ON THE DUAL OF A PBIB DESIGN, AND A NEW CLASS OF DESIGNS WITH TWO REPLICATIONS

By C. S. RAMAKRISHNAN

Indian Statistical Institute, Calcutta

1. Introduction and summary

By interchanging blocks and varieties in a given class of designs, we get a new class of designs called the dual of the original class. Bose and Nair (1939) gave examples of PBIB designs obtained by dualizing some BIB designs. Youden (1951) investigated these further. He called the dual of a BIB design by the suggestive name of Linked Block designs. Elsewhere in this issue of Sankhyā, Roy and Laha (1956) have made exhaustive study, classification and enumeration of all Linked Block designs with $r, k \leq 10$. In this paper, we give the analysis and structure of the dual of a PBIB design with two associate classes. By dualizing a simple class of designs with 2 plots per block, we also derive a new class of useful designs with two replications. These turn out to be PBIB designs with five associate classes, but the analysis of these designs by the dual method turns out to be extremely simple, whereas a straightforward analysis of these as PBIB designs with five associate classes would be tedious.

2. General remarks on the dual method

Dualization of known types of designs sometimes leads us to new designs and sometimes only yields designs already known. For example, most of the Linked Block designs given by Roy and Laha (1956) turn out to be PBIB designs with two or three associate classes. But even in these cases, the dual method gives us a simpler way of analyzing these designs. Before starting the analysis of a design, it is always worthwhile to see whether the dual is easier to be analyzed, in which case we may profitably follow the dual method of analysis. The labour involved in preparing the analysis of variance table for the dual of a design is almost the same as that involved in the analysis of the original design. But there is additional labour in getting the standard errors of treatment contrasts. We have to impose restrictions on the original design to start with, so that in the dual design we may have only a small number of types of varietal differences with different precisions, and neat expressions for the standard errors.

3. THE DUAL OF A TWO ASSOCIATE PBIB DESIGN

Consider a PBIB design with two associate classes. We adopt the standard notation (except for an asterisk) of Bose, Clatworthy and Shrikhande (1954). Let the parameters of the design be $v^*, r^*, k^*, b^*, \lambda_1^*, \lambda_2^*, n_1^*, n_2^*$. Let us also introduce the constants c_1 and c_2 defined by them. The normal equations for treatment effects $\{t_i^*\}$ take the form (again adopting the notation of the above authors)

$$r^*(k^*-1)\, t_i^* = (k^*-c_2)Q_i^* + (c_1-c_2)\underset{1}{S}(Q_i^*)$$

where Q_i^* = adjusted yield of the i-th treatment,

$\underset{1}{S}(Q_i^*)$ = sum of the Q^*'s of all the first associates of the i-th treatment.

The dual of this design will have parameters

$$v = b^*, \qquad r = k^*, \qquad k = r^*, \qquad b = v^*.$$

Let $[n_{ij}]_{b\times v}$ be the incidence matrix of this design,

T_j = total yield of the j-th treatment,

Q_j = the corresponding adjusted yield,

B_i = total yield of the i-th block,

P_i = the corresponding adjusted block total.

Then the normal equations for treatment effects $\{t_j\}$ and block effects $\{b_i\}$ take one of the two following forms (3.1) or (3.2):

$$\left.\begin{aligned} T_j &= rt_j + \sum_i n_{ij}\, b_i \\ B_i &= kb_i + \sum_j n_{ij}\, t_j \end{aligned}\right\} \qquad \ldots \quad (3.1)$$

$$\left.\begin{aligned} Q_j &= T_j - \frac{1}{k}\sum_i n_{ij} B_i = r(k-1)\,t_j - \frac{1}{k}\sum_{\substack{j'=1\\ j'\neq j}}^{j'=v} \lambda_{jj'} t_{j'} \\ P_i &= B_i - \frac{1}{r}\sum_j n_{ij}\, T_j = k(r-1)\, b_i - \frac{1}{r}\sum_{i'\neq i} \mu_{ii'}\, b_{i'} \end{aligned}\right\} \qquad \ldots \quad (3.2)$$

where $\lambda_{jj'}$ = number of blocks in which treatments j and j' occur together,

and $\mu_{ii'}$ = number of varieties common to blocks i and i'.

If the given design is the dual of the original PBIB design, the equations for block effects take the form

$$k(r-1)b_i = (r-c_2)\, P_i + (c_1-c_2)\underset{1}{S}(P_i). \qquad \ldots \quad (3.3)$$

We compute $\{b_i\}$ by the above formula.

The over-all analysis of variance table may now be set up.

TABLE 1. ANALYSIS OF VARIANCE

source	formula	s.s.	d.f.	s.s.	formula	source
blocks (ignoring treatments)	$\frac{1}{k}\Sigma B_i^2-$c.f.	B	$b-1$	B_E	$\Sigma\, b_i P_i$	blocks (eliminating treatments)
treatments (eliminating blocks)	$T-E-B$	V_E	$v-1$	V	$\frac{1}{r}\Sigma T_j^2-$c.f.	treatments (ignoring blocks)
error		E	$bk-b-v+1$	E	$T-V-B_E$	error
total		T	$bk-1$	T		total

To obtain estimates of treatment contrasts, we compute t_j from (3.1) which may be rewritten as

$$t_j = \frac{1}{r}[T_j - \sum_i n_{ij}\, b_i]. \qquad \dots \quad (3.4)$$

A check on the calculations is got by computing s.s. due to treatments (eliminating blocks) by the alternative formula $\sum_j t_j Q_j$. Estimate of any linear contrast of treatment effects is the corresponding linear function of the t_j's.

We now turn to the problem of getting the standard error of any contrast. The method usually given in books is to add one more equation to (3.2) say $\Sigma\, t_j = 0$ and invert the matrix of coefficients. It is not recognised that these can be got in a simple and elegant way through the "Q" technique of Rao (1952). Rao's method is this. Obtain any solution of the normal equations, expressing t_j as a linear function of T_j's and B_i's. Then if $c_{jj'}$ be the coefficient of $T_{j'}$ in this expression, then the variance of any linear function of the t_j's is given by

$$V\,(\Sigma\, l_j\, t_j) = (\sum_j \sum_{j'} l_j\, l_{j'}\, c_{jj'})\,\sigma^2 \qquad \dots \quad (3.5)$$

where σ^2 is the intra block error variance. Thus

$$V\,(t_j - t_{j'}) = (c_{jj} + c_{j'j'} - 2c_{jj'})\,\sigma^2.$$

Hence the number of distinct types of standard errors will depend on the number of distinct $c_{jj'}$.

In our case, following Rao's method

$$t_j = \frac{1}{r}\left\{T_j - \sum_i n_{ij}\, b_i\right\}$$

$$= \frac{1}{r}\left[T_j - \frac{r-c_2}{k(r-1)}\left\{\sum_i n_{ij}\left(B_i - \frac{1}{r}\sum_{j'} n_{ij'} t_{j'}\right)\right\}\right] +$$

$$+ \frac{c_1-c_2}{k(r-1)}\left[\sum_i n_{ij} \mathop{S}_1 \left(B_i - \frac{1}{r}\sum_{j'} n_{ij'}\, t_{j'}\right)\right].$$

Then $$c_{jj} = \text{coefficient of } T_j = \frac{1}{r}\left\{1+\frac{r-c_2}{k(r-1)}+\frac{c_1-c_2}{k(r-1)}\nu_{jj}\right\}$$

and $$c_{jj'} = \text{coefficient of } T_{j'} = \frac{1}{r^2}\left\{\frac{r-c_2}{k(r-1)}\lambda_{jj'}+\frac{c_1-c_2}{k(r-1)}\nu_{jj'}\right\}$$

where $\nu_{jj'}$ = number of times j' occurs in the first associates of all the blocks in which j occurs. Thus $c_{jj'}$ depends on $\lambda_{jj'}$, ν_{jj} and $\nu_{jj'}$.

For a general PBIB design, calculation of $\nu_{jj'}$ is tedious, and we have to impose restrictions to limit the number of distinct types of errors. A very effective restriction is imposed by taking $r = 2$.

4. A NEW CLASS OF DESIGNS

Here we derive a new class of designs with $r = 2$ by dualizing the following simple class of designs. Take v^* to be even $= 2m$, write down all pairs and omit the pairs of the form $(2i-1,\ 2i)$

(13)	(14)......		(1 2m)
(23)	(24)......		(2 2m)
	(35)......		(3 2m)
	....		....
	(2m−2 2m−1)		(2m−2 2m).

This is a Group Divisible design (Bose et al, 1954)

groups		
		$v^* = mn;\quad m = m;\ n = 2;$
1	2	$\lambda_1^* = 0;\quad \lambda_2^* = 1$
3	4	$n_1 = 1;\quad n_2 = 2(m-1)$
...	...	
2m−1	2m	$k^* = 2;\quad r^* = v-2;\quad b^* = 2m(m-1).$

Let i_a be the associate of the i-th treatment

$$i_a = i+1 \quad \text{if } i \text{ is odd}$$
$$= i-1 \quad \text{if } i \text{ is even.}$$

The dual of this design has $v = 2m(m-1)$; $k = 2(m-1)$; $r = 2$; $b = 2m$.

Following the method of § 2, normal equations for block effects $\{b_i\}$ and treatment effects $\{t_j\}$ are

$$\left.\begin{aligned} b_i &= \frac{2(k+1)}{k(k+2)}P_i - \frac{2}{k(k+2)}P_{i_a} \\ 2t_j &= T_j - b_j^{(p)} - b_j^{(q)} \end{aligned}\right\} \quad \ldots \quad (4.1)$$

where $b_j^{(p)}$, $b_j^{(q)}$ are the effects of the two blocks in which the j-th treatment occurs.

Solving for b_i and then for t_j we can get estimates of any treatment contrasts, and s.s. due to treatments. The analysis of variance table is easily set up, as in § 2.

To get the standard errors, we compute c_{jj} and $c_{jj'}$. We get

$$\left.\begin{aligned} \nu_{jj} = 0; \quad c_{jj} &= \frac{1}{2} + \frac{k+1}{k(k+2)} = c \text{ (say)} \\ c_{jj'} &= \frac{k+1}{2(k+2)k}\lambda_{jj'} - \frac{1}{2(k+2)k}\nu_{jj'} \\ V(t_j - t_{j'}) &= 2(c - c_{jj'})\sigma^2. \end{aligned}\right\} \quad \dots \quad (4.2)$$

There are five distinct types of errors for treatment differences corresponding to the following five combinations of values of $\lambda_{jj'}$ and $\nu_{jj'}$.

$$\left.\begin{array}{lll} (1) & \lambda_{jj'} = 1; & \nu_{jj'} = 0 \\ (2) & \lambda_{jj'} = 1; & \nu_{jj'} = 1 \\ (3) & \lambda_{jj'} = 0; & \nu_{jj'} = 2 \\ (4) & \lambda_{jj'} = 0; & \nu_{jj'} = 1 \\ (5) & \lambda_{jj'} = 0; & \nu_{jj'} = 0 \end{array}\right\} \quad \dots \quad (4.3)$$

Actually, the new design turns out to be a PBIB design with five associate classes, if we define association between two treatments j and j' by the above five conditions on $\lambda_{jj'}$ and $\nu_{jj'}$. We shall say that j and j' are first associates if $\lambda_{jj'} = 1$, $\nu_{jj'} = 0$; second associates if $\lambda_{jj'} = 1$, $\nu_{jj'} = 1$; and so on. An easy way of writing down the association scheme is as follows.

Let p and q be the blocks in which j occurs and p_a, q_a the respective associates of these blocks. Blocks form a Group Divisible design with $\lambda_1^* = 0, \lambda_2^* = 1$ with groups (12), (34), ..., $(2m-1\ 2m)$, and block size $k = 2(m-1)$. Consider the four blocks p, p_a, q, q_a. We shall give a method of writing down all the associates of treatment j by a mere inspection of these four blocks. The blocks p and q have one variety in common namely j. Let j_{pq_a} and j_{p_aq} be the varieties common to p, q_a and p_a, q respectively. These two varieties are those for which $\lambda_{jj'} = 1$, $\nu_{jj'} = 1$. Thus for any treatment j, there are exactly two second associates $\lambda_2 = 1$; $n_2 = 2$. The remaining $2(k-2)$ varieties of p and q (omitting the above three namely j, j_{pq_a} and j_{p_aq}) are those for which $\lambda_{jj'} = 1$, $\nu_{jj'} = 0$. These are first associates, $\lambda_1 = 1$, $n_1 = 2(k-2)$. Blocks (p_a, q_a) have exactly one variety in common and this cannot occur in either p or q. Call this $j_{p_aq_a}$. This satisfies $\lambda_{jj'} = 0$, $\nu_{jj'} = 2$. Thus there is exactly one third associate: $\lambda_3 = 0$, $n_3 = 1$. The remaining $2(k-2)$ varieties in the blocks p_a and q_a are the fourth associates $\lambda_{\beta'} = 0$, $\nu_{\beta\beta'} = 1$; $\lambda_4 = 0$, $n_4 = 2(k-2)$. The varieties not occurring in any of the above four blocks are the fifth associates: $\lambda_5 = 0$, $n_5 = 2(m-2)(m-3)$.

Following the above method, the association scheme and the parameters $[p^i_{jk}]$ can be written down. For $m = 3$ we have $n_5 = 0$. There are only four associate classes and we get a design given by Nair (1951). For $m=4$, 5, etc., we get new designs with five associate classes. For $m=6$, $k=10$ and higher values of m are not desirable.

It may be noted that the analysis of these designs as PBIB with five associate classes is tedious, whereas the dual method of analysis is extremely simple, illustrating the usefulness of the dual method of analysing designs.

5. Numerical example

Here we give numerical details of analysis of one of our new designs, design No. 2. in our list. Parameters of this design are $v = 24$, $m = 4$; $b = 8$; $k = 6$, $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \lambda_4 = \lambda_5 = 0$, $n_1 = n_4 = 8$, $n_2 = 2$, $n_3 = 1$, $n_5 = 4$.

TABLE 2. PLAN AND YIELDS

block no.	varieties and yields*											
1	(1)	1.5	(2)	3.4	(3)	3.5	(4)	7.0	(5)	6.8	(6)	7.2
2	(7)	2.8	(8)	5.7	(0)	4.0	(10)	4.9	(11)	7.6	(12)	8.1
3	(1)	3.4	(7)	4.7	(13)	4.4	(14)	4.4	(15)	5.6	(16)	6.1
4	(2)	4.4	(8)	6.1	(17)	7.8	(18)	10.3	(19)	5.3	(20)	6.0
5	(3)	7.4	(9)	0.4	(13)	4.0	(17)	11.3	(21)	8.5	(22)	9.3
6	(4)	10.2	(10)	9.7	(14)	8.0	(18)	11.0	(23)	10.5	(24)	12.3
7	(5)	11.4	(11)	9.6	(15)	7.6	(19)	6.1	(21)	9.1	(23)	10.3
8	(6)	10.2	(12)	11.8	(16)	8.0	(20)	7.7	(22)	7.3	(24)	11.1

*Varieties are indicated within brackets. Yields are written by the side.

TABLE 3. ESTIMATION OF BLOCK EFFECTS

	blocks	I	II	III	IV	V	VI	VII	VIII	sum for checks
B_i	(unadjusted block totals)	20.4	33.1	28.6	40.8	50.8	61.7	54.1	56.1	354.6
$\sum_j n_{ij}T_j$	(sum of totals of treatments in that block)	76.4	84.4	61.4	86.0	86.0	100.7	98.4	106.0	709.2
P_i	(adjusted block totals)	−8.8	−9.1	−2.1	−2.2	7.35	6.85	4.9	3.1	0
b_i	(estimate of block effects)	−2.2	−2.3	−0.5	−0.6	1.9	1.7	1.3	0.7	0

Computational checks: $\Sigma B_i = G =$ total yield, $\Sigma\Sigma n_{ij}T_j = 2G$, $\Sigma P_i = 0 = \Sigma b_i$.

TABLE 4. ANALYSIS OF VARIANCE

source	s.s.	m.s.	F	d.f.	F	m.s.	s.s.	source
blocks (ignoring treatments)	197.74			7	*16.7	10.88	76.19	blocks (eliminating varieties)
varieties (eliminating blocks)	†135.84	5.91	*9.06	23			257.39	varieties (ignoring blocks)
error	11.09	0.652		17		0.652	†11.09	error
total	344.67			47			344.67	total

† obtained by subtraction. * significant at 1%.

TABLE 5. ESTIMATION OF TREATMENT EFFECTS

	j (treatment no.)	1	2	3	4	5	6	7	8	9	10	11	12
T_j	(unadjusted treatment totals)	4.0	7.8	10.9	17.2	18.2	17.4	7.5	11.8	13.4	14.6	17.2	19.9
$\Sigma n_{ij} b_i$	(sum of the block effects in which the treatment occurs)	−2.7	−2.7	−0.3	−0.5	−0.9	−1.5	−2.8	−2.8	−0.4	−0.6	−1.0	−1.6
t_j	(estimate of treatment effects)	3.8	5.3	5.6	8.8	9.6	9.4	5.2	7.3	6.9	7.6	9.1	10.8
	j (treatment no.) (*contd.*)	13	14	15	16	17	18	19	20	21	22	23	24
T_j	(unadjusted treatment totals)	9.3	12.4	13.2	14.1	10.1	21.3	11.4	14.6	17.6	10.6	20.8	23.4
$\Sigma n_{ij} b_i$	(sum of the block effects in which the treatment occurs)	1.3	1.2	0.8	0.2	1.3	1.1	0.8	0.2	3.2	2.6	3.0	2.4
t_j	(estimate of treatment effects)	4.0	5.6	6.2	7.0	8.9	10.1	5.3	7.2	7.2	7.0	8.9	10.5

Computational checks: $\Sigma T_j = G$; $\Sigma\Sigma n_{ij} b_i = 0$; $2\Sigma t_j = G$.

TABLE 6. STANDARD ERROR OF VARIETAL DIFFERENCES

type of associates	$\lambda_{jj'}$	$\nu_{jj'}$	estimate of $V(t_j-t_{j'})$	critical difference at 5% level of significance
1	1	0	0.75	1.83
2	1	1	0.76	1.84
3	0	2	0.87	1.97
4	0	1	0.85	1.95
5	0	0	0.84	1.93

The above table can be used to test the significance of any varietal difference, first by determining the type of associates, and then comparing with the corresponding critical difference. For example $t_2-t_1 = 1.4$. This does not exceed 1.84, the critical difference for second associates, and hence is not significant.

6. PLANS OF NEW DESIGNS

Here we give plans and parameters of all designs of the new class, with $k \leqslant 10$. There are four such corresponding to $m = 3, 4, 5, 6$. We also give a table giving constants for calculating standard errors.

We tabulate $K_{jj'}$, where

$$V(t_j-t_{j'}) = K_{jj'} \cdot \sigma^2,$$

for the five types of associates.

TABLE 7. VALUES OF PARAMETERS

$r = 2;\ \lambda_1 = \lambda_2 = 1;\ \lambda_3 = \lambda_4 = \lambda_5 = 0;\ b = 2m;\ k = 2(m-1);$
$v = 2m(m-1);\ n_1 = n_4 = 4(m-2);\ n_3 = 2(m-2)(m-3)$

design no.	m	b	k	v	$n_1 = n_4$	n_3
1	3	6	4	12	4	0
2	4	8	6	24	8	4
3	5	10	8	40	12	12
4	6	12	10	60	16	24

TABLE 8. PLANS OF FOUR NEW DESIGNS

Numbers of blocks are indicated within brackets. Treatments in the block are written by the side.

DESIGN NO. 1.

(1)	1	2	3	4	(2)	5	6	7	8
(3)	1	5	9	10	(4)	2	6	11	12
(5)	3	7	9	11	(6)	4	8	10	12

TABLE 8. (Continued)

DESIGN NO. 2

(1)	1	2	3	4	5	6	(2)	7	8	9	10	11	12
(3)	1	7	13	14	15	16	(4)	2	8	17	18	19	20
(5)	3	9	13	17	21	22	(6)	4	10	14	18	23	24
(7)	5	11	15	19	21	23	(8)	6	12	16	20	22	24

DESIGN NO. 3

(1)	1	2	3	4	5	6	7	8	(2)	9	10	11	12	13	14	15	16
(3)	1	9	17	18	19	20	21	22	(4)	2	10	23	24	25	26	27	28
(5)	3	11	17	23	29	30	31	32	(6)	4	12	18	24	33	34	35	36
(7)	5	13	19	25	29	33	37	38	(8)	6	14	20	26	30	34	39	40
(9)	7	15	21	27	31	35	37	39	(10)	8	16	22	28	32	36	38	40

DESIGN NO. 4

(1)	1	2	3	4	5	6	7	8	9	10	(2)	11	12	13	14	15	16	17	18	19	20
(3)	1	11	21	22	23	24	25	26	27	28	(4)	2	12	29	30	31	32	33	34	35	36
(5)	3	13	21	29	37	38	39	40	41	42	(6)	4	14	22	30	43	44	45	46	47	48
(7)	5	15	23	31	37	43	49	50	51	52	(8)	6	16	24	32	38	44	53	54	55	56
(9)	7	17	25	33	39	45	49	53	57	58	(10)	8	18	26	34	40	46	50	54	59	60
(11)	9	19	27	35	41	47	51	55	57	59	(12)	10	20	28	36	42	48	52	56	58	60

We give below $K_{jj'}$ where $V(t_j - t_{j'}) = K_{jj'}\sigma^2$ for the five types of associates. For a randomised block experiment with $r = 2$, $V(t_j - t_{j'}) = \sigma^2$. Hence $1/K_{jj'}$ stands for the "efficiency factor" as usually understood.

TABLE 9. CONSTANTS $K_{jj'}$ FOR THE CALCULATION OF STANDARD ERROR

type of associates	design number			
	1	2	3	4
(1)	(2)	(3)	(4)	(5)
1	1.21	1.15	1.11	1.09
2	1.25	1.17	1.12	1.10
3	1.50	1.33	1.25	1.20
4	1.46	1.31	1.24	1.19
5	1.42	1.29	1.22	1.18

CONCLUSION

My sincere thanks are due to Dr. C. R. Rao for suggesting the problem and for guidance and to Shri J. Roy and Shri R. G. Laha for helpful discussions.

REFERENCES

BOSE, R. C., CLATWORTHY, W. H. and SHRIKHANDE, S. S. (1954): *Tables of Partially Balanced Designs with Two Associate Classes.* University of North Carolina.

BOSE, R. C. AND NAIR, K. R. (1939): Partially balanced incomplete block designs. *Sankhyā*, 4, 337

NAIR, K. R. (1951): Some two-replicate pbib designs. *Cal. Stat. Ass. Bull.*, 3, 174-176.

RAO, C. R. (1952): *Advanced Statistical Methods in Biometric Research.* John Wiley & Sons, New York.

ROY, J. and LAHA, R. G. (1956): Classification and analysis of linked block designs. *Sankhyā*, 17, 115-132.

YOUDEN, W. J. (1951): Linked blocks: a new class of incomplete block designs. (Abstract), *Biometrics*, 7, 124.

Paper received : February, 1956.