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Pairs of II_1 factors

V S SUNDER
Indian Statistical Institute, Bangalore 560059, India

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Abstract. A class of objects – that are best described as being actions of *group-like objects of von Neumann algebras* – is axiomatised and it is shown that there exists a bijective correspondence between isomorphism classes of such *covariant systems* and isomorphism classes of pairs of II_1 factors (M, N) satisfying $N \subset M$, $[M:N] < \infty$ and $M \cap N' = \mathbb{C}$.

Keywords. Irreducible subfactor; group-like object; covariant system.

1. Introduction

The aim of this paper is to associate with each pair (M, N) of II_1 factors satisfying $N \subseteq M$, $[M:N] < \infty$ and $M \cap N' = C$, a group-like object $\mathscr{G}_{M/N}$ (or rather, an isomorphism class of such objects) in such a way that

(i) $(M, N) \cong (\tilde{M}, \tilde{N})$ if and only if $\mathscr{G}_{M/N} \cong \mathscr{G}_{\tilde{M}/\tilde{N}}$; and

(ii) from each such object $\mathscr{G}$ that is abstractly defined, to construct a pair $(N \times \mathscr{G}, N)$ of II_1 factors (with finite index and trivial relative commutant, as above) in such a way that (a) $\mathscr{G}_{N \times \mathscr{G}/N} \cong \mathscr{G}$, and (b) $(N \times \mathscr{G}_{M/N}, N) \cong (M, N)$.

The object $\mathscr{G}$ is quite different in character from the one discussed by Ocneanu in [1] and [3]; we refer to objects such as $\mathscr{G}$ simply as covariant systems. Our notion of a covariant system (as in Definition (4.1) is defined by five axioms, three of which are similar to the group axioms, while the other two seem to correspond to the definition of a group action. Unfortunately, the author has been unable to decouple the notions of 'group-like object' and 'actions of such objects'; it is not clear whether the failure to do so is a consequence of the author's ineptness or of some inherent feature of this approach (which amounts to identifying axioms that yield the Hilbert algebra structure of the extension M purely in terms of the subfactor N and some morphisms associated with N).

The third section is devoted to 'right bases for M over N'', a slight generalization of the 'orthogonal bases' introduced in [5]; the reason for this generalization is that orthogonality gets destroyed under certain natural constructions involving products. This was most probably noticed by Ocneanu, since several results of this section occur with only slight and notational changes in [3]. This section is included here for completeness, and since there are few proofs in [3]. Thus, the first half (up till Proposition 6) of this section is a modified reproduction of parts of [5], while the second half ([Proposition 7, Proposition 11]) is essentially contained in [3].

The fourth section is the core of this paper, and is devoted to attaining the goals outlined in the first paragraph of this introduction. This section also contains descriptions of the covariant systems corresponding to G/H and $\hat{G}/\hat{H}$, where H is a (not necessarily normal) subgroup of a finite (not necessarily abelian) group G.

Finally, the author would like to attempt to describe the nature of the relationship between this work and Ocneanu's work. The two approaches (to the same problem) are very similar up to a point, after which they diverge; the common part is that part which builds on Pimsner-Popa's discussion of bases, by starting with a basis for M over N and proceeding to construct bases over N for each member M_n of the tower of the basic construction. Thus, while it is true that the author has benefited greatly from numerous conversations with Ocneanu and that this work has been strongly influenced, in spirit, by Ocneanu's work, this paper relies more heavily, in content, on the paper [5] of Pimsner and Popa.

2. Preliminaries

We shall be concerned with the category $\mathscr{C} = \{(M,N): M,\ N\ \mathrm{II}_1$ factors, $N \subseteq M$, $[M:N] < \infty\}$ and its subcategory

$$\mathscr{C}_0 = \{(M,N)\in\mathscr{C}: M \cap N' = C\},$$

where, of course a morphism $\pi:(M,N) \to (\tilde{M},\tilde{N})$ is a unital normal $*$-homomorphism $\pi: M \to \tilde{M}$ such that $\pi(N) \subseteq \tilde{N}$. We begin by recalling some facts from [1], [5] and [6].

If $(M,N)\in\mathscr{C}$, there is a notion of an extension M_1 of M by N, which has the following features:

(P1) $(M_1, M)\in\mathscr{C}$ and $[M_1:M] = [M:N]$

(P2) There exists a projection $e_0 \in M_1$ such that

(i) $e_0 x e_0 = E_N(x) e_0$, for all $x\in M$; (ii) $M_1 = \langle M, e_0\rangle$ $(=:(M \cup \{e_0\})'')$; (iii) $E_M(e_0) = \tau\cdot 1$. (Here and in the sequel, the symbol τ denotes $[M:N]^{-1}$, the symbol tr always stands for the unique normalized trace, and E_N denotes the unique tr-preserving conditional expectation of any II_1 factor onto a subfactor N.)

(P3) For each $x\in M_1$, there is a unique m in M such that $xe_0 = me_0$; in view of (P2) (iii), we necessarily have $m = \tau^{-1} E_M(xe_0)$.

If $(M,N)\in\mathscr{C}$, we shall let $\{M_n\}_{n=-1}^{\infty}$ denote the tower obtained by iterating the basic construction; thus $M_{-1} = N$, $M_0 = M$ and M_{n+1} is the extension of M_n by M_{n-1}; we shall let e_n be a projection in M_{n+1} that implements the conditional expectation of M_n onto M_{n-1}. The sequence $\{e_n\}_{n=0}^{\infty}$ of projections satisfies the following 'Jones relations':

(P4) (i) $e_m e_n = e_n e_m$ for $|m-n| > 1$; (ii) $e_n e_m e_n = \tau e_n$ if $|m-n| = 1$; (iii) $E_{M_n}(e_n) = \tau\cdot 1$

(P5) For $n \geqslant 1$, M_{2n+1} is the extension of M_n by M_{-1}, a choice of a projection in M_{2n+1} that implements the conditional expectation of M_n onto M_{-1} being given by $e_{(n)} = \tau^{-n(n+1)/2}(e_n e_{n-1}\cdots e_0)(e_{n+1}e_n\cdots e_1)\cdots(e_{2n}e_{2n-1}\cdots e_n)$.

We end this preliminary section by introducing some notation we shall employ. Symbols I, J, K, etc. will denote finite sets. We shall work with the algebra $\mathrm{Mat}_{I\times J}(N)$ of matrices, with entries in N, and rows and columns indexed by I and J respectively. An element of $\mathrm{Mat}_{I\times J}(N)$ will be denoted by $x = ((x_{ij}))$; when $I = J$, clearly $\mathrm{Mat}_{I\times I}(N)$ is II_1 factor, the equation $\mathrm{Tr}\,x = \sum_i \mathrm{Tr}\,x_{ii}$ defining a non-normalized trace on $\mathrm{Mat}_{I\times I}(N)$. Row-vectors (respectively, column vectors) will be denoted by $\mathrm{Mat}_{\cdot\times I}(N)$ (respectively, $\mathrm{Mat}_{I\times\cdot}(N)$).

In a natural fashion, we have $\mathrm{Mat}_{I\times J}(N)\subseteq \mathrm{Mat}_{I\times J}(M)\subseteq \mathrm{Mat}_{I\times J}(M_1)\subseteq\cdots$. If $a, b\in\cup_n M_n$ and $x\in\cup_n \mathrm{Mat}_{I\times J}(M_n)$, we shall let axb denote the matrix defined by $axb=((ax_{ij}b))$. Also, if $n<m$, we shall let the same symbol E_{M_n} denote the map from $\mathrm{Mat}_{I\times J}(M_M)$ into $\mathrm{Mat}_{I\times J}(M_n)$ defined by applying E_{M_n} entrywise. It must be obvious that the statements (P2)(i), (P2)(iii) and (P3) have the following matricial analogues:

(P2)(*i*)$_{\mathrm{Mat}}$ $e_0xe_0=E_N(x)e_0$ for all $x\in\mathrm{Mat}_{I\times J}(M)$;
(P2)(iii)$_{\mathrm{Mat}}$ $\mathrm{Tr}\, e_0x=\tau\,\mathrm{Tr}\, x$ for all $x\in\mathrm{Mat}_{I\times I}(M)$;
(P3)$_{\mathrm{Mat}}$ For each $x\in\mathrm{Mat}_{I\times J}(M_1)$, there is a unique $m\in\mathrm{Mat}_{I\times J}(M)$ such that $xe_0=me_0$; necessarily $m=\tau^{-1}E_M(xe_0)$.

3. Right bases

We shall assume henceforth that $(M,N)\in\mathscr{C}$ and that $\{M_n\}$ is the associated tower of the basic construction, and I will denote a finite set.

Lemma 1. *The following conditions on a column-vector* $\lambda=((\lambda_i))\in\mathrm{Mat}_{I\times\cdot}(M)$ *are equivalent*:

(i) $\lambda^*e_0\lambda=1$; (ii) $E_N(\lambda\lambda^*)=Q_\lambda$ *is a projection in* $\mathrm{Mat}_{I\times I}(N)$ *satisfying* $\mathrm{Tr}\, Q_\lambda=[M:N]$.

Proof. Fix $i_0\in I$ and define $v\in\mathrm{Mat}_{I\times I}(N)$ by $v_{ij}=\delta_{i_0j}e_0\lambda_i$, where of course δ denotes the Kronecker symbol. Then $(v^*v)_{ij}=\delta_{i_0i}\delta_{i_0j}\lambda^*e_0\lambda$, while $vv^*=Q_\lambda e_0$. Hence the assumption (i) is equivalent to the assumption that v is a partial isometry with $\mathrm{Tr}\, v^*v=1$, while (ii) amounts to requiring that v is a partial isometry with $\mathrm{Tr}\, vv^*=\tau\,\mathrm{Tr}\, Q_\lambda=1$ (cf. (P2)(iii)$_{\mathrm{Mat}}$); thus, (i) is equivalent to (ii).

DEFINITION 2

A column vector $\lambda\in\mathrm{Mat}_{I\times\cdot}(M)$ is called a right basis for M/N if λ satisfies either of the equivalent conditions of Lemma 1.

It should be remarked that this notion of right basis is a marginal generalization of the 'orthogonal basis' considered in [5]; a right basis λ (in the sense of Definition 2) is an orthogonal basis in the sense of [5] precisely when the associated projection Q_λ is diagonal. The reason for relaxing orthogonality will become evident later.

PROPOSITION 3. (*Existence of right bases*)

Let I be any finite set (with $|I|$ necessarily at least as large as $[M:N]$). and let Q be a projection in $\mathrm{Mat}_{I\times I}(N)$ *such that* $\mathrm{Tr}\, Q=[M:N]$. *Then there exists a right basis* $\lambda\in\mathrm{Mat}_{I\times\cdot}(N)$ *for M/N such that $E_N(\lambda\lambda^*)=Q$.*

Proof. Define $P\in\mathrm{Mat}_{I\times I}(N)$ by $P_{ij}=\delta_{i_0i}\delta_{i_0j}$ where $i_0\in I$ is fixed. Then $\mathrm{Tr}\, P=1$, and also $\mathrm{Tr}\, Qe_0=1$. Since Qe_0 and P are projections in the II_1 factor $\mathrm{Mat}_{I\times I}(M_1)$ with equal trace, there exists a partial isometry $v\in\mathrm{Mat}_{I\times I}(M_1)$ such that $v^*v=P$ and $vv^*=Qe_0$. Since $v^*v=P$, it follows that $v_{ij}=0$ if $j\neq i_0$; so there exist $v_i\in M_1$ such that $v_{ij}=\delta_{i_0j}v_i$; the assumption $vv^*=Qe_0$ implies that $v=e_0v$ and hence $v_i=e_0v_i$ for each i. Appeal now to (P3) to deduce the existence of $\lambda_i\in M$ such that $e_0\lambda_i=e_0v_i=v_i$. It follows at once that $q_{ij}e_0 = (vv^*)_{ij} = e_0\lambda_i\lambda_j^* e_0 = E_N(\lambda_i\lambda_j^*)e_0$; it follows that $Q = E_N(\lambda\lambda^*)$, and that λ is necessarily a right basis for M/N. □

DEFINITION 4

If λ is a right basis for M/N, define $\eta: M \to \mathrm{Mat}_{\cdot \times I}(N)$ by $\eta(x) = E_N(x\lambda^*)$; when it is necessary to indicate the dependence of η on the choice of λ, we shall write $\eta = \eta_\lambda$.

PROPOSITION 5

Let λ be as in Definition 2 and let $Q = E_N(\lambda\lambda^*)$.

(i) *If $x \in M$, then $x = \eta(x)\lambda$ and $\eta(x) = \eta(x)Q$.*
(ii) *If $x \in M$ and if $x = \xi\lambda$ where $\xi \in \mathrm{Mat}_{\cdot \times I}(N)$ satisfies $\xi = \xi Q$, then, $\xi = \eta(x)$.*

Proof. Start by noticing that if v is the partial isometry as in the proof of Lemma 1, then

$$\begin{aligned} e_0\lambda_i = v_{ii_0} = (Qe_0v)_{ii_0} = \sum_j q_{ij}e_0\lambda_j \\ = e_0(Q\lambda)_i \text{ and hence } \lambda = Q\lambda. \end{aligned} \tag{1}$$

(i) $e_0x = e_0x\cdot\lambda^*e_0\lambda = e_0E_N(x\lambda^*)\lambda$ and so (P3) ensures that $x = \eta(x)\lambda$; also,

$$\eta(x)Q = E_N(x\lambda^*)Q = E_N(x\lambda^*Q) = E_N(x\lambda^*) \text{ by (1).}$$

(ii) If $x = \xi\lambda$ where $\xi = \xi Q$, then

$$E_N(x\lambda^*) = E_N(\xi\lambda\lambda^*) = \xi E_N(\lambda\lambda^*) = \xi Q = \xi.$$ □

PROPOSITION 6. (*'Uniqueness' of right bases*)

If $\lambda \in \mathrm{Mat}_{I\times\cdot}(M)$ and $\tilde\lambda \in \mathrm{Mat}_{\tilde I\times\cdot}(M)$ are right bases for M/N, then there exists a partial isometry $u \in \mathrm{Mat}_{I\times\tilde I}(N)$ such that
(i) *$u^*u = Q_{\tilde\lambda}$, $uu^* = Q_\lambda$; (ii) $\lambda = u\tilde\lambda$, $\tilde\lambda = u^*\lambda$.*

Proof. Put $u = E_n(\lambda\tilde\lambda^*)$ and apply the previous proposition to both λ and $\tilde\lambda$. Thus, for instance, since $\tilde\lambda$ is a right basis for M/N, it follows that

$$\lambda = E_N(\lambda\tilde\lambda^*)\tilde\lambda = u\tilde\lambda.$$

The other assertions are proved as painlessly. □

For the rest of this section, assume that $(M, N) \in \mathscr{C}$ and that $\lambda \in \mathrm{Mat}_{I\times\cdot}(M)$ is a right basis for M/N.

PROPOSITION 7

(i) *$\tau^{-1/2}e_0\lambda \in \mathrm{Mat}_{I\times\cdot}(M_1)$ is a right basis for M_1/M;*

(ii) *If $\lambda^{(2)} \in \mathrm{Mat}_{I^2\times\cdot}(M_1)$ is defined by $\lambda_i^{(2)} = \tau^{-1/2}\lambda_{i_1}e_0\lambda_{i_2}$ (for $i = (i_1, i_2) \in I^2$), then $\lambda^{(2)}$ is a right basis for M_1/N.*

(iii) *More generally than (ii), if for $n = 1, 2, \ldots, \lambda^{(n)} \in \mathrm{Mat}_{I^n\times\cdot}(M_{n-1})$ is defined by*

$$\lambda_i^{(n)} = \tau^{-n(n-1)/4}\lambda_{i_1}e_0\lambda_{i_2}e_1e_0\lambda_{i_3}\cdots\lambda_{i_{n-1}}e_{n-2}e_{n-3}\cdots e_0\lambda_{i_n},$$

then $\lambda^{(n)}$ is a right basis for M_{n-1}/N.

Proof. (i) $(\tau^{-1/2}e_0\lambda)^*e_1(\tau^{-1/2}e_0\lambda)=\tau^{-1}\lambda^*e_0e_1e_0\lambda=\lambda^*e_0\lambda=1$. (cf. (P4)(ii))

(iii) We have to prove that $\lambda^{(n)*}e_{(n-1)}\lambda^{(n)}=1$ for all n, where $e_{(n)}$ is as in (P5). When $n=1$, $\lambda^{(1)}=\lambda$ and $e_{(0)}=e_0$, and $\lambda^*e_0\lambda=1$ by definition of λ.

Assume now that $\lambda^{(n)*}e_{(n-1)}\lambda^{(n)}=1$, so we need to show that $\lambda^{(n+1)*}e_{(n)}\lambda^{(n+1)}=1$. To achieve the inductive step, write $\mathbf{i}_-=(i_1,\dots,i_n)\in I^n$ whenever $\mathbf{i}=(i_1,\dots,i_{n+1})\in I^{n+1}$, and observe that the following relations hold:

$$\lambda_{\mathbf{i}}^{(n+1)}=\tau^{-n/2}\lambda_{\mathbf{i}_-}^{(n)}e_{n-1}\cdots e_0\lambda_{i_{n+1}} \tag{2}$$

and

$$e_{(n)}=\tau^{-n}e_ne_{n+1}\cdots e_{2n}e_{(n-1)}e_{2n-1}e_{2n-2}\cdots e_n; \tag{3}$$

note that $\lambda_{\mathbf{i}}^{(n)}\in M_{n-1}$ and consequently $\lambda_{\mathbf{i}}^{(n)}$ commutes with e_m for $m\geqslant n$; finally compute thus:

$$\begin{aligned}
&\lambda^{(n+1)*}e_{(n)}\lambda^{(n+1)}\\
&=\sum_{\mathbf{i}\in I^{n+1}}\tau^{-2n}\lambda_{i_{n+1}}^*e_0\cdots e_{n-1}\lambda_{\mathbf{i}_-}^{(n)*}e_n\cdots e_{2n}e_{(n-1)}e_{2n-1}\cdots e_n\lambda_{\mathbf{i}_-}^{(n)}e_{n-1}\cdots e_0\lambda_{i_{n+1}}\\
&=\sum_{i_{n+1}}\tau^{-2n}\lambda_{i_{n+1}}^*e_0\cdots e_{n-1}e_n\cdots e_{2n}\\
&\quad\times\left(\sum_{\mathbf{i}_-}\lambda_{\mathbf{i}_-}^{(n)*}e_{(n-1)}\lambda_{\mathbf{i}_-}^{(n)}\right)e_{2n-1}\cdots e_ne_{n-1}\cdots e_0\lambda_{i_{n+1}}\\
&=\sum_{i_{n+1}}\tau^{-2n}\lambda_{i_{n+1}}^*e_0\cdots e_{2n}e_{2n-1}\cdots e_0\lambda_{i_{n+1}}\\
&=\sum_{i_{n+1}}\lambda_{i_{n+1}}^*e_0\lambda_{i_{n+1}}=1,\quad\text{as desired.}
\end{aligned}$$

□

Remark 8. It might be worthwhile to point out here that once one has noticed the validity of Prop. 7(i), the formulae for $\lambda^{(n)}$ and $e_{(n)}$ are arrived at naturally. Consider the case $n=3$; first, $\tau^{-1/2}e_0\lambda$ is a right basis for M_1/M, so $\tau^{-1}e_1e_0\lambda$ is a right basis for M_2/M_1; so, if $x\in M_2$, we may write

$$\begin{aligned}
x&=\sum_{i_1}x_{i_1}(\tau^{-1}e_1e_0\lambda_{i_1})\quad(x_{i_1}\in M_1\forall i_1)\\
&=\sum_{i_1,i_2}x_{i_2i_1}(\tau^{-1/2}e_0\lambda_{i_2})(\tau^{-1}e_1e_0\lambda_{i_1})\quad(x_{i_2i_1}\in M)\\
&=\sum_{i_1,i_2,i_3}x_{i_3i_2i_1}(\lambda_{i_3})(\tau^{-1/2}e_0\lambda_{i_2})(\tau^{-1}e_1e_0\lambda_{i_1})=\sum_{\mathbf{i}\in I^3}x_{\mathbf{i}}\lambda_{\mathbf{i}}^{(3)}.
\end{aligned}$$

A second point to be made here is that even if λ is an orthogonal basis in the sense of [5], then $\lambda^{(2)}$ may fail to be an orthogonal basis; this was the reason for relaxing orthogonality and introducing Definition 2.

Finally, to see why the formula for e_n is also to be expected, note first that if $\mathbf{i}\vee\mathbf{j}=(i_1,\dots,i_n,j_1,\dots,j_m)$ where $\mathbf{i}\in I^n$, $\mathbf{j}\in I^m$, then we have the following generalization of equation (2):

$$\lambda_{\mathbf{i}\vee\mathbf{j}}^{(n+m)}=\tau^{-mn/2}\lambda_{\mathbf{i}}^{(n)}(e_{n-1}\cdots e_0)(e_n\cdots e_1)\cdots(e_{m+n-2}\cdots e_{n-1})\lambda_{\mathbf{j}}^{(m)};$$

in particular, if $\mathbf{i},\mathbf{j}\in I^n$, then

$$\lambda_{\mathbf{i}\vee\mathbf{j}}^{(2n)}=\tau^{-n^2/2}\lambda_{\mathbf{i}}^{(n)}(e_{n-1}\cdots e_0)(e_n\cdots e_1)\cdots(e_{2n-2}\cdots e_{n-1})\lambda_{\mathbf{j}}^{(n)};$$

the similarity between this expression and the formula $\lambda^{(2)}_{i_1,i_2} = \tau^{-1/2}\lambda_{i_1} e_0 \lambda_{i_2}$ should suggest to the reader that the projection in M_{2n-1} that implements the conditional expectation of M_{n-1} onto N would be of the form

$$\tau^{(\text{some power})}(e_{n-1}\cdots e_0)(e_n\cdots e_1)\cdots(e_{2n-2}\cdots e_{n-1}). \qquad \square$$

PROPOSITION 9

Define $\psi_n \in \mathrm{Mat}_{I^n \times I^{n+1}}(N)$ *by* $\psi_n = E_N(\lambda^{(n)}\lambda^{(n+1)*})$ *for* $n \geqslant 0$ *(with the convention that* $I^0 = \cdot$ *and* $\lambda^{(0)} = [1]$*); also define* $\eta_n : M_{n-1} \to \mathrm{Mat}._{\times I^n}(N)$ *by* $\eta_n(x) = E_N(x\lambda^{(n)*})$. *Then,*

(i) $\eta_{n+1}(x) = \eta_n(x)\psi_n$, *for all* $x \in M_{n-1}$;

(ii) $\eta_n(E_{M_{n-1}} x) = \eta_m(x) E_N(\lambda^{(m)}\lambda^{(n)*}) = E_N(x\lambda^{(n)*})$, *whenever* $x \in M_{m-1}$, *where* $n \leqslant m$;

(iii) *if* $n < m < p$, *them*

$$E_N(\lambda^{(p)}\lambda^{(n)*}) = E_N(\lambda^{(p)}\lambda^{(m)*}) E_N(\lambda^{(m)}\lambda^{(n)*}),$$

and consequently,

$$E_N(\lambda^{(p)}\lambda^{(n)*}) = \psi^*_{p-1}\psi^*_{p-2}\cdots\psi^*_n.$$

Proof. Observe to start with that

$$x = \eta_n(x)\lambda^{(n)} \quad \text{for all } x \in M_{n-1}$$

(cf. Prop. 5(i) and Prop. 7(iii)).

(i)
$$\eta_{n+1}(x) = E_N(x\lambda^{(n+1)*}) = E_N(\eta_n(x)\lambda^{(n)}\lambda^{(n+1)*})$$
$$= \eta_n(x)E_N(\lambda^{(n)}\lambda^{(n+1)*}) = \eta_n(x)\psi_n.$$

(ii)
$$\eta_n(E_{M_{n-1}}(x)) = E_N(E_{M_{n-1}}(x)\lambda^{(n)*}) = E_N(E_{M_{n-1}}(x\lambda^{(n)*}))$$
$$= E_N(x\lambda^{(n)*}) = E_N(\eta_m(x)\lambda^{(m)}\lambda^{(n)*})$$
$$= \eta_m(x)E_N(\lambda^{(m)}\lambda^{(n)*})$$

(iii) Fix $\mathrm{i} \in I^p$; then $\lambda^{(p)}_{\mathrm{i}} \in M_{p-1}$ and

$$E_N(\lambda^{(p)}_{\mathrm{i}}\lambda^{(n)*}) = \eta_n(E_{M_{n-1}}\lambda^{(p)}_{\mathrm{i}}) \text{ by (ii)}$$
$$= \eta_n(E_{M_{n-1}}(E_{M_{m-1}}(\lambda^{(p)}_{\mathrm{i}})))$$
$$= \eta_m(E_{M_{m-1}}(\lambda^{(p)}_{\mathrm{i}}))E_N(\lambda^{(m)}\lambda^{(n)*}) \text{ by (ii)}$$
$$= E_N(\lambda^{(p)}_{\mathrm{i}}\lambda^{(m)*})E_N(\lambda^{(m)}\lambda^{(n)*})$$

and hence

$$E_N(\lambda^{(p)}\lambda^{(n)*}) = E_N(\lambda^{(p)}\lambda^{(m)*})E_N(\lambda^{(m)}\lambda^{(n)*});$$

it follows that

$$E_N(\lambda^{(p)}\lambda^{(n)*}) = E_N(\lambda^{(p)}\lambda^{(p-1)*})E_N(\lambda^{(p-1)}\lambda^{(p-2)*})\cdots E_N(\lambda^{(n+1)}\lambda^{(n)*})$$
$$= \psi^*_{p-1}\psi^*_{p-2}\cdots\psi^*_n.$$

DEFINITION 10

If $\lambda \in \mathrm{Mat}_{I\times\cdot}(M)$ is a right basis for M/N, we shall let $\theta(=\theta_\lambda)$ denote the map $\theta: N \to \mathrm{Mat}_{I\times I}(N)$ defined by $\theta(x) = E_N(\lambda x \lambda^*)$. Further, we shall write $\theta^{(n)}$ for $\theta_{\lambda^{(n)}}$ so that $\theta^{(n)}: N \to \mathrm{Mat}_{I^n \times I^n}(N)$.

PROPOSITION 11

Let $\lambda \in \mathrm{Mat}_{I\times\cdot}(N)$ be a right basis for M/N.

(i) *θ is a faithful normal $*$-homomorphism such that $\theta(1) = Q_\lambda$-in particular, if $[M:N]$ is not an integer, then θ is not unital;*
(ii) *$\theta^{(n)}_{\mathbf{ij}}(x) = \theta_{i_1 j_1}(\theta_{i_2 j_2}(\cdots(\theta_{i_n j_n}(x))\cdots))$ for all $x \in N$ and $\mathbf{i}, \mathbf{j} \in I^n$, for all n;*
(iii) *$\psi_n \psi_n^* = \theta^{(n)}(1)$ and $\psi_n = \psi_n \theta^{(n+1)}(1)$;*
(iv) *$\theta^{(n)}(x)\psi_n = \psi_n \theta^{(n+1)}(x)$, $x \in N$, $n \geqslant 0$.*

Proof. (i) The map θ is clearly linear and σ-weakly continuous; also, it is clear that θ preserves adjoints; finally, if $x, y \in N$,

$$\begin{aligned}\theta(x)\theta(y)e_0 &= E_N(\lambda x \lambda^*)E_N(\lambda y \lambda^*)e_0 = e_0 \lambda x \lambda^* e_0 \lambda y \lambda^* e_0\\ &= e_0 \lambda x y \lambda^* e_0 = \theta(xy)e_0\end{aligned}$$

so that θ preserves products.

(ii) If $\mathbf{i}, \mathbf{j} \in I^{n+1}$ and $x \in N$, use equation (2) to deduce that

$$\begin{aligned}\theta^{(n+1)}_{\mathbf{ij}}(x) &= \tau^{-n} E_N(\lambda^{(n)}_{\mathbf{i}_-} e_{n-1} \cdots e_0 \lambda_{i_{n+1}} x \lambda^*_{j_{n+1}} e_0 \cdots e_{n-1} \lambda^{(n)*}_{\mathbf{j}_-})\\ &= \tau^{-n} E_N(\lambda^{(n)}_{\mathbf{i}_-} e_{n-1} \cdots e_1 \theta_{i_{n+1} j_{n+1}}(x) e_0 \cdots e_{n-1} \lambda^{(n)*}_{\mathbf{j}_-})\\ &= \tau^{-n} E_N(\lambda^{(n)}_{\mathbf{i}_-} \theta_{i_{n+1} j_{n+1}}(x) e_{n-1} \cdots e_0 \cdots e_{n-1} \lambda^{(n)*}_{\mathbf{j}_-})\\ &= \tau^{-1} E_N(\lambda^{(n)}_{\mathbf{i}_-} \theta_{i_{n+1} j_{n+1}}(x) e_{n-1} \lambda^{(n)*}_{\mathbf{j}_-})\\ &= E_N(\lambda^{(n)}_{\mathbf{i}_-} \theta_{i_{n+1} j_{n+1}}(x) \lambda^{(n)*}_{\mathbf{j}_-}) \text{ (by (P4)(iii)}\\ &= \theta^{(n)}_{\mathbf{i}_-\mathbf{j}_-}(\theta_{i_{n+1} j_{n+1}}(x)).\end{aligned}$$

(iii)

$$\begin{aligned}\psi_n \psi_n^* &= E_N(\lambda^{(n)} \lambda^{(n+1)*}) E_N(\lambda^{(n+1)} \lambda^{(n)*})\\ &= E_N(E_N(\lambda^{(n)} \lambda^{(n+1)*}) \lambda^{(n+1)} \lambda^{(n)*})\\ &= E_N(\lambda^{(n)} \lambda^{(n)*}) \text{ (since } \lambda^{(n)}_{\mathbf{i}} \in M_{n-1} \subseteq M_n)\\ &= \theta^{(n)}(1)\end{aligned}$$

while the equation $\psi_n = \psi_n \theta^{(n+1)}(1)$ follows from an entirely analogous computation. (Alternatively, it follows from $\psi_n \psi_n^* = \theta^{(n)}(1)$ that ψ_n is a partial isometry and $\psi_n = \psi_n \psi_n^* \psi_n = \theta^{(n)}(1)\psi_n$, and the equation $\psi_n = \psi_n \theta^{(n+1)}(1)$ is a special case of (iv).)

(iv)

$$\begin{aligned}\theta^{(n)}(x)\psi_n &= E_N(\lambda^{(n)} x \lambda^{(n)*}) E_N(\lambda^{(n)} \lambda^{(n+1)*})\\ &= E_N(E_N(\lambda^{(n)} x \lambda^{(n)*}) \lambda^{(n)} \lambda^{(n+1)*}) = E_N(\lambda^{(n)} x \lambda^{(n+1)*});\\ \psi_n \theta^{(n+1)}(x) &= E_N(\lambda^{(n)} \lambda^{(n+1)*}) E_N(\lambda^{(n+1)} x \lambda^{(n+1)*})\\ &= E_N(E_N(\lambda^{(n)} \lambda^{(n+1)*}) \lambda^{(n+1)} x \lambda^{(n+1)*})\\ &= E_N(\lambda^{(n)} x \lambda^{(n+1)*}).\end{aligned}$$

4. The axioms

For the sake of typographical convenience, we shall adopt the classical 'summation convention' regarding covariant and contravariant indices. Hence, a matrix $x \in \mathrm{Mat}_{I\times J}(N)$ will be written $x = ((x^i_j))$, so that if also $y \in \mathrm{Mat}_{J\times K}(N)$, then $xy = ((x^i_j y^j_k))$. (Caution: an index is a 'summing index' only if it appears (at least) once covariantly and (at least) once contravariantly; thus, for instance, $\delta^i_j \xi_j \neq \sum_j \delta^i_j \xi_j$.)

Sometimes, while dealing with expressions involving adjoints, we shall choose to dispense with the summation convention, and shall indicate such a departure by resorting to the familiar symbol $\sum$.

Finally, if λ is a right basis for M/N and if θ, ψ_0, ψ_1 are as before, we shall write:

$$\left.\begin{array}{l} \theta(x) = ((\theta^i_j(x)), \quad \theta^{(2)}(x) = ((\theta^{i_1}_{j_1}(\theta^{i_2}_{j_2}(x)))) \\ \psi_0 = ((\psi_i)), \quad \psi_1 = ((\psi^i_{jk})) \text{ and finally,} \\ \psi^i = \psi^*_i, \quad \psi^{jk}_i = (\psi^i_{jk})^* \text{ and} \\ \xi^i = (\xi_i)^* \text{ whenever } \xi = ((\xi_i)) \in \mathrm{Mat}_{\cdot\times I}(N) \end{array}\right\} \tag{4}$$

This section will be devoted to showing how the pair (M, N) can be recaptured from θ, ψ_0 and ψ_1 by a few 'group-like' axioms.

DEFINITION 1

Let I be a finite set, N a II_1 factor, $\theta: N \to \mathrm{Mat}_{I\times I}(N)$ a faithful normal *-homomorphism, $\psi_0 \in \mathrm{Mat}_{\cdot\times I}(N)$ and $\psi_1 \in \mathrm{Mat}_{I\times I^2}(N)$. Let us write τ for $(\mathrm{Tr}\,\theta(1))^{-1}$.

The system $\mathscr{G} = (I, N, \theta, \psi_0, \psi_1)$ will be called a 'covariant system' if the following axioms are satisfied:

Identity: $\theta^i_k(\psi_l)\psi^{kl}_j = \tau^{1/2}\theta^i_j(1) = \psi_k \psi^{ki}_j \quad \forall i, j \in I$
Associativity: $\psi^i_{pj}\psi^p_{kl} = \psi^i_{pq}\theta^p_k \psi^q_{lj} \; \forall i, j, k, l \in I$
Involution: $\theta^i_p(\psi_q \psi^q_{rk})\psi^{pr}_j = \tau^{1/2}\psi^i_{jk} \quad \forall i, j, k \in I$
Embedding: $\psi_0\psi^*_0 = 1, \quad \psi_1\psi^*_1 = \theta(1)$
Module: $a\psi_0 = \psi_0\theta(a), \quad \theta(a)\psi_1 = \psi_1\theta^{(2)}(a) \quad \forall a \in N$

(where of course $\theta^{(2)}: N \to \mathrm{Mat}_{I^2\times I^2}(N)$ is related to θ as in Prop. 3.11 (ii)).

Finally we shall write $|\mathscr{G}| = \tau^{-1}$ and call $|\mathscr{G}|$ the order of the covariant system. □

DEFINITION 2

A covariant system $\mathscr{G}$ as in Definition 1 is said to be:

(i) outer, if $\xi \in \mathrm{Mat}_{\cdot\times I}(N)$, $\xi = \xi\theta(1)$ and $a\xi = \xi\theta(a)$ for all $a \in N$ imply ξ is a scalar multiple of ψ_0; and

(ii) unimodular, if for every $\xi \in \mathrm{Mat}_{\cdot\times I}(N)$ satisfying $\xi = \xi\theta(1)$, the following equality holds: $\mathrm{Tr}\,\xi\xi^* = \tau^{-1}\,\mathrm{Tr}\,(\psi_0\psi_1)_{ij}\theta^i_k(\xi^j\xi_l)(\psi^*_1\psi^*_0)^{kl}$.

PROPOSITION 3

Let $(M, N) \in \mathscr{C}$ and let $\lambda \in \mathrm{Mat}_{I\times\cdot}(M)$ be a right basis for M/N. Define θ, ψ_0 and ψ_1 by $\theta(a) = E_N(\lambda a \lambda^*)$, $\psi_0 = E_N(\lambda^*)$, $\psi_1 = E_N(\lambda\lambda^{(2)*})$. Then $\mathscr{G}_\lambda = (I, N, \theta, \psi_0, \psi_1)$ is a unimodular covariant system which is outer precisely when $M \cap N' = \mathbf{C}$.

Proof. The conditions we have chosen to call 'Embedding' and 'Module' were verified earlier (cf. Prop 3.11 (iii) and (iv)). Before verifying the other axioms, we pause to notice that

$$\psi_i = E_N(\lambda_i^*),\ \psi_{jk}^i = E_N(\lambda_i(\tau^{-1/2}\lambda_j e_0 \lambda_k)^*) = \tau^{1/2}E_N(\lambda_i\lambda_k^*\lambda_j^*)$$
$$\psi^i = E_N(\lambda_i) \text{ and } \psi_i^{jk} = \tau^{1/2}E_N(\lambda_j\lambda_k\lambda_i^*).$$

Now compute as follows:

$$\begin{aligned}\theta_k^i(\psi_l)\psi_j^{kl} &= \sum_{k,l} E_N(\lambda_i E_N(\lambda_l^*)\lambda_k^*)\tau^{1/2}E_N(\lambda_k\lambda_l\lambda_j^*)\\ &= \tau^{1/2}\sum_{k,l} E_N(E_N(\lambda_i E_N(\lambda_l^*)\lambda_k^*)\lambda_k\lambda_l\lambda_j^*)\\ &= \tau^{1/2}\sum_{l} E_N(\lambda_i E_N(\lambda_l^*)\lambda_l\lambda_j^*)\\ &= \tau^{1/2}E_N\left(\lambda_i\left(\sum_l E_N(1\cdot\lambda_l^*)\lambda_l\right)\lambda_j^*\right)\\ &= \tau^{1/2}E_N(\lambda_i\lambda_j^*) = \tau^{1/2}\theta_j^i(1),\end{aligned}$$

and

$$\begin{aligned}\psi_k\psi_j^{ki} &= \tau^{1/2}\sum_k E_N(\lambda_k^*)E_N(\lambda_k\lambda_i\lambda_j^*) = \tau^{1/2}E_N\left(\sum_k E_N(\lambda_k^*)\lambda_k\lambda_i\lambda_j^*\right)\\ &= \tau^{1/2}E_N(\lambda_i\lambda_j^*) = \tau^{1/2}\theta_j^i(1);\end{aligned}$$

$$\begin{aligned}\psi_{pj}^i\cdot\psi_{kl}^p &= \tau\sum_p E_N(\lambda_i\lambda_j^*\lambda_p^*)E_N(\lambda_p\lambda_l^*\lambda_k^*) = \tau E_N\left(\sum_p E_N(\lambda_i\lambda_j^*\lambda_p^*)\lambda_p\lambda_l^*\lambda_k^*\right)\\ &= \tau E_N(\lambda_i\lambda_j^*\lambda_l^*\lambda_k^*)\end{aligned}$$

while

$$\begin{aligned}\psi_{pq}^i\theta_k^p(\psi_{lj}^q) &= \tau\sum_{p,q} E_N(\lambda_i\lambda_q^*\lambda_p^*)E_N(\lambda_p E_N(\lambda_q\lambda_j^*\lambda_l^*)\lambda_k^*)\\ &= \tau E_N\left(\sum_{p,q} E_N(\lambda_i\lambda_q^*\lambda_p^*)\lambda_p E_N(\lambda_q\lambda_j^*\lambda_l^*)\lambda_k^*\right)\\ &= \tau E_N\left(\sum_q \lambda_i\lambda_q^* E_N(\lambda_q\lambda_j^*\lambda_l^*)\lambda_k^*\right) = \tau E_N(\lambda_i\lambda_j^*\lambda_l^*\lambda_k^*),\end{aligned}$$

where the last step uses the obvious fact that 'λ^* is a left-basis for M/N'; finally,

$$\begin{aligned}\theta_p^i(\psi_q\psi_{rk}^q)\psi_j^{pr} &= \tau\sum_{p,q,r} E_N(\lambda_i E_N(\lambda_q^*)E_N(\lambda_q\lambda_k^*\lambda_r^*)\lambda_p^*)E_N(\lambda_p\lambda_r\lambda_j^*)\\ &= \tau\sum_{p,r} E_N\left(\lambda_i E_N\left(\sum_q E_N(\lambda_q^*)\lambda_q\lambda_k^*\lambda_r^*\right)\lambda_p^*\right)E_N(\lambda_p\lambda_r\lambda_j^*)\\ &= \tau\sum_{p,r} E_N(\lambda_i E_N(\lambda_k^*\lambda_r^*)\lambda_p^*)E_N(\lambda_p\lambda_r\lambda_j^*)\\ &= \tau\sum_r E_N(\lambda_i E_N(\lambda_k^*\lambda_r^*)\lambda_r\lambda_j^*) = \tau E_N(\lambda_i\lambda_k^*\lambda_j^*) = \tau^{1/2}\psi_{jk}^i,\end{aligned}$$

thus establishing that $\mathscr{G}_\lambda$ is a covariant system.

To see that $\mathscr{G}_\lambda$ is unimodular, note first that

$$(\psi_0\psi_1)_{ij} = \psi_p\psi^p_{ij} = \tau^{1/2}\sum_p E_N(\lambda_p^*)E_N(\lambda_p\lambda_j^*\lambda_i^*) = \tau^{1/2}E_N(\lambda_j^*\lambda_i^*);\ \text{hence,}$$

$$\begin{aligned}
&\tau^{-1}\operatorname{Tr}(\psi_0\psi_1)_{ij}\theta^i_k(\xi^j\xi_l)(\psi_1^*\psi_0^*)^{kl}\\
&= \operatorname{Tr}\sum_{i,j,k,l} E_N(\lambda_j^*\lambda_i^*)E_N(\lambda_i\xi_j^*\xi_l\lambda_k^*)E_N(\lambda_k\lambda_l)\\
&= \operatorname{Tr}\sum_{j,k,l} E_N(\lambda_j^*\xi_j^*\xi_l\lambda_k^*)E_N(\lambda_k\lambda_l) = \operatorname{Tr}\sum_{j,l} E_N(\lambda_j^*\xi_j^*\xi_l\lambda_l)\\
&= \operatorname{Tr}\sum_{j,l} E_N(\lambda_l\lambda_j^*\xi_j^*\xi_l) = \operatorname{Tr}\sum_{j,l} E_N(\lambda_l\lambda_j^*)\xi_j^*\xi_l\\
&= \operatorname{Tr}\sum_{j,l}\theta^l_j(1)\xi_j^*\xi_l\\
&= \operatorname{Tr}\xi_l\theta^l_j(1)\xi^j = \operatorname{Tr}\xi_j\xi^j\ (\text{since } \xi = \xi\theta(1))\\
&= \operatorname{Tr}\xi\xi^*,\ \text{as desired.}
\end{aligned}$$

Finally, if we write $\eta(x) = E_N(x\lambda^*)$ for x in M, note then that if $x\in M$ and $a\in N$, then $\eta(ax) = a\eta(x)$ while

$$\eta(xa) = E_N(xa\lambda^*) = E_N(\eta(x)\lambda a\lambda^*) = \eta(x)\theta(a);$$

the last assertion of the proposition follows at once. □

Remark. If α is an outer action of a finite group G on N and if $M = N\times_\alpha G$, the natural right basis $\lambda\in\operatorname{Mat}_{G\times}(M)$ for M/N is such that $s\mapsto\lambda_s$ is a group homomorphism of G into the unitary group of M such that $\lambda_s a\lambda_{s^{-1}} = \alpha_s(a)$ for all $s\in G$ and $a\in M$. The θ, ψ_0 and ψ_1 of the associated $\mathscr{G}_\lambda$ are given by $\theta^s_t(a) = \delta^s_t\alpha_t(a)$, $\psi_s = \delta_{s\varepsilon}$, where ε denotes the identity element of G, and

$$\psi^s_{tu} = \begin{cases} |G|^{-1/2}, & \text{if } s = tu\\ 0, & \text{otherwise.}\end{cases}$$

More generally, if $\alpha: G\to\operatorname{Aut}N_0$ (N_0 a II_1 factor) is an outer action of a countable group G on N_0, and if H is a subgroup of finite index in G, a right basis for $N_0\times G/N_0\times H$ is given by $\lambda = ((\lambda_{s_i}))$ where $\{s_i\}$ is a set of coset representatives, i.e., $G = \amalg_i Hs_i$. The associated covariant system $\mathscr{G}_\lambda$ is seen to be given by:

$$\theta^i_j\left(\sum_{h\in H} x_h\lambda_h\right) = \sum_{\{h: s_ihs_j^{-1}\in H\}} \alpha_{s_i}(x_h)\lambda_{s_ihs_j^{-1}},$$

$$\psi_i = \begin{cases} 0, & \text{if } s_i\notin H\\ \lambda_{s_i}, & \text{if } s_i\in H\end{cases}$$

and

$$\psi^i_{jk} = \begin{cases} 0 & \text{if } s_js_k\notin Hs_i\\ [G:H]^{-1/2}\lambda_{s_is_k^{-1}s_j^{-1}}, & \text{if } s_js_k\in Hs_i.\end{cases}$$ □

Lemma 4. *Let $\mathscr{G} = (I, N, \theta, \psi_0, \psi_1)$ be a covariant system. Then*

(i) $\psi_0 = \psi_0\theta(1)$; (ii) $\psi^{ij}_{\cdot} = \psi^{ij}_{\cdot}\theta(1)$ (i.e. $\psi^{ij}_k = \psi^{ij}_p\theta^p_k(1)$) $\forall i, j\in I$; (iii) $\theta^i_{\cdot}(a) = \theta^i_{\cdot}(a)\theta(1)$, $\forall i\in I$.

Proof. (i) follows at once from the (Module) axiom (put $a = 1$), while (iii) follows from the fact that θ is a homomorphism. As for (ii), the embedding axiom $\psi_1\psi_1^* = \theta(1)$ implies that $\psi_1^* = \psi_1^*\theta(1)$, which is the content of the parenthetical remark in (ii). □

DEFINITION 5

If $\mathscr{G}$ is a covariant system, define $\mathscr{U} = \{\xi \in \mathrm{Mat.}_{\times I}(N) : \xi = \xi\theta(1)\}$. If $\xi, \eta \in \mathscr{U}$, define

$$(\xi \cdot \eta)_i = \tau^{-1/2}\xi_j\theta_k^j(\eta_l)\psi_i^{kl},$$

and

$$(\xi^b)_i = \tau^{-1/2}\psi_j\psi_{kl}^j\theta_i^k(\xi^l).$$ □

It follows from Lemma 4 that if $\xi, \eta \in \mathscr{U}$, then $\xi \cdot \eta = (((\xi \cdot \eta)_i))$ and $\xi^b = (((\xi^b)_i))$ both belong to $\mathscr{U}$. It is easily verified, further, that if $\xi \in \mathscr{U}$ and if $a, b \in N$, then the matrix-product $a\xi\theta(b)$ also belongs to $\mathscr{U}$; thus $\mathscr{U}$ has a natural N-bimodule structure.

Lemma 6. The operations defined above endow $\mathscr{U}$ with the structure of an involutive associative algebra, and ψ_0 is the (multiplicative) identity for $\mathscr{U}$.

Proof. Clearly $\mathscr{U}$ is a vector space; further, if $\xi \in \mathscr{U}$, then for any $i \in I$,

$$\begin{aligned}(\xi \cdot \psi_0)_i &= \tau^{-1/2}\xi_j\theta_k^j(\psi_l)\psi_i^{kl} = \xi_j\theta_i^j(1) \text{ (by 'Identity')}\\ &= \xi_i;\\ (\psi_0 \cdot \xi)_i &= \tau^{-1/2}\psi_j\theta_k^j(\xi_l)\psi_i^{kl} = \tau^{-1/2}\xi_l\psi_k\psi_i^{kl} \text{ (by 'Module')}\\ &= \xi_l\theta_i^l(1) = \xi_i.\end{aligned}$$

Next, let $\xi, \eta, \zeta \in \mathscr{U}$ and $i \in I$; then

$$\begin{aligned}((\xi \cdot \eta) \cdot \zeta)_i &= \tau^{-1/2}(\xi \cdot \eta)_r\theta_s^r(\zeta_t)\psi_i^{st} = \tau^{-1}\xi_j\theta_k^j(\eta_l)\psi_r^{kl}\theta_s^r(\zeta_t)\psi_i^{st}\\ &= \tau^{-1}\xi_j\theta_k^j(\eta_l)\theta_{pq}^{(2)kl}(\zeta_t)\psi_s^{pq}\psi_i^{st} \text{ (by 'Module')}\\ &= \tau^{-1}\xi_j\theta_k^j(\eta_l)\theta_{pq}^{(2)kl}(\zeta_t)\theta_r^p(\psi_s^{qt})\psi_i^{rs} \text{ (by 'Associativity')}\\ &= \tau^{-1}\xi_j\theta_r^j(\eta_l\theta_q^l(\zeta_t)\psi_s^{qt})\psi_i^{rs}\\ &= \tau^{-1/2}\xi_j\theta_r^j((\eta \cdot \zeta)_s)\psi_i^{rs} = (\xi \cdot (\eta \cdot \zeta))_i,\end{aligned}$$

so $\mathscr{U}$ is an associative unital algebra.

Now for the involution; if $\xi \in \mathscr{U}$,

$$\begin{aligned}((\xi^b)^b)_i &= \tau^{-1/2}\psi_r\psi_{st}^r\theta_i^s((\xi^b)^t) = \tau^{-1}\psi_r\psi_{st}^r\theta_i^s(\theta_k^t(\xi_l)\psi_j^{kl}\psi^j)\\ &= \tau^{-1}\psi_r\psi_{st}^r\theta_{pk}^{(2)st}(\xi_l)\theta_i^p(\psi_j^{kl}\psi^j) = \tau^{-1}\xi_l\psi_r\psi_{pk}^r\theta_i^p(\psi_j^{kl}\psi^j) \text{ (by 'Module')}\\ &= \tau^{-1/2}\xi_l\psi_r\psi_i^{rl} \text{ (by 'Involution')}\\ &= \xi_l\theta_i^l(1) \text{ (by 'Identity')}\\ &= \xi_i \text{ (since } \xi \in \mathscr{U});\end{aligned}$$

and if $\xi, \eta \in \mathscr{U}$, then

$$
\begin{aligned}
(\eta^b \cdot \xi^b)_i &= \tau^{-1/2} (\eta^b)_j \theta^j_k(((\xi^b)_l)\psi^{kl}_i \\
&= \tau^{-3/2} \psi_m \psi^m_{np} \theta^n_j(\eta^p) \theta^j_k(\psi_r \psi^r_{st} \theta^s_l(\xi^t))\psi^{kl}_i \\
&= \tau^{-3/2} \psi_m \psi^m_{np} \theta^n_j(\eta^p) \theta^j_q(\psi_r \psi^r_{st}) \theta^{(2)qs}_{kl}(\xi^t) \psi^{kl}_i \\
&= \tau^{-3/2} \psi_m \psi^m_{np} \theta^n_j(\eta^p) \theta^j_q(\psi_r \psi^r_{st}) \psi^{qs}_k \theta^k_i(\xi^t) \text{ (by `Module')} \\
&= \tau^{-1} \psi_m \psi^m_{np} \theta^n_j(\eta^p) \psi^j_{kt} \theta^k_i(\xi^t) \text{ (by `Involution')} \\
&= \tau^{-1} \psi_m \psi^m_{np} \psi^n_{rs} \theta^{(2)rs}_{kt}(\eta^p) \theta^k_i(\xi^t) \text{ (by `Module')} \\
&= \tau^{-1} \psi_m \psi^m_{lq} \theta^l_r(\psi^q_{ps}) \theta^{(2)rs}_{kt}(\eta^p) \theta^k_i(\xi^t) \text{ (by `Associativity')} \\
&= \tau^{-1} \psi_m \psi^m_{lq} \theta^l_i(\psi^q_{ps} \theta^s_t(\eta^p) \xi^t) \\
&= \tau^{-1/2} \psi_m \psi^m_{lq} \theta^l_i((\xi \cdot \eta)^q) = ((\xi \cdot \eta)^b)_i,
\end{aligned}
$$

and the proof is complete. □

Lemma 7. Let $\mathscr{K} = Mat._{\times I}(L^2(N))$ be equipped with inner product $(\xi, \eta) = \sum_i (\xi_i, \eta_i)$ and let $\mathscr{H} = \{\xi \in \mathscr{K} : \xi = \xi\theta(1)\}$.

(i) $\mathscr{K}$ is a Hilbert space and $\mathscr{H}$ is a closed subspace of $\mathscr{K}$;
(ii) the orthogonal projection of $\mathscr{K}$ onto $\mathscr{H}$ is given by $\xi \mapsto \xi\theta(1)$;
(iii) the following equations define bounded operators on $\mathscr{H}$:

$$
\begin{aligned}
L_a \xi &= a\xi, \quad a \in N \\
R_{\theta(a)} \xi &= \xi\theta(a), \quad a \in N \\
R_x \xi &= \xi x,
\end{aligned}
$$

where x is any element of $\mathrm{Mat}_{I \times I}(N)$ *satisfying $\theta(1)x\theta(1) = \theta(1)x$.*

Proof. The first two assertions are clear while the last is a consequence of the fact that if $S \in \mathscr{L}(\mathscr{K})$ and $S(\mathscr{H}) \subseteq \mathscr{H}$, then $S|\mathscr{H} \in \mathscr{L}(\mathscr{H})$, and the fact that $L^2(N)$ is an N-bimodule. □

Lemma 8. The equation $E\xi = \xi\psi^*_0$ defines a bounded operator from $\mathscr{H}$ to $L^2(N)$ such that $E(\mathscr{U}) \subseteq N$; further, for any $\xi, \eta \in \mathscr{U}$,

$$(\xi, \eta) = \mathrm{Tr}\, E(\xi \cdot \eta^b).$$

Proof. It is clear that E is continuous and that $E(\mathscr{U}) \subseteq N$; finally if $\xi, \eta \in \mathscr{U}$, then

$$
\begin{aligned}
E(\xi \cdot \eta^b) = (\xi \cdot \eta^b)_i \psi^i &= \tau^{-1} \xi_j \theta^j_k((\psi_0 \psi_1)_{qr} \theta^q_l(\eta^r)) \psi^{kl}_i \psi^i \\
&= \tau^{-1} \xi_j \theta^j_m((\psi_0 \psi_1)_{qr}) \theta^{(2)mq}_{kl}(\eta^r) \psi^{kl}_i \psi^i \\
&= \tau^{-1} \xi_j \theta^j_m((\psi_0 \psi_1)_{qr}) \psi^{mq}_k \psi^k \eta^r \text{ (by `Module')} \\
&= \tau^{-1/2} \xi_j \psi^j_{kr} \psi^k \eta^r \text{ (by `Involution')} \\
&= \xi_j \theta^j_r(1) \eta^r = \xi_j \eta^j;
\end{aligned}
$$

Thus

$$E(\xi\cdot\eta^b)=\xi_j\eta^j \quad \text{for } \xi,\ \eta\in\mathscr{U} \tag{5}$$

so that, in particular, $\mathrm{Tr}\, E(\xi\cdot\eta^b)=(\xi,\eta)$. □

PROPOSITION 9

Let $\mathscr{G}$ be any covariant system and let $\mathscr{U}$ be constructed as above. Then $\mathscr{U}$ has the structure of a right Hilbert algebra whose associated modular operator is bounded. The modular operator is the identity if and only if $\mathscr{G}$ is unimodular.

Proof. It is clear from the definition of the product in $\mathscr{U}$ that $\xi\cdot\eta=R_y\xi$ (in the notation of Lemma 7(iii)) where $y^i_j=\tau^{-1/2}\theta^i_k(\eta_l)\psi^{kl}_j$; hence $\pi_r(\eta)=R_y$. Also, if $\xi,\ \eta,\ \zeta\in\mathscr{U}$, then by Lemma 8,

$$\langle\xi\cdot\eta,\zeta\rangle=\mathrm{Tr}\,E((\xi\cdot\eta)\cdot\zeta^b)=\mathrm{Tr}\,E(\xi\cdot(\zeta\cdot\eta^b)^b)=\langle\xi,\zeta\cdot\eta^b\rangle.$$

We need, now, to estimate $\|\xi^b\|_{\mathscr{H}}$; for this, first deduce from the embedding condition that $\psi_0\psi_1$ is a partial isometry and hence $\|\tilde{\xi}\psi_1^*\psi_0^*\|_{L^2(N)}\leqslant\|\tilde{\xi}\|_{\mathrm{Mat}.\times I^2(L^2(N)))}$ whenever $\tilde{\xi}\in\mathrm{Mat}._{\times I^2}(L^2(N))$.

Now, if $\xi\in\mathscr{U}$, then for each $i\in I$, define $\tilde{\xi}^{(i)}\in\mathrm{Mat}._{\times I^2}(L^2(N))$ by $\tilde{\xi}^i_{jk}=\theta^i_j(\xi_k)$ and notice that

$$\tilde{\xi}^{(i)}\psi_1^*\psi_0^*=\theta^i_j(\xi_k)(\psi_1^*\psi_0^*)^{jk}=(\tau^{1/2}\xi^b_i)^*;$$

hence, for any $\xi\in\mathscr{U}$,

$$\begin{aligned}\|\xi^b\|^2_{\mathscr{H}}&=\sum_i\|\xi^b_i\|^2_{L^2(N)}\leqslant\tau^{-1}\sum_i\|\tilde{\xi}^{(i)}\|^2=\tau^{-1}\sum_{i,j,k}\|\theta^i_j(\xi_k)\|^2_{L^2(N)}\\&=\tau^{-1}\,\mathrm{Tr}\,\theta^i_j(\xi_k)\theta^j_i(\xi^k)=\tau^{-1}\,\mathrm{Tr}\,\theta^i_i(\xi_k\xi^k)=\tau^{-1}\,\mathrm{Tr}\,\theta(\xi_k\xi^k)\\&=\tau^{-2}\,\mathrm{Tr}\,\xi_k\xi^k=\tau^{-2}\,\|\xi\|^2_{\mathscr{H}},\end{aligned}$$

where we used the uniqueness of the trace on N to conclude that $\mathrm{Tr}\,\theta(a)=\tau^{-1}\,\mathrm{Tr}\,(a)$ for every $a\in N$.

An easy computation shows that for any $\xi\in\mathscr{U}$,

$$\|\xi^b\|^2=\tau^{-1}\,\mathrm{Tr}\,(\psi_0\psi_1)_{ij}\theta^i_k(\xi^j\xi_l)(\psi_1^*\psi_0^*)^{kl},$$

so that the second assertion of the Proposition is a consequence of the definitions. □

Lemma 10. *The equation $\lambda^{(i)}_j=\theta^i_j(1)$ defines an element $\lambda^{(i)}$ of $\mathscr{U}$ such that $E(\xi\cdot\lambda^{(i)b})=\xi^i$ for all $\xi\in\mathscr{U}$.*

Proof. That $\lambda^{(i)}\in\mathscr{U}$ follows from Lemma 4(iii); as for the second assertion, if $\xi\in\mathscr{U}$, then it follows from equation (5) that

$$E(\xi\cdot\lambda^{(i)b})=\xi_j\lambda^{(i)j}=\xi_j\theta^j_i(1)=\xi_i$$ □

Lemma 11. *Suppose ξ is a left bounded element of $\mathscr{H}$: i.e., suppose there is a constant $K>0$ such that $\|\pi_r(\eta)\xi\|_{\mathscr{H}}\leqslant K\|\eta\|_{\mathscr{H}}$ for all $\eta\in\mathscr{U}$. Then $E\xi\in N$, and $\|E\xi\|_N\leqslant\|\pi_l(\xi)\|$.*

Proof. Since the left bounded elements of $L^2(N)$ are precisely the elements of N, both assertions of the Lemma are consequences of the following identity which we shall establish:

$$\langle \pi_l(\xi)a\psi_0, b\psi_0\rangle_{\mathscr{H}} = \langle (E\xi)a, b\rangle_{L^2(N)}, \text{ for all } a, b\in N.$$

So, let $a, b\in N$, and compute:

$$\begin{aligned}\langle \pi_l(\xi)a\psi_0, b\psi_0\rangle_{\mathscr{H}} &= \langle \pi_r(a\psi_0)\xi, b\psi_0\rangle_{\mathscr{H}} = \sum_i \tau^{-1/2}\langle \xi_j\theta^j_k(a\psi_l)\psi^{kl}_i, b\psi_i\rangle_{L^2(N)}\\ &= \sum_i \tau^{-1/2}\langle \xi_j\theta^j_p(a)\theta^p_k(\psi_l)\psi^{kl}_i, b\psi_i\rangle_{L^2(N)}\\ &= \sum_i \langle \xi_j\theta^j_p(a)\theta^p_i(1), b\psi_i\rangle_{L^2(N)}\\ &= \sum_i \langle \xi_j\theta^j_i(a), b\psi_i\rangle_{L^2(N)} = \sum_j \langle \xi_j, b\psi_i\theta^i_j(a^*)\rangle_{L^2(N)}\\ &= \sum_j \langle \xi_j, ba^*\psi_j\rangle_{L^2(N)} = \langle \xi_j\psi^j, ba^*\rangle_{L^2(N)} = \langle E\xi, ba^*\rangle_{L^2(N)}\\ &= \langle (E\xi)a, b\rangle_{L^2(N)}, \text{ as desired.}\end{aligned}$$

□

Theorem 12. *Let $\mathscr{G}$ be any covariant system, let $\mathscr{U}$ be constructed as above, and let $\tilde{M} = \pi_l(\mathscr{U})$, $\tilde{N} = \{\pi_l(a\psi_0): a\in N\}$. Then $\tilde{M}$ is a finite von Neumann algebra and $\tilde{N}$ is a II_1 subfactor of $\tilde{M}$. Further, $(\tilde{M}, \tilde{N})\in\mathscr{C}_0$ if and only if $\mathscr{G}$ is outer (in the sense of Definition* 2).

Proof. If $a\in N$ and if $\xi\in\mathscr{U}$, then

$$\begin{aligned}(\pi_l(a\psi_0)\xi)_i &= \tau^{-1/2}a\psi_j\theta^j_k(\xi_l)\psi^{kl}_i\\ &= \tau^{-1/2}a\xi_l\psi_k\psi^{kl}_i \text{ (by 'Module')}\\ &= a\xi_l\theta^l_i(1) \text{ (by 'Identity')}\\ &= a\xi_i\end{aligned}$$

and hence $\pi_l(a\psi_0) = L_a$ (cf. Lemma 7(iii)). It is clear that $a\mapsto L_a$ is a normal $*$-homomorphism of N onto $\tilde{N}$; since $L_1 = 1 \neq 0$, and since N is a factor, it follows that $a\mapsto L_a$ is a $*$-isomorphism of N onto $\tilde{N}$ and that $\tilde{N}$ is a II_1 factor.

Also, it follows directly from Proposition 9 and the general theory of Hilbert algebras that $\pi_l(\mathscr{U})''$ is a finite von Neumann algebra. We shall show now that $\pi_l(\mathscr{U})'' = \pi_l(\mathscr{U})$. So, let $x\in\pi_l(\mathscr{U})''$; since $\pi_r(\mathscr{U})\subseteq(\pi_l(\mathscr{U})'')'$, it follows that for any $\eta\in\mathscr{U}$, we have

$$x\eta = x\pi_r(\eta)\psi_0 = \pi_r(\eta)x\psi_0;$$

hence $\xi = x\psi_0$ is a left bounded element of $\mathscr{H}$ and $x = \pi_l(\xi)$. With $\lambda^{(i)}$ is in Lemma 10, note that $\lambda^{(i)b}\in\mathscr{U}$ so that $\lambda^{(i)b}$ is left-bounded and consequently $\pi_l(\xi)\lambda^{(i)b}$ is also left-bounded. The same computation as in Lemma 10 shows then that $\xi_i = E(\pi_l(\xi)\lambda^{(i)b})$; an appeal to Lemma 11 shows now that $\xi_i\in N$. Since i was arbitrary, deduce that $\xi\in\mathscr{U}$ and conclude, finally, that $x\in\pi_l(\mathscr{U})$.

Finally, if $x = \pi_l(\xi)$, $\xi\in\mathscr{U}$, an easy computation shows that $x\in\tilde{M}\cap\tilde{N}'$ if and only if $a\xi = \xi\theta(a)$ for all $a \in N$, and hence $\tilde{M} \cap \tilde{N}' = \mathbf{C}1$ if and only if the covariant system $\mathscr{G}$ is outer. If $\mathscr{G}$ is outer, then $\tilde{M} \cap \tilde{N}' = \mathbf{C}1$ and in particular, $\tilde{M}$ is a factor. Since $\tilde{M}$ is finite and since $\tilde{M}$ contains a II_1-subfactor, it must be the case that $\tilde{M}$ is a II_1 factor. Also, notice that $\pi_l(\xi) = \sum_i \pi_l(\xi_i \psi_0)\pi_l(\lambda^{(i)})$, with $\lambda^{(i)}$ as in Lemma 10; hence $\tilde{M}$ is finitely generated as a left $\tilde{N}$-module and consequently $[\tilde{M}:\tilde{N}] < \infty$, and the theorem is proved. □

We now proceed to show that every outer covariant system is automatically unimodular. We continue to let $\mathscr{G}$ denote an arbitrary covariant system, and to let $\mathscr{U}$, $\tilde{M}$, $\tilde{N}$ be as in Theorem 12.

Lemma 13. *The equation*

$$\varphi(\pi_l(\xi)) = \langle \xi, \psi_0 \rangle, \; \xi \in \mathscr{U}$$

defines a faithful normal state φ on $\tilde{M}$ such that $\varphi(ax) = \varphi(xa)$ for every $a \in \tilde{N}$ and $x \in \tilde{M}$; in particular, $\tilde{N} \subseteq \tilde{M}^{\varphi}$.

Proof. Since ψ_0 is the identity for $\mathscr{U}$, we see that $\varphi(\pi_l(\xi)) = \langle \pi_l(\xi)\psi_0, \psi_0 \rangle$ and consequently φ is normal and positive. Also $\varphi(\pi_l(\psi_0)) = \langle \psi_0, \psi_0 \rangle = 1$ (since $\psi_i \psi^i = 1$), so that φ is a state; finally, φ is faithful since

$$\varphi(\pi_l(\xi)^* \pi_l(\xi)) = \langle \xi, \xi \rangle = \| \xi \|^2.$$

Recall that a typical element of $\tilde{M}$ (resp., $\tilde{N}$) is of the form $\pi_l(\xi)$ (resp., $\pi_l(a\psi_0) = L_a$) where $\xi \in \mathscr{U}$ (resp., $a \in N$); also, notice that for ξ in $\mathscr{U}$ and a in N,

$$\begin{aligned}(\xi \cdot a\psi_0)_i &= \tau^{-1/2} \xi_j \theta^j_k(a\psi_l)\psi^{kl}_i = \tau^{-1/2} \xi_j \theta^j_p(a)\theta^p_k(\psi_l)\psi^{kl}_i \\ &= \xi_j \theta^j_p(a)\theta^p_i(1) \text{ (by `Identity')} \\ &= \xi_j \theta^j_i(a);\end{aligned}$$

i.e.,

$$\xi \cdot a\psi_0 = \xi\theta(a).$$

Hence,

$$\begin{aligned}\varphi(\pi_l(a\psi_0)\pi_l(\xi)) &= \langle a\psi_0 \cdot \xi, \psi_0 \rangle = \langle L_a \xi, \psi_0 \rangle \\ &= \langle \xi, L_{a^*}\psi_0 \rangle = \langle \xi, a^*\psi_0 \rangle = \langle \xi, \psi_0 \theta(a^*) \rangle \\ &= \langle \xi\theta(a), \psi_0 \rangle = \langle \xi \cdot a\psi_0, \psi_0 \rangle = \varphi(\pi_l(\xi)\pi_l(a\psi_0)).\end{aligned}$$

The final assertion is a consequence of the well-known characterization of $\tilde{M}^{\varphi}$ from [4]. □

PROPOSITION 14

An outer covariant system is automatically unimodular.

Proof. It is easy to see that $\mathscr{U}$ is unimodular if and only if the state φ (of Lemma 13) is a trace. To see that φ is a trace, appeal first to the Radon–Nikodym theorem of [4] to deduce the existence of a (possibly unbounded) positive self-adjoint operator h affiliated to $\tilde{M}$ such that $\varphi = \mathrm{Tr}(h\cdot)$. (Recall from Theorem 12 that the 'outer-ness' of $\mathscr{G}$ ensures that $\tilde{M}$ is a II_1 factor.)

The proposition will be proved once we establish that $h\eta\tilde{M}\cap\tilde{N}'$; for this, appeal once again to [4] to observe that the modular group σ^{φ} is given by $\sigma_t^{\varphi}(x) = h^{it}\sigma_t^{tr}(x)h^{-it} = h^{it}xh^{-it}$; infer now from Lemma 13 that $h^{it}\in\tilde{M}\cap\tilde{N}'$ for all t, whence $h\eta\tilde{M}\cap\tilde{N}'$. □

In order to complete the discussion of the relationship between outer covariant system and objects of $\mathscr{C}_0$, we need the following definitions.

DEFINITION 15

(a) Two objects (M,N) and $(\tilde{M},\tilde{N})$ *of* $\mathscr{C}_0$ are said to be isomorphic if there is a $*$-isomorphism π of M onto $\tilde{M}$ such that $\pi(N)=\tilde{N}$; in such a case, we shall write $\pi:(M,N)\xrightarrow{\cong}(\tilde{M},\tilde{N})$; (b) Two covariant systems $\mathscr{G}=(I,N,\theta,\psi_0,\psi_1)$ and $\tilde{\mathscr{G}}=(\tilde{I},\tilde{N},\tilde{\theta},\tilde{\psi}_0,\tilde{\psi}_1)$ are said to be isomorphic if there exist a $*$-isomorphism π of N onto $\tilde{N}$ and a matrix $u\in\mathrm{Mat}_{I\times\tilde{I}}(\tilde{N})$ such that

(i) $u^*u=\tilde{\theta}(1)$ and $uu^*=\pi(\theta(1))$;

(ii) $\pi(\theta(a))=u\tilde{\theta}(\pi(a))u^*$;

(iii) $\pi(\psi_0)=\tilde{\psi}_0u^*$; and

(iv) $\pi(\psi^i_{jk})=\sum_{p,q,r,s\in\tilde{I}}u_{ip}\tilde{\psi}^p_{qs}\tilde{\theta}^s_r(u^*_{kq})u^*_{jr}$,

where, in (ii) and (iii) (and in the sequel), we have used the notation $\pi(x)=((\pi(x_{ij})))$ for any matrix $x=((x_{ij}))$ with entries in N. □

DEFINITION 16

If $\mathscr{G}$ is a covariant system, and if $\tilde{M}$ and $\tilde{N}$ are as in Theorem 12, we shall call $\tilde{M}$ the 'crossed-product of N by $\mathscr{G}$' and write $\tilde{M}=N\times\mathscr{G}$; also, we shall identify N with $\tilde{N}$ with $\tilde{N}$ $(a\leftrightarrow\pi_l)a\psi_0))$ and think of N as a subfactor of $N\times\mathscr{G}$. □

PROPOSITION 17

(a) *Suppose* $(M,N),(\tilde{M},\tilde{N})\in\mathscr{C}_0$ *and suppose* $(M,N)\cong(\tilde{M},\tilde{N})$. *Let* $\lambda\in\mathrm{Mat}_{I\times\cdot}(M)$ (*resp.*, $\tilde{\lambda}\in\mathrm{Mat}_{\tilde{I}\times\cdot}(\tilde{M})$) *be a right basis for* M/N (*resp.*, $\tilde{M}/\tilde{N}$), *and let* $\mathscr{G}_\lambda$ (*resp.*, $\mathscr{G}_{\tilde{\lambda}}$) *be the covariant system constructed as in Proposition* 3. *Then* $\mathscr{G}_\lambda\cong\mathscr{G}_{\tilde{\lambda}}$.
(b) *Let* $\mathscr{G}$ *and* $\tilde{\mathscr{G}}$ *be isomorphic covariant systems, with the isomorphism being implemented by* (π,u) *as in Definition* 15(b); *then* π *extends to an isomorphism of* $N\times\mathscr{G}$ *onto* $\tilde{N}\times\tilde{\mathscr{G}}$.

Proof. (a) Suppose $\pi:(M,N)\cong(\tilde{M},\tilde{N})$. Clearly, then, $\pi(E_N(x))=E_{\tilde{N}}(\pi(x))$ for x in M; it follows (from (ii) of Lemma 1.1) that $\pi(\lambda)$ is a right basis for $\tilde{M}/\tilde{N}$. Appeal to Proposition 1.6 to find a partial isometry $u\in\mathrm{Mat}_{I\times\tilde{I}}(\tilde{N})$ such that $u^*u=E_{\tilde{N}}(\tilde{\lambda}\tilde{\lambda}^*)$, and $uu^*=E_{\tilde{N}}(\pi(\lambda\lambda^*))$. A routine computation now shows that the pair (π,u) implements an isomorphism $\mathscr{G}_\lambda\cong\mathscr{G}_{\tilde{\lambda}}$ as in Definition 15(b). (b) Conversely, if (π,u) implements an isomorphism $\mathscr{G}\cong\tilde{\mathscr{G}}$ as in Definition 15(b), it is relatively painless to verify that the map $\xi\mapsto\pi(\xi)u$ defines an isomorphism $\mathscr{U}\cong\tilde{\mathscr{U}}$ of right Hilbert algebras, which in turn yields an isomorphism $(N\times\mathscr{G},N)\cong(\tilde{N}\times\tilde{\mathscr{G}},\tilde{N})$. □

Theorem 18. (a) *If* $(M,N)\in\mathscr{C}_0$, *if* λ *is a right basis for* M/N *and if* $\mathscr{G}_\lambda$ *is the associated covariant system* (*as in Proposition* 3), *then* $(N\times\mathscr{G}_\lambda,N)\cong(M,N)$. (b) *If* $\mathscr{G}$ *is an outer* *covariant system, if λ is any right basis for $N \times \mathscr{G}/N$ and if $\mathscr{G}_\lambda$ is the associated covariant system, then $\mathscr{G} \cong \mathscr{G}_\lambda$.*

Proof. (a) An isomorphism $\pi:(N \times \mathscr{G}_\lambda, N) \xrightarrow{\cong} (M, N)$ is given by $\pi(\pi_l(\xi)) = \sum \xi_i \lambda_i$. (b) Begin by appealing to Proposition 14 and Lemma 13 to note that *the* trace on $N \times \mathscr{G}$ is given by φ. Note next, that by definition, $\varphi(\pi_l(\xi)) = \operatorname{Tr} E(\xi)$. Via the identification $N \leftrightarrow \tilde{N}$, we deduce that the restriction of E to $N \times \mathscr{G}$ is *the* trace-preserving conditional expectation of $N \times \mathscr{G}$ onto N.

Next, note that by Prop. 17(a), it suffices to prove the assertion for any one right basis for $N \times \mathscr{G}/N$. For this purpose, define $\lambda \in \operatorname{Mat}_{I \times \cdot}(N \times \mathscr{G})$ by $\lambda_i = \pi_l(\lambda^{(i)})$, with $\lambda^{(i)}$ as in Lemma 10. We need, first, to verify that $\lambda^* e_0 \lambda = 1$ where e_0 is the orthogonal projection of $\mathscr{H}$ onto $[a\psi_0 : a \in N]$.

If $\xi \in \mathscr{H}$ and $a \in N$, note that

$$\begin{aligned}\langle \xi, a\psi_0 \rangle_{\mathscr{H}} &= \sum_i \langle \xi_i, a\psi_i \rangle_{L^2(N)} = \langle \xi_i \psi^i, a \rangle_{L^2(N)} \\ &= \langle \xi \psi_0^*, a \rangle_{L^2(N)} = \langle \xi \psi_0^* \psi_0, a\psi_0 \rangle_{\mathscr{H}}\end{aligned}$$

(since $\langle \xi_0 \psi_0, \eta_0 \psi_0 \rangle_{\mathscr{H}} = \langle \xi_0, \eta_0 \rangle_{L^2(N)}$ for $\xi_0, \eta_0 \in L^2(N)$), and hence $e_0 \xi = \xi \psi_0^* \psi_0$; thus, $(e_0\xi)_i = \xi_r \psi^r \psi_i$, $\xi \in \mathscr{H}$, $i \in I$.

Now fix $\xi \in \mathscr{U}$ and compute:

$$\begin{aligned}(e_0 \lambda_i \xi)_l &= (e_0(\lambda^{(i)} \cdot \xi))_l = (\lambda^{(i)} \cdot \xi)_r \psi^r \psi_l = \tau^{-1/2} \lambda_p^{(i)} \theta_q^p(\xi_s) \psi_r^{qs} \psi^r \psi_l \\ &= \tau^{-1/2} \theta_p^i(1) \theta_q^p(\xi_s) \psi_r^{qs} \psi^r \psi_l = \tau^{-1/2} \theta_q^i(\xi_s) \psi_r^{qs} \psi^r \psi_l;\end{aligned}$$

hence,

$$\begin{aligned}\theta_k^j(e_0 \lambda_i \xi)_l \psi_t^{kl} &= \tau^{-1/2} \theta_{mq}^{(2)ji}(\xi_s) \theta_n^m(\psi_r^{qs} \psi^r) \theta_k^n(\psi_l) \psi_t^{kl} \\ &= \theta_{mq}^{(2)ji}(\xi_s) \theta_n^m(\psi_r^{qs} \psi^r) \theta_t^n(1) \text{ (by 'Identity')} \\ &= \theta_{mq}^{(2)ji}(\xi_s) \theta_t^m(\psi_r^{qs} \psi^r);\end{aligned}$$

also, in view of Proposition 14, note that the Hilbert algebra $\mathscr{U}$ is unimodular so that $\lambda_i^* = \pi_l(\lambda^{(i)})^* = \pi_l(\lambda^{(i)b})$; hence,

$$\begin{aligned}\left(\sum_i \lambda_i^* e_0 \lambda_i \xi\right)_t &= \sum_i (\lambda^{(i)b} \cdot e_0 \lambda_i \xi)_t \\ &= \sum_i \tau^{-1/2} (\lambda^{(i)b})_j \theta_k^j((e_0 \lambda_i \xi)_l) \psi_t^{kl} \\ &= \sum_i \tau^{-1/2} (\lambda^{(i)b})_j \theta_{mq}^{(2)ji}(\xi_s) \theta_t^m(\psi_r^{qs} \psi^r) \\ &= \sum_i \tau^{-1} (\psi_0 \psi_1)_{kl} \theta_j^k(\lambda^{(i)l}) \theta_{mq}^{(2)ji}(\xi_s) \theta_t^m(\psi_r^{qs} \psi^r) \\ &= \tau^{-1} (\psi_0 \psi_1)_{kl} \theta_{ji}^{(2)kl}(1) \theta_{mq}^{(2)ji}(\xi_s) \theta_t^m(\psi_r^{qs} \psi^r) \\ &= \tau^{-1} (\psi_0 \psi_1)_{kl} \theta_{mq}^{(2)kl}(\xi_s) \theta_t^m(\psi_r^{qs} \psi^r) \\ &= \tau^{-1} \xi_s (\psi_0 \psi_1)_{mq} \theta_t^m(\psi_r^{qs} \psi^r) \\ &= \tau^{-1} \xi_s \psi_j \psi_{mq}^j \theta_t^m(\psi_r^{qs} \psi^r)\end{aligned}$$

$$= \tau^{-1/2}\xi_s\psi_j\psi_i^{js} \text{ (by 'Involution')}$$
$$= \xi_s\theta_t^s(1) \text{ (by 'Identity')}$$
$$= \xi_t;$$

i.e., $\sum_i \lambda_i^* e_0 \lambda_i = 1$.

Hence λ is a right basis for $N \times \mathscr{G}/N$.

Next, if $\pi_i(a\psi_0) \in \tilde{N}$, then

$$E(\lambda^{(i)} \cdot a\psi_0 \cdot \lambda^{(j)b}) = (\lambda^{(i)} \cdot a\psi_0)_j \quad \text{(by Lemma 10)}$$
$$= (\lambda^{(i)}\theta(a))_j = \theta_k^i(1)\theta_j^k(a) = \theta_j^i(a);$$

this shows that θ_λ (the θ of $\mathscr{G}_\lambda$) may be identified with θ. This shows in particular that $[N \times \mathscr{G}:N] = (\operatorname{Tr}\theta_\lambda(1))^{-1} = (\operatorname{Tr}\theta(1))^{-1} = \tau^{-1}$.

To complete the proof, we now verify that also $\psi_{0\lambda} = \psi_0$ and $\psi_{1\lambda} = \psi$, so that, in fact, $\mathscr{G}_\lambda = \mathscr{G}$.

$E(\lambda^{(i)b}) = E(\psi_0 \cdot \lambda^{(i)b}) = \psi_i$ (by Lemma 10) and

$$(\psi_{1\lambda}^*)_k^{ij} = \tau^{1/2}E(\lambda^{(i)} \cdot \lambda^{(j)} \cdot \lambda^{(k)b})$$
$$= \tau^{1/2}(\lambda^{(i)} \cdot \lambda^{(j)})_k \text{ (by Lemma 10)}$$
$$= \lambda_p^{(i)}\theta_q^p(\lambda_r^{(j)})\psi_k^{qr}$$
$$= \theta_p^i(1)\theta_q^p(\theta_r^j(1))\psi_k^{qr}$$
$$= \theta_{qr}^{(2)ij}(1)\psi_k^{qr}$$
$$= \psi_k^{ij} \text{ (by Prop. 3.11 (iii))},$$

and the proof is complete. □

We digress now to make a few observations on the duality theory in $\mathscr{C}$. The 'duality theorem' is the following result (cf. [5]): if $(M, N) \in \mathscr{C}$ and if $N \subseteq M \subseteq M_1 \subseteq M_2 \subseteq \cdots$ is the tower of the basic construction, then

$$(M_2, M_1) \cong (\operatorname{Mat}_{[M:N]}(M), \operatorname{Mat}_{[M:N]}(N)),$$

where by $\operatorname{Mat}_r(N)$ (r a positive real number) we mean $Q(\operatorname{Mat}_{I\times I}(N))Q$, where I is a finite set and Q is a projection in $\operatorname{Mat}_{I\times I}(N)$ satisfying $\operatorname{Tr} Q = r$.

In view of Proposition 3.7(i), it is natural, then, to let $\mathscr{G}_{\hat{\lambda}}$ denote the 'dual covariant system' of $\mathscr{G}_\lambda$, where, if λ is a right basis for M/N, we let $\hat{\lambda} = \tau^{-1/2}e_0\lambda$. This leads naturally to the following definition.

DEFINITION 19

If $\mathscr{G} = (I, N, \theta, \psi_0, \psi_1)$ is a covariant system, the dual of $\mathscr{G}$ is the covariant system $\hat{\mathscr{G}} = (I, N \times \mathscr{G}, \hat{\theta}, \hat{\psi}_0, \hat{\psi}_1)$ defined by

$$\hat{\theta}_j^i(\pi_l(\xi)) = \tau^{-1/2}\theta_m^i(\xi_n)\psi_j^{mn}, \quad \xi \in \mathscr{U}$$
$$\hat{\psi}_i = \tau^{1/2}\lambda_i^*, \text{ where } \lambda_i = \pi_l(\lambda^{(i)}) \text{ (cf. Lemma 10)}$$

and $\hat{\psi}_{jk}^i = \theta_k^i(1)\psi_j$. □

It is an immediate consequence of the duality theorem stated above that $(N \times \mathcal{G}) \times \hat{\mathcal{G}} \cong \mathrm{Mat}_{r-1}(N)$; this statement is more similar, in form, to Takesaki's duality theorem.

In view of the known (and fairly easily established) fact $(R, R^G) \cong (R \times G, R)$ for outer actions of finite groups on the hyperfinite Π_1 factor R, we conclude this minor digression into duality theory with an identification of the covariant system associated with (N, N^G), at least in the case of a specific model of an outer G-action on the hyperfinite II_1 factor.

Example 20. Let G be any finite group, and let $\Omega = G^{Z_+}$ be equipped with the usual Borel structure and Haar measure. Then Ω admits a natural measure preserving G action thus: if $s \in G$, $\alpha = (\alpha_0, \alpha_1, \ldots) \in \Omega$, then $s \cdot \alpha = (s\alpha_0, s\alpha_1, \ldots)$. Let R denote the 'tail equivalence relation': $(\alpha, \beta) \in R$ iff $\alpha_j = \beta_j$ for all but finitely many j. Let N denote the von Neumann algebra of bounded left multiplication operators on $L^2(R)$ (the measure on R coming from Haar measure on Ω and counting measure on equivalence classes). Then N is the hyperfinite II_1 factor; if $x \in N$, we write $x(\alpha, \beta)$ for $(x\xi)(\alpha, \beta)$ where ξ is the indicator function of the diagonal. It is well-known and easily established that the equation $(\zeta_s x)(\alpha, \beta) = x(s^{-1} \cdot \alpha, s^{-1} \cdot \beta)$ defines an outer action of G on N; thus $x \in N^G$ iff $x(\alpha, \beta) = x(s^{-1} \cdot \alpha, s^{-1} \cdot \beta)$ for almost every (α, β), for each s in G.

Let us denote by $\hat{G}$ a set consisting of exactly one member from each equivalence class of irreducible unitary representations of G. If $\pi \in \hat{G}$, write $\mathcal{H}_\pi$ for the Hilbert space on which π represents G, and let $d_\pi = \dim \mathcal{H}_\pi$. Further, fix an orthonormal basis $\{\xi_i : 1 \leqslant i \leqslant d_\pi\}$ for $\mathcal{H}_\pi$. Now, define $I = \{(\pi, i, j) : \pi \in \hat{G},\ 1 \leqslant i, j \leqslant d_\pi\}$, and for each $(\pi, i, j) \in I$, define an element $\lambda_{(\pi,i,j)}$ of N by

$$\lambda_{(\pi,i,j)}(\alpha, \beta) = \delta_{\alpha\beta} \langle \pi(\alpha_0^{-1})\xi_i, \xi_j \rangle \sqrt{d_\pi}.$$

In order to verify that $\lambda = ((\lambda_{(\pi,i,j)})) \in \mathrm{Mat}_{I \times 1}(N)$ is a right basis for N/N^G, note first that the conditional expectation of N onto N^G is implemented by the projection e_0 defined on $L^2(R)$ by $(e_0\eta)(\alpha, \beta) = 1/|G| \sum_{s \in G} \eta(s^{-1} \cdot \alpha, s^{-1} \cdot \beta)$; so, if $(\pi, i, j) \in I$ and $x \in N$, we have

$$(\lambda^*_{(\pi,i,j)} e_0 \lambda_{(\pi,i,j)} x)(\alpha, \beta) = \frac{d_\pi}{|G|} \langle \pi(\alpha_0)\xi_j, \xi_i \rangle \sum_{s \in G} \langle \pi(\alpha_0^{-1}s)\xi_i, \xi_j \rangle x(s^{-1} \cdot \alpha, s^{-1} \cdot \beta);$$

it follows that

$$\begin{aligned}
&\left(\sum_{(\pi,i,j) \in I} \lambda^*_{(\pi,i,j)} e_0 \lambda_{(\pi,i,j)} x \right)(\alpha, \beta) \\
&= \frac{1}{|G|} \sum_{s \in G} x(s^{-1} \cdot \alpha, s^{-1} \cdot \beta) \sum_{(\pi,i,j)} d_\pi \langle \pi(s)\xi_i, \pi(\alpha_0)\xi_j \rangle \langle \pi(\alpha_0)\xi_j, \xi_i \rangle \\
&= \frac{1}{|G|} \sum_{s \in G} x(s^{-1} \cdot \alpha, s^{-1} \cdot \beta) \sum_{\pi,i} d_\pi \langle \pi(s)\xi_i, \xi_i \rangle \\
&= \frac{1}{|G|} \sum_{s \in G} x(s^{-1} \cdot \alpha, s^{-1} \cdot \beta) \sum_{\pi} d_\pi \chi_\pi(s) = x(\alpha, \beta),
\end{aligned}$$

where we have written χ_π for the character of π, used the fact that $\sum d_\pi \chi_\pi$ is the character χ_r associated with the regular representation, and the fact that

$$\chi_r(s) = \begin{cases} |G|, & \text{if } s = \text{identity of } G \\ 0, & \text{otherwise} \end{cases}.$$

Hence $\lambda = ((\lambda_{(\pi,i,j)}))$ is a right basis for N/N^G. A fairly straightforward calculation (which makes use of the orthogonality relations for the matrix entries of the irreducible representations) shows that the constituents of the covariant system $\mathscr{G}_\lambda$ are given as follows:

$$(\theta^{(\pi,i,j)}_{(\pi',i',j')}(x))(\alpha,\beta) = \delta_{\pi\pi'}\delta_{ii'}\langle \pi(\alpha_0^{-1}\beta_0)\xi_{j'},\xi_j\rangle x(\alpha,\beta), \quad x\in N^G,$$

$$\psi_{(\pi,i,j)} = \begin{cases} 1, & \text{if } \pi \text{ is the trivial representation,} \\ 0, & \text{otherwise} \end{cases}$$

and

$$(\psi^{(\pi,i,j)}_{(\pi_1,i_1,j_1),(\pi_2,i_2,j_2)})(\alpha,\beta)$$
$$= \delta_{\alpha\beta}\left(\frac{d_\pi d_{\pi_1} d_{\pi_2}}{|G|}\right)^{1/2} \langle \varphi_{(\pi,\pi(\alpha_0^{-1})\xi_i,\xi_j)}, \varphi_{(\pi_1\otimes\pi_2,(\pi_1\otimes\pi_2)(\alpha_0^{-1})(\xi_{i_1}\otimes\xi_{i_2}),\xi_{j_1}\otimes\xi_{j_2})}\rangle$$

where we have used the notation $\langle \varphi_1,\varphi_2\rangle = 1/|G|\sum_{s\in G}\varphi_1(s)\overline{\varphi_2(s)}$ for functions on G and $\varphi_{(\pi,\xi,\eta)}(s) = \langle \pi(s)\xi,\eta\rangle$. (Note, in particular, that the matrix ψ_1 carries the Clebsch-Gordon data.)

More generally, if H is a subgroup of G (where G is still assumed finite), and if N is as above, a right basis for N^H/N^G may be constructed as follows: let $\hat{G}^H$ be a set consisting of one irreducible (unitary) representation π of G from each unitary equivalence class with the property that π/H contains the trivial representation of H; thus if $\mathscr{H}_\pi$ is the representation space of π, then $V_\pi = \{\xi\in\mathscr{H}_\pi : \pi(H)\xi = \xi\} \neq 0$. For each π in $\hat{G}_H$, fix orthonormal bases $\{\xi_j : 1 \leqslant j \leqslant d_\pi\}$ and $\{\eta_i : 1 \leqslant i \leqslant d^H_\pi\}$ for $\mathscr{H}_\pi$ and V_π respectively. Finally, let

$$I = \{(\pi,i,j) : \pi\in\hat{G}^H,\ 1 \leqslant i \leqslant d^H_\pi,\ 1 \leqslant j \leqslant d_\pi\},$$

and define $\lambda_{(\pi,i,j)}\in N^H$ by

$$\lambda_{(\pi,i,j)}(\alpha,\beta) = \delta_{\alpha\beta}\sqrt{d_\pi}\langle \pi(\alpha_0^{-1})\eta_i,\xi_j\rangle.$$

It may then be verified, using the fact that

$$\sum_{\substack{\pi\in\hat{G}^H \\ 1\leqslant i\leqslant d^H_\pi}} \frac{d_\pi}{[G:H]}\langle \pi(s)\eta_i,\eta_i\rangle = \begin{cases} 0, & \text{if } s\notin H \\ 1, & \text{if } s\in H, \end{cases}$$

that $\lambda = ((\lambda_{(\pi,i,j)}))$ is a right basis for N^H/N^G. The associated covariant system may be easily computed. □

Finally, we would like to make the following remarks:

(1) If $(M,N)\in\mathscr{C}$ and if λ is a right basis for M/N, let $\lambda^{(n)}$, $\theta^{(n)}$ be defined as in §3. Let $\mathscr{U}_n = \{\xi\in\mathrm{Mat}_{I^n\times\cdot}(N) : \xi = \xi\theta^{(n)}(1)\}$. Since $\lambda^{(n)}$ is a right basis for M_{n-1}, it follows that there is a bijection between M_{n-1} and $\mathscr{U}_n$; we think of M_{n-1} as $\pi_l(\mathscr{U}_n)$ where $\mathscr{U}_n$ is the Hilbert algebra defined in the way $\mathscr{U}(=\mathscr{U}_1)$ was defined in this section. Let M_∞ denote the II_1 factor obtained as the inverse limit of the tower $N \subseteq M \subseteq M_1 \subseteq M_2 \subseteq \cdots$. The preceding observations lead to an interpretation of the standard form of M_∞ as 'the space of bounded N-valued martingales defined on the space of paths in Γ', as follows: one defines 'the space of square-integrable martingales'

$$\mathscr{H}_\infty = \{(\xi_n)_{n=1}^{\infty} : \xi_n \in \mathrm{Mat}_{I^n \times .}(L^2(N)),\ \xi_n = \xi_n \theta^{(n)}(1),$$
$$\xi_n = \xi_{n+1}\psi_n^* \forall n \quad \text{and } \sup_n \|\xi_n\|_2 < \infty\}$$

where the ψ_n are defined as in Proposition 9; one next defines

$$\mathscr{U}_\infty = \left\{ (\xi_n)_{n=1}^{\infty} \in \mathscr{H}_\infty : \xi_n \in \mathscr{U}_n \quad \text{for all } n \atop \text{and } \sup_n \|\pi_l(\xi_n)\| < \infty \right\}.$$

Then $\mathscr{H}_\infty$ admits a $\mathscr{U}_\infty$ action as follows: if $\xi \in \mathscr{U}_\infty$, $\eta \in \mathscr{H}_\infty$, we have

$$(\xi \cdot \eta)_n = \lim_{m \to \infty} (\pi_l(\xi_m)\eta_m)\psi_{m-1}^* \psi_{m-2}^* \cdots \psi_n^*;$$

if we denote the map $\eta \mapsto \xi \cdot \eta$ by $\pi_l(\xi)$, it can finally be shown that $M_\infty = \pi_l(\mathscr{U}_\infty)$.
(2) It should be noted that the definition of a covariant system clearly makes sense when N is any finite von Neumann algebra, and that a large proportion of the results continue to hold there. In fact the author can prove the following result: there exists a covariant system $\mathscr{G} = (I, N, \theta, \psi_0, \psi_1)$ with N a finite-dimensional von Neumann algebra if and only if there exists a rectangular matrix A with non-negative integral entries such that τ is the Perron-Frobenius eigenvalue of AA^t. It is the author's belief that it should be póssible to prove the 'if' part of the above assertion, with 'finite-dimensional von Neumann algebra' replaced by 'the hyperfinite II_1 factor', by using the model for AF-algebras discussed in [7] to explicitly write down the outer covariant system which does the job.

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