SOLUTIONS TO SOME FUNCTIONAL EQUATIONS AND THEIR APPLICATIONS TO CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS

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SUMMARY. Three sets of results are contained in this paper. The first is on a new matrix product. If A and B are two matrices of orders $p \times r$ and $q \times r$ respectively, and if $\alpha_1, \ldots, \alpha_r$ are column vectors of A and $\beta_1, \ldots, \beta_r$ are those of B then the new product $A \odot B$ is the partitioned matrix

$$(\alpha_1 \otimes \beta_1 \vdots \alpha_2 \otimes \beta_2 \vdots \cdots \vdots \alpha_r \otimes \beta_r)$$

where $\otimes$ denotes the Kronecker product. Propositions involving the new product of matrices are stated.

The second is on the solution of functional equations of two types. One is of the form

$$\sum_{u=1}^{p} c_{ju}\, \psi_u(e_u' t) + \sum_{i=1}^{r} b_{ji}\, \phi_i(\alpha_i' t) = g_j \text{ (constant)}, \quad j = 1, \ldots, q$$

involving a vector variable t where e_u are unit vectors of an identity matrix of order p, α_i are given column vectors and ψ_u, ϕ_i are unknown continuous functions. Another is of the form

$$\sum_{j=1}^{s} d_{ij}\, \phi(b_j t) = g_i, \qquad i = 1, \ldots, q$$

involving an unknown function ϕ of a single variable t. Conditions under which the unknown functions in these two types of equations are polynomials of an assigned degree are given.

The third, on the characterization of normal and gamma distributions, extends the earlier work of the authors (Rao, 1967 and Khatri and Rao, 1968*). We consider two sets of functions $L_1, \ldots, L_q$ and $M_1, \ldots, M_p$ of independent random variables $X_1, \ldots, X_n$ with the condition

$$E(L_i | M_1, \ldots, M_p) = g_i \text{ (constant)}$$

for $i = 1, \ldots, q$. When L_i and M_j are linear, the X_i have normal distributions. When L_i are linear in the reciprocals of the variables and M_j are linear in the variables, the X_i have gamma or conjugate gamma distributions. When the X_i variables are non-negative, L_i are linear in the variables and M_j are linear in the logarithms of the variables, the X_i have gamma distributions. These results are proved under some conditions on the compounding coefficients for $p > 1$, and in the case of $p = 1$ with the further condition that the X_i are identically distributed.

1. Introduction

Linnik (1964) considered a functional equation in two variables t_1, t_2 of the type

$$\phi_1(t_1 + b_1 t_2) + \ldots + \phi_r(t_1 + b_r t_2) = \xi_1(t_1) + \xi_2(t_2) \qquad \ldots \quad (1.1)$$

defined for $|t_1| < \partial$, $|t_2| < \partial$, for some $\partial > 0$, where $\phi_1, \ldots, \phi_r$ and ξ_1, ξ_2 are unknown continuous functions, and showed, by an extremely elegant method, that all the functions involved in (1.1) must be polynomials provided only that $b_1, \ldots, b_r$ are all different. In a recent paper Rao (1966) considered a slightly extended form of (1.1)

$$\phi_1(t_1 + b_1 t_2) + \ldots + \phi_r(t_1 + b_r t_2) = \xi_1(t_1) + \xi_2(t_2) + Q(t_1, t_2) \qquad \ldots \quad (1.2)$$

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defined for $|t_1| < \partial$, $|t_2| < \partial$, where Q is a quadratic function in t_1, t_2 and showed that each function involved in (1.2) is a polynomial of degree not more than max $(2, r)$ provided that $b_1, \ldots, b_r$ are different. In the case of (1.1), without the quadratic function, the degree of each polynomial is found to be utmost r.

We now consider a functional equation in $p(>)2$ variables $t_1, \ldots, t_p$ of the type

$$\phi_1(\alpha_1' t)+\ldots+\phi_r(\alpha_r' t) = \xi_1(t_1)+\ldots+\xi_p(t_p) \qquad \ldots \quad (1.3)$$

defined for $|t_i| < \partial$, $i = 1, \ldots, p$, where t represents the column vector of variables $t_1, \ldots, t_p$ and $\alpha_1, \ldots, \alpha_r$ are given column vectors. Our object is to determine the conditions on $\alpha_1, \ldots, \alpha_r$ under which each function in (1.3) is a polynomial and to find an upper bound to the maximum degree of the polynomials. It is shown that more precise estimates of the maximum degree than in the case (1.2) can be found depending on the nature of the vectors $\alpha_1, \ldots, \alpha_r$. The case where the maximum degree is utmost unity (see Lemma 4) is of special interest and is considered in some detail. Conditions under which the maximum degree is $k < r$ are given in Lemma 5. Thus, an increase in the number of variables in Linnik's equation (1.1) places a restriction on the degree of the polynomials.

As a generalisation of the equation (1.3), we consider multiple equations of the form

$$\sum_{u=1}^{p} c_{ju}\psi_u(e_u' t)+\sum_{i=1}^{r} b_{ji}\phi_i(\alpha_i' t) = g_j, \quad j = 1, \ldots, q \qquad \ldots \quad (1.4)$$

defined for $|t_i| < \partial$, $i = 1, \ldots, p$, where $\psi_1, \ldots, \psi_p$; $\phi_1, \ldots, \phi_r$ are unknown continuous functions, g_j are constants, e_u are unit column vectors of the identity matrix I_p of order p and $\alpha_1, \ldots, \alpha_r$ are given column vectors. In Lemmas 6, 7 and 8, we determine the conditions under which the functions involved in (1.4) are polynomials of a degree not exceeding a given number.

Finally, we consider multiple equations of the form

$$\sum_{j=1}^{n} d_{ij}\phi(b_j t) = g_i \text{ (constant)}, \qquad i = 1, \ldots, q \qquad \ldots \quad (1.5)$$

in a single variable t defined for $|t| < \partial$, where ϕ is an unknown function. This is a generalization of the single equation

$$a_1\phi(b_1 t)+\ldots+a_n\phi(b_n t) = 0 \qquad \ldots \quad (1.6)$$

considered by Rao (1967). It is shown that when $\phi(t)$ is of the form $c+t\psi(t)$ where $\psi(t)\to a$ (constant) as $t\to 0$, then $\phi(t)$ is a linear function under some conditions on the coefficients.

We use the solutions of the equations (1.3), (1.4) and (1.5) in characterizing normal and gamma distributions. These results extend those obtained in earlier papers by Rao (1967) and Khatri and Rao (1968).

In Section 2 of the paper we define a new product of matrices and consider its properties. The solutions of the functional equations (1.3), (1.4) and (1.5) are discussed in Section 3 and the main theorems on characterization of the normal and the gamma distributions are given in Sections 4 and 5.

2. A NEW PRODUCT OF MATRICES

Let $A = (a_{ij})$ and B be any two matrices. Then the Kronecker product $A \otimes B$ is defined by

$$A \otimes B = (a_{ij}B). \qquad \ldots \quad (2.1)$$

If A is $p \times q$ matrix and B is $m \times n$ matrix, then the order of $A \otimes B$ is $pm \times qn$.

Now we shall consider two matrices A of order $p \times r$ and B of order $q \times r$ and denote the column vectors of A by $\alpha_1, \ldots, \alpha_r$ and those of B by $\beta_1, \ldots, \beta_r$.

Definition : The new product $A \odot B$ is defined to be the partitioned matrix

$$A \odot B = (\alpha_1 \otimes \beta_1 \vdots \alpha_2 \otimes \beta_2 \vdots \ldots \vdots \alpha_r \otimes \beta_r) \qquad \ldots \quad (2.2)$$

which is of order $pq \times r$.

We state some propositions involving the new product of matrices, which follow from the definition or which can be easily established.

(i) It is easy to see that if C is of order $s \times r$ with column vectors $\gamma_1, \ldots, \gamma_r$,

then $$A \odot B \odot C = (\alpha_1 \otimes \beta_1 \otimes \gamma_1 \vdots \ldots \vdots \alpha_r \otimes \beta_r \otimes \gamma_r) \qquad \ldots \quad (2.3)$$

is of order $pqs \times r$ and

$$(A \odot B) \odot C = A \odot (B \odot C) \qquad \ldots \quad (2.4)$$

and so on.

Further $A \odot B$ and $B \odot A$ differ only in a permutation of rows. Hence the six possible orders of multiplying three matrices, A, B, C, lead to matrices which differ only in a permutation of rows.

(ii) Let T_1 be a matrix of order $m \times p$ and T_2 of order $n \times q$. Then

$$(T_1 \otimes T_2)(A \odot B) = T_1 A \odot T_2 B. \qquad \ldots \quad (2.5)$$

(iii) If $\alpha_1, \ldots, \alpha_r$ and $\beta_1, \ldots, \beta_r$ are all non-null vectors, then $A \odot B$ has no null column. If A has a null column vector, then the corresponding column vector in $A \odot B$ is null. Conversely if $A \odot B$ has a null column vector, then the corresponding column vector in A or B must be null.

(iv) If two non-null columns in $A \odot B$ are proportional, then the two corresponding non-null column vectors in A as well as in B will be proportional and conversely.

(v) Let all the column vectors of B corresponding to independent column vectors of A be non-null. Then rank $(A \odot B) \geqslant$ rank A. Similarly, if all the column vectors of A corresponding to independent column vectors of B are non-null, then rank $(A \odot B) \geqslant$ rank B.

(vi) If rank $(A \odot B) = r$ and the i_1-th, i_2-th, ..., i_u-th column vectors of B are proportional, then the i_1-th, i_2-th, ..., i_u-th column vectors of A are linearly independent, and all column vectors of A and B are non-null vectors.

(vii) If rank $A = r$, the number of columns of A, and s is the number of null column vectors in B, then rank $(A \odot B) = r-s$.

Definition : Let A^* be the matrix obtained from $A \odot A$ by deleting the p rows involving the square terms (i.e., by deleting the 1st, $(p+2)$-th, ..., p^2-th rows), where A is of order $p \times r$.

(viii) If rank $A^* = r$, then

(a) no two columns of A are dependent, and

(b) each column of A contains at least two non-zero elements.

Note: We observe that while rank $(A \odot A) \geqslant$ rank A, it is not possible to make a general statement regarding the relative magnitudes of the ranks of A and A^*. We give some examples to show that rank A^* may be less than, greater than or equal to rank A.

Consider the matrices

$$A_1 = \begin{bmatrix} 1 & \cdot & \cdot & 1 \\ -1 & \cdot & \cdot & 1 \\ \cdot & 1 & 1 & \cdot \\ \cdot & -1 & 1 & \cdot \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 2 & \cdot \\ 1 & \cdot & 1 & 1 \\ \cdot & 1 & 1 & -1 \\ 1 & \cdot & 1 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 1 & 2 \\ & 1 & 1 \\ 1 & \cdot & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 1 & -1 \\ \cdot & 1 & 1 \\ 1 & \cdot & \cdot \end{bmatrix}, \quad A_5 = \begin{bmatrix} 1 & 1 & -1 \\ \cdot & 1 & 1 \\ 1 & \cdot & 1 \end{bmatrix}$$

By actual computation we find

(a) rank $A_1 = 4$, rank $A_1^* = 2$, and rank $(A_1 \odot A_1^*) = 4$.

(b) rank $A_2 = 2$, rank $A_2^* = 3$, and rank $(A_2 \odot A^*_2) = 4$.

(c) rank $A_3 = 2$, rank $A_3^* = 3$,

(d) rank $A_4 = 3$, rank $A_4^* = 2$,

(e) rank $A_5 = 3$, rank $A_5^* = 3$.

Definition : Let us denote, for any positive integer s,

$$(A \odot)^s A^* = (A \odot)^{s-1} A \odot A^*$$
$$= \underbrace{A \odot A \odot \ldots A \odot}_{s \text{ times}} A^* \qquad \ldots \quad (2.0)$$

(ix) If no two column vectors of $(A \odot)^s A^*$ or $A^*(\odot A)^s$ are proportional, then

(a) no two column vectors of A are proportional, and

(b) each column vector of A has at least two non-zero entries.

(x) Rank $(A \odot)^s A^* \geq$ rank $(A \odot)^t A^*$ for $s \geq t \geq 0$.

(xi) Rank of $A \odot A \geq$ rank A where A is of order $p \times r$, but if no two column vectors of A are proportional to each other, then rank of $A \odot A \geq$ min $(r, 1+\text{rank } A)$.

3. Solutions to some functional equations

First we quote a lemma proved in an earlier paper (Lemma 2 in Rao, 1966) which is used in proving the main results of this section.

Lemma 1 : *Let A be $p \times r$ matrix such that the i-th column vector of A is not a multiple of any other column vector of A or of any column vector of B of order $p \times m$, and the first element of the i-th column vector of A is non-zero (without loss of generality). Then there exists a $2 \times p$ matrix*

$$H = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & h_2 & \dots & h_p \end{bmatrix} \qquad \dots \quad (3.1)$$

such that the matrices

$$C_1 = HA, \quad C_2 = HB \qquad \dots \quad (3.2)$$

of orders $2 \times r$ and $2 \times m$ respectively satisfy the property that the i-th column vector of C_1 is not a multiple of any other column vector of C_1 or of any column vector of C_2.

3.1. *Functional equation* (1.3). Consider the functional equation

$$\phi_1(\alpha_1' t)+\dots+\phi_r(\alpha_r' t) = \xi_1(t_1)+\dots+\xi_p(t_p) \qquad \dots \quad (3.3)$$

defined for $|t_i| < \partial$, $i = 1, \dots, p$, where t is a column vector of the variables $t_1, \dots, t_p$ and $\alpha_1, \dots, \alpha_r$ are the column vectors of a given matrix A (of order $p \times r$). The functions $\phi_1, \dots, \phi_r, \xi_1, \dots, \xi_p$ are unknown except that they are continuous. The object is to determine the form of these functions under different conditions on the elements of A.

Lemma 2 : *Let α_i, the i-th column vector of A, be not proportional to any other column of A or to any column of I_p an identity matrix of order p. Then the function ϕ_i is a polynomial of maximum degree r.*

Proof : Without loss of generality we take the first element of i-th column of A as non-zero. By Lemma 1, there exists a matrix

$$H = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & h_2 & \dots & h_p \end{bmatrix} \qquad \dots \quad (3.4)$$

such that the i-th column of $B = HA$ is not proportional to any other column of HA or to any column of H. Let

$$t' = (t_1, \ldots, t_p) = (u_1, u_2)H. \quad \ldots \quad (3.5)$$

Then the equation (3.3) becomes

$$\begin{aligned} &\phi_1(b_{11}u_1+b_{12}u_2)+\ldots+\phi_r(b_{r1}u_1+b_{r2}u_2) \\ &= \xi_1(u_1)+\xi_2(h_2u_2)+\ldots+\xi_p(h_pu_2) \\ &= \xi_1(u_1)+\eta(u_2) \end{aligned} \quad \ldots \quad (3.6)$$

valid for some interval round the origin of u_1 and u_2. In (3.6), (b_{i1}, b_{i2}) is not proportional to (b_{j1}, b_{j2}), $j \neq i$ or to $(1, 0)$ or to $(0, 1)$. Hence the term $\phi_i(b_{i1}u_1+b_{i2}u_2)$ cannot be combined with any other ϕ_j. Then by using Linnik's lemma as stated by Rao (1966), ϕ_i is a polynomial of degree r utmost.

Lemma 3 : *If no column of A is proportional to any other column of A or to any column of I_p, then $\phi_1, \ldots, \phi_r$ and $\xi_1, \ldots, \xi_p$ are all polynomials of degree r utmost.*

Proof : By Lemma 2, $\phi_1, \ldots, \phi_r$ are all polynomials of degree r utmost, and hence $\xi_1, \ldots, \xi_p$ are all polynomials of degree r utmost.

Lemma 4 : *Consider the matrix A^* of order $p(p-1)\times r$ as defined in Section 2. If rank A^* is r, then $\phi_1, \ldots, \phi_r$ and $\xi_1, \ldots, \xi_p$ are all linear functions.*

Proof : The proof consists of two parts. By (viii) of Section 2, rank $A^* = r$ implies that no column of A is a multiple of any other column of A or of any column of I_p. Hence using Lemma 3, all ϕ_i and all ξ_i are polynomials of degree r utmost.

Now let

$$\begin{aligned} \phi_i(u) &= \lambda_{ir}u^r+\ldots+\lambda_{i1}u+\lambda_{i0}, \qquad i = 1, \ldots, r, \\ \xi_j(u) &= \mu_{jr}u^r+\ldots+\mu_{j1}u+\mu_{j0}, \qquad j = 1, \ldots, p \end{aligned} \quad \ldots \quad (3.7)$$

and denote $\lambda_i' = (\lambda_{1i}, \ldots, \lambda_{ri})$. Using the functional forms (3.7) in (3.3) and collecting the coefficients of t_it_j, $i \neq j$ we find

$$A^*\lambda_2 = 0 \quad \ldots \quad (3.8)$$

which implies that $\lambda_2 = 0$, since A^* has full rank equal to r by assumption. Thus the second degree terms in the polynomial forms (3.7) are absent.

Now collecting the coefficients of $t_i^{\pi_1}t_j^{\pi_2}t_k^{\pi_3}$, $i \neq j \neq k$ and $(\pi_1+\pi_2+\pi_3) = 3$ with at least two π's non-zero, we find

$$(A \odot A^*)\lambda_3 = 0 \quad \text{or} \quad (A^* \odot A)\lambda_3 = 0. \quad \ldots \quad (3.9)$$

By (x) of Section 2 rank $(A \odot A^*) = s$ since rank $A^* = s$. Thus $\lambda_3 = 0$, or the third degree terms are absent. Similarly collecting coefficients of $t_i^{\pi_1}t_j^{\pi_2}t_k^{\pi_3}t_l^{\pi_4}$, $i \neq j \neq k \neq l$ and $(\pi_1+\pi_2+\pi_3+\pi_4) = 4$, with at least two π's non-zero, we find

$$[(A \odot)^2A^*]\lambda_4 = 0 \quad \text{or} \quad [A^*(\odot A)^2]\lambda_4 = 0 \quad \ldots \quad (3.10)$$

and by the same argument used to show $\lambda_3 = 0$ we have $\lambda_4 = 0$ and so on. Thus, all terms of degree higher than unity are absent in the polynomials (3.7) which proves that $\phi_1, \ldots, \phi_r$ can utmost be of degree one and so must be $\xi_1, \ldots, \xi_p$.

Lemma 5 : *Consider a non-negative integer* $s < r-1$. *If rank* $[(A\odot)^s A^*] = r$, *then* $\phi_1, \ldots, \phi_r, \xi_1, \ldots, \xi_p$ *are all polynomials of degree* $(s+1)$ *utmost.*

Proof: Since rank $(A\odot)^s A^* = r$, no two columns of $(A\odot)^s A^*$ are proportional. Hence using (ix) of Section 2, no two columns of A are proportional and each column of A has at least two non-zero entries. Then Lemma 2 shows that the functions $\phi_1, \ldots, \phi_r, \xi_1, \ldots, \xi_p$ are all polynomials of degree r utmost.

Since rank $(A\odot)^s A^* = r$, by arguments similar to those of Lemma 4, the $(s+2)$-th degree terms in the polynomials are absent. Further, the condition, rank $(A\odot)^s A^* = r \Longrightarrow$ rank $[(A\odot)^{s+1} A^*] = r$. Then the $(s+3)$-th degree terms are absent and so on, so that the maximum degree of $\phi_1, \ldots, \phi_r$ can be $(s+1)$ utmost. This proves Lemma 5.

Corollary : *Let* A *be of rank* $r (< p)$ *such that each column has at least two non-zero entries. Then* $\phi_1, \ldots, \phi_r, \xi_1, \ldots, \xi_p$ *are all quadratic functions.*

Proof: Note that in this case A^* has all the column vectors non-null. Hence rank $(A\odot A^*) \geqslant$ rank $A = r$ which is true only if rank $(A\odot A^*) = r$. Then by Lemma 5, we get the result.

3.2. *Functional equation* (1.4). Consider a partitioned matrix

$$\begin{bmatrix} \underset{(q\times p)}{C} & \underset{(q\times r)}{B} \\ \underset{(p\times p)}{I} & \underset{(p\times r)}{A} \end{bmatrix} \qquad \ldots \quad (3.11)$$

and represent the i-th column vector of A by α_i and the (u, i)-th element of A by a_{ui}. Similarly $\beta_i, b_{ji}, \gamma_u, c_{ju}$ are defined for the matrices B and C respectively. The column vectors of I_p are denoted by $e_1, \ldots, e_p$.

Consider the q equations in p unknowns $t' = (t_1, \ldots, t_p)$

$$\sum_{u=1}^{p} c_{ju}\, \psi_u(e_u' t) + \sum_{i=1}^{r} b_{ji}\, \phi_i(\alpha_i' t) = g_j \text{ (constant)}, \quad j = 1, \ldots, q \qquad \ldots \quad (3.12)$$

defined for $|t_i| < \partial$, $i = 1, \ldots, p$, where $\psi_1, \ldots, \psi_p; \phi_1, \ldots, \phi_r$ are continuous functions.

Lemma 6 : *Let*

(a) *each column of* C *and* B *have at least one non-zero entry, and*

(b) *each column of* A *be not proportional to any other column of* A *or to any column of* I_p.

Then $\psi_1, \ldots, \psi_p$ *and* $\phi_1, \ldots, \phi_r$ *are all polynomials of degree* r *utmost.*

Proof: The proof follows on the same lines as those of Lemmas 2 and 3. The condition (b) of Lemma 6 can be replaced by the more general condition (b').

(b') Suppose that a column α_{i_1} of A is proportional to other columns α_{i_2}, α_{i_3}, ..., of A and some e_m. Then it should be possible to find constants $a_1, \ldots, a_q$ such that in the equation

$$\sum_j a_j [\sum_u c_{ju} \psi_u(e_u' t) + \sum_i b_{ji} \phi_i(\alpha_i' t) - g_j] = 0 \quad \ldots \quad (3.13)$$

the coefficients of functions involving the arguments $\alpha_{i_2}' t$, $\alpha_{i_3}' t$, ..., $e_m' t$ are all zero and the coefficient of the function involving the argument $\alpha_{i_1}' t$ is not zero. Observe that the equation (3.13) is obtained from the equations (3.12) by multiplying the j-th equation by a_j and adding over j.

Lemma 7: *Let in* (3.12) *rank* $(B \odot A^*) = r$. *Then* $\psi_1, \ldots, \psi_p, \phi_1, \ldots, \phi_r$ *are linear functions.*

Lemma 8: *Let in* (3.13), *rank* $[B \odot (A \odot)^s A^*] = r$ (*where* $s < r-1$). *Then* $\psi_1, \ldots, \psi_p, \phi_1, \ldots, \phi_r$ *are polynomials of degree* $(s+1)$ *utmost.*

Note that on account of (vi) and (viii) or (ix) of Section 2, the condition (b') of Lemma 6 will be satisfied. Hence, proofs of Lemmas 7 and 8 are similar to those of Lemmas 4 and 5.

3.3. *Functional equation* (1.5). Consider the q equations involving an unknown function ϕ and a single variable t

$$\sum_{j=1}^{n} d_{ij} \phi(b_j t) = g_i, \quad i = 1, \ldots, q \quad \ldots \quad (3.14)$$

defined for $|t| < \delta$, where $b_1, \ldots, b_n$ are different without loss of generality. By multiplying the i-th equation of (3.14) by a_i' and adding over i, we obtain a compound equation

$$a_1 \phi(b_1 t) + \ldots + a_n \phi(b_n t) = h$$

where

$$a_j = \sum_i a_i' d_{ji}, \quad h = \sum_i a_i' g_i \quad \ldots \quad (3.15)$$

Lemma 9: *Let there exist constants* $a_1', \ldots, a_q'$ *such that the coefficients* $a_1, \ldots, a_q$ *satisfy the following conditions.*

(a) $\sum a_i b_i = 0$, *and*

(b) *if* $a_1, \ldots, a_s (s \leqslant n)$ *are non-zero without loss of generality, then there is only one element in the set* $(|b_1|, \ldots, |b_s|)$ *which exceeds the others. If* $|b_1| > \max (|b_2|, \ldots, |b_s|)$, *without loss of generality, then* $a_i b_i$, $i = 2, \ldots, s$ *have the same sign but different from that of* $a_1 b_1$. *Further let* $\phi(t) = c + t\psi(t)$ *where* $\psi(t) \to$ *constant as* $t \to 0$. *Then* $\phi(t)$ *is a linear function of* t.

We observe that, if some of the b_i are the same we can rewrite the equations (3.14) by combining some of the terms such that in the resulting equations the b_i are all different though with a lesser number of terms. The conditions of the lemma can then be stated in terms of coefficients of the reduced equations.

The proof is similar to that of Lemma 2 given by Rao (1967), using the compound equation (3.15).

4. Characterization of the normal law

Let $X_1, \ldots, X_n$ be independent random variables, not necessarily identically distributed. Consider a linear function

$$a_1X_1+\ldots+a_nX_n \qquad \ldots \quad (4.1)$$

with all non-zero coefficients, which by suitable scaling can be written as

$$L = X_1+\ldots+X_n. \qquad \ldots \quad (4.2)$$

Further let

$$\begin{array}{l} b_{11}X_1+\ldots+b_{1n}X_n \\ \quad . \qquad \ldots \qquad . \\ b_{p1}X_1+\ldots+b_{pn}X_n \end{array} \qquad \ldots \quad (4.3)$$

be p linearly independent functions, which by a suitable transformation can be written in a canonical form

$$\begin{array}{l} M_1 = X_1+c_{11}X_{p+1}+\ldots+c_{1,n-p}X_n \\ \quad . \quad . \quad . \quad \ldots \quad . \\ M_p = X_p+c_{p1}X_{p+1}+\ldots+c_{p,n-p}X_n. \end{array} \qquad \ldots \quad (4.4)$$

Denote the matrix of the c_{ij} coefficients in the equation (4.4) by C which is of order $p\times(n-p)$ and let $\mathbf{c}_i$ be the i-th column vector of C.

Theorem 1 : *Let $p > 1$ and each column vector of C be not proportional to any other column vector of C or to a column of the identity matrix. Further let $X_1, \ldots, X_n$ have finite first moments. Then the condition*

$$E(L\,|\,M_1, \ldots, M_p) = 0 \qquad \ldots \quad (4.5)$$

is necessary and sufficient that $X_1, \ldots, X_n$ are all normally distributed.

Proof: The condition (4.5) is equivalent to

$$E\left(Le^{it_1M_1+\ldots+it_pM_p}\right) = 0 \qquad \ldots \quad (4.6)$$

which gives the functional equation

$$\psi_1(t_1)+\ldots+\psi_p(t_p)+\psi_{p+1}(\mathbf{c}_1'\,t)+\ldots+\psi_n(\mathbf{c}_{n-1}'t) = 0 \qquad \ldots \quad (4.7)$$

valid for $|t_i| < \partial$, $i = 1, \ldots, p$, where $\psi_i = \phi_i'/\phi_i$, ϕ_i being the characteristic function of X_i. Using Lemma 2, we find each ψ_i is a polynomial and hence X_i is normal.

Theorem 2 : *Suppose that* $\mathbf{c}_i$ *is proportional to* $\mathbf{c}_{i_1}, \mathbf{c}_{i_2}, \ldots,$ *and to* $\mathbf{e}_s$, *the s-th column of a unit matrix. Let*

$$\mathbf{c}_{i_1} = \lambda_{i_1}\mathbf{c}_i, \mathbf{c}_{i_2} = \lambda_{i_2}\mathbf{c}_i, \ldots, \mathbf{e}_s = \lambda_s \mathbf{c}_i. \qquad \ldots \quad (4.8)$$

If $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_s$ *are all of the same sign and at least one column of* C *contains two non-zero entries, then* $X_i, X_{i_1}, X_{i_2}, \ldots$ *are all normally distributed when* (4.5) *holds and the first moments of the variables exist.*

The proof follows on the same lines as in Theorem 3 of Rao (1967).

We now consider two sets of q and $p \geqslant 2$ linear functions which are all independent and hence can be written in canonical forms

$$L_j = \sum_{u=1}^{p} c_{ju}X_u + \sum_{t=p+1}^{n} b_{j,t-p}X_t, \qquad j = 1, \ldots, q$$

$$M_i = X_i + \sum_{u=p+1}^{n} a_{i,u-p}X_u, \qquad i = 1, \ldots, p \qquad \ldots \quad (4.9)$$

and examine the restrictions on the coefficients under which the conditions

$$E(L_j \mid M_1, \ldots, M_p) = 0, \quad j = 1, \ldots, q \qquad \ldots \quad (4.10)$$

imply normality of the variables $X_1, \ldots, X_n$.

Theorem 3 : *Let* $A = (a_{ij})$ *be of order* $p \times n-p$, $B = (b_{ij})$ *be of order* $q \times n-p$ *and* $C = (c_{ij})$ *be of order* $q \times p$. *Then under the restrictions on the elements of* A, B, C, *mentioned in Lemma 6, the condition* (4.10) *and the existence of the first moments of the variables imply normality of the variables* $X_1, \ldots, X_n$.

Proof : The result follows from the functional equations obtained from the conditions

$$E[L_j \exp(it_1 M_1 + \ldots + it_p M_p)] = 0, \qquad j = 1, \ldots, q \qquad \ldots \quad (4.11)$$

by applying Lemma 6.

The result is, however, true under more general situations than those considered in Lemma 6. When some of the vectors in the matrix A are proportional, a theorem similar to Theorem 2 could be stated.

The case of $p = 1$ needs special discussion which we state in Theorem 4.

Theorem 4 : *Let* $X_1, \ldots, X_n$ *be independent and identically distributed random variables, and*

$$L_i = \sum_{j=1}^{n} d_{ij}X_j, \quad i = 1, \ldots, q \qquad \ldots \quad (4.12)$$

be q linearly independent functions and

$$M = b_1X_1+\ldots+b_nX_n \quad \ldots \quad (4.13)$$

be a function linearly independent of $L_1, \ldots, L_q$. Then under the conditions of Lemma 9 on the coefficients b_i and d_{ij}, the conditions

$$E(L_i \mid M) = 0, \quad i = 1, \ldots, q \quad \ldots \quad (4.14)$$

imply that X_i is normally distributed.

Proof: Let $\phi(t)=f'(t)/f(t)$ where $f(t)$ is the characteristic function of X_i. Then the condition (4.14) gives

$$\sum_j d_{ij}\phi(b_jt) = 0, \qquad i = 1, \ldots, q \quad \ldots \quad (4.15)$$

valid for $|t| < \delta$. It is shown by R. N. Pillai* that the condition (4.15) implies that $E(X_i^2) < \infty$, so that ϕ is of the form $t\psi(t)$ where $\psi(t) \to a$ (constant) as $t \to 0$. Then applying Lemma 9, we find ϕ is a linear function and hence X_i is normally distributed.

5. Characterization of the gamma distribution

Let $X_1, \ldots, X_n$ be independent random variables, not necessarily identically distributed. We consider two situations where X_i are non-negative and when X_i are arbitrary. When X_i are non-negative, let

$$Y_i = \log X_i$$

$$M_i = a_{i1}Y_1+\ldots+a_{in}Y_n \quad \ldots \quad (5.1)$$

and when X_i are arbitrary, let

$$M_i' = a_{i1}X_1+\ldots+a_{in}X_n. \quad \ldots \quad (5.1')$$

In a previous paper (Khatri and Rao, 1968), it was proved, under some conditions, that

$$E\left(\sum_{j=1}^{n} b_{ij}e^{Y_j} \mid M_1, \ldots, M_{n-q}\right) = g_i, \text{ (constant)}, \quad i = 1, \ldots, q \leqslant n-2 \quad \ldots \quad (5.2)$$

when X_i are non-negative or

$$E(\Sigma\, b_{ij}X_j^{-1} \mid M_1', \ldots, M_{n-q}') = g_i', \text{ (constant)}, \quad i = 1, \ldots, q \leqslant (n-2) \quad \ldots \quad (5.2')$$

when X_i are arbitrary, implies that $X_1, \ldots, X_n$ have gamma or conjugate gamma distributions. Now, we consider the conditions under which

$$E(\Sigma b_{ij}e^{Y_j} \mid M_1, \ldots, M_p) = g_i \text{(constant)}, \quad i = 1, \ldots, q; \quad q+p \leqslant n \quad \ldots \quad (5.3)$$

when X_i are non-negative, or

$$E(b_{ij}X_j^{-1} \mid M_1', \ldots, M_p') = g_i', \text{(constant)}, \quad i = 1, \ldots, q; \quad q+p \leqslant n \quad \ldots \quad (5.3')$$

*Published in this issue, see pp. 145–146.

when X_i are arbitrary implies that $X_1, ..., X_n$ have gamma or conjugate gamma distributions. It may be noted that the generalization consists in reducing the conditioning variables to p from the full complement of $n-q$ considered in the earlier paper by the authors (Khatri and Rao, 1968).

We observe that the linear functions $M_1, ..., M_p$ can be considered to be linearly independent in which case they may be represented in the canonical form

$$\begin{aligned} M_1 &= Y_1+a_{11}Y_{k+1}+...+a_{1,\,n-p}Y_n \\ &\cdot \quad \cdot \quad \cdots \quad \cdot \\ M_p &= Y_p+a_{p1}Y_{p+1}+...+a_{p,\,n-p}Y_n. \end{aligned} \qquad \text{... (5.4)}$$

The functions $M'_1, ..., M'_p$ have a similar representation. Let A denote the matrix (a_{ij}). We shall first consider the case of $q = 1$.

Theorem 5: *Let $Y_1, ..., Y_n$; $M_1, ..., M_p$ and $M'_1, ..., M'_p$ be as considered in (5.1), (5.1′) and (5.4) respectively. Further let $1 < p \leqslant (n-1)$ and the rank of A^* defined in Section 2 be $r = (n-p)$. Then the condition*

$$E(e^{Y_1}+...+e^{Y_n}\,|\,M_1, ..., M_p) = g \text{ (constant)}, \qquad \text{... (5.5)}$$

when X_i are non-negative and $E(X_i) < \infty$ for all i, implies that $X_1, ..., X_n$ have gamma distributions. The condition

$$E(X_1^{-1}+...+X_n^{-1}\,|\,M'_1, ..., M'_p) = g \text{ (constant)} \qquad \text{... (5.5′)}$$

when X_i are arbitrary and $E(X_i^{-1}) \neq 0$ and $|E(X_i^{-1})| < \infty$ for all i, implies that $X_1, ..., X_n$ have gamma distributions when $E(X_i) > 0$ and conjugate gamma distribution when $E(X_i) < 0$.

Proof: Let

$$\phi_j(t) = \Big[\int e^{Y_j}e^{itY_j}\,dF(Y_j)\Big]\Big/\Big[\int e^{itY_j}dF(Y_j)\Big] \qquad \text{... (5.6)}$$

when X_i are non-negative or

$$\phi_j(t) = \Big[\int x_j^{-1}e^{itx_j}\,dF(x_j)\Big]\Big/\Big[\int e^{itx_j}dF(x_j)\Big] \qquad \text{... (5.6′)}$$

when X_i are arbitrary and $E(X_i^{-1}) \neq 0$. Then the condition (5.5) or (5.5′) is equivalent to

$$\phi_1(t_1)+...+\phi_p(t_p)+\phi_{p+1}(\alpha'_1t)+...+\phi_n(\alpha'_{n-p}t) = \text{constant} \qquad \text{... (5.7)}$$

valid for $|t_i| < \partial$, $i = 1, ..., p$, where α_i is the i-th column vector of A and $t' = (t_1, ..., t_p)$. We now apply Lemma 4 which shows that, under the condition rank $A^* = r = n-p$, the functions $\phi_1 ..., \phi_n$ are all linear in t. In such a case, it is shown in the earlier paper of Khatri and Rao (1968) that X_i has a gamma distribution for each i.

It may be noted that in the conditions (5.5) and (5.5′) we could have chosen the more general function under the expectation,

$$a_1 e^{Y_1}+\ldots+a_n e^{Y_n}, \quad a_i \neq 0, \quad i = 1, \ldots, n \qquad \ldots \quad (5.8)$$

or

$$a_1 X_1^{-1}+\ldots+a_n X_n^{-1}, \quad a_i \neq 0, \quad i = 1, \ldots, n \qquad \ldots \quad (5.8')$$

and obtained the same result.

Theorem 6 : *Let $X_1, \ldots, X_n$ be independent and non-negative random variables with finite expectations and $Y_1, \ldots, Y_n$ be as defined in (5.1). Consider*

$$L_i = c_{i1} e^{Y_1}+\ldots+c_{ip} e^{Y_p}+b_{i1} e^{Y_{p+1}}+\ldots+b_{i\,n-p} e^{Y_n}, \quad i = 1, \ldots, q \qquad \ldots \quad (5.9)$$

$$M_j = Y_j+a_{j1} Y_{p+1}\ldots+a_{j\,n-p} Y_n, \qquad j = 1, \ldots, p > 1. \qquad \ldots \quad (5.10)$$

If the matrices $C = (c_{ij})$, $B = (b_{ij})$ and $A = (a_{ij})$ satisfy the conditions of Lemma 7, and

$$E(L_i \mid M_1, \ldots, M_p) = g_i (\text{constant}), \quad i = 1, \ldots, q, \qquad \ldots \quad (5.11)$$

then $X_1, \ldots, X_n$ have gamma distributions.

Proof: It is seen that condition (5.11) gives rise to a functional equation of the form (3.12) and hence an application of Lemma 7 yields the desired result.

Theorem 7 : *Let $X_1, \ldots, X_n$ be independent variables with non-zero and finite expectations for $X_1^{-1}, \ldots, X_n^{-1}$, and*

$$L_i = c_{i1} X_1^{-1} +\ldots+c_{ip} X_p^{-1}+b_{i1} X_{p+1}^{-1}+\ldots+b_{i\,n-p}\, X_n^{-1}, \quad i = 1, \ldots, q, \qquad \ldots \quad (5.12)$$

$$M_j' = X_j+a_{j1} X_{p+1}+\ldots+a_{j\,n-p}\, X_n, \qquad j = 1, \ldots, p > 1. \qquad \ldots \quad (5.13)$$

If the matrices $C = (c_{ij})$, $B = (b_{ij})$ and $A = (a_{ij})$ satisfy the conditions of Lemma 7, and

$$E(L_i \mid M_1', \ldots, M_p') = g_i \ (\text{constant}), \quad i = 1, \ldots, q \qquad \ldots \quad (5.14)$$

then X_i has a gamma or a conjugate gamma distribution according as $E(X_i) > 0$ or < 0, $i = 1, \ldots, n$.

Theorem 8 : *Let $X_1, \ldots, X_n$ be non-negative independent and identically distributed variables. Consider*

$$L_i = \Sigma\, d_{ij}\, e^{Y_j}, \qquad i = 1, \ldots, q \qquad \ldots \quad (5.15)$$

$$M = b_1 Y_1+\ldots+b_n Y_n \qquad \ldots \quad (5.16)$$

where the coefficients b_i and d_{ij} satisfy the conditions of Lemma 9. Then the conditions

$$E(L_i \mid M) = g_i \ (\text{constant}), \qquad i = 1, \ldots, q \qquad \ldots \quad (5.17)$$

and $E(X_i \log X_i)$ is bounded imply that X_i has a gamma distribution.

Theorem 9 : *Let $X_1, \ldots, X_n$ be independent and identical variables (not necessarily non-negative) such that $E(1/X_i)$ exists and is non-zero. Further let*

$$L_i' = \Sigma\ d_{ij}\ X_j^{-1}, \qquad i = 1, \ldots, q$$

and

$$M' = b_1 X_1+\ldots+b_n X_n$$

where the coefficients b_i and d_{ij} satisfy the conditions of Lemma 9. Then the conditions

$$E(L'_i \mid M') = g'_i \quad (constant), \quad i = 1, \ldots, q$$

imply that X_i has a gamma or a conjugate gamma distribution according as $E(X_i) > 0$ or < 0.

The proofs of Theorems 8 and 9 follow on the same lines as the other theorems.

REFERENCES

RAO, C. RADHAKRISHNA (1966): Characterisation of the distribution of random variables in linear structural relations. *Sankhyā*, Series A, 28, 251-260.

——— (1967): On some characterisations of the normal law. *Sankhyā*, Series A, 29, 1-14.

KHATRI, C. G. and RAO, C. RADHAKRISHNA (1968): Some characterisations of the gamma distribution. *Sankhyā*, Series A, 30, 157-166.

LINNIK, YU. V. (1964): *Decomposition of Probability Distribution*, Oliver and Boyd.

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