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An Extension of the Characteristic Sequences Method to the Case of Repeated Roots

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Abstract—**An extension of the characteristic sequences method to the case of repeated roots has been investigated. The aim of the present note is to complete the study and the related design of systems whose mathematical modeling in state-space form or transfer-function form is not available. A comparative study between the characteristic sequences method and the characteristic locus method shows that the two methods are closely related to each other and the former one is a time-domain counterpart of the latter one.**

I. Introduction

The analysis and design of multivariable systems in terms of the time-domain input/output method is called the characteristic sequences method (CSM) [3]. This method works quite well in time domain. It has been found that the characteristic sequences method is closely related to the characteristic locus method (CLM) [3] and in fact the former one is a time-domain counterpart concept of the latter one.

The aim of the present note is to extend the existing concept of the characteristic sequences method to the case of repeated roots. A detailed theoretical derivation has been given to complete the study and the related design of systems in terms of time-domain input/output method, i.e., the characteristic sequences method.

II. Derivation of Algorithms for the Case of Repeated Roots

The concept of distinct roots' CWS/CVS, as given in the Appendix may be extended to the treatment of repeated roots in the following way.

The Case of Repeated Roots Associated with Simple Jordan Canonical Form

To begin, the first two roots of $G(1)$ are repeated with a view to writing the eigenvectors at the second sampling instant as

$$W_j(2) = [W_1(1), W_2(1)]\,\alpha_j(2) + W'_{1,2}(1)\beta_j(2) \tag{2.1}$$

where $\alpha_j(2)$ = two-dimensional arbitrary constant vectors, $\beta_j(2) = (m-2)$ dimensional fixed constant vectors, and $W'_{1,2}(1) = [W_3(1) \cdots W_m(1)]$.

The reason for the above projection of $W_j(2)$ becomes clear from the equations which are going to be used for the calculation of $\beta_j(i)$.

Considering the equation for CWS/CVS at the second sampling instant from the equation (A4) of the Appendix we get

$$T_j(2)W_j(1) + W'_{1,2}(1)[\Delta\Lambda''_1(1)][V'_{1,2}(1)]'[W'_{1,2}(1)]\beta_j(2) = 0 \tag{2.2}$$

where

$$V'_{1,2}(1) = \begin{bmatrix} V_3(1) \\ \vdots \\ V_m(1) \end{bmatrix}'$$

and

$$\Delta\Lambda''_1(1) = \begin{bmatrix} g_1 - g_3 & & \bigcirc \\ & \ddots & \\ \bigcirc & & g_1 - g_m \end{bmatrix}.$$

So,

$$\beta_j(2) = -[\Delta\Lambda''_1(1)]^{-1}[V'_{1,2}(1)]'T_j(2)W_j(1). \tag{2.3}$$

Evidently, the equation (2.3) comes as a substitute for $\beta_j(2)$ of the equation (2.1) and we get

$$W_j(2) = [W_1(1), W_2(1)]\,\alpha_j(2) - W'_{1,2}(1)[\Delta\Lambda''_1(1)]^{-1} \cdot [V'_{1,2}(1)]'T_j(2)W_j(1). \tag{2.4}$$

We have seen that the number of repeated eigenroots of $G(1)$ indicates the number of eigenvectors of $G(1)$ which can be taken out to be multiplied with the arbitrary constant $\alpha_j(i)$, and the remaining eigenvectors of $W(1)$ will be multiplied with the fixed element of the vector $\beta_j(i)$. Thus, the vector $W_j(i)$ is computed every time as linear combinations of eigenvectors at initial time. So far no explicit algorithm for the computation of the dual CVS matrix S_v has been proposed. This, however, is not necessary when one realizes that S_v is the dual matrix sequences of S_W, which means $S_v \otimes S_W = S_E$ or $S_v = [S_W]^{-1}$. Thus, S_v may be obtained by the inversion of S_W and this may be simply executed using the formula given in [3].

But if all the roots of $G(1)$ are repeated, then $T_j^+(1)$ will no longer be valid, and in that case the equation (A4) of the Appendix will be changed as follows:

$$\left.\begin{aligned} &T_j(2)W_j(1) = 0 \\ &T_j(3)W_j(1) + T_j(2)W_j(2) = 0 \\ &\vdots \\ &T_j(K)W_j(1) + T_j(K-1)W_j(2) + \cdots + T_j(2)W_j(K-1) = 0 \end{aligned}\right\}. \tag{2.5}$$

It is quite obvious that the set of equations (2.5) is exactly the same as the equation (A4) of the Appendix with one appropriate sampling shift.

The Case of Repeated Roots Associated with Nonsimple Canonical Form

Now let us consider that $G(1)$ has nonsimple Jordan canonical form. Then we decompose it with the help of Jordan block, and its expression will be as follows:

$$G(1) = W(1)\{J(1)\}V(1) \tag{2.6}$$

where

$$J(1) = \text{diag}\{J_j(1)\}, \qquad j = 1, \cdots, m$$

and $J_j(1)$ represents a Jordan canonical block. More precisely, $G(1) = W(1)[\Lambda(1) + \epsilon]V(1)$ where

$$\Lambda(1) = \text{diag}\{\lambda_j(1)\}, \qquad j = 1 \cdots m$$

and where ϵ is the zero matrix with 1's appearing on the super diagonal above those diagonal elements that correspond to the nonsimple repeated

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roots. Hence, the spectral decomposition of S_G will be as follows:

$$S_G = S_W \otimes \left[S_\Lambda \oplus \begin{pmatrix} S_o & S_e & S_o & & \cdots & \\ S_o & S_o & S_e & S_o & \cdots & \\ S_o & S_o & S_o & S_o & \cdots & \\ \vdots & & & & & \\ S_o & S_o & S_o & S_o & \cdots & S_o \end{pmatrix} \right] \otimes S_v \quad (2.8)$$

provided we can define S_v from S_W [3].

Now if we can establish a closed-loop relationship, with respect to the open-loop relation as given in (2.8), in the same way as other input/output methods, then for stability purposes we can treat our every Jordan block as a set of SISO operators and we may carry out the rest of the design study in the usual manner [3]. Let us consider the closed-loop relation in the following way:

$$\begin{aligned} S_R &= (S_E \oplus S_G)^{\Box 1} \otimes S_G \\ &= S_W \otimes [S_E \oplus S_\Lambda \oplus S_\epsilon]^{\Box 1} \otimes [S_\Lambda \oplus S_\epsilon] \otimes S_V \\ &= S_W \otimes S_{\Lambda_{R_J}} \otimes S_V \end{aligned} \quad (2.9)$$

where $S_{\Lambda_{R_J}}$ is closed-loop CWS matrix. Now we concentrate on $S_{\Lambda_{R_J}}$ and try to achieve an expression similar to that obtained for the case of distinct roots.

For the sake of convenience of calculation let us consider a WMS whose first instant matrix has its first two roots repeated and associated with a nonsimple Jordan block. Clearly so, by (2.8) we have

$$S_G = S_W \otimes \left[S_\Lambda \oplus \left(\begin{array}{cc|c} S_o & S_e & \\ S_o & S_o & S_o \\ \hline \multicolumn{2}{c|}{S_o} & S_o \end{array} \right) \right] \otimes S_V \quad (2.10)$$

and from (2.9) we obtain

$$\begin{aligned} S_R &= (S_E \oplus S_G)^{\Box 1} \otimes S_G \\ &= \left[\frac{S_e}{(S_e \oplus S_{\lambda_1}) \otimes (S_e \oplus S_{\lambda_2})} \right] \otimes S_W \\ &\quad \otimes \left[\begin{array}{cc|c} S_{\lambda_1} \otimes (S_{\lambda_2} \oplus S_e) & S_e & \\ S_e & S_{\lambda_2} \otimes (S_{\lambda_1} \oplus S_e) & S_o \\ \hline & & \dfrac{S_{\lambda_3} \otimes (S_e \oplus S_{\lambda_1}) \otimes (S_e \oplus S_{\lambda_2})}{S_e \otimes (S_e \oplus S_{\lambda_3})} \\ \multicolumn{2}{c|}{S_o} & \ddots \\ & & \dfrac{S_{\lambda_m} \otimes (S_e \oplus S_{\lambda_1}) \otimes (S_e \oplus S_{\lambda_2})}{S_e \otimes (S_e \oplus S_{\lambda_m})} \end{array} \right]. \end{aligned} \quad (2.11)$$

Looking at the expression of (2.10) and (2.11) we can say that the case of nonsimple repeated roots gives rise to the same simple open-loop and closed-loop relations between the CWS and CVS as in the case of distinct roots. Moreover, we can also extend the above ideas to (2.8) and (2.9) which are a more generalized form of (2.10) and (2.11).

Now for the calculation of eigenvalues and eigenvectors at every instant we change the equation (A4) of the Appendix in the following way:

$$\left. \begin{aligned} &T_j(1) W_j(1) = -W_{j-1}(1) \\ &T_j(2) W_j(1) + T_j(1) W_j(2) = -W_{j-1}(1) \\ &\vdots \\ &T_j(K) W_j(1) + T_j(k-1) W_j(2) + \cdots + T_j(1) W_j(K) = -W_{j-1}(1) \end{aligned} \right\} \quad (2.12)$$

where

$$T_j(i) = = g_j(i) I - G(i).$$

The arguments and the subscripts are same as in the equation (A4) of the Appendix. If we choose the second sampling instant and try to calculate the eigenvalue at that instant we get the following result:

$$g_j(2) = V_j^t(1) G(2) W_j(1) - V_j^t(1) W_{j-1}(1). \quad (2.13)$$

For the calculation of the CVS we can obtain the value of $W_j(2)$ by writing

$$W_j(2) = [W_1(1), W_2(1)] \alpha_j(2) + W'_{1,2}(1) \beta_j(2) \quad (2.14)$$

where we assume that $G(1)$ has its first two eigenvalues repeated. Substituting the above in the second condition of (2.12) multiplying both sides by $V_{1,2}(1)^t$ we get

$$\begin{aligned} \beta_j(2) = &-[\Delta \Lambda_1''(1)]^{-1} [V'_{1,2}(1)]^{-1} [T_j(2)] W_j(1) \\ &-[\Delta \Lambda_1''(1)]^{-1} [W_{j-1}(1)] [V'_{1,2}(1)]^t \end{aligned} \quad (2.15)$$

so that $W_j(2)$ now becomes

$$\begin{aligned} W_j(2) = &[W_1(1), W_2(1)] \alpha_j(2) - W'_{1,2}(1) [\Delta \Lambda_1''(1)]^{-1} \\ &\cdot [V'_{1,2}(1)]^t T_j(2) W_j(1) - W'_{1,2}(1) [\Delta \Lambda_1''(1)]^{-1} \\ &\cdot [W_{j-1}(1)] [V'_{1,2}(1)]^t. \end{aligned} \quad (2.16)$$

Therefore, we can follow the above procedures for the calculations of the rest of CWS/CVS. From the above derivations and discussions, we have made so far, it is quite implicit that CWS and CVS can be treated in an analogous manner to characteristic gain and characteristics direction, respectively [5].

III. Conclusion

The idea of repeated roots has been developed in a systematic manner. Interesting results have been derived and unique relationships have been established. It has been found that even in the case of nonsimple repeated roots the CSM bears the same simple open-loop and closed-loop relations between CWS and CVS as in the case of distinct roots. The calculation of CWS and CVS are also similar to those of distinct roots' CWS and CVS except for a few differences. It is believed that with the help of some additional research effort the time-domain approach for multivariable plant will become a powerful tool along with other established methods such as the characteristic locus method.

Appendix

When a system with m-inputs and m-outputs is given in terms of input/output time-domain data, then as per [3] we can define m characteristic weighting sequences (CWS) and to each characteristic weighting sequence S_{g_i} there corresponds a characteristic vector sequence (CVS), i.e., S_{W_i} defined by the equation

$$[S_{g_i} \otimes S_E - S_G] \otimes S_{W_i} = S_o, \qquad i = 1, \cdots, m. \quad (A1)$$

We can define dual of S_{W_i} which can be represented by $S^t_{V_i}$. Now if we do the spectral decomposition of the weighting matrix sequences (WMS) we get

$$S_G = S_W \otimes S_{\Lambda_G} \otimes S_V \tag{A2}$$

where S_V is the dual characteristic vector sequences matrix. The relationship between open-loop and closed-loop characteristic sequences can be derived by inserting the decomposition of $S_Q = S_W \otimes S_{\Lambda_Q} \otimes S_V$ into the closed-loop input/output relations for the discrete system as given in [3]. By suitable preconvolution we get

$$\left[S_E \oplus S_W \otimes S_{\Lambda_Q} \otimes S_V \right] \otimes S_R = S_W \otimes S_{\Lambda_Q} \otimes S_V$$
$$\text{where} \quad S_R = S_W \otimes S_{\Lambda_R} \otimes S_V$$
$$\text{and} \quad S_{\Lambda_R} = \left[S_E \oplus S_Q \right]^{\Box 1} \otimes S_{\Lambda_Q}. \tag{A3}$$

Thus, S_{Λ_R} may be identified as closed-loop characteristic weighting sequences.

Now for the calculation of the CWS and CVS we consider (A1) at each sampling instant and thus obtain

$$T_j(k)W_j(1) + T_j(k-1)W_j(2) + \cdots + T_j(1)W_j(k) = 0 \tag{A4}$$

where

$$T_j(i) = \begin{cases} g_j(i)I - G(i) & \text{for } i \geqslant 1 \\ 0 & \text{for } i < 1 \end{cases} \tag{A5}$$

and i denotes the particular sampling instant, while the subscript $j = 1, 2, \cdots, m$ identifies the particular CWS and CVS under consideration. Clearly from the above $g_j(1)$ and $W_j(1)$ emerge as the eigenvalue/eigenvector pairs of $G(1)$. Premultiplication of (A4) for $k = 2$ by the dual eigenvector $V^t_j(1)$ eliminates the unknown vector $W_j(2)$ and yields the value of S_{g_j} at $k = 2$ as

$$g_j(2) = V^t_j(1) G(2) W_j(1). \tag{A6}$$

Knowledge of $g_j(2)$ however does now permit the calculation of $W_j(2)$ as

$$W_j(2) = T^+_j(1) G(2) W_j(1) + \alpha_j(2) W_j(1) \tag{A7}$$

where $\alpha_j(2)$ is an arbitrary constant and $T^+_j(1)$ is the commuting g_2-penrose inverse of $T_j(1)$ as mentioned in [3].

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Invariance of the Strict Hurwitz Property for Polynomials with Perturbed Coefficients

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***Abstract*—Given a strictly Hurwitz polynomial $f(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0$, it is of interest to know how much the coefficients a_i can be perturbed while simultaneously preserving the strict Hurwitz property. For systems with $n \leqslant 4$, maximal intervals of the a_i are given in a recent paper by Guiver and Bose [1]. In this note, a theorem of Kharitonov is exploited to obtain a general result for polynomials of any degree.**

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I. Introduction

In a large variety of applications, design parameters are chosen so that a certain polynomial will have all its roots in the strict left half-plane. Since the design is typically based upon a mathematical model, the possibility arises that the true values of the coefficients for the polynomial in question may differ from the *assumed values* which are used in carrying out the design; e.g., if state equations are developed for a system whose elements are known within a ± 10 percent tolerance, then the resulting characteristic polynomial can have coefficients which vary around some set of nominal values. Consequently, it is of interest to derive maximal intervals, centered about the nominal values of the coefficients, having the following property: the polynomial remains strictly Hurwitz[1] for all variations of the coefficients within these intervals. In [1], Guiver and Bose derive maximal intervals for fourth degree polynomials. The objective of this note is to provide a general result for a polynomial of arbitrary degree. Our proof is executed quite simply via application of a theorem of Kharitonov [2] which is not well-known[2] in western literature. It is felt that the application of Kharitonov's theorem to this particular problem will suggest numerous applications in the robustness area.

II. Notation, Formulation, and Assumptions

The polynomial

$$f(\lambda) \triangleq \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0 \tag{1}$$

is assumed to be strictly Hurwitz. To allow for different weightings on perturbations in the a_i above, we take as given (as in [1]) a set of nonnegative weights $\omega_i, \bar{\omega}_i$; $i = 0, 1, 2, \cdots, n-1$. Consequently, we can define allowable variations in the coefficients quite simply. Given any $\epsilon > 0$, the polynomial

$$g(\lambda) = \lambda^n + \gamma_{n-1}\lambda^{n-1} + \gamma_{n-2}\lambda^{n-2} + \cdots + \gamma_1\lambda + \gamma_0 \tag{2}$$

is said to be ϵ-admissible if its coefficients γ_i satisfy

$$a_i - \omega_i\epsilon < \gamma_i < a_i + \bar{\omega}_i\epsilon; \qquad i = 0, 1, 2, \cdots, n-1.$$

The set of all ϵ-admissible polynomials is denoted by $\mathscr{P}_\epsilon$. We seek the largest value of ϵ, call it $\epsilon_{\max}$, such that all polynomials in $\mathscr{P}_\epsilon$ are strictly Hurwitz.

Note that when $\epsilon = 0$, $\mathscr{P}_0$ consists of the single polynomial $f(\lambda)$. Also, observe that having $\epsilon_{\max}$ enables us to generate maximal intervals $(a_i - \omega_i\epsilon_{\max},\ a_i + \bar{\omega}_i\epsilon_{\max})$ for Hurwitz invariance. The quantity $\epsilon_{\max}$ can be viewed as a *measure of robustness.*

To complete this section, we define a notation for the *Hurwitz testing matrix*; i.e., let l be the largest integer less than or equal to $n/2$, and, for a given polynomial $g(\lambda)$ as in (2), define the following $n \times n$ matrix:

$$\mathscr{H}(\gamma_0, \gamma_1, \cdots, \gamma_{n-1}) \triangleq \begin{bmatrix} \gamma_{n-1} & \gamma_{n-3} & \gamma_{n-5} & \cdots & \gamma_{n-1-2l} & 0 & 0 & \cdots & 0 \\ 1 & \gamma_{n-2} & \gamma_{n-4} & \cdots & \gamma_{n-2l} & 0 & 0 & \cdots & 0 \\ 0 & \gamma_{n-1} & \gamma_{n-3} & \cdots & \gamma_{n+1-2l} & \gamma_{n-1-2l} & 0 & \cdots & 0 \\ 0 & 1 & \gamma_{n-2} & \cdots & \gamma_{n+2-2l} & \gamma_{n-2l} & 0 & \cdots & 0 \\ 0 & 0 & \gamma_{n-1} & \cdots & \gamma_{n+3-2l} & \gamma_{n+1-2l} & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \gamma_{n+4-2l} & \gamma_{n+2-2l} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \end{bmatrix}. \tag{3}$$

We also recall that the Routh–Hurwitz criterion [3]: $g(\lambda)$ is strictly

[1] All roots have strictly negative real parts.
[2] In the opinion of this author.