	.0584
	1	.00759	.0136	.0181	.0269	.0344	.118
20	2	.0227	.0344	.0422	.0564	.0677	.174
	3	.0436	.0607	.0714	.0902	.105	.228

		Probability of acceptance					
n	*c*	.25	.20	.15	.10	.05	.01
3	0	.370	.415	.669	.536	.632	.785
	1	.674	.713	.756	.804	.865	.941
	0	.242	.275	.316	.369	.451	.602
5	1	.454	.490	.532	.584	.657	.778
	2	.641	.673	.710	.753	.810	.894
	0	.159	.182	.211	.250	.312	.438
8	1	.303	.330	.364	.406	.471	.590
	2	.433	.462	.496	.538	.600	.707
	0	.129	.149	.173	.206	.259	.369
10	1	.247	.271	.300	.337	.394	.504
	2	.355	.381	.411	.450	.507	.612
	0	.109	.126	.146	.175	.221	.319
12	1	.209	.230	.255	.288	.339	.440
	2	.301	.324	.351	.386	.438	.537
	0	.0883	.102	.119	.142	.181	.264
	1	.170	.187	.208	.236	.279	.368
15	2	.245	.264	.287	.317	.363	.453
	3	.317	.337	.362	.393	.440	.529
	0	.0670	.0773	.0905	.109	.139	.206
	1	.129	.142	.159	.181	.216	.289
20	2	.187	.202	.220	.245	.283	.358
	3	.242	.259	.278	.304	.344	.421