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ON PRINCIPAL GRAPHS AND WEAK DUALITY

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Abstract.

The main result is that if a finite tree occurs as a principal graph of a subfactor N of a II_1 factor M of index greater than 4, if the contragredient maps on the principal graphs are trivial, and if no vertex has degree greater than 3, then the tree must contain a subgraph isomorphic to what is denoted by $E_6^{(1)}$ in [GHJ] (and by T in this paper). The proof uses a notion that we call "weak duality" of graphs. The result needed is that certain kinds of graphs can be weakly dual only to themselves. This paper also contains a proof of the assertion that the distinguished vertex $*$ of a principal graph is essentially determined up to a parity-preserving automorphism of the bipartite graph.

1. Introduction.

This paper is devoted to a study of the properties of a pair $(\mathscr{G}, \mathscr{H})$ of pointed finite bipartite graphs wich arise as a part of Ocneanu's paragroup invariant of a finite index subfactor. After setting up some notation and recalling some basic facts about subfactors, we proceed to analyse the role of the distinguished vertex of the graph $\mathscr{G}$; in particular, we describe the extent of uniqueness of the vertex $*$ up to a graph automorphism. In the last section, we single out a property possessed by a pair of principal graphs of a subfactor for which the contragredient maps are trivial, which we term 'weak duality'. The main result here is that if a pair of graphs $\mathscr{G}$ and $\mathscr{H}$ are weakly dual, then $\mathscr{G}$ is necessarily isomorphic to $\mathscr{H}$ if $\mathscr{G}$ satisfies some conditions – at most triple points, no double bonds, and the absence of two specific kinds of subgraphs. This is the technically complicated part of the paper, although, when suitably combined with an observation by Ocneanu, it leads fairly easily to a proof of what has been referred to, in the Abstract, as the main result.

2. Notation and other preliminaries.

In this paper, we will be dealing with bipartite graphs. If $\mathscr{G}$ is such a graph, its vertex set $V(\mathscr{G})$ admits a partition $V(\mathscr{G}) = \mathscr{G}^0 \coprod \mathscr{G}^1$, the vertices of the former

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(resp., latter) being referred to as even (resp. odd) vertices. (We use the symbol $\coprod$ here and elsewhere to indicate a disjoint union, and the even vertices of $\mathcal{G}$ are written with a superscript 0 and odd vertices with superscript 1.) We shall also find it convenient to encode the data of the graph $\mathcal{G}$ by the matrix G with rows (resp., columns) indexed by $\mathcal{G}^0$ (resp., $\mathcal{G}^1$), with the (β^0, ξ^1) entry equal to the number of bonds joining the even vertex β^0 to the odd vertex ξ^1. It is clear that the adjacency matrix $A(\mathcal{G})$, whose rows and columns are indexed by $V(\mathcal{G})$ is given in block form by

$$\begin{bmatrix} 0 & G \\ G^t & 0 \end{bmatrix}.$$

It follows that the non-zero eigenvalues of $A(\mathcal{G})$ are precisely the numbers $\pm\lambda$, where λ^2 is a non-zero eigenvalue of G^tG; further, the restrictions of the Perron-Frobenius eigenvector of $A(\mathcal{G})$ to $\mathcal{G}^1$ and $\mathcal{G}^0$ are the Perron-Frobenius eigenvector of G^tG and a suitable $(= \|\mathcal{G}\|^{-1})$ multiple of its image under G respectively.

We now recall some basic facts concerning the principal graphs associated to a finite-index inclusion $N \subset M$ of II_1 factors (usually assumed to be hyperfinite). Most of these facts stem from one of three sources – the work of Jones, Ocneanu and Popa – and we will not take the trouble of painstakingly ascribing each to a specific author.

Let $N \subset M$ be a pair of hyperfinite II_1 factors, with finite index. Then the *basic construction* of Jones yields a canonical tower $N \subset M \subset M_1 \subset M_2 \subset \ldots \subset M_n \subset \ldots$ and a semi-canonical tunnel $M_0 = M \supset M_{-1} = N \supset M_{-2} \supset M_{-3} \supset \ldots \supset M_{-n} \supset \ldots$ In any case, the doubly indexed grid $\{A_{k,n} = M'_{-n} \cap M_k: k, n \geqq 0\}$ is canonical. It is well-known that the inclusion data of this grid is encoded, in a precise manner, by a pair $\{\mathcal{G}, \mathcal{H}\}$ of bipartite graphs which satisfy several conditions. (Recall that the subfactor N is said to have finite depth precisely when either (equivalently both) of the principal graphs is (are) finite.)

3. The distinguished vertex *.

Any graph $\mathcal{G}$ that arises as a principal graph is, in addition, a pointed connected graph; i.e., it contains an even vertex – always denoted by $*_{\mathcal{G}}$ – which has the distinguished feature of being an even vertex at which the Perron-Frobenius eigenvector of the adjacency matrix $A(\mathcal{G})$ is minimal. The content of the next proposition is to note that the above "feature" determines the vertex essentially uniquely. Although fairly straightforward, this proposition has been included here because the authors have not seen a proof of this in the literature.

PROPOSITION 1. *Let $\mathcal{G}$ be a principal grph of a subfactor $N \subset M$, of finite depth.*

Then the vertex $*_{\mathscr{G}}$ is uniquely determined up to a "parity-preserving" automorphism of the bipartite graph $\mathscr{G}$.

PROOF. (In this proof and elsewhere, when we have to use the language of hypergroups, we shall use the results as well as the terminology of [S] and [SV].) If $\mathscr{G}$ is a hypergroup, the set $\mathscr{G}_{(0)} = \{\beta \in \mathscr{G}: d_\beta = 1\}$ is a *sub-group* of the hypergroup $\mathscr{G}$, where of course, $\alpha \mapsto d_\alpha$ denotes the dimension function of $\mathscr{G}$ (which is the unique "positive homomorphism" from the hypergroup ring $\mathbf{Z}\mathscr{G}$ to $\mathbf{R}$). Also since the uniqueness of the dimension function implies that $d_{\bar{\alpha}} = d_\alpha$, it follows that

$$\alpha \in \mathscr{G}_{(0)} \Rightarrow 1 = d_\alpha^2 = d_\alpha \cdot d_\alpha = \sum_{\gamma \in \mathscr{G}} \langle \alpha \cdot \bar{\alpha}, \gamma \rangle d_\gamma = 1 + \sum_{1 \neq \gamma \in \mathscr{G}} \langle \alpha \cdot \bar{\alpha}, \gamma \rangle d_\gamma$$

and hence that $\langle \alpha \cdot \bar{\alpha}, \gamma \rangle = 0$ for $1 \neq \gamma \in \mathscr{G}$. In other words, $\alpha \cdot \bar{\alpha} = 1$. It is immediate that $\mathscr{G}_{(0)}$ is a group.

On the other hand, if the graph $\mathscr{G}$ is a principal graph, then in the terminology of [SV], the set $\mathscr{G}^0$ has the structure of a hypergroup which acts on the set $\mathscr{G}^1$. If $\alpha \in \mathscr{G}^0_{(0)}$, and if L_0 and L_1, respectively, denote the matrices (with respect to the natural bases) of left multiplication by α on $\mathbf{Z}\mathscr{G}^0$ and $\mathbf{Z}\mathscr{G}^1$, the already established equation $\alpha \cdot \bar{\alpha} = 1$ implies that the matrices L_0 and L_1 are orthogonal matrices; as these are non-negative integral matrices, they must be permutation matrices. Hence, the map $\lambda \mapsto f(\lambda) = \alpha \cdot \lambda$ defines a parity-preserving permutation of $V(\mathscr{G})$ such that $f(1) = \alpha$, where 1 denotes the (unique) identity element of the hypergroup $\mathscr{G}^0$.

To complete the proof, we need to verify that the map f defines an automorphism of the graph $\mathscr{G}$, i.e., $G(\beta, \xi) = G(f(\beta), f(\xi))$, for all $\beta \in \mathscr{G}^0$, $\xi \in \mathscr{G}^1$. However, we have $G(\beta, \xi) = \langle \beta \cdot \lambda, \xi \rangle$, where λ denotes the sum (with multiplicities taken into account in case of multiple bonds) of the neighbours in $\mathscr{G}^1$ of $*_{\mathscr{G}}$ (where we think of λ as an element of $\mathbf{Z}\mathscr{G}^1$). Hence,

$$G(f(\beta), f(\xi)) = \langle (\alpha \cdot \beta) \cdot \lambda, (\alpha \cdot \xi) \rangle = \langle (\bar{\alpha} \cdot \alpha) \cdot \beta \cdot \lambda, \xi \rangle = G(\beta, \xi)$$

as desired.

REMARK 2. Note that we have *not* proved that the distinguished vertex $*$ is determined uniquely up to an automorphism of the graph; what we have shown is that once we have pre-determined the parity, then among the even vertices, the vertex $*$ is uniquely determined up to an automorphism. This prompts the question: is it possible for a graph $\mathscr{G}$ that arises as a principal graph to admit vertices α and ξ of different parity at both of which the Perron-Frobenius eigenvector of the adjacency matrix $A(\mathscr{G})$ assumes the minimal value, and yet such that there is no (necessarily, parity reversing) automorphism which maps α to ξ? (The graph A_{2n} is a non-example.)

EXAMPLE 3. We are grateful to Bhaskar Bagchi for this combinatorial example which, besides being pretty, illustrates two different phenomena; viz., (i) there is a bipartite graph $\mathscr{G}$ at all of whose even vertices the Perron-Frobenius eigenvector takes on the same value and yet the automorphism group of $\mathscr{G}$ does not act transitively on the set of even vertices; (hence by the above proposition $\mathscr{G}$ cannot arise as a principal graph;) (ii) there is a second pointed bipartite graph $\mathscr{H}$ which is not isomorphic to $\mathscr{G}$ although the two graphs are "weakly dual" – cf. Definition (4).

Both the graphs $\mathscr{G}$, $\mathscr{H}$ have $15 = \binom{6}{2}$ odd vertices indexed by the 15 edges of the complete graph K_6. Each graph has 10 even vertices indexed by certain subgraphs of K_6 (isomorphic to $C_3 \coprod C_3$ or C_6, where C_k denotes a k-cycle – thus C_6 is a hexagon, etc.). In both graphs, an odd vertex is adjacent to an even vertex precisely when the relevant edge belongs to the relevant subgraph.

The even vertices of $\mathscr{H}$ correspond to all the $10 = \left(\binom{6}{3}\right)/2$ subgraphs of K_6 isomorphic to $C_3 \coprod C_3$.

The graph $\mathscr{G}$ also has ten even vertices; these correspond to six subgraphs isomorphic to $C_3 \coprod C_3$ and four subgraphs isomorphic to C_6. They are: $\{(124)\coprod(356),\ (125)\coprod(346),\ (136)\coprod(245),\ (145)\coprod(236),\ (134)\coprod(256),\ (146)\coprod(235)\}$ and $\{(123456), (126453), (156423), (153426)\}$ – where we have used the obvious notation $(v_1 \ldots v_k)$ to denote the k-cycle that successively passes through the vertices $v_1, \ldots, v_k$.

Both graphs share the following properties: (i) each odd vertex has degree 4 and each even vertex has degree 6 (and hence the value of the Perron-Frobenius eigenvector at a vertex depends only on the parity of the vertex); (ii) given any two distinct odd vertices, the number of paths of length two which join them is 1 or 2 according as the corresponding edges (in K_6) share a common vertex or not. These facts ensure that the graphs are weakly dual as asserted, provided the distinguished vertices of both $\mathscr{G}$ and $\mathscr{H}$ are taken to be the same graph isomorphic to $C_3 \coprod C_3$.

Note now that the graph $\mathscr{G}$ has two kinds of even vertices. We assert that any automorphism of $\mathscr{G}$ leaves invariant the set of all even vertices of either kind. Since the Perron-Frobenius eigenvector is constant on the set of even vertices of $\mathscr{G}$, this example does indeed illustrate the claimed features. (In fact $\mathrm{Aut}(\mathscr{H})$ is isomorphic to S_6 while $\mathrm{Aut}(\mathscr{G})$ can be seen to be a group of order 48.)

To prove the assertion, first observe that $\mathscr{G}^1$ is identified with the edges of K_6, which in turn constitute the vertices of $L(K_6)$-the so-called line graph of K_6. (Recall that two vertices are adjacent in the line graph $L(K)$ precisely when the corresponding edges in K have a vertex in common.) Suppose now that we are given an automorphism σ of $\mathscr{G}$; by property (i) above of G, the automorphism σ must preserve parity; so $\sigma|_{\mathscr{G}^1}$ yields a self-map of the vertices of $L(K_6)$; the fact that $\mathscr{G}$ has the property (ii) of the previous paragraph is seen to imply that this map must preserve adjacency in the graph $L(K_6)$, and is hence an automorphism of $L(K_6)$. It is not hard to see that every automorphism of $L(K_6)$ is induced by an automorphism of K_6. Thus, $\sigma \mapsto \sigma|_{\mathscr{G}^1}$ yields a map from $\mathrm{Aut}(\mathscr{G})$ to $\mathrm{Aut}(K_6) = S_6$, which is clearly a monomorphism. Since no automorphism of K_6 can map a subgraph of the form $C_3 \coprod C_3$ onto a subgraph of the form C_6, the proof of the assertion is complete.

4. Weak duality.

If two pointed bipartite graphs $\mathscr{G}, \mathscr{H}$ arise as the two principal graphs corresponding to a finite-index subfactor, then the sets $\mathscr{G}^0$, $\mathscr{H}^0$ of even vertices of the two graphs are naturally equipped with involutions corresponding to the contragredient mapping at the level of bimodules. We shall be concerned with subfactors for which both these involutions are trivial; for brevity, we shall simply say that the subfactor has trivial contragredient maps when this happens. For such a subfactor – i.e., one with trivial contragredient maps – it follows from the description of the grid $\{A_{k,n}\}$ discussed earlier, that the graphs $\mathscr{G}$ and $\mathscr{H}$ are "weakly dual" in the sense of the next definition.

DEFINITION 4. Two pointed finite connected bipartite graphs $(\mathscr{G}, *_{\mathscr{G}})$ and $(\mathscr{H}, *_{\mathscr{H}})$ are said to be "weakly dual" if the following conditions are satisfied:

(1) $\mathscr{G}^1 = \mathscr{H}^1$.

(2) $G^t(*_{\mathscr{G}}) = H^t(*_{\mathscr{H}})$ (i.e. the neighbours of $*$ *in* $\mathscr{G}$ and $\mathscr{H}$ are the same).

(3) $G^tG(\xi^1, \eta^1) = H^tH(\xi^1, \eta^1)$ for all ξ^1, $\eta^1 \in \mathscr{G}^1$, (i.e. the number of paths, of length 2, between ξ^1 and η^1 is the same in $\mathscr{G}$ and $\mathscr{H}$).

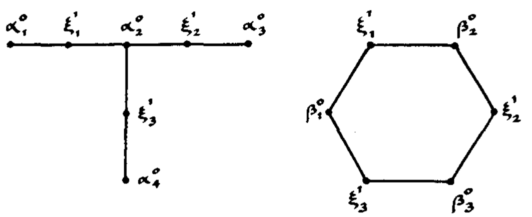
(This would be the appropriate place to acknowledge our gratitude to Uffe Haagerup for leading us to think along these lines; he had, in oral communication, pointed out that if the graph $\mathscr{G}$ "looks like an A_n up to a certain distance from $*_{\mathscr{G}}$", so also must $\mathscr{H}$ – cf. Remark 7 (2).)

REMARK. Note that when $\mathscr{G}$ and $\mathscr{H}$ are a pair of principal graphs, the identification of the odd vertices in (1) is via the contragredient map τ. When the contragredient map on the even vertices is nontrivial the graphs do not satisfy condition (3), but they satisfy $G^t\tau_{\mathscr{G}^0}G = H^t\tau_{\mathscr{H}^0}H$.

We turn now to pairs of graphs which are not isomorphic but which are weakly dual. The simplest known example comes from the principal graphs for the inclusion $N \subset M$, when M is the crossed-product of N with a non-abelian group of outer automorphisms of N. In this example, as is well-known, the graph $\mathscr{H}$ has multiple bonds while $\mathcal{G}$ does not. An example of graphs without multiple bonds is furnished by Example 3. Another such, but smaller, example is given below.

EXAMPLE 5. We begin by discussing a pair of graphs which are "almost" weakly dual, but just fail to be so; nevertheless they have near relatives which do furnish an example of a pair of graphs which are weakly dual but not isomorphic. The non-example is discussed here mainly because of the key role these two graphs play in Proposition 6.

(a) Consider the following pair of graphs, with even and odd vertices labelled as indicated (fig. 1).



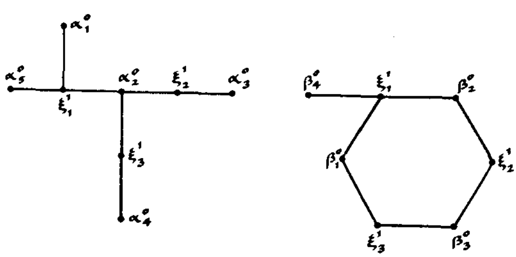
$G = T$ $\qquad$ $H = C_6$

Figure 1.

(Here and in the sequel, we write T to denote the "T-graph" each of whose arms is two edges long. This graph is denoted by $E_6^{(1)}$ in [GHJ], but we use the notation T because it is more suggestive.)

It is easily verified that the condition $G'G = H'H$ is satisfied. Since every even vertex in $\mathcal{H}$ has degree 2 while none in $\mathcal{G}$ does, clearly these two graphs cannot be wealy dual (by condition (2)).

(b) Consider these graphs with labelling as indicated (fig. 2).

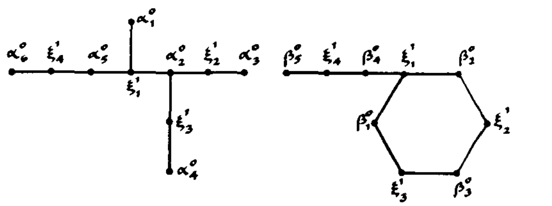


$G = A_2 \# T$ $\qquad$ $H = A_2 \# C_6$

Figure 2.

(Here and elsewhere, we use the symbol # to denote "connected sum", whereby we mean that a pair of vertices, one from each of the graphs in question, has been identified; to be sure, there are several ways of forming such a connected sum.) Again, the condition $G^tG = H^tH$ is satisfied. While the vertices α_5^0 and β_4^0 have the same degree, what fails now is that the minimum value of the Perron-Frobenius eigenvector of $A(\mathscr{G})$ occurs not at α_5^0 but at the vertices α_3^0 and α_4^0.

(c) Finally, the desired example comes from the following graphs, with labelling as indicated (fig. 3).



$G = A_4 \# T$ $H = A_4 \# C_6$

Figure 3.

Here, it is the case that $*_{\mathscr{G}} = \alpha_6^0$ and $*_{\mathscr{H}} = \beta_5^0$.

(d) It goes without saying that by extending the A-part of the graphs more and more, the graphs $\mathscr{G}$ and $\mathscr{H}$ generate a whole sequence of pairs of non-isomorphic graphs – namely $A_{2n} \# T$ and $A_{2n} \# C_6$ – which are weakly dual.

We are now ready to prove the following proposition which gives some criteria on a bipartite graph $\mathscr{G}$ which ensure that the only graph, up to isomorphism, which is weakly dual to $\mathscr{G}$ is $\mathscr{G}$ itself. (Observe that in view of Example 5 (c), (d), the conditions (3) and (4) in the proposition are almost necessary.)

PROPOSITION 6. *Suppose $\mathscr{G}$ is a finite connected bipartite graph satisfying the following conditions:*

(1) *no vertex of $\mathscr{G}$ has degree greater then 3;*

(2) *$\mathscr{G}$ does not have double bonds;*

(3) *$\mathscr{G}$ has no 6-cycles; and*

(4) *$\mathscr{G}$ has no subgraph isomorphic to T such that each of the vertices of degree 1 in T is an even vertex in $\mathscr{G}$ whose degree in $\mathscr{G}$ is still 1.*

Then the identification $\mathscr{G}^1 = \mathscr{H}^1$ extends to a graph isomorphism of $\mathscr{G}$ to $\mathscr{H}$.

Before proceeding to the proof proper, we set up some notation. We shall use the notation $(\xi_0 - \xi_1 - \ldots - \xi_n) \in \mathscr{G}$ to signify that $\xi_0, \xi_1, \ldots, \xi_n$ are vertices of the graph $\mathscr{G}$ such that ξ_{i-1} is adjacent to ξ_i in $\mathscr{G}$ for $1 \leqq i \leqq n$.

The set of neighbours of α in $\mathscr{G}$ will be denoted by $\mathscr{N}_\alpha^{\mathscr{G}}$. In the following, since we shall be dealing with a pair of weakly dual graphs $\mathscr{G}$ and $\mathscr{H}$, which have the with a pair of weakly dual graphs $\mathscr{G}$ and $\mathscr{H}$, which have the y $\mathscr{N}_\alpha$ when α is an even vertex of either $\mathscr{G}$ or $\mathscr{H}$.

We shall also employ the following notation: for vertices ξ, η in $\mathscr{G}$:

(i) the symbol $\mathscr{G}(\xi, \eta)$ will denote the set of common neighbours in $\mathscr{G}$ of ξ and η, i.e. $\mathscr{G}(\xi, \eta) = \mathscr{N}_\xi^{\mathscr{G}} \cap \mathscr{N}_\eta^{\mathscr{G}}$; (note that, in the absence of double bonds, $|\mathscr{G}(\xi^1, \eta^1)| = G^tG(\xi^1, \eta^1)$, whence $|\mathscr{G}(\xi^1, \xi^1)|$ is the degree of ξ^1 in $\mathscr{G}$;)

(ii) the symbol $\mathscr{I}^{\mathscr{G}}(\xi)$ will denote the set of degree one neighbours of ξ in $\mathscr{G}$; i.e., $\mathscr{I}^{\mathscr{G}}(\xi) = \{\beta \in \mathscr{N}_\xi^{\mathscr{G}} : \deg_{\mathscr{G}}(\beta) = 1\}$;

(iii) the symbol Λ will denote the set of triple points (i.e., vertices of degree 3) in $\mathscr{G}^0$; suppose $\Lambda = \{\lambda_1^0, \lambda_2^0, \ldots, \lambda_l^0\}$, $l \geqq 0$.

Proof. It is not hard to see that the above conditions (1), (2) and (4) of the proposition imply the conditions (1′), (2′) and (4′) below. (To be precise, conditions (1) and (2) are together equivalent to conditions (1′) and (2′); while condition (4) is equivalent to (4′).) What we shall prove is that conditions (1′), (2′), (3) and (4′) suffice to ensure the validity of the conclusion of the Proposition. (We have, however, chosen to state the proposition as we have, since we fell that this formulation is more "visual" and easier to verify.)

(1′) $(G^tG)(\xi^1, \xi^1) \leqq 3$, for all ξ^1 in $\mathscr{G}^1$;

(2′) for all $\beta^0 \in \mathscr{G}^0$ $\deg(\beta^0) \leqq 3$; and

(4′) for all $\lambda^0 \in \Lambda$ there exists $\xi_{\lambda^0}^1 \in \mathscr{N}_{\lambda^0}$ such that $\mathscr{I}^{\mathscr{G}}(\xi_{\lambda^0}^1) = \emptyset$.

In the proof we would have occasion to use the following condition (3′) which can be seen to be implied by (3).

(3′) If $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1\}$ is a subset of $\mathscr{G}^1$ such that between any two vertices in Ω there is a path of length 2 in $\mathscr{G}$, i.e., $(G^tG)(\omega_i^1, \omega_j^1) \neq 0$ for $i \neq j$, then, Ω is the set of neighbours of some triple point in $\mathscr{G}$, i.e., $\Omega = \mathscr{N}_{\lambda^0}$ for some $\lambda^0 \in \Lambda$.

We break the proof, which is somewhat involved, into the following steps.

Step 1. $\mathscr{H}$ has no double bonds.

Reason: $H^tH(\xi^1, \xi^1) = G^tG(\xi^1, \xi^1) \leqq 3$ for all $\xi^1 \in \mathscr{G}^1$.

Step 2. Each vertex in $\mathscr{H}$ has degree at most 3.

Reason: For the same reason as in Step 1, this is clear for the odd vertices. Suppose, now, that there is an even vertex δ^0 in $\mathscr{H}^0$ such that $\deg(\delta^0) \geqq 4$.

Case (1). There is an even vertex δ^0 in $\mathscr{H}^0$ such that $\deg(\delta^0) > 4$. Then δ^0 has at least five neighbours, ξ_1^1, ξ_2^1, ξ_3^1, ξ_4^1, and ξ_5^1. The path $(\xi_i^1 - \delta^0 - \xi_j^1)$ in $\mathscr{H}$ ensures that $G^tG(\xi_1{}^i, \xi_j^1) = H^tH(\xi_i^1, \xi_j^1) \neq 0$ for all i and j. By (3′), for any choice of distinct i, j and k, $\{\xi_i^1, \xi_j^1, \xi_k^1\} = \mathscr{N}_{\lambda^0}$ for some $\lambda^0 \in \Lambda$, i.e. each ξ_i^1 is adjacent to $\left(\binom{4}{2} = \right)$ 6 distinct triple points in $\mathscr{G}$, which contradicts (1).

Case (2). Suppose there is an even vertex δ^0 in $\mathscr{H}^0$ such that $\deg(\delta^0) = 4$. Then δ^0 has four neighbours ξ_1^1, ξ_2^1, ξ_3^1, and ξ_4^1. By the same reasoning as above, for distinct i, j and k, $\{\xi_i^1, \xi_j^1, \xi_k^1\} = \mathscr{N}_{\lambda^0}$ for some $\lambda^0 \in \Lambda$. So there are 4 triple points λ_1^0, $\lambda_2^0, \lambda_0^3, \lambda_4^0$ in $\mathscr{G}$ such that $\mathscr{N}_{\lambda_i^0} = \{\xi_j^1 : j \neq i\}$. Let $\mathscr{G}'$ be the induced subgraph of $\mathscr{G}$ on the vertices $\{\xi_j^1 : 1 \leqq j \leqq 4\} \cup \{\lambda_i^0 : 1 \leqq i \leqq 4\}$. Since each λ_i^0 and ξ_j^1 has degree 3 in $\mathscr{G}'$, the conditions (1) and (2) imply that $\mathscr{G}'$ is a connected component of $\mathscr{G}$, and hence $\mathscr{G}' = \mathscr{G}$ by the assumed connectedness of $\mathscr{G}$.

Since $\mathscr{G}$ and $\mathscr{H}$ are weakly dual, we have the following:

(i) $\mathscr{H}^1 = \{\xi_1^1, \xi_2^1, \xi_3^1, \xi_4^1\}$.

(ii) $H^t H(\xi_i^1, \xi_j^1) = 2$ for $1 \leqq i \neq j \leqq 4$.

(iii) $H^t H(\xi_i^1, \xi_i^1) = 3$ for $1 \leqq \mathrm{i} \leqq 4$.

We proceed to deduce that there must exist another even vertex $\delta_1^0 \neq \delta^0$ of degree 4 in $\mathscr{H}$ such that $\mathscr{N}_{\delta_1^0} = \mathscr{N}_{\delta^0} = \{\xi_1^1, \xi_2^1, \xi_3^1, \xi_4^1\}$.

By (ii) there are unique even vertices κ_{ij}^0, distinct from δ^0, such that $(\xi_i^1 - \kappa_{ij}^0 - \xi_j^1)$ are in $\mathscr{H}$. Then for any ξ_i^1 and $j \neq i$, we have $(\xi_i^1 - \delta^0)$, and $(\xi_i^1 - \kappa_{ij}^0)$, are in $\mathscr{G}$. But $\deg(\xi_i^1) \leqq 3$. Therefore for each i, $\kappa_{ij}^0 = \kappa_{ik}^0$ for some $j \neq k$. Now $(\xi_j^1 - \kappa_{ij}^0 = \kappa_{ik}^0 - \xi_k^1)$ is in $\mathscr{H}$. But κ_{jk}^0 is the unique vertex other than δ^0 such that $(\xi_j^1 - \kappa_{jk}^0 - \xi_k^1)$ is in $\mathscr{H}$. So $\kappa_{ij}^0 = \kappa_{ik}^0 = \kappa_{jk}^0$, which is then a vertex of degree at least three. Hence each ξ_i^1 is connected to a $\kappa_i^0 \neq \delta^0$ such that $\deg(\kappa_i^0) \geqq 3$. We now show that all the κ_i^0 are the same.

Now, for $1 \leqq i, j \leqq 4$, we see that,

$$|\mathscr{N}_{\kappa_i^0} \cap \mathscr{N}_{\kappa_j^0}| = |\mathscr{N}_{\kappa_i^0}| + |\mathscr{N}_{\kappa_j^0}| - |\mathscr{N}_{\kappa_i^0} \cup \mathscr{N}_{\kappa_j^0}|$$
$$\geqq 3 + 3 - |\mathscr{H}^1| = 2.$$

Let $1 \leqq i \neq j \leqq 4$. Then there exist $1 \leqq k \neq l \leqq 4$ such that $\xi_k^1, \xi_l^1 \in \mathscr{N}_{\kappa_i^0} \cap \mathscr{N}_{\kappa_j^0}$. Then since $\mathscr{H}(\xi_k^1, \xi_l^1) \supset \{\delta^0, \kappa_i^0, \kappa_j^0\}$ and $\delta^0 \neq \kappa_i^0, \kappa_j^0$, the property (ii), stated above, implies that $\kappa_i^0 = \kappa_j^0$.

So there does indeed exist $\delta_1^0 \neq \delta^0 \in \mathscr{H}^0$, such that $\deg(\delta_1^0) = 4$, and $\mathscr{N}_{\delta_1^0} = \{\xi_1^1, \xi_2^1, \xi_3^1, \xi_4^1\}$.

By (iii) there must exist even vertices β_1^0, β_2^0, β_3^0, β_4^0 in $\mathscr{H}^0$, such that $(\xi_1{}^i - \beta_i^0)$ are in $\mathscr{H}$ and $\deg \beta_i^0 = 1$.

Thus the graphs $\mathscr{G}$ and $\mathscr{H}$ are fully determined. Observe that all the even vertices of $\mathscr{G}$ $(\lambda_i^0, 1 \leqq i \leqq 4)$ have degree 3, while the even vertices of $\mathscr{H}$ have degree either 4 $(\deg(\delta^0) = \deg(\delta_1^0) = 4)$ or 1 $(\deg(\beta_i^0) = 1$ for all $i)$. So there can be no choice of $*_{\mathscr{G}}$ and $*_{\mathscr{H}}$ such that $\mathscr{G}^t(*_{\mathscr{G}}) = \mathscr{H}^t(*_{\mathscr{H}})$.

This completes the proof of Step 2.

Let $\mathscr{M} = \{\mu_1^0, \mu_2^0, \ldots, \mu_m^0\}$, $m \geqq 0$, be the set of triple points in $\mathscr{H}^0$.

Consider the following partition of the sets of even vertices of $\mathscr{G}$ and $\mathscr{H}$ respectively, obtained by considering the degrees of the even vertices:

$$\mathscr{G}^0 = \coprod_{\xi^1 \in \mathscr{G}^1} \mathscr{I}^{\mathscr{G}}(\xi^1) \coprod_{\substack{\xi^1, \eta^1 \in \mathscr{G}^1 \\ \xi^1 \neq \eta^1}} ((\mathscr{G}(\xi^1, \eta^1) \backslash \Lambda) \coprod \Lambda.$$

$$\mathscr{H}^0 = \coprod_{\xi^1 \in \mathscr{H}^1} \mathscr{I}^{\mathscr{H}}(\xi^1) \coprod_{\substack{\xi^1, \eta^1 \in \mathscr{H}^1 \\ \xi^1 \neq \eta^1}} ((\mathscr{H}(\xi^1, \eta^1) \backslash \mathscr{M}) \coprod \mathscr{M}.$$

To establish that $\mathscr{G}$ is isomorphic to $\mathscr{H}$, it is enough to set up bijections between the corresponding components of the above partition for $\mathscr{G}^0$ and $\mathscr{H}^0$, which preserve neighbours – i.e., if $f: \mathscr{H}^0 \to \mathscr{G}^0$ is the resulting "grand bijection", then $\mathscr{N}_{\alpha^0} = \mathscr{N}_{f(\alpha)^0}$ for all α^0 in $\mathscr{H}^0$.

Step 3. In order that there exist a bijection between $\mathscr{G}^0$ and $\mathscr{H}^0$ as in the preceding sentence, it is necessary and sufficient that the following conditions (A) – equivalently (A′) – and (B) are satisfied:

(A) There is a bijection $f: \mathscr{M} \mapsto \Lambda$ so that $\mathscr{N}_\mu = \mathscr{N}_{f(\mu)}$ for all μ in $\mathscr{M}$.

(A′) For any three element subset $\mathscr{N}$ of $\mathscr{G}^1$, $|\{\lambda_i^0 \in \Lambda: \mathscr{N}_{\lambda_i^0} = \mathscr{N}\}| = |\{\mu_i^0 \in \mathscr{M}: \mathscr{N}_{\mu_i^0} = \mathscr{N}\}|$.

(B) $|\mathscr{I}^{\mathscr{G}}(\xi^1)| = |\mathscr{I}^{\mathscr{H}}(\xi^1)|$ for all ξ^1 in $\mathscr{G}^1$.

Reason: The necessity of the conditions (A) and (B) is easy to see, as is the equivalence of the conditions (A) and (A′). (One way of seeing that (A) $\Leftrightarrow$ (A′) is by appealing to the "marriage lemma".)

As for sufficiency, suppose the conditions (A′) and (B) are met. Note that $|\mathscr{G}(\xi^1, \eta^1)| = G^tG(\xi^1, \eta^1) = H^tH(\xi^1, \eta^1) = |\mathscr{H}(\xi^1, \eta^1)|$; on the other hand, the condition (A′) implies that, for all ξ^1, η^1 in $\mathscr{G}^1$, we have the equality

$$|\{\lambda_i^0 \in \Lambda: (\xi^1 - \lambda_i^0 - \eta^1) \in \mathscr{G}\}| = |\{\mu_i^0 \in \mathscr{M}: (\xi^1 - \mu_i^0 - \eta^1) \in \mathscr{H}\}|.$$

Therefore, for all ξ^1, η^1 in $\mathscr{G}^1$, we have

$$|\mathscr{G}(\xi^1, \eta^1) \backslash \Lambda| = |\mathscr{H}(\xi^1, \eta^1) \backslash \mathscr{M}|,$$

which establishes a bijection between the vertices of degree two connecting ξ^1 and η^1 in $\mathscr{G}$ and $\mathscr{H}$. This completes the proof of Step 3.

Hence, in order to complete the proof of the proposition, we only need to verify the validity of (A′) and (B). The proof of (A′) will be achieved in Steps 4 and 5, while Step 6 will prove (B).

Step 4. For $\mathscr{N} = \{\xi_1^1, \xi_2^1, \xi_3^1\} \subseteq \mathscr{G}^1$, $|\{\lambda^0 \in \Lambda: \mathscr{N}_{\lambda^0} = \mathscr{N}\}| \geqq |\{\mu^0 \in \mathscr{M}: \mathscr{N}_{\mu^0} = \mathscr{N}\}|$.

Reason : We consider three cases according to the number of triple points $\mu^0 \in \mathscr{M}$ such that $\mathscr{N} = \mathscr{N}_{\mu^0}$, (which cannot exceed 3 since the odd vertices can have degree at most 3, in either graph).

Case (*i*). Let $\mathscr{N} = \mathscr{N}_{\mu^0}$ for some $\mu^0 \in \mathscr{M}$. Then $G^tG(\xi_i^1, \xi_j^1) = H^tH(\xi_i^1, \xi_j^1) \neq 0$ for all i, j. By (3′) $\mathscr{N}_{\mu^0} = \mathscr{N}_{\lambda^0}$ for some $\lambda^0 \in \Lambda$.

Case (ii). Suppose there exist $\mu_1^0, \mu_2^0 \in \mathcal{M}$ such that $\mu_1^0 \neq \mu_2^0$ and $\mathcal{N}_{\mu_1^0} = \mathcal{N}_{\mu_2^0} = \mathcal{N}$. By (i) above, there exists $\lambda_1^0 \in \Lambda$ such that $\mathcal{N}_{\lambda_1^0} = \mathcal{N}_{\mu_1^0} = \mathcal{N}$. Since there are at least two triple points in $\mathcal{H}$ each of whose set of neighbours equals $\mathcal{N}$, $G^tG(\xi_i^1, \xi_j^1) = H^tH(\xi_i^1, \xi_j^1) \geqq 2$. Therefore, there exist $\kappa_{ij}^0 \neq \lambda_1^0$ in $\mathcal{G}^0$, such that $(\xi_i^1 - \kappa_{ij}^0 - \xi_j^1)$ are in $\mathcal{G}$ for all distinct i and j.

If all the κ_{ij}^0's were distinct, $(\xi_1^1 - \kappa_{12}^0 - \xi_2^1 - \kappa_{23}^0 - \xi_3^1 - \kappa_{31}^0 - \xi_1^1)$ would from a 6-cycle in $\mathcal{G}$. Therefore for some $j \neq k$, $\kappa_{ij}^0 = \kappa_{ik}^0$, which then is a triple point, λ_2^0, in $\mathcal{G}^0$, such that $\mathcal{N}_{\lambda_2^0} = \mathcal{N}$.

Case (iii). Suppose there exist three distinct points $\mu_1^0, \mu_2^0, \mu_3^0 \in \mathcal{M}$ such that $\mathcal{N}_{\mu_i} = \mathcal{N}$ for all i. By (i) above there exists $\lambda_1^0 \in \Lambda$ such that $\mathcal{N}_{\lambda_1^0} = \mathcal{N}_{\mu_1^0} = \mathcal{N}$. Since there are three triple points in $\mathcal{H}$ each of whose set of neighbours equals $\mathcal{N}$, we have $H^tH(\xi_1^1, \xi_2^1) = 3$. So there exist distinct κ_1^0, κ_2^0, distinct from λ^0, such that $(\xi_1^1 - \kappa_1^0 - \xi_2^1), (\xi_1^1 - \kappa_2^0 - \xi_2^1)$ are in $\mathcal{G}$. Now $G^tG(\xi_1{}^1, \xi_3^1) = 3$ and $\mathrm{Deg}(\xi_1^1) \leqq 3$. Therefore $(\xi_1^1 - \kappa_1^0 - \xi_3^1)$ and $(\xi_1^1 - \kappa_2^0 - \xi_3^1)$ are in $\mathcal{G}$. So we have $\{\lambda_1^0, \lambda_2^0 = \kappa_1^0, \lambda_3^0 = \kappa_2^0\} \in \Lambda$ such that $\mathcal{N}_{\lambda_i^0} = \mathcal{N}$.

Step 5. End of proof of (A′).

For all $\xi^1 \in \mathcal{G}^1$

$$(1) \qquad |\mathcal{G}(\xi^1, \xi^1)| = \sum_{\xi^1 \neq \eta^1} |\mathcal{G}(\xi^1, \eta^1)| - |\{\lambda_i^0 \in \Lambda : (\lambda_i^0 - \xi^1) \in \mathcal{G}\}| + |\mathcal{I}^{\mathcal{G}}(\xi^1)|.$$

and,

$$(2) \quad |\mathcal{H}(\xi^1, \xi^1)| = \sum_{\xi^1 \neq \eta^1} |\mathcal{H}(\xi^1, \eta^1)| - |\{\mu_i^0 \in \mathcal{M} : (\mu_i^0 - \xi^1) \in \mathcal{H}\}| + |\mathcal{I}^{\mathcal{H}}(\xi^1)|.$$

(Reason: While $|\mathcal{G}(\xi^1, \eta^1)|$ counts the number of even vertices β^0 such that $(\xi^1 - \beta^0 - \eta^1)$ is in $\mathcal{G}$, the first summation on the right side counts such vertices of degree two precisely once and vertices of degree three twice.)

If ξ^1 is such that $|\mathcal{I}^{\mathcal{G}}\xi^1)| = 0$, then

$$|\mathcal{G}(\xi^1, \xi^1)| = \sum_{\eta^1 \neq \xi^1} |\mathcal{G}(\xi^1, \eta^1)| - |\{\lambda_i^0 \in \Lambda : (\lambda_i^0 - \xi^1) \in \mathcal{G}\}| \text{ and}$$

$$|\mathcal{H}(\xi^1, \xi^1)| = \sum_{\eta^1 \neq \xi^1} |\mathcal{H}(\xi^1, \eta^1)| - |\{\mu_i^0 \in \mathcal{M} : (\mu_i^0 - \xi^1) \in \mathcal{H}\}| + |\mathcal{I}^{\mathcal{H}}(\xi^1)|.$$

Now $|\mathcal{G}(\xi^1, \xi^1)| = |\mathcal{H}(\xi^1, \xi^1)|$, and $|\mathcal{G}(\xi^1, \eta^1)| = |\mathcal{H}(\xi^1, \eta^1)|$.
Therefore,

$$|\{\lambda_i^0 \in \Lambda : (\lambda_i^0 - \xi^1) \in \mathcal{G}\}| = |\{\mu_i^0 \in \mathcal{M} : (\mu_i^0 - \xi^1) \in \mathcal{H})\}| - |\mathcal{I}^{\mathcal{H}}(\xi^1)|.$$

And hence,

$$|\{\lambda_i^0 \in \Lambda : (\lambda_i^0 - \xi^1) \in \mathcal{G}\}| \leqq |\{\mu_i^0 \in \mathcal{M} : (\mu_i^0 - \xi^1) \in \mathcal{H}\}|.$$

So, we have,

$$0 \geqq |\{\lambda_i^0 \in \Lambda : (\lambda_i^0 - \xi^1) \in \mathscr{G}\}| - |\{\mu_i^0 \in \mathscr{M} : (\mu_i^0 - \xi^1) \in \mathscr{H}\}|$$
$$= |\{\lambda_i^0 \in \Lambda : \xi^1 \in \mathscr{N}_{\lambda_i^0}\}| - |\{\mu_i^0 \in \mathscr{M} : \xi^1 \in \mathscr{N}_{\mu_i^0}\}$$
$$= \sum_{\mathscr{N} \ni \xi^1} (|\{\lambda_i^0 \in \Lambda : \mathscr{N}_{\lambda_i^0} = \mathscr{N}\}| - |\{\mu_i^0 \in \mathscr{M} : \mathscr{N}_{\mu^{0i}} = \mathscr{N}\}|$$
$\geqq 0$ (since each term in the sum is positive by Step 4 above)

Hence each term in the sum is zero, i.e.,

$$|\{\lambda_i^0 \in \Lambda : \mathscr{N}_{\lambda_i^0} = \mathscr{N}\}| = |\{\mu_i^0 \in \mathscr{M} : \mathscr{N}_{\mu_i^0} = \mathscr{N}\}$$

for all $\mathscr{N} \subseteq \mathscr{G}^1$, containing an element ξ^1 such that $|\mathscr{I}^{\mathscr{G}}(\xi^1)| = 0$.

But, by (4′), every $\mathscr{N}_\lambda$ has an element ξ_λ^1 with $|\mathscr{I}^{\mathscr{G}}(\xi_\lambda^1)| = 0$. If $\mathscr{N} \neq \mathscr{N}_\lambda$ for any λ, there is no triple point in $\mathscr{G}$ whose set of neighbours equals $\mathscr{N}$ and so, by Step 4, there is no such triple point in $\mathscr{H}$ either. So we have (A′).

Step 6. Proof of (B).

By (A′) we know that for any ξ^1 in $\mathscr{G}^1$

$$|\{\lambda_i^0 \in \Lambda : (\lambda_i^0 - \xi^1) \in \mathscr{G}\}| = \sum_{\mathscr{N} \ni \xi^1} |\{\lambda_i^0 \in \Lambda : \mathscr{N}_{\lambda_i^0} = \mathscr{N}\}|$$
$$= \sum_{\mathscr{N} \ni \xi^1} |\{\mu_i^0 \in \mathscr{M} : \mathscr{N}_{\mu_i^0} = \mathscr{N}\}|$$
$$= |\{\mu_i^0 \in \mathscr{M} : (\mu_i^0 - \xi^1) \in \mathscr{H}\}|$$

Therefore by comparing (1) and (2) we have

$$|\mathscr{I}^{\mathscr{G}}(\xi^1)| = |\mathscr{I}^{\mathscr{H}}(\xi^1)|$$

The proof of the proposition is finally complete.

REMARK 7. (1) Each of the graphs A_n, D_n and E_n satisfies the four hypotheses of the last proposition. (For $n > 8$, we write $E_n = A_{n-8} \# E_8$, where the $*$'s of the two graphs are identified.)

(2) For a bipartite graph $\mathscr{G}$, there is a natural induced metric on $V(\mathscr{G})$. For each integer $n \geqq 0$, write $\mathscr{G}_{0]} = \{*\}$, $\mathscr{G}_{1]} = \mathscr{N}_*^{\mathscr{G}} \cup \{*\}$, etc.) It is clear that if $\mathscr{G}$ and $\mathscr{H}$ are weakly dual, so are $\mathscr{G}_{2n]}$ and $\mathscr{H}_{2n]}$. In particular, we recapture Haagerup's observation: if $\mathscr{G}$ and $\mathscr{H}$ are the principal graphs of a finite-index subfactor and if $\mathscr{G}_{n]} = A_{n+1}$, then $\mathscr{H}_{n]} = A_{n+1}$ for even n. (To be sure, it must be verified that the contragredient map is trivial on the even vertices; but for this it is enough to note that for all n the set $\mathscr{G}_{2n+2]}^0 - \mathscr{G}_{2n]}^0$ is invariant under the involution of the even vertices.) The above statement is also valid for odd n; this follows from the case of even n and the connectedness of the principal graphs.

(3) It is tempting to call the subgraph conditions – cf. (3) and (4) of Proposition

6 – a "double of a star-triangle" relation; more precisely, is there more than just a superficial similarity between the two notions?

We now recall the following observation made by Ocneanu (see [K] for the statement and [OK] for a proof).

OBSERVATION 8. Suppose a graph $\mathscr{G}$ satisfies the following conditions:
(1) $\mathscr{G}$ does not contain a subgraph isomorphic to C_4;
(2) $\mathscr{G}$ contains a triple point; and
(3) $\|G\| > 2$.
Then it is not possible to construct a commuting square of the following form:

$$\begin{array}{ccc} C & \overset{G^t}{\subset} & D \\ {}_G\cup & & \cup_{G^t} \\ A & \overset{G}{\subset} & B \end{array}$$

In particular, there does not exist a finite depth subfactor $N \subset M$ with trivial contragredient maps, both of whose principal graphs are $\mathscr{G}$.

The above observation, in conjunction with the preceding proposition, has the following interesting consequence.

THEOREM 9. *Let $N \subset M$ be a pair of* II_1 *factors such that* $[M: N] > 4$. *Assume that the subfactor N has trivial contragredient maps. If one of the associated principal graphs $\mathscr{G}$ is a finite tree, each of whose vertices has degree at most three, then $\mathscr{G}$ contains a subgraph isomorphic to T such that the vertices of degree one in T are even vertices of degree one in $\mathscr{G}$.*

PROOF. Suppose now that a graph $\mathscr{G}$ arises as in the statement of the theorem. The hypothesis ensures that $\mathscr{G}$ satisfies conditions (1), (2) and (3) of Proposition 6, as well as conditions (1) and (3) of Observation 8. Since $\|G\| > 2$, the graph $\mathscr{G}$ is not A_n for any n. Since $\mathscr{G}$ is assumed to have at most triple points, it follows that $\mathscr{G}$ also satisfies condition (2) of Observation 8.

On the other hand, it must be obvious that a finite graph cannot arise as a principal graph of a subfactor with trivial contragredient maps, if it satisfies conditions (1)–(4) of Proposition 6 as well as conditions (1)–(3) of Observation 8.

Hence it must be the case that $\mathscr{G}$ violates condition (4) of Proposition 6, and the proof is complete.

REMARK 10. The hypothesis about trivial contragredient maps is essential. It has been shown by Haagerup (in an unpublished manuscript) that there exists a subfactor whose principal graphs are as shown below, where the non-trivial contragredient mapping in the graph $\mathscr{G}$ is indicated by the dotted line (fig. 4).

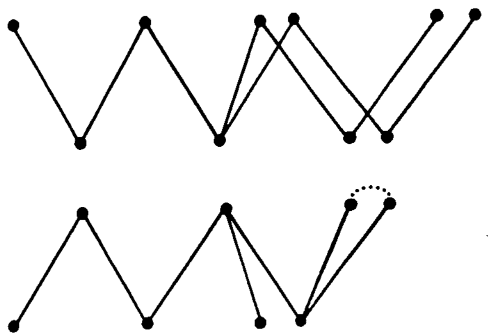


Figure 4.

The above theorem shows that no T-graph – i.e., a graph with a unique vertex of degree 3, with all other vertices of degree at most two – with norm greater than 2 can arise as a principal graph of a subfactor with trivial contragredient maps.

We conclude by describing some more graphs which cannot arise as a principal graph of a subfactor with trivial contragredient maps, as these are trees which satisfy condition (4) of Proposition 6 (and which have norm greater than 2 and have no vertex of degree greater than 3):

(i) any version of a connected sum of A_n, $n \geqq 2$ and E_8 in which a vertex of degree one from A_n has been identified with one of the vertices of degree one in E_8 or one of their degree two neighbours (fig. 5);

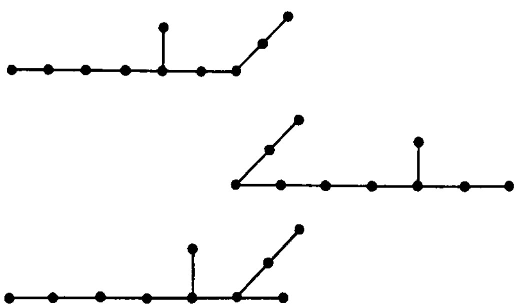


Figure 5.

(ii) the Cayley tree and many other subgraphs of the Bethe lattice (fig. 6).

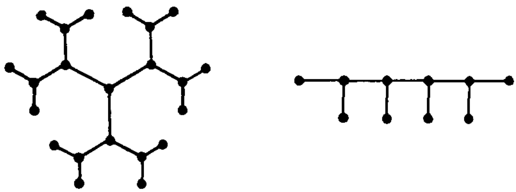


Figure 6.

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