TWO ASSOCIATE PARTIALLY BALANCED DESIGNS INVOLVING THREE REPLICATIONS

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1. Summary

Partially Balanced Incomplete Block (PBIB) designs with two associate classes and involving a few replications are of practical importance. Bose (1951) obtained the complete class of two associate PBIB designs involving two replications. Bose and Clatworthy (1955) recently derived all such designs with $k > r = 3$ and $\lambda_1 = 1$ and $\lambda_2 = 0$.

In the present paper two associate PBIB designs involving three replications are completely enumerated.

2. Introduction

PBIB designs were introduced by Bose and Nair (1939) and extended by Nair and Rao (1942). Recently for PBIB designs with two associate classes a less demanding definition has been given by Bose and Clatworthy (1955) which is as follows: An arrangement of v treatments in b blocks each of k plots is said to be a PBIB design with two associate classes if

(i) each treatment occurs in r blocks and no treatment occurs more than once in each block.

(ii) any two treatments are either first or second associates.

(iii) each treatment has n_1 first associates and n_2 second associates.

(iv) for any pair of treatments which are i-th associates the number p^i_{11} of treatments which are simultaneously the first associate of both the treatments is independent of the pair of treatments with which we start; $i = 1, 2$.

(v) any pair of treatments which are i-th associates occur together in exactly λ_i blocks; $i = 1, 2$.

In such a case it has been shown that the number p^i_{jk} of treatments which are simultaneously the j-th associates of a treatment θ and k-th associates of a treatment ϕ where θ and ϕ are themselves i-th associates is independent of θ and ϕ. These parameters satisfy the following conditions:

$$\left.\begin{aligned} vr &= bk \\ v &= n_1+n_2+1 \\ \lambda_1 n_1+\lambda_2 n_2 &= r(k-1) \end{aligned}\right\} \qquad \dots \quad (2.1)$$

$$\left.\begin{aligned} p^1_{11}+p^1_{12}+1 = p^2_{11}+p^2_{12} &= n_1 \\ p^1_{12}+p^1_{22} = p^2_{12}+p^2_{22}+1 &= n_2 \\ n_1 p^1_{12} &= n_2 p^2_{11} \\ n_1 p^1_{22} &= n_2 p^2_{12} \,. \end{aligned}\right\} \qquad \dots \quad (2.2)$$

To avoid triviality, we shall take $n_1, n_2 > 0$ and $\lambda_1 \neq \lambda_2$ or, without loss of generality, $\lambda_1 > \lambda_2$.

For a PBIB design with two associate classes, let the 'incidence matrix' be denoted by $N = (n_{ij})$ where $n_{ij} = 1$ or 0 according as the j-th treatment occurs in the i-th block or not $i = 1, 2, \ldots, b$; $j = 1, 2, \ldots, v$. It has been shown by Connor and Clatworthy (1954) that

$$|N'N| = rk(r-x_1)^{\alpha_1}(r-x_2)^{\alpha_2} \quad \ldots \quad (2.3)$$

where

$$x_i = \frac{1}{2}\left[(\lambda_1+\lambda_2)+(\lambda_1-\lambda_2)\{-\gamma+(-1)^i\sqrt{\Delta}\}\right] \quad \ldots \quad (2.4)$$

$$\alpha_i = [(v-1)\{(-1)^i\gamma+\sqrt{\Delta}+1\}-2n_i]/2\sqrt{\Delta} \quad \ldots \quad (2.5)$$

$$(i = 1, 2)$$

where

$$\left.\begin{aligned} \gamma &= p^2_{12}-p^1_{12} \\ \beta &= p^2_{12}+p^1_{12} \\ \Delta &= \gamma^2+2\beta+1. \end{aligned}\right\} \quad \ldots \quad (2.6)$$

The numbers α_1, α_2 must be positive integers and in general

$$r > x_2 > x_1 \quad \ldots \quad (2.7)$$

but if $b < v$

$$r = x_2 \quad \ldots \quad (2.8)$$

and

$$b > v-\alpha_2 = \alpha_1+1. \quad \ldots \quad (2.9)$$

For designs with $b < v$ the relation $r = x_2$ may be written alternatively as

$$(\lambda_1-\lambda_2)\{(r-\lambda_2)p^1_{12}-(r-\lambda_1)p^2_{12}\} = (r-\lambda_1)(r-\lambda_2) \quad \ldots \quad (2.10)$$

a result first obtained by Nair (1943). In general, we have however

$$(r-\lambda_1)(r-\lambda_2) > (\lambda_1-\lambda_2)\{(r-\lambda_2)p^1_{12}-(r-\lambda_1)p^2_{12}\} \quad \ldots \quad (2.11)$$

A two associate PBIB design will be said to be 'connected' if the matrix $C = rI-\frac{1}{k}N'N$ is of rank $(v-1)$. Since the latent roots of the matrix C are 0 and $r(1-1/k)+x_i/k$ of multiplicity α_i $(i = 1, 2)$ it follows that a necessary and sufficient condition for connectivity is

$$r(k-1)+x_i > 0 \quad \text{for } i = 1, 2 \quad \ldots \quad (2.12)$$

3. Three replicate two associate PBIB designs with $k > 3$

For three replicate two associate PBIB designs the parameters (λ_1, λ_2) can take only the following different pairs of values: (i) (3, 2) (ii) (3, 1) (iii) (3, 0) (iv) (2,1) (v) (2, 0) and (vi) (1, 0). Of these six types (iii) is not connected and (i) and (ii) can be derived from corresponding Balanced Incomplete Block (BIB) designs by replacing each treatment by a group of treatments. This is discussed in sub-section 3.1 If we consider designs with $k > 3$, then of the remaining possibilities (iv), (v) and (vi), the type (vi) has been completely enumerated by Bose and Clatworthy (1955). There it is shown that:—*Three replicate two associate PBIB designs with $k > 3$ and $\lambda_1 = 1$, $\lambda_2 = 0$ must belong to one of the following classes:*

(a) *Designs obtained by dualising BIB designs with parameters $k^* = 3$ and $\lambda^* = 1$*

(b) *Lattice designs with three replications*

(c) *The design with parameters*

$$v = 45, b = 27, r = 3, k = 5, n_1 = 12, n_2 = 32$$

$$\begin{bmatrix} p^1_{11} = 3 & p^1_{12} = 18 \\ & p^1_{22} = 24 \end{bmatrix} \quad \begin{bmatrix} p^2_{11} = 3 & p^2_{12} = 0 \\ & p^2_{22} = 22 \end{bmatrix}$$

We shall show in sub-section 3.3 that the case (v) is impossible and obtain in sub-section 3.2 all designs of the type (iv).

3.1. *Designs with* $\lambda_1 = 3$. It is well-known that if in a BIB design D^* with parameters

$$v^* = m, \ b^* = b, \ r^* = 3, \ k^* = p, \ \lambda^* = \lambda (\lambda = 1, 2)$$

each treatment is replaced by a group of n treatments the resulting design D is a two associate PBIB with the following parameters

$$\begin{aligned} v &= mn, \ b = b, \ r = 3, \ k = pn \\ \lambda_1 &= 3, \ \lambda_2 = \lambda \\ n_1 &= n-1, n_2 = n(m-1), p^1_{12} = 0 \end{aligned}$$

Conversely, it can be easily verified that any two associate PBIB design with $r = 3$ and $\lambda_1 = 3$, $\lambda_2 = \lambda$ $(\lambda = 1, 2)$ can always be obtained in like manner from a BIB design of the type D^*. It is also easy to check that any two associate PBIB design with $r = 3$, $\lambda_1 = 3$ and $\lambda_2 = 0$ is disconnected.

It is also known that there are only three BIB designs of the type D^*. Namely

$$v^* = 4, \ b^* = 6, \ k^* = 2, \ r^* = 3, \ \lambda^* = 1 \qquad \text{... (3.1)}$$

$$v^* = b^* = 7, \ k^* = r^* = 3, \ \lambda^* = 1 \qquad \text{... (3.2)}$$

and

$$v^* = b^* = 4, \ k^* = r^* = 3 \ \lambda^* = 2 \qquad \text{... (3.3)}$$

Therefore we have the following:

Theorem 3.1. *All two associate PBIB designs with three replications and $\lambda_1 = 3$ are derivable from the BIB designs (3.1), (3.2) or (3.3) by replacing each treatment by a group of n treatments.*

3.2. *Designs with $\lambda_1 = 2$ and $\lambda_2 = 1$.*

From (2.1) we get
$$bk = 3v = 3(n_1+n_2+1)$$
$$3(k-1) = 2n_1+n_2.$$

Eliminating n_2 we have

$$b = 9-t \quad \dots \quad (3.4)$$

$$v = 3k-kt/3 \quad \dots \quad (3.5)$$

where t is given by

$$n_1 = kt/3-2. \quad \dots \quad (3.6)$$

Since $k > 3$, that is $b < v$, we have from (2.10)

$$2p^1_{12}-p^2_{12} = 2. \quad \dots \quad (3.7)$$

Using the relations

$$p^1_{12} = n_2 p^2_{11}/n_1 \quad \text{and} \quad p^2_{12} = n_1 - p^2_{11}$$

and substituting in (3.7) we get

$$p^2_{11} = \frac{t(kt-6)}{9(6-t)}. \quad \dots \quad (3.8)$$

From (2.4) and (2.8) we get

$$3 = r = x_2 = \frac{1}{2}(3-\gamma+\sqrt{\Delta})$$

so that
$$\sqrt{\Delta} = 3+\gamma.$$

Substituting in (2.5), we get

$$\alpha_1 = \frac{n_1+2n_2}{3+\gamma}.$$

But
$$3+\gamma = 3+p^2_{12}-p^1_{12} = p^1_{12}+1 \quad \text{because of (3.7)}$$
$$= n_2 p^2_{11}/n_1+1$$

and substituting the values of n_1, n_2 and p^2_{11},

we get $$\alpha_1 = \frac{9k(6-t)^2}{(kt+6)(9-t)-kt^2}. \qquad \ldots \quad (3.9)$$

Let us write

$$kt/3 = u, \text{ or, } k = 3u/t. \qquad \ldots \quad (3.10)$$

Since n_1 is a positive integer, from (3.6), we find that u must be a positive integer greater than 2. From (3.8) we get, in terms of u

$$p^2_{11} = \frac{t(u-2)}{3(6-t)} = \frac{2-u}{3}+\frac{2(u-2)}{6-t} \qquad \ldots \quad (3.11)$$

Since $3p^2_{11}$ is a positive integer, so must be

$$\frac{6(u-2)}{6-t} = w, \text{ (say.)}$$

Then $$u = w+2-\frac{tw}{6} \qquad \ldots \quad (3.12)$$

so that

$$p^2_{11} = \frac{wt}{18}. \qquad \ldots \quad (3.13)$$

We also find from (3.8) that the only admissible values of t are 1, 2, 3, 4 and 5. For each such value of t, w must make $wt/18$ positive integral, and the corresponding value of u given by (3.12) should make

$$\alpha_1 = \frac{3k(6-t)^2}{(u+2)(9-t)-ut} \qquad \ldots \quad (3.14)$$

a positive integer which because of (2.9) must not be greater than $b-1$.

Let us take up the different possible values of t one by one. Consider the case $t = 1$. In this case $p^2_{11} = w/18 = a$ (say), so that $w = 18a$. Therefore we get $u = 15a+2$ and $k = 3u = 3(15a+2)$ so that

$$\alpha_1 = \frac{15(15a+2)}{7a+2} = 7+\frac{10(11a+1)}{7a+2}.$$

But here $b = 8$ and $\alpha_1 > b-1$ for all positive values of a. Hence no design is possible.

For the case $t = 2$ we have $p^2_{11} = w/9 = a$, say. Then $w = 9a$, $u = 6a+2$, $k = 3u/2 = 3(3a+1)$ so that

$$\alpha_1 = \frac{24(3a+1)}{5a+4} = 6+\frac{42a}{5a+4}.$$

But here $b = 7$ and $\alpha_1 > b-1$ for all positive values of a. Hence in this case also no design is available.

Now take the case $t = 3$. Here $p^2_{11} = w/6 = a$ say. Then $w = 6a$, $u = 3a+2$, $k = u = 3a+2$ so that

$$\alpha_1 = \frac{3(3a+2)}{a+2}.$$

Here $b = 6$ and since

$$b-1-\alpha_1 = \frac{4(1-a)}{a+2}$$

has to be non-negative, the only permissible value of a is $a = 1$. Fortunately this value of a makes $\alpha_1 = 5$ a positive integer. For this value of a, we have $k = 5$, $v = 10$, $n_1 = 3$ and $p^2_{11} = 1$. We have merely proved that the above parameters satisfy all the known necessary restrictions for the existence of the corresponding design. This design however does exist and is listed in Bose, Clatworthy and Shrikhande (1954) as Design number T9.

For $t = 4$, $p^2_{11} = 2w/9$ which must therefore be an even integer $= 2a$, say. Then $w = 9a$, $u = 3a+2$ and $k = 3u/4 = (9a+6)/4 = 2a+1+(a+2)/4$. Since this is integral a itself must be of the form $4c+2$. Thus $p^2_{11} = 4(2c+1)$, $w = 9(4c+2)$, $u = 4(3c+2)$ and $k = 3(3c+2)$. Thus

$$\alpha_1 = \frac{6(3c+2)}{2c+3}$$

Here $b = 5$ and therefore $b-1-\alpha_1 = \dfrac{-10c}{2c+3}$ which is non-negative only when $c = 0$. If $c = 0$, $\alpha_1 = 4$ is integral. For this case, we get $k = 6$, $v = 10$, $n_1 = 6$ and $p^2_{11} = 4$. This design is known to exist and listed as Design number T15 in Bose, Clatworthy and Shrikhande (1954).

When $t = 5$, we get $p^2_{11} = 5w/18$ which must be divisible by 5, $p^2_{11} = 5a$ say. Then $w = 18a$, $u = 3a+2$ and $k = 3u/5 = (9a+6)/5 = 2a+1-(a-1)/5$. Hence a must be of the form $a = 5c+1$ so that k may be integral. Then $p^2_{11} = 5(5c+1)$, $w = 18(5c+1)$, $u = 5(3c+1)$ amd $k = 3(3c+1)$ so that

$$\alpha_1 = \frac{3(3c+1)}{1-5c}$$

The only value of c for which this is positive integral is $c = 0$. But if $c = 0$, $k = 3$ which contradicts our assumption that $k > 3$. Hence no such design exists. We may summarise the results of this section in the following.

Theorem 3.2. *There are only two PBIB designs with two associate classes and* $r = 3, \lambda_1 = 2, \lambda_2 = 1$ and $k > 3$, *namely*:

$$T9 : v = 10, k = 5, n_1 = 3, p_{11}^2 = 1$$

and

$$T15 : v = 10, k = 6, n_1 = 6, p_{11}^2 = 4.$$

3.3. *Impossibility of designs with* $\lambda_1 = 2$ *and* $\lambda_2 = 0$. Suppose that such a design exists. Then from (2.10) we get

$$2(3p_{12}^1 - p_{12}^2) = 3.$$

But since p_{12}^1 and p_{12}^2 are non-negative integers, clearly such a thing is impossible. Therefore, we get the

Theorem 3.3. *No PBIB design with two associate classes and* $r = 3$, $\lambda_1 = 2$, $\lambda_2 = 0$ *and* $k > 3$ *exists.*

4. Three replicate two associate PBIB designs with $k = 3$

4.1. *The case* $\lambda_1 = 2, \lambda_2 = 1$. Here, we get from (2.1) $2n_1 + n_2 = 6$ and therefore there are only two possibilities namely (i) $n_1 = n_2 = 2$ and (ii) $n_1 = 1, n_2 = 4$.

In case (i), if we set $p_{12}^1 = t$, we get from (2.2)) $p_{11}^1 = 1 - t$ and $p_{22}^2 = t - 1$. Hence the only permissible value of t is $t = 1$. Then we have $v = b = 5$, $p_{12}^1 = 1$ and $\alpha_1 = 2$ which is integral.

Writing the digits 1, 2, ..., 5 for the treatments, a plan for the design (4.1) obtained by taking the following triplets as blocks:

$$(1, 2, 3); (1, 2, 4); (1, 3, 5); (2, 4, 5); (3, 4, 5) \qquad \dots \quad (4.1)$$

In case (ii) we have from (2.2)

$$p_{11}^1 + p_{12}^1 = 0$$

Therefore

$$p_{11}^1 = p_{12}^1 = p_{11}^2 = 0,$$

and we get $v = b = 6$ and $\alpha_1 = 2$ is integral. A plan of the design is

$$(1, 2, 3); (1, 2, 4); (1, 5, 6); (2, 5, 6); (3, 4, 5); (3, 4, 6) \qquad \dots \quad (4.2)$$

where the triplets form the blocks.

We may summarise these results in the following

Theorem 4.1. *There are only two two-associate PBIB designs with $r = k = 3$ and $\lambda_1 = 2$, $\lambda_2 = 1$ namely*

$$v = b = 5,\ n_1 = n_2 = 2,\ p^1_{12} = 1$$

and
$$v = b = 6,\ n_1 = 1,\ n_2 = 4,\ p^1_{12} = 0$$

with plans given in (4.1) and (4.2) respectively.

4.2. *The case $\lambda_1 = 2$, $\lambda_2 = 0$.* Here we get $n_1 = 3$ and from (2.11) $2(3p^1_{12}-p^2_{12}) \leqslant 3$ so that the permissible values are $3\,p^1_{12}-p^2_{12} \leqslant 1$. Setting $p^1_{12} = t$, we get from (2.2) $p^2_{12} = 3-\dfrac{3t}{n_2}$. Hence $3t\,(1+1/n_2) \leqslant 4$ and the only possibility is $t = 1$ and $n_2 = 3$. This gives $v = b = 7$ and $\alpha_1 = 3-3/2\sqrt{2}$ which is not integral. Hence, the

Theorem 4.2. *There is no two associate PBIB design with $r = k = 3$ and $\lambda_1 = 2$, $\lambda_2 = 0$.*

4.3. *The case $\lambda_1 = 1$, $\lambda_2 = 0$.* Here $n_1 = 6$ and from (2.11) we get $3p^1_{12}-2p^2_{12} \leqslant 6$. Setting $p^1_{12} = t$, we get from (2.2) $p^2_{12} = 6-\dfrac{6t}{n_2}$ and therefore the following restriction on t and n_2.

$$t(1+4/n_2) \leqslant 6 \qquad \dots \quad (4.3)$$

Since $p^1_{11} = 5-t$, the permissible values of t are 0, 1, 2, 3, 4 and 5. Let us take up these values of t one by one.

If $t = 0$, write $n_2 = a$. Then we get $v = 7+a$, $\gamma = 6$, $\beta = 6$ and $\sqrt{\Delta} = 7$ so that $\alpha_1 = a/7$ on simplification. Since this is integral a must be a multiple of 7, $a = 7(c-1)$ say. Then $v = b = 7c$ and the values of x_1, x_2 defined in (2.4) come out to be -6 and $+1$ respectively.

But
$$r = k = 3$$

and therefore
$$r(k-1) + x_1 = 6+(-6) = 0$$

which contradicts the condition (2.12) for connectivity. Therefore there is no connected design in this case.

If $t = 1$, since 6 must be divisible by n_2, the permissible values of n_2 are 1, 2, 3 and 6. The only value of n_2 that makes α_1 integral is $n_2 = 1$ which gives $v = b = 8$, $p^1_{12} = 1$, $p^2_{12} = 0$. This design exists and is listed as Design number R5 in Bose, Clatworthy and Shrikhande (1954).

If $t = 2$, we get from (4.3) $4/n_2 \leqslant 2$ and since $6t = 12$ must be divisible by n_2 the only permissible value are $n_2 = 2, 3, 4, 6,$ and 12. Amongst these, the values of n_2 that make α_1 integral are $n_2 = 2$ and $n_2 = 3$ respectively. If $n_2 = 2$, we get

$v = b = 9$, $p_{12}^2 = 0$; this is a triple lattice design. If $n_2 = 3$, we get $v = b = 10$ and this is listed as Design number T6 in Bose, Clatworthy and Shrikhande (1954).

When $t = 3$ we get from (4.3) $4/n_2 \leqslant 1$. Since $6t = 18$ must be divisible by n_2, permissible values of n_2 are $n_2 = 6$, 9 and 18. For $n_2 = 6$ and 9, α_1 is integral. When $n_2 = 6$, we get $v = b = 13$, $p_{12}^2 = 3$: this has the Design number C1 in Bose, Clatworthy and Shrikhande (1954). When $n_2 = 9$ we get, $v = b = 16$ and $p_{12}^2 = 4$. This design exists and is listed as Design number LS14 in Bose, Clatworthy and Shrikhande (1954).

When $t = 4$, we must have $4/n_2 \leqslant 1/2$ and n_2 must be a factor of 24. Thus $n_2 = 8$, 12 and 24; Thus $n_2 = 8$, 12 and 24; of these the only value that makes α_1 integral is $n_2 = 8$. When $n_2 = 8$, we get $v = b = 15$, $p_{12}^2 = 3$. This is Design number T28 in Bose, Clatworthy and Shrikhande (1954).

In the case $t = 5$, $4/n_2 < 1/5$ and n_2 must be a factor of 30. Thus $n_2 = 30$ but this value does not make α_1 integral. Therefore there is no design in this class.

We may summarise our results in the following

Theorem 4.3. *The class of two-associate PBIB design with* $r = k = 3$ *and* $\lambda_1 = 1$, $\lambda_2 = 0$ *contains only the designs numbered* R5, T6, C1, LS 14 *and* T28 *in* Bose, Clatworthy and Shrikhande (1954) *and the triple lattice design with* $v = b = 9$.

It is interesting to note that Bose, Clatworthy and Shrikhande (1954) give another design, namely Design number S11 with parameters $v = b = 19$, $r = k = 3$, $\lambda_1 = 1$, $\lambda_2 = 0$, $n_1 = 6$ $n_2 = 12$

$$\begin{bmatrix} p_{11}^1 = 1 & p_{12}^1 = 14 \\ & p_{22}^1 = 8 \end{bmatrix} \quad \begin{bmatrix} p_{11}^2 = 2 & p_{12}^2 = 4 \\ & p_{22}^2 = 7 \end{bmatrix}$$

and two other designs numbered S12 and S13 are derived from it by respectively duplicating and triplicating this design. This is not covered by our theorem. That a two associate PBIB design with these parameters is impossible can be easily demonstrated by calculating the value of α_1 which terms out to be $\alpha_1 = 9+3/\sqrt{17}$ which is not integral.

5. Three replicate two associate PBIB designs with $k = 2$

5.1. *The case* $\lambda_1 = 2$, $\lambda_2 = 1$. In this case we get from (2.1) the relation $2n_1+n_2 = 3$, so that the only solution is $n_1 = n_2 = 1$ and therefore $v = 3$. But $b = rv/k = 9/2$ becomes fractional. Hence no such design is possible.

5.2. *The case* $\lambda_1 = 2$, $\lambda_2 = 0$. Such designs are clearly impossible since (2.1) leads to $n_1 = 3/2$.

5.3. *The case* $\lambda_1 = 1$, $\lambda_2 = 0$. In this case we get from (2.1) $n_1 = 3$. Setting $p_{12}^1 = t$, we get from (2.2) $p_{11}^1 = 2-t$ and $p_{12}^2 = 3-3t/n_2$. Therefore the permissible values of t are 0,1 and 2. But from (2.11) we get on simplification

$$t(1+2/n_2) \leqslant 4. \qquad \ldots \quad (5.1)$$

Thus if $t = 0$ we get on simplification $\alpha_1 = \frac{v}{4} - 1$ and therefore v must be a multiple of 4, $v = 4a$, say. Then $n_2 = 4(a-1)$ and the values of x_1, x_2 defined in (2.4) turn out to be -3 and 1 respectively. But this makes

$$r(k-1)+x_1 = 3+(-3) = 0$$

violating the condition (2.12) of connectivity. Hence no connected design is possible with $t = 0$. If $t = 1$ since n_2 divides $3t$, $n_2 = 1, 3$, but none of these makes α_1 an integer. If $t = 2$, n_2 must divide 6 and at the same time satisfy (5.1). Thus $n_2 = 2$, 3 and 6. Of these values only $n_2 = 2$ and $n_2 = 6$ make α_1 integral. If $n_2 = 2$ we get $v = 6$, $b = 9$, $p^2_{12} = 0$. This design is immediately recognised to be the dual of the simple 3^2 lattice design and the blocks are given by the pairs

$$(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5) (3, 6) \quad \ldots \quad (5.2)$$

where the integers 1, 2, ..., 6 denote the six treatments.

If $n_2 = 6$, we get $v = 10$, $b = 15$, $p^2_{12} = 2$. A plan for such a design is given below:

$$\left.\begin{array}{lllll} (1, 8), & (2, 6), & (3, 5), & (4, 5), & (5, 10) \\ (1, 9), & (2, 7), & (3, 7), & (4, 6), & (6, 9) \\ (1, 10), & (2, 10), & (3, 9) & (4, 8), & (7, 8) \end{array}\right\} \quad \ldots \quad (5.3)$$

where the treatments are indicated by the integers 1, 2, ..., 10 and the pairs are the blocks.

We therefore have the following

Theorem 5.1. *There are only two two-associate PBIB designs with $r = 3$ and $k = 2$ namely*

$$v = 6, b = 9, \lambda_1 = 1, \lambda_2 = 0, n_1 = 3, n_2 = 2, p^2_{12} = 0$$

and

$$v = 10, b = 15, \lambda_1 = 1, \lambda_2 = 0, n_1 = 3, n_2 = 6, p^2_{12} = 2$$

with plans given by (5.2) *and* (5.3) *respectively.*

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