# ON THE EXISTENCE AND UNIQUENESS OF A NEW CLASS OF INDEX NUMBERS

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*SUMMARY*. Geary (1958) and Khamis, (1969, 1970) proposed a new class of index numbers for consistent price and quantity comparisons. These are based on the concepts of 'exchange rate' of the currency and 'average price' of a commodity. These index numbers emerge from the solution of a system of linear equations. In this paper, we obtain the index numbers using only the concept of 'exchange rate'. We also derive necessary and sufficient conditions for the existence of these index numbers. We prove that unique set of meaningful index numbers exists if and only if the groups under comparison cannot be divided into two nonempty subclasses such that there is no commodity in use common to both of these classes. Finally, we tackle a practical problem which arises due to the absence of price and quantity composition of the expenditure on some items. We illustrate this by taking the expenditure data on Rural India from the 18th round of National Sample Survey.

A new class of index numbers was proposed by Khamis (1969; 1970) which leads to consistent inter-country comparisons. These index numbers are based on a set of $M+N$ homogeneous equations (1), formulated by using the concepts of 'average price' and 'exchange rate'. These equations were first proposed by Geary (1958). Geary observed in his paper that nontrivial solution exists for the system (1) depending on one of the arbitrary parameters. Khamis (1970) derives some sufficient conditions for the existence of unique positive solution for the system (1).

In this paper, we arrive at the required equations by using only the concept of 'exchange rate' and some definitional identities. This way of formulating the system implies that this approach is unique if one wishes to make use of the concept of 'exchange rate'. Further, we derive necessary and sufficient condition for the existence of positive solution for (1) in terms of data on prices and quantities, which are easily verifiable. We prove that unique positive solution exists if and only if the set of groups involved in the comparison cannot be divided into two non-empty subclasses such that there is no commodity which is consumed in both the classes. This result is proved by using some properties of dominant diagonal matrices, non-negative matrices and graphs. This condition justifies the intuitive feeling that the comparisons are meaningful only when the groups under comparison are related in a particular manner and our result gives an objective criterion for doing this. Results in this paper are stronger than Khamis' presented in (1970) where only sufficient conditions for the existence of positive solutions are derived.

Finally, we consider a practical problem, which arises due to the absence of price and quantity composition of the expenditure on some items. We tackle this problem by distributing this expenditure to various commodity groups, by suitably modifying the method described in Section 1. We illustrate this by taking the expenditure data on Rural India from the 18th round of NSS.

## 1. Definitions

Let $p_{ij}$ and $q_{ij}$ represent the price and quantity of the $i$-th commodity in $j$-th group respectively, for $i = 1, ..., N$ and $j = 1, ..., M$. Let $R_j$ and $P_i$ represent the exchange rate of $j$-th currency and average price of $i$-th commodity respectively. Then they are defined by the following equations

$$R_j = \frac{\sum_{i=1}^{N} P_i q_{ij}}{\sum_{i=1}^{N} p_{ij} q_{ij}} \qquad \text{for} \quad j = 1, ..., M$$

$$P_i = \frac{\sum_{j=1}^{M} R_j p_{ij} q_{ij}}{\sum_{j=1}^{M} q_{ij}} \qquad \text{for} \quad i = 1, ..., N \qquad ... \quad (1)$$

Denote*

$$e_{ij} = p_{ij} q_{ij};\ E_j = \sum_{i=1}^{N} e_{ij},\ v_{ij} = e_{ij}/E_j;$$

$$Q_i = \sum_{j=1}^{M} q_{ij};\ q^*_{ij} = q_{ij}/Q_i \quad \text{and} \quad R^*_j = R_j E_j$$

Equations (1) can be simplified by substituting the values of $P_i$ in the values of $R_j$ and put in the form

$$BR = 0 \qquad ... \quad (2)$$

where $B = ((b_{kl}))$ and $b_{kl} = \delta_{kl} - \sum_{i=1}^{N} q^*_{ik} v_{il}$; $\delta_{kl} = l$ if $k = l$ and $= 0$ if $k \neq l$, and

$$R = [R^*_1 \dots R^*_M]'.$$

If we are to form the equations (1) and hence (2), we must assume that

*Assumption* 1 : For all $i$ and $j$, (i) $p_{ij} > 0$, (ii) $Q_i > 0$ and (iii) $E_j > 0$. This assumption can be interpreted easily. In most of the practical situations this is satisfied.

For all practical purposes, it is enough to work with system (2) since any solution for (1) is a solution for (2) and vice versa. So in our paper we consider only (2).

### Alternative formulation

We will obtain (2) by considering some sort of consistent conditions. Suppose that $R_j$ represents the exchange rate for one unit of money in $j$-th group for $j = 1, ..., M$. Under the same notation as above $R_j\ E_j$ represents the worth of the money expenditure

*We have deviated from Khamis' (1970) notation for convenience.

$E_j$ in the $j$-th group in a common currency unit. Similarly $\sum_{j=1}^{M} e_{ij}R_j$ represents the worth of the total money expenditure on the $i$-th commodity in all the groups taken together in a common currency unit. $q^{*}_{ik} \sum_{j=1}^{M} e_{ij}R_j$ represents the share of the $k$-th group in $\sum_{j=1}^{M} e_{ij}R_j$. This can be defined for commodities, $i = 1, \ldots, N$. This above definitions imply that, for $k = 1, \ldots, M$

$$R_K E_K = \sum_{i=1}^{N} q^{*}_{ik} \sum_{j=1}^{M} e_{ij}R_j$$

$$= \sum_{i=1}^{N} q^{*}_{ik} \sum_{j=1}^{M} \frac{e_{ij}}{E_j} E_j R_j$$

$$R^{*}_K = \sum_{i=1}^{N} q^{*}_{ik} \sum_{j=1}^{M} v_{ij}R^{*}_j$$

$$= \sum_{j=1}^{M} R^{*}_j \sum_{i=1}^{N} q^{*}_{ik}v_{ij}$$

Therefore

$$R^{*}_K - \sum_{j=1}^{M} R^{*}_j \sum_{i=1}^{N} q^{*}_{ik}v_{ij} = 0 \qquad \text{for } k = 1, \ldots, M.$$

These equations can be rewritten and be put in the form

$$BR = 0.$$

We could arrive at the system of equations, by the definition of $R_j$'s. This way of formulating the problem seems to be more simple. We need not define average prices to arrive at these equations. This way of formulating the problem has another important advantage. This formulation implies that any one who wants to introduce the 'exchange rate' concept would, eventually, arrive at system (2). This increases the need for this approach in the index number problem.

However the motivation for solving (2) is the following. Once we arrive at solution for (2), we can arrive at $P_i$'s from (1). Using the values of $R^{*}_j$'s and $P_i$'s we can define the price and quantity index numbers in the same way as Khamis (1969) did, where the reader would find various properties of these index numbers.

## 2. Existence and uniqueness

Consider the equations

$$BR = 0.$$

The matrix $B$ has the following properties.

(i) The element $b_{kl} > 0$ if $k = l$ and $b_{kl} < 0$ if $k \neq l$ for all $k$ and $l$.

(ii) Each column sum in $B$ is zero. Consider the 1-th column.

We have $$b_l = \sum_{k=1}^{M} b_{kl}$$

$$= \left( 1 - \sum_{i=1}^{N} q_{il} \; v_{il} \right) - \sum_{k=1}^{M} \sum_{i=1}^{N} q^{*}_{ik} v_{il}$$

$$= 1 - \sum_{k=1}^{N} \sum_{i=1}^{N} q^{*}_{ik} v_{il}$$

$$= 1 - \sum_{i=1}^{N} v_{il} \sum_{k=1}^{M} q^{*}_{ik}$$

$$= 0 \left( \text{since} \sum_{k=1}^{M} q^{*}_{ik} = 1 \text{ and } \sum_{i=1}^{N} v_{il} = 1 \right)$$

By property (2) we have that $B$ is singular.

Let $$C = B' = (c_{kl})$$

where $$c_{kl} = \delta_{kl} - \sum_{i=1}^{N} v_{ik} q^{*}_{il} \text{ and } \delta_{kl} = 1 \text{ if } k = l \text{ and } =0 \text{ if } k \neq l.$$

So $c_{kl} \geqslant 0$ for $k = 1$ and $c_{kl} \leqslant 0$ for $k \neq l$ and each row sum in $C$ is zero and hence

$$|c_{kk}| = \sum_{\substack{l=1 \\ \neq k}}^{M} |c_{kl}|, \text{ for all } K.$$

Also $c_{kl} = 0$ if and only if $c_{lk} = 0$. These properties follow by assumption 1.

We will try to solve (2) in the following manner. We put one of the $R^{*}_{j}$'s to be 1 and arrive at the solution for the rest of the $R^{*}_{j}$'s. Without loss of generality put $R^{*}_{m} = 1$. Then we have

$$\bar{B}\,\bar{R} = y \qquad \ldots \quad (3)$$

where $\bar{B}$ is the matrix obtained by deleting the $M$-th row and the $M$-th column. $\bar{R}$ is obtained by deleting the $M$-th element in $R$ and $y$ is obtained by deleting the last element in the last column of $B$. Define $A = \bar{B}'$.

Some pertinent definitions are given below.

*Definition* 1 : Let $G$ be the graph with the groups as vertices and join vertices $k$ and $l$ if there is a commodity $i$ for which $q_{ik}$ and $q_{il}$ are positive. Call $G$ the 'adjacent graph' of the given data.

*Definition* 2 : The graph $G$ is said to be connected if we can pass from any vertex to any other vertex.

*Definition* 3 : We say that the $i$-th row of the matrix $A$ is dominated by its diagonal if $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$.

*Definition* 4 : A matrix $A_{n \times n}$ is said to possess property $P$ if the removal of any $K$ rows, $1 \leqslant K \leqslant n-1$, and the corresponding columns, leaves the matrix with at least one row dominated by its diagonal element.

*Definition* 5 : A matrix $A_{n \times n}$ is said to be dominant diagonal if there exist positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that for all $i$

$$\lambda_i \, |a_{ii}| > \sum_{j \neq i} \lambda_j \, |a_{ij}|. \qquad \ldots \quad (4)$$

*Definition* 6 : Wey say that a matrix is a $P$-matrix if all the principal minors are positive.

*Definition* 7 : A non-negative matrix $A_{n \times n} = (a_{ij})$ is said to be indecomposable if there is no nonempty proper subset $J$ of $\{1, \ldots, n\}$ such that $a_{ij} = 0$ for $i \epsilon J$ and $j \epsilon J$.

*Definition* 8 : Let $A_{n \times n} = (a_{ij}) \geqslant 0$. Then let $G^*$ be the graph adjoint to the matrix $A$ with the vertex set $\{1, \ldots, n\}$ and join $i$-th vertex to $j$-th vertex if there exists a chain of vertices $\{K_0, K_1, \ldots, K_l\}$, $K_0 = i$ and $K_l = j$ such that for any consecutive $K_s$ and $K_{s+1}$, $a_{K_s K_{s+1}} > 0$.

Observe that the graph is a directed graph.

Our aim is to arrive at the necessary and sufficient conditions for the existence and uniqueness of the non-negative solution for the system (3). In arriving at these conditions we will prove a sequence of lemmas tending to the ultimate goal.

Lemma 2.1 : *A necessary and sufficient condition that the matrix $C$ has property $P$ is that the adjacent graph $G$ is connected.*

*Proof* : Suppose first that $G$ is connected. If the rows numbered $i_1, \ldots, i_k$ and the corresponding columns of $C$ are deleted let the new matrix be $C_l$. Now since $G$ is connected, there is an edge joining one of $i_1, \ldots, i_k$ to some vertex $l$ outside. Then in the $l$-th row of $C$ there is a non-zero element in one of the columns $i_1, \ldots, i_k$ and since

$$|C_{ll}| = \sum_{j \neq l} |C_{lj}|$$

it follows that the $l$-th row of $C$ is dominated by its diagonal element. This proves sufficiency.

Next let $C$ have property $P$. If $G$ is not connected, then the vertices can be partitioned into non empty subsets $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots j_l\}$ such that there is no edge joining an $i$ and a $j$. Now we have $C_{i_\alpha j_\beta} = 0$ for $\alpha = 1, \ldots, k$ and $\beta = 1, \ldots, l$. Thus the matrix obtained from $C$ by deleting the rows numbered $i_1, \ldots, i_k$ and the corresponding columns have no row dominated by its diagonal element. This contradiction proves the necessity.

Let $A$ be the matrix obtained by deleting the last row and the last column of $C$. Then we have

Lemma 2.2 : *$A$ is dominant diagonal if and only if $A$ has property $P$ and $A$ has a row dominated by its diagonal element.*

*Proof* : First let $A$ be dominant diagonal. Then there exist positive numbers $\lambda_1, \ldots, \lambda_{m-1}$ satisfying (4) above. Suppose now that $A$ does not have property $P$. Then there exist $i_1, \ldots, i_k$ with $1 \leqslant k \leqslant M-2$ such that the matrix obtained from $A$ by deleting the rows numbered $i_1, \ldots, i_k$ and the corresponding columns does not have any row which is dominated by its diagonal element.

Thus
$$|a_{ii}| = \sum_{j \neq i, i_1, \ldots, i_k} |a_{ij}| \quad \text{for} \quad i \neq i_1, i_2, \ldots, i_k.$$

Let
$$\lambda_i = \min \{\lambda_j : j \neq i_1, \ldots, i_k\}$$

Then we have

$$\sum_{j \neq i, i_1, \ldots, i_k} \lambda_j |a_{ij}| \geqslant \sum_{j \neq i, i_1, \ldots, i_k} \lambda_i |a_{ij}| = \lambda_i |a_{ii}| > \sum_{j \neq i, i_1, \ldots, i_k} \lambda_j |a_{ij}|$$

a contradiction which proves that $A$ has property $P$. Since $A$ is dominant diagonal there exists a row which is dominated by its diagonal element. This proves the only if part.

Conversely let $A$ have property $P$ and let a row be dominated by its diagonal element. Then we construct positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_{M-1}$ satisfying (4) above.

Let $i_1, \ldots, i_k$ be the rows of $A$ dominated by their diagonal elements. Then choose any numbers $\mu_{i_k}, \ldots, \mu_{i_k}$ such that

$$\frac{\sum_{j \neq i_\alpha} |a_{i_\alpha j}|}{|a_{i_\alpha i_\alpha}|} < \mu_{i_\alpha} < 1 \quad \text{for } \alpha = 1, \ldots, K.$$

Let $\mu^{(1)} = \max(\mu_{i_1}, \ldots, \mu_{i_k})$ and define

$$\lambda_{i_1} = \lambda_{i_2} = \ldots = \lambda_{i_k} = \mu(1).$$

In our construction all the $\lambda$'s will be less than unity. Hence for $\alpha = 1, \ldots, k$,

$$\lambda_{i_\alpha} |a_{i_\alpha i_\alpha}| = \mu^{(1)} |a_{i_\alpha i_\alpha}| > \mu_{i_\alpha} |a_{i_\alpha i_\alpha}| > \sum_{j \neq i_\alpha} |a_{i_\alpha j}| > \sum_{j \neq i_\alpha} \lambda_j |a_{i_\alpha j}|$$

Now delete the rows numbered $i_1, \ldots, i_k$ and the corresponding columns from $A$ and let $j_1, \ldots, j_l$ be the rows which are dominated by their diagonal elements in the new matrix. Then choose numbers $\mu_{j_1}, \ldots, \mu_{j_l}$ such that for $\beta = 1, \ldots, l$,

$$\frac{\sum\limits_{j \neq j_\beta, i_1, \ldots, i_k} |a_{j_\beta j}| + \sum\limits_{j = i_1, \ldots, i_k} \lambda_j |a_{j_\beta j}|}{|a_{j_\beta j_\beta}|} < \mu_{j_\beta} < 1.$$

This choice is possible since

$$|a_{j_\beta j_\beta}| = \sum_{j \neq j_\beta, i_1, \ldots, i_k} |a_{j_\beta j}| + |\sum_{j = i_1, i_2, \ldots, i_k} |a_{j_\beta j}|$$

$$\sum_{j = i_1, i_2, \ldots, i_k} |a_{j_\beta j}| > 0 \quad \text{and} \quad \lambda_j < 1 \quad \text{for} \quad j = i_1, i_2, \ldots, i_k.$$

Let $\mu^{(2)} = \max(\mu_{j_1}, \ldots, \mu_{j_l})$ and define

$$\lambda_{j_1} = \lambda_{j_2} = \ldots = \lambda_{j_l} = \mu^{(2)}.$$

Then for $\beta = 1, \ldots, l$

$$\begin{aligned} \lambda_{j_\beta} |a_{j_\beta j_\beta}| &= \mu^{(2)} |a_{j_\beta j_\beta}| > \mu_{j_\beta} |a_{j_\beta j}| \\ &> \sum_{j \neq j_\beta, i_1, \ldots, i_k} |a_{j_\beta j}| + \sum_{j = i_1, \ldots, i_k} \lambda_j |a_{j_\beta j}| \\ &\geqslant \sum_{j \neq j_\beta} \lambda_j |a_{j_\beta j}|. \end{aligned}$$

Now deleting the rows numbered $j_1, \ldots, j_l$ and the corresponding columns we get some rows which are dominated by their diagonal elements. The corresponding $\lambda$'s can be defined similarly and the process is repeated until all rows are exhausted. It is evident that the resulting numbers $\lambda_1, \ldots, \lambda_{M-1}$ are less than unity and satisfy condition (4) above. This completes the proof of the lemma.

Lemma 2.3 : *A has property P and A has at least one row dominated by its diagonal element if and only if C has property P.*

*Proof* : If part is trivial, for, the removal of rows numbered $i_1, \ldots, i_k$ and the corresponding columns of $A$ is equivalent to the removal of the rows numbered $i_1, i_2, \ldots, i_k, M$ and the corresponding columns from $C$.

To prove the only if part suppose that $A$ has property $P$, $A$ has a row dominated by its diagonal element and $C$ does not have property $P$.

Let $i_1, \ldots, i_k$ be such that the removal of rows $i_1, \ldots, i_k$ and the corresponding columns from $C$ gives a matrix with row dominated by its diagonal element. Then evidently $M \neq i_1, \ldots, i_k$. Let now $\{M, j_1, \ldots, j_l\} = \{1, \ldots, M\} - \{i_1, \ldots, i_k\}$. Now by definition of $i_1, \ldots, i_k$, we have

$$c_{j_\alpha i_\beta} = 0 \quad \text{for} \quad \alpha = 1, \ldots, l \text{ and } \beta = 1, \ldots, k$$

$$c_{m i_\beta} = 0 \quad \text{for} \quad \beta = 1, \ldots, k.$$

But then $C_{i_s j_n} = 0$ and $C_{i_s m} = 0$ and so the deletion of the rows and columns numbered $M, j_1, \ldots, j_l$ in $C$ leaves a matrix with no row dominated by its diagonal element, a contradiction to the hypothesis. This completes the proof of the lemma.

From Lemmas 2.2 and 2.3 we get

Theorem 2.4 : *$A$ is dominant diagonal if and only if $C$ has property $P$.*

Theorem 2.5 : *A matrix $A = (a_{ij})$ having a positive dominant diagonal is a $P$-matrix.*

*Proof :* See Nikaido, (1968), pp. 386-87.

If $A$ is a matrix whose typical element $a_{kl}$ and $a_{kl} = \rho\delta_{kl} - d_{kl}$ where $\delta_{kl} = 0$ for $k \neq l$ and $\delta_{kl} = l$ for $k = l$, and $d_{kl} \geqslant 0$ for all $k$ and $l$ and $\rho > 0$

$$Ax = y \qquad \ldots \quad (5)$$

is a system of equations, we have

Theorem 2.6 : *The following statements are equivalent.*

(i) *System (5) has a set of non-negative solutions $x_i \geqslant 0$ $(i = 1, \ldots, n)$ for some set of positive $y_i > 0$ $(i = 1, \ldots, n)$.*

(ii) *System (5) is solvable in the non-negative unknowns $x_i \geqslant 0$ $(i = 1, \ldots, n)$ for any set of non-negative $y_i \geqslant 0$ $(i = 1, \ldots, n)$.*

(iii) *All the principal minors are positive.*

*Proof :* See, Nikaido (1968), pp. 90-93.

As a result of the above theorem, we get

Corollary 2.7 : *If $A$ is a $P$-matrix and of the form required by (5) then $A^{-1}$ exists and is non-negative.*

*Proof :* $A^{-1}$ exists since $A$ is a $P$-matrix. $A^{-1}$ is non-negative follows from (ii) of the above theorem.

If $A^{-1}$ exists and is non-negative we have that $B^{-1}$ exists and is non-negative which assures that $\bar{R}$ is non-negative.

Our aim is to show that existence of $A^{-1}$ implies that the graph $G$ is connected. Let us prove the following.

Lemma 2.8 : *If $A^{-1}$ exists then graph $G$ is connected.*

*Proof :* We have that $A$ is non-singular. Now suppose that $G$ is not connected. Then the vertices can be partitioned into non-empty subsets $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_l\}$ such that there is no edge between these two sets. Observe that $\{M\}$ cannot be equal to either of the sets, for, the matrix $A$ which is obtained by deleting the row and column numbered $M$ from the matrix $C$, would not have any row dominated by its diagonal element. This implies that $A$ is singular.

Further the fact that there are no edges between the sets $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_l\}$ implies that $A$ can be written in the form

$$\begin{bmatrix} A_1 & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & A_2 \end{bmatrix}$$

where one of the matrices $A_1$ and $A_2$ has all the row sums equal to zero which means that $A$ is singular. This contradiction proves the theorem.

In view of the above theorem, we have that if $\overline{B}$ inverse exists, which means that $A^{-1}$ exists, then $G$ is connected.

Theorem 2.9 : *$A$ is non-negative invertible if and only if the 'adjacent graph' $G$ of the data is connected.*

*Proof :* Follows trivially from the lemmas, theorems and corollaries proved before.

Theorem 2.9 implies that unique non-negative solution exists for $R_j^*$ ($j = 1, M-1$), given $R_M^* = 1$, since $A = \overline{B}'$ is obtained by deleting the $M$-th row and $M$-th column of $B'$. This theorem holds for the matrix obtained by deleting any row and the corresponding column of $B'$. This means that any $(M-1)$ equations in system (2) are linearly independant. Thus we are able to get unique solutions for the ratios of $R_j^*$'s. The ratios are same no matter what $R_j^*$ is chosen to be unity. We shall derive the condition for the existence of positive solution for $R_j^*$'s. We state two theorems, proofs of which appear in Nikaido (1968, pp. 107-109).

Theorem 2.10 : *A matrix $A = (a_{ij}) \geqslant 0$ is indecomposable if and only if the 'adjoint graph' $G^*$ of $A$ is connected.*

Theorem 2.11 : *If the matrix $(\rho I - A)$, $\rho > 0$ and $A \geqslant 0$ is non-negatively invertible for an indecomposable $A$, its inverse $(\rho I - A)^{-1}$ is a positive matrix.*

Let us now state and prove the following Main Theorem of the paper.

Main Theorem : *Unique positive solution for system (2) exists if and only if the 'adjacent graph' $G$ of the data is connected.*

*Proof :* Necessity follows from Theorem 2.9. To prove the sufficiency, let the 'adjacent graph' $G$ of the data be connected. Then we can find at least two vertices whose removal would result in connected graph (Berge, 1962). Let $k$ be one such vertex. Let $A = \overline{B}'$, where $\overline{B}$ is obtained by deleting the $k$-th row and $k$-th column. It is enough if we prove that $A^{-1}$ is positive. Observe that $A$ is a matrix of the form $(I-D)$ and $D \geqslant 0$. Further the 'adjoint graph' $G^*$ of this matrix is connected since it has the property that $a_{ij} = 0$ if and only if $a_{ji} = 0$. Hence the 'adjoint graph' of $D$ is connected. By Theorems 2.10 and 2.11, we have that $A^{-1}$ is positive. This completes the proof of the theorem.

Thus we have arrived at the necessary and sufficient condition for the existence of unique positive solution for the system (2). Further the conditions are easily verifiable.

## 3. Some empirical results

It is evident that formulation of system (1) and system (2) needs the information regarding the prices and quantities of all items in all groups. However, in many practical problems, as considered in Mukherjee (1969), we come a cross a situation where the price and quantity information is absent for some items. In this case it would be difficult to form the equations (2). One way of forming these equations is to ignore the item for which price and quantity counterparts of the expenditure are not known. This is in fact equivalent to assuming that prices of these items in this group have the same trend as the prices of other items taken together. We will try to get a more satisfactory solution to this problem.

The traditional way of accounting for the expenditure of the item is to distribute this expenditure over the other commodities according to some criterion. For example, we may distribute this expenditure over other items according to the value ratios of other items. We will explore the possibility of introducing similar ideas into our analysis. Without loss of generality we may combine all items for which the data is missing and call it 'composite commodity'.

Suppose we start with a grouping of commodities, say $K$ groups represented by the sets $A_1, A_2, \ldots, A_K$ such that $\bigcup_{i=1}^{K} A_i$ and $A_i \bigcap A_j = \phi$ for $i \neq j$. where $A$ is the set of all commodities. The expenditure on the 'composite commodity' can be split up into the expenditure related to various groups. In some cases, it could be that a part of this expenditure is not related to any of the groups. For example, in the National Sample Survey data on Rural India's consumption pattern, price and quantity components for the service items are absent and these items form a group by themselves and hence cannot be related to any other groups.

Distribution of the expenditure on 'composite commodity' over all groups, implies the assumption that the exchange rate of currency in a particular commodity group is also the exchange rate relating to the part of the expenditure on composite commodity. If we want to have a similar set up as (1), we need to define some new variables. We will try to distinguish between exchange rate of currency in various commodity groups. Let $R_j^k$ ($k = 1, \ldots, K$, $j = 1, \ldots, M$) be the exchange rate of currency in $j$-th group restricted over the $k$-th commodity group. So restricting to the $k$-th commodity group, we have

$$R_j^k = \frac{\sum\limits_{i \epsilon A_k} P_j^k q_{ij}}{\sum\limits_{i \epsilon A_k} p_{ij} q_{ij}} \qquad \text{for} \quad j = 1, \ldots, M$$

where $P_i^k$ represents the average price of $i$-th commodity taking into account only $R_j^k$ and given by

$$P_i^k = \frac{\sum\limits_{j=1}^{M} R_j^k e_{ij}}{\sum\limits_{j=1}^{M} q_{ij}} \qquad \text{for all } i \epsilon A_k.$$

The above definitions of $R_j^k$ and $P_i^k$ for $j = 1, \ldots, M$, $i \in A_k$ will lead to a system which is similar to (1) for $k = 1, \ldots, K$.

If this system is to be solvable we should have that each system separately satisfies the necessary and sufficient conditions derived in the earlier section. If every system is solvable then our problem is to arrive at overall purchasing power of currency in each group. Easiest way of doing this is to get a weighted average of $R_i^k$'s for each $j$, the weights decided by the value ratios. However, we will try to justify this in a more objective fashion. Consider the equations in (1). Recalling the definitions of $R_j$'s, we have for $j = 1, \ldots, M$

$$R_j = \frac{\sum_{i=1}^{N} P_i q_{ij}}{\sum_{i=1}^{N} p_{ij} q_{ij}}$$

$$= \frac{\sum_{k=1}^{K} \sum_{i \in A_k} P_i q_{ij}}{\sum_{i=1}^{N} p_{ij} q_{ij}}$$

$$= \sum_{k=1}^{K} \frac{\sum_{i \in A_k} P_i q_{ij}}{\sum_{i \in A_k} p_{ij} q_{ij}} \quad \frac{E_j^k}{E_j}$$

where $E_j^k = \sum_{i \in A_k} p_{ij} q_{ij}$. But for the $P_i$'s in these equations, the value $R_j$ is same as $\sum_{k=1}^{K} R_j^k \frac{E_j^k}{E_j}$.

Now we are well equipped to tackle the problem of 'composite commodity'. We distribute the expenditure on the composite commodity to various commodity groups with the implicit assumption that general price level in the $k$-th group reflects the price level concerning the expenditure allocated to $k$-th group. This means that $R_j^k$ will be the same for the group $k$, whatever be the share of the expenditure on composite commodity allocated to $k$-th group. But the essential difference is reflected in the value of $R_j$ which is a weighted average of $R_j^k$'s.

Let us illustrate these points by considering a numerical example. We consider the decile groups of Rural sector of India as our groups. We take the National Sample Survey 18th round data on the consumption expenditure in these groups for the illustration. We take the 56 items of consumption for which the price and quantity information is available for analysis. The following table shows the expenditure on the 56 items and the residual expenditure for which the price and quantity information is not available.

TABLE 1

Rs. per capita per month

| fractile group | expenditure on 56 items | residual | total |
|---|---|---|---|
| (1) | (2) | (3) | (4) |
| 0–10 | 6.6548 | 1.6852 | 8.34 |
| 10–20 | 9.3373 | 2.1527 | 11.49 |
| 20–30 | 11.1079 | 2.5821 | 13.69 |
| 30–40 | 12.7240 | 2.7461 | 15.47 |
| 40–50 | 14.1066 | 3.3639 | 17.47 |
| 50–60 | 15.8946 | 3.9154 | 19.81 |
| 60–70 | 17.7173 | 4.8627 | 22.58 |
| 70–80 | 20.2055 | 6.0545 | 26.26 |
| 80–90 | 23.9850 | 8.2350 | 32.23 |
| 90—100 | 34.5642 | 21.2458 | 55.81 |

We have calculated the exchange rates taking into account only 56 items. To take into account the residual item, we have grouped the 56 items into three groups, (i) food, (ii) fuel and light and (iii) clothing. Then the residual expenditure is distributed over these commodity groups so that we may assume that the average price level in the corresponding residual item. Rest of the residual item which is remaining is supposed to be corresponding to the service items and there is no justification in distributing it over the three groups considered. So we assume that the price of this commodity is governed by an average price obtained by combining the three groups purchasing powers in the manner described above. Table 2 represents the distribution of actual and residual expenditure over the various commodity groups.

TABLE 2

Rs. per capita per month

| fractile group | food | | fuel and light | | clothing | | remaining residual (services) | total |
|---|---|---|---|---|---|---|---|---|
| | actual | residual allocated | actual | residual allocated | actual | residual allocated | | |
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) |
| 0–10 | 5.9014 | 1.1671 | 0.6294 | 0.1471 | 0.1240 | 0.0701 | 0.3109 | 8.34 |
| 10–20 | 8.3133 | 1.3843 | 0.7925 | 0.1673 | 0.2316 | 0.1139 | 0.4972 | 11.49 |
| 20–30 | 9.7736 | 1.5251 | 0.9873 | 0.1943 | 0.3470 | 0.1676 | 0.6951 | 13.69 |
| 30–40 | 11.3469 | 1.4039 | 0.9524 | 0.1661 | 0.4256 | 0.2464 | 0.9287 | 15.47 |
| 40–50 | 12.4386 | 1.6410 | 1.0725 | 0.2554 | 0.5955 | 0.3253 | 1.1417 | 17.47 |
| 50–60 | 13.9579 | 1.8842 | 1.1405 | 0.2442 | 0.7962 | 0.3967 | 1.3903 | 19.81 |
| 60–70 | 15.4496 | 2.1513 | 1.2103 | 0.3590 | 1.0574 | 0.5547 | 1.7952 | 22.58 |
| 70–80 | 17.5090 | 2.4357 | 1.3396 | 0.3436 | 1.3569 | 0.7811 | 2.4941 | 26.26 |
| 80–90 | 20.2704 | 2.8377 | 1.5809 | 0.3650 | 2.1437 | 1.2490 | 3.7833 | 32.23 |
| 90—100 | 28.0068 | 4.7296 | 2.1220 | 0.5778 | 4.4354 | 2.7941 | 13.1446 | 55.81 |

Now we will present the $R_j^k$'s calculated for the three groups and the overall purchasing power $R_j$ obtained by calculating the weighted average, along with the $R_j$'s taking into account only the 56 items, taking the purchasing power of currency in the first decile group to be unity.

TABLE 3

| fractile group | food | fuel and light | clothing | exchange rate for currency: weighted average | exchange rate for currency: based on 56 items |
|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) |
| 0– 10 | 1.000000 | 1.000000 | 1.000000 | 1.000000 | 1.000000 |
| 10– 20 | 0.970153 | 0.908822 | 0.991014 | 0.965509 | 0.964692 |
| 20– 30 | 0.948358 | 0.868573 | 0.932449 | 0.940475 | 0.940848 |
| 30– 40 | 0.945371 | 0.872806 | 0.956361 | 0.933310 | 0.940051 |
| 40– 50 | 0.934769 | 0.848008 | 0.917623 | 0.926756 | 0.927221 |
| 50– 60 | 0.919917 | 0.843338 | 0.974110 | 0.911747 | 0.912272 |
| 60– 70 | 0.921276 | 0.840770 | 0.857001 | 0.910210 | 0.912611 |
| 70– 80 | 0.912252 | 0.807593 | 0.857443 | 0.899908 | 0.902636 |
| 80– 90 | 0.911300 | 0.826575 | 0.793610 | 0.801464 | 0.897935 |
| 90–100 | 0.893130 | 0.763937 | 0.713271 | 0.854490 | 0.866122 |

With the help of the above tables and discussion we are in a position to present the price index numbers based on system of equations (1), using only 56 items, using residual item also, Laspeyres' and Paasche's index numbers for comparison purposes. We present these index numbers with the first decile group as the base.

Table 4 shows that the above method is useful when the expenditure on 'composite commodity' is high. In the groups 70-80, 80-90 and 90-100 there is significant difference These new index numbers lie in the range given by Laspeyres' and Paasche's index numbers calculated for these expenditure groups.

TABLE 4

| decile group | new index nos. based on 56 items | new index numbers taking residual item | Laspeyres' | Paasche's |
|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) |
| 0– 10 | 100.00 | 100.00 | 100.00 | 100.00 |
| 10– 20 | 103.66 | 103.57 | 103.61 | 103.44 |
| 20– 30 | 106.31 | 106.33 | 106.48 | 106.06 |
| 30– 40 | 106.38 | 107.14 | 106.59 | 106.15 |
| 40– 50 | 107.85 | 107.90 | 108.33 | 107.77 |
| 50– 60 | 109.62 | 109.68 | 109.39 | 109.62 |
| 60– 70 | 109.57 | 109.86 | 109.69 | 109.62 |
| 70– 80 | 110.78 | 111.12 | 111.41 | 110.80 |
| 80– 90 | 111.37 | 112.18 | 110.90 | 111.55 |
| 90–100 | 115.46 | 117.03 | 112.43 | 115.62 |

ACKNOWLEDGEMENTS

I am greatly indebted to Professor M. Mukherjee for a number of suggestions, encouragement and guidance. I gratefully acknowledge many constructive and interesting discussions with Dr. A. Ramachandra Rao and Shri S. Bhaskara Rao during the preparation of this paper.

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*Paper received : October, 1970.*
*Revised : April, 1971.*