Essays on Incentive Compatibility on Restricted Domains

Anup Pramanik

Thesis submitted to the Indian statistical Institute in partial fulfilment of the requirements for the award of the degree of Doctor of Philosophy

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To Maa

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Chapter 1

Introduction

This thesis comprises three chapters that investigate the structure of dominant strategy incentive compatible (strategy-proof) social choice functions on restricted domains. The first chapter deals with standard mechanism design problem in quasi-linear environments with multidimensional non-convex type spaces. It identifies a class of type spaces called *ordinal type spaces* and shows that the simple and familiar *2-cycle monotonicity* condition is necessary and sufficient for implementation. This result covers the single-peaked domain. The second chapter deals with the standard social choice problem (no monetary transfers) but one where agents have indifference in their preference orderings generated via an exogenous partition of the set of alternatives. A domain is dictatorial if a strategy-proof and unanimous social choice function defined on this domain is dictatorial. The chapter explores the relationship between agent partitions and dictatorial domains. The final chapter considers the standard social choice model with linear preferences. It provides new results on dictatorial domains in this model.

We provide a brief description of each chapter below.

1.1 Multidimensional Mechanism Design with Ordinal Restrictions

In this chapter, we study multidimensional mechanism design in private values and quasi-linear utility environments when the set of alternatives is finite and the allocation rule is deterministic. A standard goal in mechanism design is to investigate conditions that are necessary and sufficient for implementing an allocation rule. An allocation rule in such an environment is implementable if there exists a payment rule such that truth-telling is a dominant strategy for the agents in the resulting mechanism. Our main result is that in a large class of multidimensional type spaces that satisfy some ordinal restrictions, implementability is equivalent to a simple condition called *2-cycle monotonicity*. By virtue of revenue equivalence, which holds in these type spaces, we are able to characterize the entire class of dominant strategy incentive compatible mechanisms. The 2-cycle monotonicity condition requires the following: given the types of other agents, if the alternative chosen by the allocation rule is a when agent i reports its type to be t and the alternative chosen by the allocation rule is b when agent i reports its type to be s, then it must be that

$$t(a) - t(b) \geq s(a) - s(b),$$

where for any alternative x, $t(x)$ and $s(x)$ denote the values of alternative x in types t and s respectively.

Rochet (1987) showed that a significantly stronger condition called cycle monotonicity is necessary and sufficient for implementability in any type space - see also Rockafellar (1970). Myerson (1981) formally establishes that in the single object auction set up, where the type is single dimensional, 2-cycle monotonicity is necessary and sufficient for implementation. When the type space is multidimensional, if the set of alternatives is finite and the type space is convex, 2-cycle monotonicity implies cycle monotonicity (Bikhchandani et al., 2006; Saks and Yu, 2005; Ashlagi et al., 2010). Though convexity is a natural geometric property satisfied in many economic environments, it excludes many interesting type spaces. A primary objective of this chapter is to formulate restrictions on type spaces without the convexity assumption made in the literature and answer the question of implementability in such multidimensional type spaces. Indeed, our restrictions allow many interesting multidimensional non-convex type spaces. Prominent type spaces covered by our formulation are the single peaked type spaces and its generalizations. We show that 2-cycle monotonicity is necessary and sufficient for implementability in all these type spaces.

Our method to impose *ordinal* restriction on type spaces is quite simple. Such a method of imposing ordinal restriction is usually followed in the mechanism design literature without transfers (a la strategic voting or social choice theory literature). To see how such restrictions can be imposed in a cardinal environment like ours, note that a type in our environment is a vector in $\mathbb{R}^{|A|}$, where A is the set of alternatives. Now, let us restrict attention to *strict* types, where value of no two alternatives is the same. Such a type must induce a complete and strict ordering on A. We put restrictions by allowing only a subset of orderings that can be induced by any type. The set of all strict types in $\mathbb{R}^{|A|}_{++}$ that induce an ordering belonging to a set of permissible orderings define a *strict type space*. To allow for indifferences, we take the closures of such type spaces. We call such type spaces *ordinal type spaces*. A prominent ordinal type space where our 2-cycle monotonicity characterization holds is the type space induced by all single peaked preference ordering on a tree graph, where the graph consists of alternatives as nodes and a preference ordering must be single peaked along paths of the tree. Single peakedness on a tree is a generalization of classical single peaked preference orderings due to Demange (1982). We also show that for a large class of permissible preference orderings, the 2-cycle monotonicity characterization holds if the type space consists of only *strict types* induced by these orderings. We give an example to show that our result does not hold if we allow for indifference in such type spaces. If we assume a mild continuity condition on the allocation rules, then our result holds with indifference.

1.2 Strategy-proof Social Choice in the Exogenous Indifference Class Model

This chapter deals with the standard social choice problem (no monetary transfers) but one where agents have indifference in their preference orderings in a specific way. In particular, we investigate *the exogenous indifference class model* (first introduced in Barbera and Ehlers (2011)). In this model, the indifference classes of agents' preferences is exogenously given. Specifically, every individual has an exogenous partition of the set of alternatives. An individual is always indifferent between alternatives a and b iff both a and b belong to the same element of her partition set. But an individual's ranking of the different elements of her partition set, is complete.

This framework includes several well-studied models as special cases. For instance, the case of private goods and selfish preferences is one where an individual is indifferent between all alternatives that give her the same commodity bundle. It includes the one-sided matching model studied in Svensson (1999), Papai (2000). It also includes the Gibbard-Satterthwaite framework where the elements of the partition are all singletons. Further examples are provided in Sato (2009).

In this model, we examine the relationship between dictatorship results in this model and the structure of indifference classes across agents. Our results are formulated in terms of the *pairwise partition graph* induced by the indifference classes. Fix a pair of agents i and j and their indifference classes. The partition graph for this pair is a bipartite graph whose vertices are i and j's indifference classes. There are no edges between the vertices representing the indifference classes of a given agent; vertices for i and j' have an edge if the indifference classes representing these vertices have *no common alternative*. We show that a necessary condition for strategy-proofness and unanimity to imply dictatorship in the domain induced by a partition is that each associated pairwise partition graph is connected with the degree of every vertex being at least two. If we replace unanimity by efficiency, this requirement can be weakened to the graphs being connected (with possibly isolated vertices).

We are unfortunately, unable to show that these necessary conditions are sufficient for dictatorship. However we are able to identify a number of stronger conditions that are sufficient. The first of these is the existence of at least two common indifference classes with no restrictions on their size - a result which clearly generalizes that of Sato. In addition we have three sufficient conditions for the case of two voters. One applies to the case where there is exactly one common indifference class while another shows that strategy-proofness and unanimity imply dictatorship when the partition graph is a cycle. Finally, we show that with the stronger assumption of efficiency, strategy-proofness implies dictatorship when the partition graph is connected with possibly isolated vertices. The last condition implies that a full characterization is obtained in the case of two agents for domains where strategy-proofness and efficiency imply dictatorship.

1.3 Further Results on Dictatorial Domains

The seminal result of Gibbard (1973) and Satterthwaite (1975) states that a surjective and strategy-proof social choice function (scf) defined over the complete domain, is dictatorial (provided that there are at least three alternatives). Aswal et al. (2003) show that the assumption of a complete domain is far from being necessary for this result. They say that two alternatives a and b are *connected* if there exists a preference in the domain where a is ranked first and b, second and another preference where the reverse is true. They consider the following graph: each alternative is a vertex and there is an edge between a pair of vertices if the two alternatives represented by the vertices, are connected. A domain is *linked* if there exist an arrangement of the vertices such that the first three are mutually connected and each vertex is connected to at least two in the set of vertices that precedes it. Their main result is that every linked domain is dictatorial.

In this chapter we generalize the linked domain result of Aswal et al. (2003) in two ways. The first way is to weaken the notion of connectedness between a pair of alternatives to *weak connectedness* while retaining the "connection structure" of the induced graph as in linkedness. The second way is to strengthen the notion of connectedness but weakening the "connection structure" on the induced graph.

The notion of weak connectedness is the following: two alternatives a and b are weakly connected if there exists a (possibly empty) set of alternatives B and *four* orderings in the domain such that there is a *reversal* between B and b when a is top-ranked and there is a reversal between B and a when b is top-ranked. Reversality requires alternatives between a and b to belong to B in the case where B is better than b. Similarly, alternatives between b and a to belong to B in the case where B is better than a. A domain is called a β domain if we can arrange all the alternatives (vertices in the induced graph) in a way that the first three are mutually weakly connected and each alternative is weakly connected to at least two in the set of alternatives (vertices) that precedes it. Our first result is that β domains are dictatorial. These domains are obviously supersets of linked domains. It is also possible to find β domains that are smaller than any linked domain.

Property T between a and b requires the following "intermediateness" property in addition to weak connectedness: for any alternative c other than a and b, there exists two orderings in the domain, one where c is above b while a at the top and another where c is above a while b at the top. A domain is called a γ domain if its induced graph is connected in the usual graph-theoretic sense, i.e. there exists a path between any two alternatives (vertices). Our second result is that all γ domains whose induced graph *is not* a star-graph, are dictatorial domains. The same result holds in the star-graph case with mild additional conditions. These results generalize results on circular domains in Sato (2010) and Chatterji et al. (2013). Finally, we apply our result to a facility location problem in a restricted environment.

Chapter 2

Multidimensional Mechanism Design with Ordinal Restrictions

2.1 Introduction

An enduring theme in mechanism design is to investigate conditions that are necessary and sufficient for implementing an allocation rule. We investigate this question in private values and quasi-linear utility environments when the set of alternatives is finite and the allocation rule is deterministic (i.e., does not randomize). An allocation rule in such an environment is implementable if there exists a payment rule such that truth-telling is a dominant strategy for the agents in the resulting mechanism. Our main result is that in a large class of multidimensional type spaces that satisfy some ordinal restrictions, implementability is equivalent to a simple condition called *2-cycle monotonicity*. By virtue of revenue equivalence, which holds in these type spaces, we are able to characterize the entire class of dominant strategy incentive compatible mechanisms. The 2-cycle monotonicity condition requires the following: given the types of other agents, if the alternative chosen by the allocation rule is a when agent i reports its type to be t and the alternative chosen by the allocation rule is b when agent i reports its type to be s, then it must be that

$$t(a) - t(b) \geq s(a) - s(b),$$

where for any alternative x, $t(x)$ and $s(x)$ denote the values of alternative x in types t and s respectively.

One of the earliest papers to pursue this question was Rochet (1987), who proved a very general result. He showed that a significantly stronger condition called cycle monotonicity is necessary and sufficient for implementability in any type space - see also Rockafellar (1970). Myerson (1981) formally establishes that in the single object auction set up, where the type is single dimensional, 2-cycle monotonicity is necessary and sufficient for implementation (in Myerson's set up, this is true even if we consider randomized allocation rules) - see also Spence (1974). When the type space is multidimensional, if the set of alternatives is finite and the type space is convex, 2-cycle monotonicity implies cycle monotonicity (Bikhchandani et al., 2006; Saks and Yu, 2005; Ashlagi et al., 2010). Though convexity is a natural geometric property satisfied in many economic environments, it excludes many interesting type spaces. Moreover, how far this result extends to type spaces that do not satisfy convexity remain an intriguing question - we discuss this issue in detail in Section 2.2. A primary objective of this chapter is to formulate restrictions on type spaces without the convexity assumption made in the literature and answer the question of implementability in such multidimensional type spaces. Indeed, our restrictions allow many interesting multidimensional non-convex type spaces. Prominent type spaces covered by our formulation are the single peaked type spaces [1] and its generalizations. In all these type spaces, we show that 2-cycle monotonicity is necessary and sufficient for implementability. To our knowledge, this chapter is the first to identify such a large class of interesting non-convex type spaces where 2-cycle monotonicity characterizes implementability.

We use a novel method to impose *ordinal* restriction on type spaces. Such a method of imposing ordinal restriction is usually followed in the mechanism design literature without transfers (a la strategic voting or social choice theory literature). To see how such restrictions can be imposed in a cardinal environment like ours, note that a type in our environment is a vector in $\mathbb{R}^{|A|}$, where A is the set of alternatives. Now, let us restrict attention to *strict* types, where value of no two alternatives is the same. Such a type must induce a complete and strict ordering on A. We put restrictions by allowing only a subset of orderings that can be induced by any type. We discuss such restrictions in detail in Section 2.3. The set of all strict types in $\mathbb{R}^{|A|}_{++}$ that induce an ordering belonging to a set of permissible orderings define a *strict type space*. To allow for indifferences, we take the closures of such type spaces. We call such type spaces *ordinal type spaces*. A prominent ordinal type space where our 2-cycle monotonicity characterization holds is the type space induced by all single peaked preference ordering on a tree graph, where the graph consists of alternatives as nodes and a preference ordering must be single peaked along paths of the tree. Single peakedness on a tree is a generalization of classical single peaked preference orderings due to Demange (1982). A detailed definition is given in Section 2.3. We also show that for a large class of permissible preference orderings, the 2-cycle monotonicity characterization holds if the type space consists of only *strict types* induced by these orderings. We give an example to show

[1] Roughly, a single peaked type is defined using a strict and complete order on the set of alternatives. A type is single peaked if the values of alternatives decrease as we go to the *left* or *right* (where left and right are defined with respect to the given order) of the peak (the highest valued alternative).

that our result does not hold if we allow for indifference in such type spaces. If we assume a mild continuity condition on the allocation rules, then our result extends to such type spaces when we allow for indifferences. Thus, we highlight an important consequence of restricting attention to only strict type spaces.

Though we identify many different ordinal type spaces, the proof methodology we employ for them is quite similar. This shows that our general methodology is quite robust and can be potentially applied to other type spaces that we do not discuss in the chapter.

We are aware of only one instance where such an idea of ordinal restriction was pursued in this literature. The *order-based* type space considered in Bikhchandani et al. (2006) is defined by considering a *weak partial* order on the set of alternatives and *every* type must induce this order. Firstly, an order-based type space is convex. Second, we consider *a set of strict and complete* orders and a type in our type space must induce one of the orders in this set. In that sense, our type space restrictions are different from the order-based type space, and neither is stronger than the other.

A characterization of implementability using 2-cycle monotonicity is useful because the cycle monotonicity condition, which can be used to characterize implementability in any type space, is a difficult condition to use and interpret. On the other hand, 2-cycle monotonicity is a simpler condition and the appropriate extension of the monotonicity condition used by Myerson (1981) to characterize implementability in the single object auction model. For this reason, 2-cycle monotonicity is often referred to as *weak monotonicity* (Bikhchandani et al., 2006; Saks and Yu, 2005) or *monotonicity* (Ashlagi et al., 2010). In his paper, Rochet (1987) likens the implementability question to the rationalizability question in revealed preference theory. Quoting Rochet:

> *Condition (3)* [2] *is thus the analogue of the Strong Axiom of Revealed Preferences (SARP), and our theorem is the analogue of Afriat's result [Afriat, (1965)], which shows how to compute, for any set of data satisfying SARP, a utility function which rationalizes the data. In the one dimensional context, one can restrict oneself to cycles of order 2: condition (3) for 2-cycles is the analogue of the Weak Axiom of Revealed Preferences (WARP).*

We also characterize the set of payment rules that can implement an implementable allocation rule. We do this by establishing revenue equivalence in a large class of ordinal type spaces. Revenue equivalence is a property which stipulates that two payment rules that implement the same allocation rule must differ by a constant. We show that the revenue equivalence result holds in a much larger class of ordinal type spaces than the 2-cycle

[2] Condition (3) in Rochet (1987) is the cycle monotonicity condition.

monotonicity result. [3] By characterizing the implementable allocation rules using 2-cycle monotonicity and payments by revenue equivalence, we characterize the set of dominant strategy incentive compatible mechanisms in our multidimensional type spaces with ordinal restrictions.

The rest of the chapter is organized as follows. In Section 2.2, we define the model. In Section 2.3, we define the single peaked type space on a tree and state our main result. Section 2.4 defines a broad class of ordinal type spaces consisting of strict types where the 2-cycle monotonicity characterization holds. We describe the set of payment rules that implement an implementable allocation rule in Section 2.5 using revenue equivalence. Section 2.6 introduces another ordinal type space where the methodology used in our earlier proofs can be used to derive the 2-cycle monotonicity characterization. We relate our results to the literature and conclude in Section 2.7. The Appendix contains all omitted proofs.

2.2 Implementation and Cycle Monotonicity

We consider a model with a single agent. As is well known in this literature, this is without loss of generality. All our results generalize easily to a model with multiple agents. The single agent is denoted by i. The set of alternatives for agent i is denoted by A. In an n-agent model, A denotes the possible allocations of agent i. [4] The type (private information) of agent i is a vector $t \in \mathbb{R}^{|A|}$. If agent i has type t, then $t(a)$ will denote the value of agent i for alternative a. We assume private values and quasi-linear utility. This means that if alternative a is chosen and agent i with type t makes a payment of p, then his net utility is given by $t(a) - p$.

Not all possible vectors in $\mathbb{R}^{|A|}$ can be a type of agent i. Let $D \subseteq \mathbb{R}^{|A|}$ be the type space of agent i - these are the permissible types of agent i. An **allocation rule** is a mapping $f : D \to A$. We will assume that f is onto. This is standard in the literature - if f is not onto, then all the results can be restated in terms of range of A.

A **payment rule** of agent i is a mapping $p : D \to \mathbb{R}$. A **mechanism** consists of an allocation rule and a payment rule.

Definition **2.1** *An allocation rule f is* **implementable** *if there exists a payment rule p such that for every $s, t \in D$, we have*

$$s(f(s)) - p(s) \geq s(f(t)) - p(t).$$

[3] The exact connection of our revenue equivalence result with the literature is established later.

[4] For instance, in a model with n agents and n objects, where each agent can be assigned exactly one object and there is no externality in allocations across agents, A will be the set of objects and *not* the set of *matchings*.

In this case, we will say that p implements f and (f, p) is an **incentive compatible** *mechanism.*

The primary objective of this chapter is to give a simple necessary condition on the allocation rule that is also sufficient for implementability in a large class of interesting type spaces. For this, we revisit a classic condition that is already known to be necessary and sufficient for implementability in *any* type space.

DEFINITION **2.2** *An allocation rule f is K-***cycle monotone***, where $K \geq 2$ is a positive integer, if for every finite sequence of types $(t^1, t^2, \ldots, t^k)$, with $k \leq K$, we have*

$$\sum_{j=1}^{k} \left[t^j(f(t^j)) - t^j(f(t^{j-1})) \right] \geq 0, \tag{2.1}$$

where $t^0 \equiv t^k$. An allocation rule f is **cyclically monotone** *if it is K-cycle monotone for every positive integer $K \geq 2$.*

It is well known that implementability is equivalent to cycle monotonicity (Rochet, 1987; Rockafellar, 1970). This result is very general - it works on any type space D and does not even require A to be finite. [5] However, cycle monotonicity is a difficult condition to use and interpret since it requires verifying non-negativity of Inequality 2.1 for arbitrary length sequences of types. In a series of papers, it has been established that a significantly weaker condition than cycle monotonicity is sufficient for implementation in various interesting type spaces. Bikhchandani et al. (2006) showed that 2-cycle monotonicity is sufficient for implementability if D is an *order-based* type space - this includes many interesting type spaces in the context of multi-object auctions. Saks and Yu (2005) show that 2-cycle monotonicity is sufficient for implementation if D is convex - this extends the result in Bikhchandani et al. (2006) because an order-based type space is convex. Ashlagi et al. (2010) extend this result to show that if the closure of D convex, then 2-cycle monotonicity is sufficient for implementation.

However, Mishra and Roy (2013) show that there are interesting non convex type spaces where 2-cycle monotonicity is not sufficient for implementation. Further, they identify an interesting class of non-convex type spaces where 3-cycle monotonicity is sufficient for implementation but 2-cycle monotonicity is not sufficient.

Interestingly, Ashlagi et al. (2010) establish a surprising result by allowing for *randomization*, i.e., an allocation rule picks a probability distribution over alternatives. They show

[5] When the set of alternatives is finite, this result can be slightly strengthened to say that implementability is equivalent to $|A|$-cycle monotonicity (Mishra and Roy, 2013).

that if every 2-cycle monotone randomized allocation rule is also cyclically monotone in a type space D of dimension at least 2, then the closure of D must be convex.

It is not clear how far this result is true if f is allowed to be deterministic. Vohra (2011) contains a simple example of a non-convex type space with four alternatives where every *deterministic* allocation rule satisfying 2-cycle monotonicity is implementable. In his example, Vohra (2011) considers the sale of two objects α and β to agents. The set of alternatives is the set of all subsets of $\{\alpha, \beta\}$. The restriction on values of agents is the following: $t(\{\alpha, \beta\}) = \max(t(\{\alpha\}), t(\{\beta\}))$ for each i. Hence, each agent desires at most one object, though he may be assigned both the objects. The type space here is non-convex. To see this, consider two types of agent i

$$t(\emptyset) = 0, t(\{\alpha\}) = 3, t(\{\beta\}) = 4, t(\{\alpha, \beta\}) = 4$$
$$s(\emptyset) = 0, s(\{\alpha\}) = 5, s(\{\beta\}) = 4, s(\{\alpha, \beta\}) = 5.$$

A convex combination of $(0.5, 0.5)$ of these two types generates values 4 for objects α and β but a value of 4.5 for the bundle of objects $\{\alpha, \beta\}$. This violates the restriction on the type space.

Note that if we allow at most one object to be assigned to an agent, then the type space becomes convex, and we can apply earlier result to conclude that 2-cycle monotonicity is sufficient for implementation. However, by allowing the alternative $\{\alpha, \beta\}$, but still having a restriction that agents desire at most one object, we get to a non-convex type space. The result in Vohra (2011) shows that 2-cycle monotonicity is sufficient for implementation in such an example. It is not clear on how to extend the proof of this example if there are more than two objects.

2.2.1 A Motivating Example

Since the type space in the example in Vohra (2011) seems to be a slight modification of a convex type space, it is still unclear whether there are interesting non-convex type space where 2-cycle monotonicity is sufficient for implementation. The result in Ashlagi et al. (2010) shows that if every 2-cycle monotone randomized allocation rule is implementable in a multidimensional type space, then it must be convex. This shows that there is a significant gap in understanding implementability of deterministic allocation rules in non-convex multidimensional type spaces. We give below a motivating example to show that there are interesting non-convex type spaces where the current results are silent. Our results will apply to such type spaces.

Consider a general scheduling problem as follows. A number of firms procure products/parts from a supplier over a time horizon. In each time period, the supplier can only supply to one firm. Every firm i has a time period τ^* where it gets the maximum value from getting its products supplied. The firms have *single peaked* preference over time, i.e., for any time periods τ, τ', if $\tau < \tau' < \tau^*$ or $\tau > \tau' > \tau^*$, then a firm values supply of its products at time period τ' to time period τ (this may be due to inventory carrying cost and delivery delay costs).

The type space in this example is non-convex. To see this, suppose there are just three time periods $\{1, 2, 3\}$ and consider two single peaked types of agent (firm) i: $s := (6, 4, 3)$ (peak value is period 1) and $t := (3, 4, 6)$ (peak value is period 3). A convex combination $\frac{s+t}{2}$ produces the type $(4.5, 4, 4.5)$, which is no longer single peaked.

In such non-convex type spaces, we characterize implementability using 2-cycle monotonicity and apply revenue equivalence to obtain a complete characterization of dominant strategy incentive compatible mechanisms. Thus, there are interesting type spaces where earlier results are silent and our results provide sharp characterizations of implementability and incentive compatibility.

2.3 The Type Space

We consider the problem of choosing an alternative (a location) over a tree network. Our network G is given by a finite set of nodes A and a set of undirected edges E between these nodes. The set A is the set of alternatives or outcomes from which one of the alternatives must be chosen. We will assume that G is a tree, i.e., a graph whose edges do not form any cycles and there is a unique path between every pair of alternatives/nodes. The private information or type of each agent is a vector $t \in \mathbb{R}^{|A|}_+$. The set of possible types (type space) of each agent will be determined by G.

We define the type space by imposing ordinal restriction on type spaces. Notice that each type induces a weak ordering on the set of alternatives. We call a type t strict if $t(a) \neq t(b)$ for all $a \neq b$. A strict type induces a linear order on the set of alternatives. Let $\mathcal{P}$ be the set of all linear orders over A. Given a linear order $P \in \mathcal{P}$, we denote the k-th ranked alternative in P as $P(k)$. Given any pair of alternatives $a, b \in A$, there exists a unique path in G between a and b, and we denote this unique path as $\Pi(a, b)$. A linear order $P \in \mathcal{P}$ is **single peaked** with respect to G if for every $a \in A$ and every $b \in \Pi(a, P(1))$, we have bPa. Let $\mathcal{D} \subseteq \mathcal{P}$ be the set of all single peaked linear orders in $\mathcal{P}$.

Definition **2.3** *The* **strict single peaked type space** T^G *(with respect to G) is the set of all non-negative type vectors that induce a linear order in $\mathcal{D}$, i.e.,*

$$T^G := \{t \in \mathbb{R}^{|A|}_+ : t \text{ induces } P \text{ for some } P \in \mathcal{D}\}.$$

The **single peaked type space** *is $cl(T^G)$, where $cl(T^G)$ denotes the closure of the set T^G.*

The main result of the chapter is the following.

Theorem **2.1** *An allocation rule $f : cl(T^G) \to A$ is implementable if and only if it is 2-cycle monotone.*

Remark. In many contexts, it is natural to assume that there is an alternative whose value is always zero (for instance, in auction problems, the alternative of not getting any object gives zero value to the agent). Though we do not explicitly allow this in our model, all our proofs can be modified straightforwardly to accommodate the fact that there is an alternative which is worst ranked and has value zero at every type.

2.3.1 Proof of Theorem 2.1

The proof of Theorem 2.1 will be done using a series of Lemmas. These lemmas will reveal the underlying structure of the type space. Further, we will show how these steps can be used in other type spaces to extend Theorem 2.1.

Denote by $D \equiv cl(T^G)$. First, by Rochet (1987), if $f : D \to A$ is implementable, then it is 2-cycle monotone. Next, again by Rochet (1987), if f is cyclically monotone, then it is implementable. So, we will show that if f is 2-cycle monotone, then it is cyclically monotone. In the remainder of the section, we assume that f is 2-cycle monotone.

For every $a \in A$, define $D(a)$ as follows.

$$D(a) := \{t \in D : f(t) = a\}.$$

Since f is onto, $D(a)$ is non-empty. Next, for every $s, t \in D$, define $\ell(s, t)$ as follows.

$$\ell(s, t) := t(f(t)) - t(f(s)).$$

Notice that 2-cycle monotonicity is equivalent to requiring that for every $s, t \in D$, we have $\ell(s, t) + \ell(t, s) \geq 0$. Now, for every $a, b \in A$, define $d(a, b)$ as follows.

$$d(a, b) := \inf_{t \in D(b)} \big[t(b) - t(a)\big].$$

We state below a well known fact - see, for instance, Lemma 6 in Bikhchandani et al. (2006).

Lemma **2.1** *For every $a, b \in A$, $d(a, b) + d(b, a) \geq 0$.*

Proof: Suppose $d(a, b) + d(b, a) = -\epsilon < 0$ for some $a, b \in A$. This means, there is a $s \in D(b)$ and $t \in D(a)$ such that $[s(b) - s(a)] + [t(a) - t(b)] < 0$. But this means that $\ell(s, t) + \ell(t, s) < 0$, a contradiction to 2-cycle monotonicity. ■

For any $a, b \in A$, we say a and b are G**-neighbors** if the unique path between a and b in G is a direct edge between a and b in G. The following facts will be useful throughout the proofs. These facts are true due to the single peakedness of the type space.

FACT **2.1** *For any $a, b \in A$, if a and b are G-neighbors, then there exists a linear order $P \in \mathcal{D}$ such that $P(1) = a, P(2) = b$.*

Fact 2.1 says that if a and b are G-neighbors then there is some ordering where they are ranked first and second.

FACT **2.2** *For any $a, c \in A$ and $b \in \Pi(a, c)$ [6] such that b is a G-neighbor of a, there exists a linear order $P \in \mathcal{D}$ such that $\{P(1), P(2)\} = \{a, b\}$, xPc for all $x \in \Pi(a, c) \setminus \{c\}$, and cPx for all $x \notin \Pi(a, c)$.*

Fact 2.2 says that if a and c are any pair of alternatives with b being a G-neighbor of a in $\Pi(a, c)$, then there is some ordering where a and b are first and second ranked, followed by all the other alternatives in $\Pi(a, c)$, and followed by the remaining alternatives outside $\Pi(a, c)$. The first step of the proof of Theorem 2.1 is the following lemma.

LEMMA **2.2** *If a, b are G-neighbors, then $d(a, b) + d(b, a) = 0$.*

Proof: Consider $a, b \in A$ such that a and b are G-neighbors. By Lemma 2.1, $d(a, b) + d(b, a) \geq 0$. Assume for contradiction $d(a, b) + d(b, a) = \epsilon > 0$. Then, either $d(a, b) > \frac{\epsilon}{2}$ or $d(b, a) > \frac{\epsilon}{2}$. Suppose $d(a, b) > \frac{\epsilon}{2}$ - a similar proof works if $d(b, a) > \frac{\epsilon}{2}$. Then, there is a type $s \in D(b)$ such that $d(a, b) \leq s(b) - s(a) < d(a, b) + \epsilon_1$, for any $\epsilon_1 > 0$ arbitrarily close to zero, in particular $\epsilon_1 < \frac{\epsilon}{2}$. Hence, $s(b) - s(a) > \frac{\epsilon}{2}$. We now choose a $\delta \in (2\epsilon_1, s(b) - s(a))$ but arbitrarily close to $2\epsilon_1$. Since a and b are G-neighbors, by Fact 2.1, there exists a $P \in \mathcal{D}$ such that b is top ranked and a is second ranked. We can construct a type $u \in D$ that induces P and

$$u(x) = \begin{cases} s(x) + \delta & \text{if } x = a \\ s(x) + \frac{\delta}{2} & \text{if } x = b \\ \leq \min(s(x), s(a)) & \text{if } x \notin \{a, b\}, \end{cases}$$

Notice that since $s(b) > s(a)$, we have $u(b) > u(a)$ for sufficiently small $\delta > 2\epsilon_1$. Also, alternatives other than a and b are ordered according to P but their values are not increased.

We will now argue that $f(u) = a$. First, if $f(u) = x \notin \{a, b\}$, we have $u(x) - u(b) \leq s(x) - s(b) - \frac{\delta}{2} < s(x) - s(b)$, which violates 2-cycle monotonicity. Second, if $f(u) = b$, we have $u(b) - u(a) = s(b) - s(a) - \frac{\delta}{2} < d(a, b) - (\frac{\delta}{2} - \epsilon_1) < d(a, b)$, which violates the definition of $d(a, b)$. Hence, $f(u) = a$.

[6] With a slight abuse of notation, we let $\Pi(a, c)$ to denote the set of alternatives (including a and c) in the unique path from a to c in G.

But this implies that $d(b,a) \leq u(a) - u(b) = s(a) - s(b) + \frac{\delta}{2} \leq -d(a,b) + \frac{\delta}{2}$. Hence, $d(b,a) + d(a,b) \leq \frac{\delta}{2}$. Since δ, ϵ_1 can be chosen arbitrarily close to zero, this contradicts the fact that $d(a,b) + d(b,a) = \epsilon > 0$. ■

The next step is to show that for any pair of alternatives a and c, there is some alternative $b \in \Pi(a,c)$ such that a version of the *reverse triangle inequality* holds between a, b, and c using $d(\cdot,\cdot)$.

Lemma **2.3** *For any pair of alternatives $a, c \in A$ such that a and c are not G-neighbors, there exists an alternative $b \in \Pi(a,c)$ such that*

$$d(a,b) + d(b,c) \leq d(a,c).$$

Proof: Fix $a, c \in A$ such that a and c are not G-neighbors. Choose an $\epsilon > 0$ and arbitrarily close to zero and a $t \in D_c$ such that $d(a,c) \leq t(c) - t(a) \leq d(a,c) + \epsilon$. We consider two cases.

Case 1. $t(c) \geq t(a)$. Choose $b \in \Pi(a,c)$ such that b is a G-neighbor of c. By single peakedness, for every $x \in \Pi(a,c)$, we have $t(x) \geq t(a)$. Then, we can construct a new type in which b and c occupy the top two ranks. We construct such a new type s as follows. Choose $\epsilon' > 0$ but arbitrarily close to zero and let $\delta := t(c) - t(b) - d(b,c) + 2\epsilon'$. Note that since $t \in D_c$, we have $t(c) - t(b) \geq d(b,c)$, and this implies that $\delta > 0$.

$$s(x) = \begin{cases} t(x) + \epsilon' & \text{if } x = c \\ t(a) & \text{if } x \in \Pi(a,c) \setminus \{b,c\} \\ t(x) + \delta & \text{if } x = b \\ \leq \min(t(x), t(a)) & \text{if } x \notin \Pi(a,c). \end{cases}$$

By Fact 2.2, we can define s such that it is in $cl(T^G)$. We argue that $f(s) = b$. First, suppose $f(s) = x \notin \{b,c\}$. Then, $s(x) - s(c) < t(x) - t(c)$, and this contradicts 2-cycle monotonicity. Next, suppose $f(s) = c$. Then, $d(b,c) \leq s(c) - s(b) = t(c) - t(b) - \delta + \epsilon' = d(b,c) - \epsilon' < d(b,c)$, a contradiction. Hence, $f(s) = b$.

Now, $d(a,b) \leq s(b) - s(a) = [t(b) - t(a) + \delta] = t(c) - t(a) - d(b,c) + 2\epsilon' \leq d(a,c) - d(b,c) + 2\epsilon' + \epsilon$. Since ϵ and ϵ' can be chosen arbitrarily close to zero, we conclude that $d(a,b) + d(b,c) \leq d(a,c)$.

Case 2. $t(c) < t_i(a)$. Let $b \in \Pi(a,c)$ be the G-neighbor of a. Define the subset of alternatives C as follows: $C := \{c' \in \Pi(b,c) : t(c') = t(c) \text{ and } \forall\ c'' \in \Pi(c',c), t(c'') = t(c)\}$. In other words, C is the set of "contiguous" alternatives in $\Pi(b,c)$ starting from c which have the same value as $t(c)$ in type t. Now, construct a new type s as follows. Choose an $\epsilon' > 0$ but arbitrarily close to zero. Note that ϵ' can be chosen sufficiently close to zero such that for all $x \in \Pi(a,c) \setminus C$, we have $t(x) > t(c) + \epsilon'$. Also, choose $\delta = t(c) - t(b) - d(b,c) + 2\epsilon'$. As before, $\delta > 0$.

$$s(x) = \begin{cases} t(c) + \epsilon' & \text{if } x \in \Pi(a,c) \setminus \{a,b\} \\ t(x) & \text{if } x = a \\ t(x) + \delta & \text{if } x = b \\ \leq \min(t(x), s(c)) & \text{if } x \notin \Pi(a,c). \end{cases}$$

Again, by Fact 2.2, such a type s can be found in $cl(T^G)$. If $f(s) = x \notin C \cup \{b\}$, then $s(x) - s(c) < t(x) - t(c)$, which violates 2-cycle monotonicity. Hence, $f(s) \in C \cup \{b\}$. We consider two subcases.

Case 2a. Suppose $f(s) = c' \in C$. Then, $d(c,c') \leq s(c') - s(c) = 0$. But $f(t) = c$ implies that $d(c',c) \leq t(c) - t(c') = 0$. This implies that $d(c,c') + d(c',c) \leq 0$. By Lemma 2.1, $d(c,c') + d(c',c) = 0$. Since, $d(c,c') \leq 0$ and $d(c',c) \leq 0$, we conclude that $d(c,c') = d(c',c) = 0$. Further, since $f(s) = c'$, $d(a,c') \leq s(c') - s(a) = t(c) - t(a) + \epsilon' \leq d(a,c) + \epsilon + \epsilon'$. Since ϵ and ϵ' can be chosen arbitrarily small, $d(a,c') \leq d(a,c)$. Hence, $d(a,c') + d(c',c) \leq d(a,c)$, where we used the fact that $d(c',c) = 0$. This completes the proof of this case.

Case 2b. Suppose $f(s) = b$. Then, $d(a,b) \leq s(b) - s(a) = t(b) - t(a) + \delta = t(c) - t(a) - d(b,c) + 2\epsilon' \leq d(a,c) - d(b,c) + 2\epsilon' + \epsilon$. Since ϵ and ϵ' can be chosen arbitrarily small, $d(a,b) + d(b,c) \leq d(a,c)$. This completes the proof of this case. ■

Lemmas 2.2 and 2.3 are the foundations of our proof. The next lemma (and many subsequent lemmas) is a consequence of these two lemmas. Lemmas 2.2 and 2.3 are the only place where we use the fact that the type space is $cl(T^G)$. This implies that as long as we can prove analogues of Lemmas 2.2 and 2.3 in a type space, Theorem 2.1 continues to hold. Now, consider the following lemma.

Lemma **2.4** *For any pair of alternatives $a_1, a_k \in A$, let $\Pi(a_1,a_k) = (a_1, a_2, \ldots, a_k)$ with $k > 2$. Then, the following are true.*

$$d(a_1,a_2) + d(a_2,a_3) + \ldots + d(a_{k-1},a_k) \leq d(a_1,a_k)$$
$$d(a_k,a_{k-1}) + d(a_{k-1},a_{k-2}) + \ldots + d(a_2,a_1) \leq d(a_k,a_1).$$

Proof: Consider any pair of alternatives $a_1, a_k \in A$ and let $(a_1, a_2, \ldots, a_k)$ be the sequence of alternatives on $\Pi(a_1,a_k)$. We do the proof using induction on k. If $k = 3$, then the claim is true due to Lemma 2.3. Suppose the claim is true for all $k < K$. If $k = K$, then by Lemma 2.3, there is an alternative $a_r \in \{a_2, \ldots, a_{K-1}\}$ such that $d(a_1, a_r) + d(a_r, a_K) \leq d(a_1, a_K)$. The paths $(a_1, \ldots, a_r)$ and $(a_r, \ldots, a_K)$ each contain less than K nodes. By our induction hypothesis, $d(a_1, a_2) + \ldots + d(a_{r-1}, a_r) \leq d(a_1, a_r)$ and $d(a_r, a_{r+1}) + \ldots + d(a_{K-1}, a_K) \leq d(a_r, a_K)$. Hence, $d(a_1, a_2) + \ldots + d(a_{K-1}, a_K) \leq d(a_1, a_K)$.

A similar argument shows that $d(a_k, a_{k-1}) + d(a_{k-1}, a_{k-2}) + \ldots + d(a_2, a_1) \leq d(a_k, a_1)$. ■

The following lemma is well known - see, for instance, Heydenreich et al. (2009).

LEMMA **2.5** *Suppose for every sequence of alternatives $(a_1, \ldots, a_k)$, we have*

$$\sum_{j=1}^{k} d(a_j, a_{j+1}) \geq 0,$$

where $a_{k+1} \equiv a_1$. Then, f is cyclically monotone.

Proof: Consider any sequence of types $(t^1, \ldots, t^k)$ such that $f(t^j) = a_j$ for all $j \in \{1, \ldots, k\}$. Then, $[t^2(a_2) - t^2(a_1)] + \ldots + [t^k(a_k) - t^k(a_{k-1})] + [t^1(a_1) - t^1(a_k)] \geq d(a_1, a_2) + \ldots + d(a_{k-1}, a_k) + d(a_k, a_1) \geq 0$, where we use $d(a, a) = 0$ for any $a \in A$. So, f is cyclically monotone. ■

LEMMA **2.6** *Suppose $\Pi(a_1, a_k) \equiv (a_1, \ldots, a_k)$. Then,*

$$\sum_{j=1}^{k} d(a_j, a_{j+1}) \geq 0,$$

where $a_{k+1} \equiv a_1$.

Proof: If $k = 2$, then the claim is true by 2-cycle monotonicity. Else, $k > 2$ and a_j and a_{j+1} are G-neighbors for all $j \in \{1, \ldots, k-1\}$. By Lemma 2.4, $d(a_k, a_1) \geq d(a_k, a_{k-1}) + \ldots + d(a_2, a_1)$. Hence,

$$\begin{aligned} d(a_1, a_2) + d(a_2, a_3) + \ldots + d(a_{k-1}, a_k) + d(a_k, a_1) &\geq \sum_{j=1}^{k-1} \left[d(a_j, a_{j+1}) + d(a_{j+1}, a_j) \right] \\ &= 0, \end{aligned}$$

where the last equality follows from the fact that a_j and a_{j+1} are G-neighbors for all $j \in \{1, \ldots, k-1\}$ and Lemma 2.2. ■

At this point, it will be useful to consider another graph G^f. [7] The set of nodes in G^f is the set of alternatives A. It is a complete directed graph. Hence, for every pair of alternatives

[7] In Heydenreich et al. (2009), this graph is called the *allocation graph*.

$a, b \in A$, there is an edge from a to b and an edge from b to a. The path from an alternative a to another alternative b in G^f is a directed path. Note that for every path $(a_1, a_2, \ldots, a_k)$ in G^f from a_1 to a_k, the corresponding undirected path may or may not exist in G. For any pair of alternatives $a_1, a_k \in A$, denote by $dist^f(a_1, a_k)$ the shortest path length from a_1 to a_k in G^f.

The next lemmas shows that the shortest path in G^f between a pair of alternatives a and b is the unique path $\Pi(a, b)$ in G.

LEMMA **2.7** *For any pair of alternatives $a, b \in A$, let $\Pi(a, b) \equiv (a \equiv a_1, a_2, \ldots, b \equiv a_k)$. Then,*

$$\sum_{j=1}^{k-1} d(a_j, a_{j+1}) = dist^f(a, b).$$

Proof: Fix $a, b \in A$ and choose a shortest path from a to b in G^f. Let this path be $(a'_1, \ldots, a'_h)$, where $a'_1 \equiv a$ and $a'_h \equiv b$. Now, take any edge (x, y) in this path. If x and y are not G-neighbors, then we can pick the path $\Pi(x, y) \equiv (x, c_1, \ldots, c_r, y)$ in G from x to y, and $d(x, y) \geq d(x, c_1) + d(c_1, c_2) + \ldots + d(c_{r-1}, c_r) + d(c_r, y)$ by Lemma 2.4. Combining the paths $\Pi(a'_j, a'_{j+1})$ for all $j \in \{1, \ldots, k-1\}$, we get the path $\Pi(a, b)$ from a to b in G, which we denote by $(a_1, \ldots, a_k)$ with $a \equiv a_1$ and $b \equiv a_k$, and some cycles in G. By Lemma 2.6, these cycles have non-negative length (according to weights defined in G^f). Hence, $dist^f(a, b) \geq \sum_{j=1}^{k-1} d(a_j, a_{j+1})$. By definition, $dist^f(a, b) \leq \sum_{j=1}^{k-1} d(a_j, a_{j+1})$. Hence, $dist^f(a, b) = \sum_{j=1}^{k-1} d(a_j, a_{j+1})$. ■

This leads to the final lemma in the proof of Theorem 2.1.

LEMMA **2.8** *Every cycle of G^f has non-negative length.*

Proof: Consider a cycle $(a_1, \ldots, a_k, a_1)$ in G^f. By Lemma 2.7, the unique path $\Pi(a_1, a_k) \equiv (a_1, b_1, \ldots, b_r, a_k)$ in G satisfies $d(a_1, b_1) + d(b_1, b_2) + \ldots + d(b_{r-1}, b_r) + d(b_r, a_k) = dist^f(a_1, a_k) \leq d(a_1, a_2) + \ldots + d(a_{k-1}, a_k)$. This shows that

$$d(a_1, a_2) + \ldots + d(a_{k-1}, a_k) \geq d(a_1, b_1) + d(b_1, b_2) + \ldots + d(b_{r-1}, b_r) + d(b_r, a_k).$$

Now, consider the path $(a_k, b_r, \ldots, b_1, a_1)$ from a_k to a_1. By Lemma 2.4,

$$d(a_k, a_1) \geq d(a_k, b_r) + d(b_r, b_{r-1}) + \ldots + d(b_2, b_1) + d(b_1, a_1).$$

Adding the previous two inequalities, we get

$$\begin{aligned}\sum_{j=1}^{k} d(a_j, a_{j+1}) \geq & [d(a_1, b_1) + d(b_1, a_1)] + [d(b_1, b_2) + d(b_2, b_1)] + \dots \\ & + [d(b_{r-1}, b_r) + d(b_r, b_{r-1})] + [d(a_k, b_r) + d(b_r, a_k)] \\ & = 0,\end{aligned}$$

where $a_k \equiv a_1$ and the last equality follows from Lemma 2.2 and the fact that consecutive alternatives on the path $(a_1, b_1, \dots, b_r, a_k)$ are G-neighbors. ■

Lemmas 2.8 and 2.5 establish that f is cyclically monotone, and hence, implementable. This completes the proof of Theorem 2.1.

2.4 Type Spaces with Ordinal Restrictions: The Difference Indifference Makes

We now investigate how far we extend Theorem 2.1. For this, we formally define the notion of an ordinal type space. Let $\mathcal{D} \subseteq \mathcal{P}$ be some subset of strict orderings of the set of alternatives A. We will refer to $\mathcal{D}$ as a *domain*. We denote by $T(\mathcal{D})$ the set of *all* strict types in $\mathbb{R}^{|A|}_{++}$ that are consistent with the strict orderings in $\mathcal{D}$, i.e.,

$$T(\mathcal{D}) := \{t \in \mathbb{R}^{|A|}_{++} : t \text{ is consistent with some } P \in \mathcal{D}\}.$$

Definition **2.4** *A type space $D \subseteq \mathbb{R}^{|A|}_{+}$ is* **ordinal** *if there exists $\mathcal{D} \subseteq \mathcal{P}$ such that $D = T(\mathcal{D})$.*

In our previous section, we assumed that $\mathcal{D}$ is the set of all single peaked preferences with respect to a tree graph and showed that if the type space is $cl(T(\mathcal{D})$, then every 2-cycle monotone allocation rule is implementable.

We first give an example to illustrate how tight this result is. Below, we consider an example with three alternatives and $\mathcal{D}$ that contains one more preference ordering than the set of all single peaked orderings. We show that Theorem 2.1 fails in this domain.

Example **2.1**

Let $A = \{a, b, c\}$ and $\mathcal{D}$ consists of the following orderings shown in Table 2.1.

P^1	P^2	P^3	P^4	P^5
a	b	b	c	c
b	a	c	b	a
c	c	a	a	b

Table 2.1: A K-connected domain satisfying top lifting

Notice that $\mathcal{D} \setminus \{P^5\}$ is the set of all single peaked linear orderings with respect to the graph with edges $\{a,b\},\{b,c\}$ (i.e., the line graph with a and c as endpoints). Let $D = T(\mathcal{D})$. We now define an allocation rule $f : cl(D) \to A$ as follows.

$$f(t) = \begin{cases} a & \text{if } t(a)-t(b) \geq 1, t(a) > t(b) \geq t(c) \\ b & \text{if } t(a)-t(b) < 1, t(a) \geq t(b) \geq t(c), t(a) \neq t(b) \neq t(c) \\ b & \text{if } t(b) \geq t(a) \geq t(c), t(a) \neq t(b) \neq t(c) \\ b & \text{if } t(b) \geq t(c) \geq t(a), t(a) \neq t(b) \neq t(c) \\ c & \text{if } t(c) \geq t(b) \geq t(a), t(a) \neq t(b) \neq t(c) \\ c & \text{if } t(c) > t(a) \geq t(b) \\ a & \text{if } t(c) = t(a) > t(b), t(a)-t(b) \geq 1 \\ c & \text{if } t(c) = t(a) > t(b), t(a)-t(b) < 1 \\ c & \text{if } t(a) = t(b) = t(c). \end{cases}$$

Note that $d(a,b) = -1, d(b,a) = 1, d(b,c) = d(c,b) = d(c,a) = d(a,c) = 0$. Hence, f is 2-cycle monotone. But $d(a,b)+d(b,c)+d(c,a) = -1$ implies that f is not 3-cycle monotone, and hence, not implementable.

2.4.1 Strict Types, Ordinal Connectedness, and Lifting

Surprisingly, if we restrict attention to strict types, Theorem 2.1 can be shown to be true in a larger class of type spaces (including the interior of the type space discussed in Example 2.2). In this section, our objective is to define a class of ordinal type spaces consisting of strict types where Theorem 2.1 continues to hold [8].

We will focus on ordinal type spaces that are *ordinally connected* in some way. We define two notions of connectedness and a *lifting* property here. We use the notation $P(k)$ to denote the k-th ranked alternative in ordering P. Let $\mathcal{D} \subseteq \mathcal{P}$.

DEFINITION **2.5** *An alternative $a \in A$ can be* **top lifted** *at an ordering $P \in \mathcal{D}$, if for every*

[8]If the set of preference orderings in $\mathcal{D}$ equal the set of all preference orderings $\mathcal{P}$, then $cl(T(\mathcal{D}))$ is a convex type space, and using Saks and Yu (2005), we can immediately conclude that Theorem 2.1 holds. The type spaces that we will cover are not necessarily convex and this makes the results novel.

$b \in A$ with aPb, there exists an ordering $P' \in \mathcal{D}$ such that (a) $P'(1) = a$ and (b) if bPc for any alternative c then $bP'c$.

Notice that P' may be different for different b in the definition above. Also, if $P(1) = a$, then a can be top lifted using P itself, and hence, the condition is vacuously satisfied.

Table 2.2 illustrates the idea of top lifting. Suppose $A = \{x, y, z, x', y', a, b\}$ and P, as shown in Table 2.2, is in $\mathcal{D}$. Consider $a, b \in A$. Note that in P' and P'', a is the top ranked alternative. For any alternative $a' \in \{z, b\}$ the alternatives that were worse than a' in P continue to be worse than a' in P'. For any alternative $a' \in \{x', y'\}$ the alternatives that were worse than a' in P continue to be worse than a' in P'. Hence, if $P', P'' \in \mathcal{D}$, a can be top lifted at P.

P	P'	P''
x	$[a]$	$[a]$
y	x	x
$[a]$	z	z
z	b	b
b	y	y
x'	y'	x'
y'	x'	y'

Table 2.2: Top Lifting Property

Definition **2.6** *A domain of preference orderings $\mathcal{D}$ satisfies* **top lifting** *if for every $a \in A$ and every $P \in \mathcal{D}$, a can be top lifted at P.*

We now define two notions of connectedness. We generalize the notion of G-neighbors we had defined earlier. Two alternatives a and b are **neighbors** in a domain of preference ordering $\mathcal{D}$ if there is an ordering $P \in \mathcal{D}$ such that $P(1) = a$ and $P(2) = b$, and another ordering $P' \in \mathcal{D}$ such that $P'(1) = b$ and $P'(2) = a$. Now, construct an undirected graph $G(\mathcal{D})$ as follows. The set of nodes in $G(\mathcal{D})$ is the set of alternatives A. For any $a, b \in A$, there is an edge $\{a, b\}$ in $G(\mathcal{D})$ if and only if a and b are neighbors.

Definition **2.7** *A domain of preference orderings $\mathcal{D}$ is* **ordinally connected** *if $G(\mathcal{D})$ is a connected graph, i.e., for every pair of alternatives $a, b \in A$ there is a path in $G(\mathcal{D})$ between a and b.*

Note that we do *not* require that every pair of alternatives are neighbors. We only require that the neighborhood graph is connected. Example 2.1 gives a domain that is ordinally connected, but in the closure of the ordinal type space induced from it, the 2-cycle monotonicity characterization result does not hold. Below, we strengthen the notion of ordinal connectedness. Our strengthening will still include the domain in Example 2.1. However, we will not allow for indifferences, i.e., only consider strict types. In that case, we will show that our 2-cycle monotonicity characterization holds.

We need some preliminary definitions. For any ordering $P \in \mathcal{D}$ and for any alternative $b \equiv P(k) \neq P(1)$, an ordering P' is a **local** b-lift of P if $P'(k-1) = P(k) = b$, $P'(k) = P(k-1)$, and $P'(j) = P(j)$ for all $j \notin \{k, k-1\}$. For any ordering P and any alternative b, we say an ordering P' is in the Kemeny b-path from P if there exists a sequence of orderings $(P \equiv P^1, P^2, \ldots, P^k \equiv P')$ such that for all $j \in \{1, \ldots, k-1\}$, P^{j+1} is a local b-lift of P^j.

Definition **2.8** *A pair of alternatives $a, b \in A$ are* **K-neighbors**[9] *if*

1. *for every ordering $P \in \mathcal{D}$ with $P(1) = a$ and every P' that is in the Kemeny b-path from P, we have $P' \in \mathcal{D}$ and*
2. *for every ordering $P \in \mathcal{D}$ with $P(1) = b$ and every P' that is in the Kemeny a-path from P, we have $P' \in \mathcal{D}$.*

In words, a and b are K-neighbors if whenever a is the top ranked, the preference ordering obtained by lifting b one position up belongs to the domain and whenever b is top ranked, the preference ordering obtained by lifting a one position up belongs to the domain. Note that if a domain satisfies top lifting, then for any alternative $a \in A$, there is at least one preference ordering P such that $P(1) = a$. Next, in such domains if a and b are K-neighbors, then they are neighbors. Now construct an undirected graph $K(\mathcal{D})$, where the set of nodes is A and there is an edge between a pair of alternatives $a, b \in A$ if and only if a and b are K-neighbors.

Definition **2.9** *A domain of preference orderings $\mathcal{D}$ is* **K-connected** *if the graph $K(\mathcal{D})$ is connected.*

Clearly, a K-connected domain is ordinally connected. However, ordinally connected domains need not be K-connected if $|A| \geq 4$. The following example illustrates that.

Example **2.2**

Suppose $A = \{a, b, c, d\}$. Consider a domain $\mathcal{D}$ consisting of the six preference orderings shown in Table 2.3. Clearly, this domain is ordinally connected - the graph $G(\mathcal{D})$ is a line graph with edges $\{a, b\}, \{b, c\}, \{c, d\}$. But this domain is not K-connected. To see this,

P^1	P^2	P^3	P^4	P^5	P^6
a	b	b	c	c	d
b	a	c	b	d	c
c	c	a	a	a	a
d	d	d	d	b	b

Table 2.3: Connected but not K-connected

consider the preference ordering P^5. The ordering P', where $P'(1) = c, P'(2) = d, P'(3) = b, P'(4) = a$ is a local b-lift of P^5 but is not in $\mathcal{D}$. Hence, b and c are not K-neighbors.

Definition **2.10** *A type space D is* **ordinally admissible** *if there is a domain of preference orderings $\mathcal{D}$ that is K-connected and satisfies top lifting such that $D = T(\mathcal{D})$.*

Our main result is that in ordinally admissible type spaces, 2-cycle monotonicity is necessary and sufficient for implementation. However, notice that an ordinally admissible type space consists of strict types only.

Theorem **2.2** *Suppose D is an ordinally admissible type space. Then, $f : D \to A$ is implementable if and only if it is 2-cycle monotone.*

The proof of Theorem 2.2 is given in the Appendix. The proof follows the exact steps that we followed to prove Theorem 2.1. But now with strict types, we show that our steps work in more general type spaces.

Although indifferences in ordinally admissible type spaces may break Theorem 2.2 (as Example 2.1 shows), if we assume a certain amount of continuity of the allocation rule, Theorem 2.2 is true.

Definition **2.11** *Let $D \equiv T(\mathcal{D})$ be an ordinally admissible type space. An allocation rule $f : cl(D) \to A$ satisfies* **condition** *C^* if for every $t \in cl(D) \setminus D$, there exists a sequence of types $\{t^k\}_k$ in D such that $\lim_{k\to\infty} t^k = t$ and $f(t) = f(t^k)$ for all t^k in the sequence.*

With this additional condition, Theorem 2.2 now extends to domains with indifferences.

Theorem **2.3** *Suppose $T \equiv T(\mathcal{D})$ is an ordinally admissible type space and $f : cl(D) \to A$ be an allocation rule satisfying condition C^*. Then, f is implementable if and only if it is 2-cycle monotone.*

[9]The 'K' in 'K-neighbors' stands for Kemeny.

Proof: The proof is a direct consequence of Theorem 2.2 and condition C^*. Suppose f is 2-cycle monotone but not implementable. Then, it is not cycle monotone and there exists a sequence of types $t^1, \ldots, t^k$ such that

$$\sum_{j=1}^{k}[t^j(f(t^j)) - t^j(f(t^{j+1})] < 0, \tag{2.2}$$

where $t^{k+1} \equiv t^1$. By condition C^*, there exists a sequence of types $s^1, \ldots, s^k$ such that for all $j \in \{1, \ldots, k\}$, $f(s^j) = f(t^j)$, $s^j \in T(\mathcal{D})$ and s^j is arbitrarily close to t^j. By Theorem 2.2, we know that f restricted to $T(\mathcal{D})$ satisfies cycle monotonicity. Hence,

$$\sum_{j=1}^{k}[s^j(f(t^j)) - s^j(f(t^{j+1})] \geq 0.$$

Since s^j is arbitrarily close to t^j, this contradicts Inequality 2.2. ■

The section highlights that there are ordinal type spaces where violation of 2-cycle monotonicity can only occur at boundary. The usual revenue maximization problems (a la Myerson (1981)) seek to maximize expected payment of an agent. Now, note that for any implementable allocation rule $f : cl(T(\mathcal{D})) \to A$, we can take its restriction in $T(\mathcal{D})$ and extend it to an implementable allocation rule $\bar{f} : cl(T(\mathcal{D})) \to A$ satisfying condition C^*. We know by revenue equivalence that the payment of an implementable allocation rule is uniquely determined upto a constant. Usually, this constant is determined by the individual rationality constraints and unique for all the implementable allocation rule if we are interested in expected revenue maximization. Hence, f and $\bar{f}$ must differ in payments only at boundary points. Since the boundary points have Lebesgue measure zero, the expected payment of f and $\bar{f}$ must be the same. Hence, as far as revenue maximization is concerned, condition C^* is without loss of generality.

2.4.2 Examples

We give various examples of $\mathcal{D}$ that satisfy the K-connectedness and the lifting property. The first example is a convex type space, and hence, the earlier results already imply Theorem 2.2 in this type space. The remaining examples are of non-convex multidimensional type spaces, and hence, the earlier results are silent on such examples. The single peaked type space on a tree also satisfies K-connectedness and the lifting property, but since Theorem 2.1 covers it, we do not discuss it here.

1. Complete type space. In the complete type space, $\mathcal{D}$ is the set of all possible orderings over A. So, K-connectedness holds. Further, top lifting is satisfied since all

possible orderings are in $\mathcal{D}$. Note that the complete type space covers the multi-object auction model with unit demand. To see this, let A be the set of heterogeneous objects and agent i can be assigned exactly one object from A. Then, the complete type space assumption requires that any vector of non-negative valuations can be assigned to the objects.

2. Semi single peaked type space. In semi single peaked type space, there is an exogenously given ordering $\succ$ on the set of alternatives A. We say an ordering P is single-peaked to left if for any pair of alternatives $a, b \in A$, $b \succ a$ and $a \succ P(1)$ implies aPb. Similarly, an ordering P is single-peaked to right if for any pair of alternatives $a, b \in A$, $P(1) \succ a$ and $a \succ b$ implies aPb. Hence, semi single peakedness requires single peakedness to one of the sides of the peak.

   The set of admissible orderings $\mathcal{D}$ is a semi single peaked domain if it consists of either all left single peaked orderings or all right single peaked orderings. Consider a semi single peaked domain $\mathcal{D}$ and assume that it consists of all right single peaked orderings. Then, alternatives a and b are neighbors if $B(a, b) = \emptyset$, where $B(a, b)$ is the set of alternatives between a and b according to $\succ$. Note that a can be top ranked in an ordering and any alternative $b \in L(a)$ can be second ranked. However, if b is top ranked, a can be second ranked only if $B(a, b) = \emptyset$. For this reason, $G(\mathcal{D})$ is a line graph, which is connected. It is also easy to see that every neighbor of a is also a K-neighbor. This is because if b is a neighbor of a, then it can be lifted to any rank from a given preference ordering. Hence, $K(\mathcal{D})$ is also connected.

   We can verify that the semi single peaked domain satisfies top lifting. To see this, consider an ordering $P \in \mathcal{D}$, where aPb. If $b \in L(a)$, we can construct an ordering where a is top ranked and b is second ranked by lowering all the alternatives (except a) below b but maintaining single peaked to the right of a. If $b \in R(a)$, then alternatives to the left of a can be lowered sufficiently to make a the peak and it will automatically maintain single peakedness to the right of a.

   The interior of the type space of Example 2.1 is semi single peaked. To see this, consider the exogenous ordering $a \succ b \succ c$. The set of all right single peaked preference orderings with respect to $\succ$ is exactly the preference orderings shown in Example 2.1. As we had shown, Theorem 2.1 does not apply to this type space but Theorem 2.2 applies.

3. Single peaked type space with characteristics. This is a generalization of the single peaked type space. We are now exogenously given a set of orderings $\mathcal{S}$ over the set of alternatives. The domain $\mathcal{D}$ consists of all orderings that are single peaked with respect to some $\succ \in \mathcal{S}$. If $\mathcal{S}$ is a singleton, this is precisely the single

peaked domain. Suppose the set of alternatives are objects. An element of $\mathcal{S}$ can be interpreted as a "characteristic" of the objects. Depending on the characteristic used by an agent to rank the objects, his preference must be single peaked with respect to that characteristic.

Consider an example with $A = \{a, b, c, x, y\}$ and let $\mathcal{S} = \{\succ_1, \succ_2\}$, where $a \succ_1 b \succ_1 c \succ_1 x \succ_1 y$ and $y \succ_2 a \succ_2 b \succ_2 x \succ_2 c$. Figure 2.1 shows the graph $G(\mathcal{D})$ for this domain. The edges are derived from the single peaked restrictions on each characteristics.

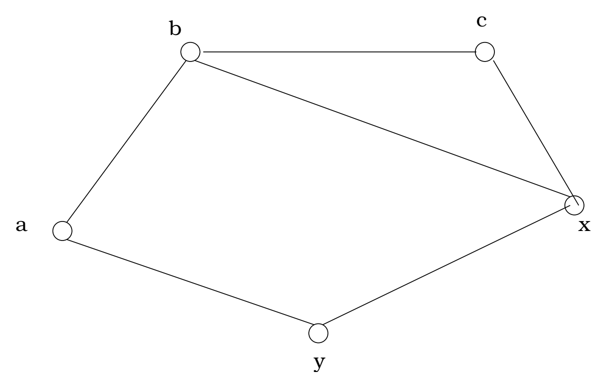

Figure 2.1: Graph $K(\mathcal{D})$ for the single peaked domain with two characteristics.

In general, the graph $K(\mathcal{D})$ is connected since $\mathcal{D}$ contains the single peaked domain, which is connected. Also, $\mathcal{D}$ satisfies the top lifting property. To see this, consider any $a, b \in A$ and suppose there is a preference ordering P where aPb. Since P is single peaked with respect to some $\succ \in \mathcal{S}$, we can apply the arguments for the single peaked domain to show that there is some ordering P' that is single peaked with respect to $\succ$ such that top lifting holds for a and b at P.

2.5 Payments and Revenue Equivalence

It is well know that if f is implementable, then the following payment rule implements f. Fix a type $s \in D$ and set $p(t) = 0$ for all t with $f(t) = f(s)$. For all $t \in D$ such that $f(t) \neq f(s)$, set $p(t)$ equal to $dist^f(f(s), f(t))$. If f is cyclically monotone, then, p implements f - see for instance, Vohra (2011) and Kos and Messner (2013).

The characterization of the set of *all* payment rules that implement an allocation rule is done using the revenue equivalence principle.

Definition **2.12** *An allocation rule f satisfies* **revenue equivalence** *if for all payment* *rules p, q that implement f, there exists a constant $\alpha \in \mathbb{R}$ such that for all $t \in D$*

$$p(t) = q(t) + \alpha.$$

The revenue equivalence holds in ordinal type spaces under weaker conditions than the conditions we have discussed for the 2-cycle monotonicity characterization. We are going to show revenue equivalence in ordinally connected type spaces. To remind, a set of preference orderings $\mathcal{D}$ is ordinally connected, if the graph $G(\mathcal{D})$ is connected.

THEOREM **2.4** *Suppose $\mathcal{D}$ is ordinally connected and $D = T(\mathcal{D})$ or $D = cl(T(\mathcal{D}))$. Then every implementable allocation rule $f : D \to A$ satisfies revenue equivalence.*

The proof of Theorem 2.4 is in the Appendix. The proof essentially follows by showing a counterpart of Lemma 2.2 for ordinally connected type spaces and then using standard results from the literature. We remark that Chung and Olszewski (2007) and Heydenreich et al. (2009) have shown that if D is a topologically connected subset of $\mathbb{R}^{|A|}$, then every implementable allocation rule satisfies revenue equivalence in such a type space. However, since $\mathcal{D}$ consists of strict orderings, $T(\mathcal{D})$ is not topologically connected and hence, our result is not a direct corollary of their results. However, both these papers provide sufficient conditions using which revenue equivalence can be checked in our ordinal type spaces. The proof uses such conditions.

In the classical literature on multidimensional mechanism design a type space is usually a subset of $\mathbb{R}^{|A|}$ possessing some geometrical properties. The state-of-art in that literature is that if the type space is a topologically connected subset of $\mathbb{R}^{|A|}$, then every implementable allocation rule satisfies revenue equivalence (Chung and Olszewski, 2007; Heydenreich et al., 2009). On the other hand, if the type space is convex, then every 2-cycle monotone allocation rule is implementable. Some parallel between our results and these results can be drawn as follows. Theorem 2.4 shows that in ordinally connected type spaces, every implementable allocation rule satisfies revenue equivalence. On the other hand, Theorem 2.2 shows that in ordinally admissible type spaces (which requires K-connectedness and top lifting), every 2-cycle monotone allocation rule is implementable.

An ordinally connected type space allows us to be precise on the nature of the shortest paths between any pair of nodes in G^f. Suppose f is implementable. Now, for any pair of alternatives $a, b \in A$, consider any path $(a_1, \ldots, a_k)$ in $G(\mathcal{D})$, where $a_1 \equiv a$ and $b \equiv a_k$.

Then, $dist^f(a,b) = \sum_{j=1}^{k-1} d(a_j, a_{j+1})$. This follows from the fact that

$$\begin{aligned}
0 &= dist^f(a,b) + dist^f(b,a) \\
&\leq \sum_{j=1}^{k-1} d(a_j, a_{j+1}) + \sum_{j=k-1}^{1} d(a_{j+1}, a_j) \\
&= \sum_{j=1}^{k-1} \big[d(a_j, a_{j+1}) + d(a_{j+1}, a_j)\big] \\
&= 0,
\end{aligned}$$

where the first equality follows from Theorem 2.4 and the last equality from Lemma 2.2. Since in many examples, we know the structure of $G(\mathcal{D})$, this allows us to know the payments in these type spaces explicitly. Moreover, it can be shown that these particular payments occupy a central role among the set of all payments - Kos and Messner (2013) contain a detailed discussion on this topic in very general type spaces.

2.6 Domains with Free Triple at the Top

So far, we have discussed domains that are K-connected (the single peaked domain on a tree in Theorem 2.1 is also a K-connected domain). In this section, we identify an ordinal domain that is not K-connected and still every 2-cycle monotone allocation rule is implementable in the type space induced by this domain. [10] Our domain uses the following definition. We say alternatives $a, b, c \in A$ are a free triple at the top in domain $\mathcal{D}$ if for every $x, y, z \in \{a, b, c\}$ with $x \neq y \neq z$, there exists $P \in \mathcal{D}$ such that $P(1) = x, P(2) = y, P(3) = z$. In other words, there exists six distinct orderings where a, b, c occupy top three ranks.

Definition **2.13** *A domain $\mathcal{D}$ satisfies* **free triple at the top (FTT)** *if every three distinct alternatives in A are a free triple at the top in $\mathcal{D}$.*

We make some observations about FTT domains. First, an FTT domain is ordinally connected since any pair of alternatives can be ranked first and second. However, it need not be K-connected. To see this consider the following example with $A = \{a, b, c, d, e\}$. Suppose whenever a, b, c occupy the top 3 positions, d is better than e. It is easy to construct an FTT domain that satisfies this restriction. But such a domain will not be K-connected since

[10] If we did not focus on ordinal type spaces, then a a type space not satisfying K-connectedness but where 2-cycle monotonicity implies implementability is easy to find - this follows from the fact in any convex type space, 2-cycle monotonicity implies implementability.

K-connectedness will require the existence of an ordering where e is fourth ranked and d is fifth ranked. [11]

We are now ready to state the main result of this section.

Theorem 2.5 *Suppose $D = T(\mathcal{D})$ or $D = cl(T(\mathcal{D}))$, where $\mathcal{D}$ is an FTT domain. Then, $f : D \to A$ is implementable if and only if it is 2-cycle monotone.*

The proof is given in the appendix. After an initial step, the proof uses the general methodology developed in the proof of Theorem 2.1.

2.7 Relation to the Literature

We discuss specific literature and its relation to our results. As discussed earlier, in the one dimensional model of single object auctions, Myerson (1981) characterizes implementable allocation rules using a monotonicity condition, which is equivalent to 2-cycle monotonicity - see also Spence (1974). The cycle monotonicity characterization in Rochet (1987) can be thought of as an extension of Myerson's characterization to multidimensional models. The recent literature on multidimensional mechanism design started with the paper of Jehiel et al. (1999) who observed that besides 2-cycle monotonicity, an integral condition is required to ensure Bayesian implementability in multidimensional environments with randomization. However, if the set of alternatives is finite, the allocation rule is deterministic and the type space is convex, only 2-cycle monotonicity is sufficient (Bikhchandani et al., 2006; Saks and Yu, 2005; Ashlagi et al., 2010; Gui et al., 2004; Cuff et al., 2012). [12] Our results are extensions of these results to non-convex type spaces. Mishra and Roy (2013) also consider a non-convex type space, which they call rich dichotomous type space, and show that 3-cycle monotonicity is sufficient for implementability in their type space but 2-cycle monotonicity is not sufficient.

A parallel literature in multidimensional mechanism design pursues type spaces where revenue equivalence result in Myerson (1981) holds. Contributions to this are

[11] This also motivates the following FTT domain. Suppose the alternatives are some public projects. There is a "social ranking" $\succ$ of these projects. The type of an agent consists of some component of his private preference and the remaining of the social ranking. In particular, the agent ranks any *three* projects as his top three and then follows the social ranking for the remaining projects. Such a domain will satisfy the FTT assumption.

[12] There are many papers which characterize different extensions of implementability in convex type spaces using 2-cycle monotonicity and additional technical conditions - for Bayes-Nash implementation, see Jehiel et al. (1999) and Muller et al. (2007); for randomized implementation, see Archer and Kleinberg (2008); for implementation with general value functions, see Berger et al. (2010) and Carbajal and Ely (2013); for extension of cycle monotonicity to general environments, see Rahman (2011).

Krishna and Maenner (2001); Milgrom and Segal (2002); Chung and Olszewski (2007); Heydenreich et al. (2009); Carbajal (2010); Kos and Messner (2013). We use a characterization in Heydenreich et al. (2009) to prove revenue equivalence in our type spaces.

Most of the type space restrictions in multidimensional mechanism design is geometric (using assumptions like convexity or connectedness in topological spaces). Our ordinally admissible domain formulation is influenced by a vast literature in strategic social choice theory where transfers are not allowed. For instance, the connectedness and lifting properties we discuss have close resemblance to similar properties being used to identify *dictatorial domains* (Aswal et al., 2003), *median domains* (Chatterji et al. (2013), Nehring and Puppe (2007a), Nehring and Puppe (2007b)), *tops-only domains* (Chatterji and Sen, 2011; Weymark, 2008) in social choice theory. We find it interesting to observe that such conditions could be used in multidimensional mechanism design models with transfers to derive sufficient conditions for implementability. Since most of our non-convex type spaces are single peaked type spaces or their generalizations, we will like to point out that strategic social choice theory, starting with Moulin (1980) and Sprumont (1991), have a long tradition of studying these type spaces without monetary transfers. Other notable papers in this literature are Otten et al. (1996), Klaus et al. (1998) and Bossert and Peters (2009). However, allowing for transfers in many of these type spaces is practical in many of these models. Hence, our results extend this literature to the case of transfers. At the same time, our characterizations using 2-cycle monotonicity are only implicit characterizations, unlike the characterizations in the strategic social choice theory, which are more explicit in describing the form of the implementable allocation rules. The counterpart to such explicit characterizations in the multidimensional mechanism design with transfers literature is Roberts' theorem (Roberts, 1979), who showed that *affine maximizers* are the only implementable allocation rules in the complete type space. We leave such characterizations in single peaked domains for future research.

An important objective in mechanism design is to design revenue maximizing mechanisms. While Myerson (1981) solved this problem for the sale of a single object, the problem remains unsolved for multidimensional problems - see a recent take on this topic in Manelli and Vincent (2007); Hart and Nisan (2012); Hart and Reny (2012). However, as Myerson illustrates, there are two important steps in solving the optimal auction problem: (a) characterizing the implementable allocation rules using a monotonicity property and (b) establishing revenue equivalence to pin down the payments. Though the eventual optimization problem remains illusive in the multidimensional type spaces, the literature has made significant progress in advancing these two steps for multidimensional type spaces. Our results add to this literature and we hope that these advances will eventually help us solve the revenue maximization problem in the multidimensional type spaces.

Appendix: Omitted Proofs

Proof of Theorem 2.2

We follow the steps in Theorem 2.1. Obviously, some of the steps of Theorem 2.1 continues to apply in this type space. So, we only do the steps that do not apply. Lemma 2.1 continues to hold. The first step of the proof of Theorem 2.2 is the analogue of Lemma 2.2.

Lemma **2.9** *If a, b are neighbors, then $d(a,b) + d(b,a) = 0$.* [13]

Proof: Consider $a, b \in A$ such that a and b are neighbors. By Lemma 2.1, $d(a,b)+d(b,a) \geq 0$. Assume for contradiction $d(a,b) + d(b,a) = \epsilon > 0$. Then, either $d(a,b) > \frac{\epsilon}{2}$ or $d(b,a) > \frac{\epsilon}{2}$. Suppose $d(a,b) > \frac{\epsilon}{2}$ - a similar proof works if $d(b,a) > \frac{\epsilon}{2}$. Then, there is a type $s \in D(b)$ such that $d(a,b) \leq s(b) - s(a) < d(a,b) + \epsilon_1$, for any $\epsilon_1 > 0$ arbitrarily close to zero, in particular $\epsilon_1 < \frac{\epsilon}{2}$. Hence, $s(b) - s(a) > \frac{\epsilon}{2}$. We now choose a $\delta \in (2\epsilon_1, s(b) - s(a))$ but arbitrarily close to $2\epsilon_1$. Since a and b are neighbors, there exists a $P \in \mathcal{D}$ such that b is top ranked and a is second ranked. We can construct a type $u \in D$ that induces P and

$$u(x) = \begin{cases} s(x) + \delta & \text{if } x = a \\ s(x) + \frac{\delta}{2} & \text{if } x = b \\ \leq \min(s(x), s(a)) & \text{if } x \notin \{a, b\}, \end{cases}$$

Notice that since $s(b) > s(a)$, we have $u(b) > u(a)$ for sufficiently small $\delta > 2\epsilon_1$. Also, alternatives other than a and b are ordered according to P but their values are not increased.

We will now argue that $f(u) = a$. First, if $f(u) = x \notin \{a, b\}$, we have $u(x) - u(b) \leq s(x) - s(b) - \frac{\delta}{2} < s(x) - s(b)$, which violates 2-cycle monotonicity. Second, if $f(u) = b$, we have $u(b) - u(a) = s(b) - s(a) - \frac{\delta}{2} < d(a,b) - (\frac{\delta}{2} - \epsilon_1) < d(a,b)$, which violates the definition of $d(a,b)$. Hence, $f(u) = a$.

But this implies that $d(b,a) \leq u(a) - u(b) = s(a) - s(b) + \frac{\delta}{2} \leq -d(a,b) + \frac{\delta}{2}$. Hence, $d(b,a) + d(a,b) \leq \frac{\delta}{2}$. Since δ, ϵ_1 can be chosen arbitrarily close to zero, this contradicts the fact that $d(a,b) + d(b,a) = \epsilon > 0$. ■

The next lemma establishes the counterpart of Lemma 2.3. For any pair of alternatives, a, c we will consider paths in $K(\mathcal{D})$ between a and c. Since we assume $\mathcal{D}$ to be K-connected, there is at least one path between a and c. A path between a and c is direct if it involves only a and c. A path between a and c is indirect if it is not direct. By definition, there is a direct path between a and c if and only if a and c are K-neighbors.

Lemma **2.10** *For every pair of alternatives $a, c \in A$ and any indirect path $\Pi(a,c)$ between a and c in $K(\mathcal{D})$, there exists an alternative b in this path such that $d(a,b) + d(b,c) \leq d(a,c)$.*

[13] Note that we do not require a and b to be K-neighbors.

Proof: Fix $a, c \in A$ and an indirect path $\Pi(a, c)$ between a and c in $K(\mathcal{D})$. Choose an $\epsilon > 0$ arbitrarily close to zero and a $t \in D(c)$ such that $d(a, c) \leq t(c) - t(a) < d(a, c) + \epsilon$. We consider two cases.

Case 1. $t(c) > t(a)$. Choose an alternative b in $\Pi(a, c)$ such that b is a K-neighbor of c. Let the ordering induced by t be P. By top lifting, there exists an ordering $P' \in \mathcal{D}$ such that (a) $P'(1) = c$ and (b) if aPc' for any alternative c' then $aP'c'$. Hence, we can construct a type $t' \in D$ that induce P' and $t'(c) = t(c) + \epsilon'$, $t'(a) = t(a)$, and $t'(x) \leq t(x)$ for all $x \notin \{a, c\}$, where $\epsilon' > 0$ but arbitrarily close to zero such that $\epsilon' < \epsilon$. Since $t'(c) > t(c)$ and $t'(x) \leq t(x)$ for all $x \neq c$, we have $[t'(x) - t'(c)] + [t(c) - t(x)] < 0$ for all $x \neq c$, and hence, by 2-cycle monotonicity, $f(t') = c$. Further, $d(a, c) \leq t'(c) - t'(a) < d(a, c) + \epsilon$.

Let $\delta = t'(c) - t'(b) - d(b, c) + \epsilon''$, for some $\epsilon'' > 0$ but arbitrarily close to zero. Since $f(t') = c$, we have $t'(c) - t'(b) \geq d(b, c)$. Hence, $\delta > 0$ but arbitrarily close to $t'(c) - t'(b) - d(b, c)$. Now, we construct a new type s as follows. Choose an $\bar{\epsilon} > 0$ but arbitrarily close to zero.

$$s(x) = \begin{cases} t'(x) + \bar{\epsilon} & \text{if } x = c \\ t'(x) + \delta + \bar{\epsilon} & \text{if } x = b \\ t'(x) & \text{if } x \in A \setminus \{b, c\} \end{cases}$$

Since c is top at t' and b is the neighbor of c in $\Pi(c, a)$, by K-connectedness, the ordering induced by s belongs to $\mathcal{D}$. Hence, $s \in D$.

We argue that $f(s) = b$. First, suppose $f(s) = x \notin \{b, c\}$. Then, $s(x) - s(c) < t'(x) - t'(c)$, and this contradicts 2-cycle monotonicity. Next, suppose $f(s) = c$. Then, $d(b, c) \leq s(c) - s(b) = t'(c) - t'(b) - \delta < d(b, c)$, a contradiction. Hence, $f(s) = b$.

Now, $d(a, b) \leq s(b) - s(a) = [t'(b) - t'(a) + \delta] + \bar{\epsilon}$. Since $\delta = [t'(c) - t'(b)] - d(b, c) + \epsilon''$, we have $d(a, b) \leq [t'(c) - t'(a)] - d(b, c) + \bar{\epsilon} + \epsilon'' < d(a, c) + \epsilon - d(b, c) + \bar{\epsilon} + \epsilon''$. This implies that $d(a, b) + d(b, c) < d(a, c) + \epsilon + \bar{\epsilon} + \epsilon''$. Since ϵ, $\bar{\epsilon}$ and ϵ'' can be chosen arbitrarily close to zero, we conclude that $d(a, b) + d(b, c) \leq d(a, c)$.

Case 2. $t(c) < t(a)$. Choose an alternative b in $\Pi(a, c)$ such that b is a K-neighbor of a. Let the ordering induced by t be P. By top lifting, there exists an ordering $P' \in \mathcal{D}$ such that (a) $P'(1) = a$ and (b) if cPc' for any alternative c' then $cP'c'$. Hence, we can construct a type $t' \in D$ that induce P' and $t'(c) = t(c) + \epsilon'$, $t'(a) = t(a)$, and $t'(x) \leq t(x)$ for all $x \notin \{a, c\}$, where $\epsilon' > 0$ but arbitrarily close to zero, in particular, $\epsilon' < \epsilon$. Since $t'(c) > t(c)$ and $t'(x) \leq t(x)$ for all $x \neq c$, we have $[t'(x) - t'(c)] + [t(c) - t(x)] < 0$ for all $x \neq c$, and hence, by 2-cycle monotonicity, $f(t') = c$. Further, $d(a, c) \leq t'(c) - t'(a) < d(a, c) + \epsilon$.

Let $\delta = t'(c) - t'(b) - d(b, c) + \epsilon''$, for some $\epsilon'' > 0$ but arbitrarily close to zero. Since $f(t') = c$, we have $t'(c) - t'(b) \geq d(b, c)$. Hence, $\delta > 0$ but arbitrarily close to $t'(c) - t'(b) - d(b, c)$.

Now, we construct a new type s as follows. Choose an $\bar{\epsilon} > 0$ but arbitrarily close to zero.

$$s(x) = \begin{cases} t'(x) + \bar{\epsilon} & \text{if } x = c \\ t'(x) + \delta + \bar{\epsilon} & \text{if } x = b \\ t'(x) & \text{if } x \in A \setminus \{b, c\} \end{cases}$$

Since a is top at t' and b is the neighbor of a in $\Pi(c, a)$, by K-connectedness, the ordering induced by s belongs to $\mathcal{D}$ for small enough $\bar{\epsilon}$. Hence, $s \in D$.

We argue that $f(s) = b$. First, suppose $f(s) = x \notin \{b, c\}$. Then, $s(x) - s(c) < t'(x) - t'(c)$, and this contradicts 2-cycle monotonicity. Next, suppose $f(s) = c$. Then, $d(b, c) \leq s(c) - s(b) = t'(c) - t'(b) - \delta < d(b, c)$, a contradiction. Hence, $f(s) = b$.

Now, $d(a, b) \leq s(b) - s(a) = [t'(b) - t'(a) + \delta] + \bar{\epsilon}$. Since $\delta = [t'(c) - t'(b)] - d(b, c) + \epsilon''$, we have $d(a, b) \leq [t'(c) - t'(a)] - d(b, c) + \bar{\epsilon} + \epsilon'' < d(a, c) + \epsilon - d(b, c) + \bar{\epsilon} + \epsilon''$. This implies that $d(a, b) + d(b, c) < d(a, c) + \epsilon + \bar{\epsilon} + \epsilon''$. Since ϵ, $\bar{\epsilon}$ and ϵ'' can be chosen arbitrarily close to zero, we conclude that $d(a, b) + d(b, c) \leq d(a, c)$. ■

Once we have established these major lemmas. The remaining Lemmas in the proof of Theorem 2.1 goes through. We state their generalizations and skip the proof since it mirrors the proof given in their counterparts for the proof of Theorem 2.1.

Lemma **2.11** *For any pair of alternatives $a_1, a_k \in A$, let $(a_1, a_2, \ldots, a_k)$ be an indirect path $\Pi(a_1, a_k)$ between a_1 and a_k in $K(\mathcal{D})$. Then, the following are true.*

$$d(a_1, a_2) + d(a_2, a_3) + \ldots + d(a_{k-1}, a_k) \leq d(a_1, a_k)$$
$$d(a_k, a_{k-1}) + d(a_{k-1}, a_{k-2}) + \ldots + d(a_2, a_1) \leq d(a_k, a_1).$$

Lemma **2.12** *Suppose for every sequence of alternatives $(a_1, \ldots, a_k)$, we have*

$$\sum_{j=1}^{k} d(a_j, a_{j+1}) \geq 0,$$

where $a_{k+1} \equiv a_1$. Then, f is cyclically monotone.

Lemma **2.13** *Suppose $(a_1, \ldots, a_k)$ is a path in $K(\mathcal{D})$. Then,*

$$\sum_{j=1}^{k} d(a_j, a_{j+1}) \geq 0,$$

where $a_{k+1} \equiv a_1$.

At this point, it will be useful to consider another graph G^f. [14] The set of nodes in G^f is the set of alternatives A. It is a complete directed graph. Hence, for every pair of alternatives $a, b \in A$, there is an edge from a to b and an edge from b to a. The path from an alternative a to another alternative b in G^f is a directed path. Note that for every path $(a_1, a_2, \ldots, a_k)$ in G^f from a_1 to a_k, the corresponding undirected path may or may not exist in $K(\mathcal{D})$. For any pair of alternatives $a_1, a_k \in A$, denote by $dist^f(a_1, a_k)$ the shortest path length from a_1 to a_k in G^f.

LEMMA **2.14** *For any pair of alternatives $a, b \in A$, there exists a path $(a_1, a_2, \ldots, a_k)$ in $K(\mathcal{D})$, where $a \equiv a_1$ and $b \equiv a_k$, such that*

$$\sum_{j=1}^{k-1} d(a_j, a_{j+1}) = dist^f(a, b).$$

Again, the proof of Lemma 2.14 mirrors the proof of Lemma 2.7 and is skipped. This leads to the final lemma in the proof of Theorem 2.2.

LEMMA **2.15** *Every cycle of G^f has non-negative length.*

The proof of Lemma 2.15 is similar to the proof of Lemma 2.8 and is skipped. Lemmas 2.15 and 2.12 establish that f is cyclically monotone, and hence, implementable. This completes the proof of Theorem 2.2.

Proof of Theorem 2.4

We start by observing that though we proved Lemma 2.2 by assuming the type space to be $T(\mathcal{D})$, where $\mathcal{D}$ is ordinally connected, the same proof goes through even if we assume the type space to be $cl(T(\mathcal{D}))$.

Now, the remainder of the proof can be easily done using existing results. Heydenreich et al. (2009) showed that an implementable allocation rule f satisfies revenue equivalence if and only if $dist^f(a, b) + dist^f(b, a) = 0$ for all $a, b \in A$. We show that this property is satisfied in our ordinally connected domains. To see this, fix a pair of alternatives, $a, b \in A$. Since f is cyclically monotone, $dist^f(a, b) + dist^f(b, a) \geq 0$ - the union of a shortest path from a to b and a shortest path from b to a gives rise to cycles, which have non-negative length due to cycle monotonicity.

But take any path in $G(\mathcal{D})$ from a to b (since the type space is ordinally connected, such a path will always exist). Let this path be $(a \equiv a_0, a_1, \ldots, a_k \equiv b)$. Now, consider the path $(b, a_k, \ldots, a_1, a_0 \equiv a)$. The sum of lengths of these paths is $\sum_{j=0}^{k-1} d(a_j, a_{j+1}) = 0$, where the

[14] In Heydenreich et al. (2009), this graph is called the *allocation graph*.

equality followed from Lemma 2.2. Since $dist^f(a, b) + dist^f(b, a)$ is less than or equal to the sum of lengths of these paths, and we already know that $dist^f(a, b) + dist^f(b, a) \geq 0$, we conclude that $dist^f(a, b) + dist^f(b, a) = 0$.

Proof of Theorem 2.5

We already know that if f is implementable, then it is 2-cycle monotone. So, we only show that if f is 2-cycle monotone, then it is implementable. We do the proof in two steps. As in the proof of Theorem 2.4, we note here that Lemma 2.2 holds in a type space of the form $T(\mathcal{D})$ or $cl(T(\mathcal{D}))$ as long as $\mathcal{D}$ is ordinally connected.

Step 1. We say a domain $\mathcal{D}$ satisfies **free pair at the top (FPT)** if every pair of alternatives is a neighbor in $\mathcal{D}$ (i.e., they can be ranked first and second at some ordering in $\mathcal{D}$). Clearly, a domain that satisfies FTT also satisfies FPT. Further, an FPT domain is ordinally connected. Hence, if $\mathcal{D}$ satisfies FPT, then for every $a, b \in A$, $d(a, b) + d(b, a) = 0$. Now, consider a sequence of alternatives $(a_1, \ldots, a_k)$. By Lemma 2.5, if we show

$$\sum_{j=1}^{k} d(a_j, a_{j+1}) \geq 0,$$

where $a_{k+1} \equiv a_1$, then f is implementable. We show this using induction on k. If $k = 2$ or $k = 3$, then we are done using 3-cycle monotonicity. If $k > 3$, we pick any $j' \in \{1, \ldots, k-2\}$ and choose alternatives $a_{j'}, a_{j'+2}$. Note that $d(a_{j'}, a_{j'+2}) + d(a_{j'+2}, a_{j'}) = 0$. Now, the big cycle can be broken into sum of two smaller cycles using the pair of edges $(a_{j'}, a_{j'+2})$ and $(a_{j'+2}, a_{j'})$ as follows:

$$\begin{aligned}\sum_{j=1}^{k} d(a_j, a_{j+1}) &= \Big[\sum_{j=1}^{j'} d(a_j, a_{j+1}) + d(a_{j'}, a_{j'+2}) + \sum_{j=j'+2}^{k} d(a_j, a_{j+1})\Big] \\ &\quad + \Big[d(a_{j'}, a_{j'+1}) + d(a_{j'+1}, a_{j'+2}) + d(a_{j'+2}, a_{j'})\Big] \\ &\geq 0,\end{aligned}$$

where the last inequality followed from the fact that the cycles $(a_1, \ldots, a_{j'}, a_{j'+2}, \ldots, a_k, a_1)$ and $(a_{j'}, a_{j'+1}, a_{j'+2}, a_{j'})$ has less than k nodes and our induction hypothesis applies.

Step 2. In this step, we show that in an FTT domain, every 2-cycle monotone allocation rule is 3-cycle monotone. Consider any triple of alternatives $a, b, c \in A$. We need to show that $d(a, b) + d(b, c) + d(c, a) \geq 0$. If $\max(d(a, b), d(b, c), d(c, a)) \geq 0$, then we are done. Suppose, without loss of generality, $d(c, a) < 0$. Using the fact that $d(x, y) + d(y, x) = 0$ for all $x, y \in A$ (due to FTT and Lemma 2.2), it is sufficient to show that $d(c, b) + d(b, a) \leq d(c, a)$.

To show this, we consider a type t such that $f(t) = a$ and $d(c,a) \leq t(a) - t(c) < d(c,a) + \epsilon$ for some $\epsilon > 0$ but arbitrarily close to zero. Hence, $t(a) < t(c)$. Since FTT is satisfied, there exists a preference ordering P such that $P(1) = c, P(2) = a$, and $P(3) = b$. Further, we can construct a type s that induces P as follows: $s(a) = t(a) + \epsilon_1, s(c) = t(c)$, and $s(x) < t(x)$ for all $x \notin \{a, c\}$, where $\epsilon_1 > 0$ and arbitrarily close to zero but in particular much smaller than ϵ. By 2-cycle monotonicity, $f(s) = a$ - if $f(s) = y \neq a$, then $s(y) - s(a) < t(y) - t(a)$, violating 2-cycle monotonicity. Also, ϵ_1 can be chosen sufficiently small such that $d(c,a) \leq s(a) - s(c) < d(c,a) + \epsilon$.

Now, let $\delta = s(a) - s(b) - d(b,a)$. Note that since $s(a) - s(b) \geq d(b,a)$, we have $\delta \geq 0$. We construct a type $\hat{s}$ such that $\hat{s}(b) = s(b) + \delta + 2\epsilon'$, $\hat{s}(a) = s(a) + \epsilon'$, and $\hat{s}(x) = s(x)$ for all $x \notin \{a, b\}$, where $\epsilon' > 0$ but arbitrarily close to zero. We argue that $f(\hat{s}) \notin \{a, b\}$. This is because if $f(\hat{s}) = x \notin \{a, b\}$, then $\hat{s}(x) - \hat{s}(a) < s(x) - s(a)$, contradicting 2-cycle monotonicity. Next, if $f(\hat{s}) = a$, then $d(b,a) \leq \hat{s}(a) - \hat{s}(b) = s(a) - s(b) - \delta - \epsilon' < d(b,a)$, a contradiction.

Hence, $f(\hat{s}) = b$. This implies that $d(c,b) \leq \hat{s}(b) - \hat{s}(c) = s(b) - s(c) + \delta + 2\epsilon' = s(a) - s(c) - d(b,a) + 2\epsilon' < d(c,a) - d(b,a) + \epsilon + 2\epsilon'$. Since ϵ and ϵ' can be chosen arbitrarily close to zero, we get $d(c,b) + d(b,a) \leq d(c,a)$. This completes the proof.

Chapter 3

Pairwise Partition Graphs and Strategy-proof Social Choice in the Exogenous Indifference Class Model

3.1 Introduction

The seminal work of Gibbard (1973) and Satterthwaite (1975) showed that all deterministic strategy-proof social choice functions with a range of at least three alternatives and defined over the complete domain, is dictatorial. A large body of literature has since focused on relaxing the underlying assumptions of this result. A natural way to do this is to impose domain restrictions. Indeed, many real life problems have inherent domain restrictions. This chapter is a contribution to this stand of literature.

It is well known that the structure of strategy-proof social choice functions becomes more complex when indifference is permitted in individual preferences. [1] In this chapter, we investigate a model of domain restrictions involving indifference, *the exogenous indifference class model* first introduced in Barbera and Ehlers (2011). In this model, the indifference classes of agents' preferences is exogenously given. In particular, every individual has an exogenous partition of the set of alternatives. An individual is always indifferent between alternatives a and b iff both a and b belong to the same element of her partition set. But an individual's ranking of the different elements of her partition set, is complete.

This framework includes several well-studied models as special cases. For instance, the

[1] See for instance, the literature of strategy-proofness in classical exchange economies originating from Hurwicz (1972) and Satterthwaite and Sonnenschein (1981).

case of private goods and selfish preferences is one where an individual is indifferent between all alternatives that give her the same commodity bundle. It includes the one-sided matching model studied in Svensson (1999),Papai (2000). It also includes the Gibbard-Satterthwaite framework where the elements of the partition are all singletons. Further examples are provided in Sato (2009).

The goal of Barbera and Ehlers (2011) was to study the Arrovain aggregation issue in the exogenous indifference class model. Sato (2009) examined the same model from the perspective of strategic voting. This is an interesting model in this respect as well because it covers both the complete domain over which strategy-proofness implies dictatorship (the Gibbard-Sattherwaite Theorem) as well as the private good allocation model for which it is well-known that a rich class of strategy-proof social choice functions exist (Papai (2000)). Sato showed that the number of common indifference classes is critical to the existence of strategy-proof and non-dictatorial social choice functions. He assumed that common indifference classes are singletons and obtained two results. First, an onto and strategy-proof social choice function is dictatorial whenever there are at least three common indifference classes. Second, the same result holds when the number of common indifference classes is two, provided that unanimity is strengthened to efficiency.

In this chapter, we further examine the relationship between dictatorship results in this model and the structure of indifference classes across agents. Our results are formulated in terms of the *pairwise partition graph* induced by the indifference classes. Fix a pair of agents i and j and their indifference classes. The partition graph for this pair is a bipartite graph whose vertices are i and j's indifference classes. There are no edges between the vertices representing the indifference classes of a given agent; vertices for i and j' have an edge if the indifference classes representing these vertices have *no common alternative*. We show that a necessary condition for strategy-proofness and unanimity to imply dictatorship in the domain induced by a partition is that each associated pairwise partition graph is connected with the degree of every vertex being at least two. This requirement can be weakened to the graphs being connected (with possibly isolated vertices), if unanimity is replaced by efficiency.

We are unfortunately, unable to show that these necessary conditions are sufficient for dictatorship. However we are able to identify a number of stronger conditions that are sufficient. The first of these is the existence of at least two common indifference classes with no restrictions on their size - a result which clearly generalizes that of Sato. In addition we have three sufficient conditions for the case of two voters. One applies to the case where there is exactly one common indifference class while another shows that strategy-proofness and unanimity imply dictatorship when the partition graph is a cycle. Finally, we show that with the stronger assumption of efficiency, strategy-proofness implies dictatorship when the partition graph is connected with possibly isolated vertices. The last condition implies that we have a necessary and sufficient condition for dictatorship for the case of efficiency when there are exactly two voters.

We now proceed to details.

3.2 The Model

Let $A = \{a, b, c, \ldots\}$ denote a finite set of alternatives with $|A| = m$. Let $I = \{1, ..., n\}$, $n \geq 2$ be a finite set of agents. We impose restrictions on the domain of preferences following Barbera and Ehlers (2011). Each agent i has a partition S_i of A that is exogenously specified and independent of preferences. A typical element of S_i is s_i^j where $j = 1, \ldots J$. An ordering $R(S_i)$ over A *respects* S_i if (i) a pair of alternatives belonging to the same element of S_i are indifferent to each other and (ii) otherwise one is strictly preferred to the other. Let $I(S_i)$ and $P(S_i)$ denote the symmetric and assymetric components of $R(S_i)$ respectively. Let $\mathcal{R}(S_i)$ be the set of all orderings respecting S_i. Let $S = (S_1, \ldots, S_n)$ be an n-tuple of partitions, one for each agent. The admissible preference domain is $\mathcal{R}(S) = \prod_{i \in N} \mathcal{R}(S_i)$.

Observation **3.1** For a given partition S_i, $\mathcal{R}(S_i)$ consists of all possible strict orderings over the elements of S_i. If all agents have the same partition, the model reduces to the standard voting model where the elements of the partition can be thought of as an alternative. The essence of the problem is that partitions across agents can differ. Suitable choices of S_i's yield the usual private goods allocation model with selfish preferences, the universal domain with strict orderings as well as several interesting intermediate cases as shown in Sato (2009).

Example **3.1** Let $A = \{a, b, c, d, e\}$ and $I = \{1, 2\}$. Let $S_1 = \{ab, c, de\}$ and $S_2 = \{ab, cd, e\}$. Note that Table 3.1 and Table 3.2 below represent $\mathcal{R}(S_1)$ and $\mathcal{R}(S_2)$ respectively.

R_1^1	R_1^2	R_1^3	R_1^4	R_1^5	R_1^6
ab	ab	c	c	de	de
c	de	ab	de	ab	c
de	c	de	ab	c	ab

Table 3.1: $\mathcal{R}(S_1)$

R_2^1	R_2^2	R_2^3	R_2^4	R_2^5	R_2^6
ab	ab	cd	cd	e	e
cd	e	ab	e	ab	cd
e	cd	e	ab	cd	ab

Table 3.2: $\mathcal{R}(S_2)$

Whenever S or S_i is fixed, $\mathcal{R}(S)$, $R(S)$, $\mathcal{R}(S_i)$, and $R(S_i)$ will be simply written as $\mathcal{R}$, R, $\mathcal{R}_i$, and R_i respectively. Fix a partition S_i. An indifference class of agent i is an element of S_i. For any $R_i \in \mathcal{R}_i$ and $k \in \{1, \ldots, J\}$, the k^{th} ranked indifference class in R_i is denoted by $r_k(R_i)$ (in other words, there are $k-1$ elements of S_i ranked strictly higher than $r_k(R_i)$). For any $B \subseteq A$ and $R_i \in \mathcal{R}_i$, $M(R_i, B)$ is the set of maximal elements in B according to R_i i.e. $M(R_i, B) = \{a \in B : aR_ib \textit{ for all } b \in B\}$.

Let $V(S) = \bigcap_{i=1}^n S_i$ denote the set of common indifference classes for the partition n-tuple S. Also let $v(S) = |V(S)|$ denote the number of common indifference classes. Note that in Example 3.1, $V(S) = \{ab\}$ and $v(S) = 1$. Moreover, in the private goods allocation model with selfish preferences $v(S) = 0$ while in the standard voting model $v(S) = m$.

Definition **3.1** *A social choice function (scf) is a mapping $f : \mathcal{R} \to A$.*

Each agent's preference ordering is private information, i.e. known only to herself. These preferences must therefore be elicited by the mechanism designer. If a is *strategy-proof*, then no agent can benefit by misrepresenting her preferences irrespective of her beliefs about the preference announcement of other agent.

Definition **3.2** *A scf $f : \mathcal{R} \to A$ is manipulable by agent i at a profile $R \in \mathcal{R}$ via R_i' if*

$$f(R_i', R_{-i})P_if(R).$$

A scf f is strategy-proof if it is not manipulable by any agent at any profile.

The following additional properties of scfs are standard.

Definition **3.3** *A scf f is unanimous if for all $R \in \mathcal{R}$*

$$f(R) \in \bigcap_{i=1}^n r_1(R_i) \textit{ whenever } \bigcap_{i=1}^n r_1(R_i) \neq \emptyset.$$

A unanimous scf always respects consensus whenever it exists, i.e. if all agents agree on some set of alternatives as their best, then the scf must pick an element from this set. A stronger condition than unanimity is efficiency.

Definition **3.4** *A scf f is efficient if for all $R \in \mathcal{R}$, for all $a, b \in A$*

$$\begin{bmatrix} aR_ib \textit{ for all } i \in I \\ aP_ib \textit{ for some } j \in I \end{bmatrix} \Rightarrow [f(R) \neq b]$$

If agents preferences are strict but sufficiently "rich", strategy-proofness and unanimity implies efficiency. In our setting where indifference is permitted, this proposition typically does not hold. We therefore consider the consequences of unanimity and efficiency separately.

Definition **3.5** *A scf f is dictatorial if there exists an agent i such that $f(R) \in r_1(R_i)$ for all $R \in \mathcal{R}$.*

An important feature of our model is that a scf is not fully specified even when an agent is a dictator since this agent will typically have more than one best alternative at any profile.

An important subclass of dictatorial scfs are *serial dictatorships* which we define below.

DEFINITION **3.6** *A priority σ is a one-to-one map $\sigma : I \to I$. Let R be a profile and define the sets $T_{\sigma(1)}(R), T_{\sigma(2)}(R), \ldots, T_{\sigma(n)}(R)$ inductively as follows:*

$$\begin{aligned}
T_{\sigma(1)}(R) &= M(R_{\sigma(1)}, A)\\
T_{\sigma(2)}(R) &= M(R_{\sigma(2)}, T_{\sigma(1)}(R))\\
&\vdots\\
T_{\sigma(n)}(R) &= M(R_{\sigma(n)}, T_{\sigma(n-1)}(R))
\end{aligned}$$

A serial dictatorship with respect to σ satisfies $f^{\sigma}(R) \in T_{\sigma(n)}(R)$ for all R.

A priority is a linear arrangement of agents where $\sigma(1)$ is the first agent, $\sigma(2)$ the second and so on with $\sigma(n)$ being the last. A serial dictatorship (with respect to σ) works as follows: at any profile R, agent $\sigma(1)$ picks her maximal elements in A, $\sigma(2)$ picks her maximal elements in $\sigma(1)$'s maximal elements and so on. Note that a serial dictatorship is not fully specified because $T_{\sigma(n)}(R)$ may contain more than one alternative.

3.3 RESULTS

Our goal is to investigate the structure of indifference classes across agents for which unanimity (or efficiency) and strategy-proofness imply dictatorship. Our first step is to provide necessary conditions for dictatorship. Subsequently, we provide various sufficient conditions.

3.3.1 Necessary Conditions for Dictatorship

These condition will be formulated in terms of graphs arising from the set S. We briefly review some basic graph-theoretic concepts.

A Graph G is a pair of finite sets V and E where V is the set of vertices or nodes and E is the set of edges. An edge is a non-ordered pair of vertices. If $e = \{u, v\}$ is an edge, i.e. $e \in E$, then u and v are adjacent vertices and u and e are incident as are v and e. The *degree of a vertex* v, $deg_G(v)$ is the number of edges incident with v. An *isolated vertex* is a vertex with degree zero. A *path* uv is a finite sequence of distinct vertices $(v_1, v_2, \ldots, v_k)$ where $v_1 = u$, $v_k = v$ and $\{v_i, v_{i+1}\} \in E$ for all $i \in \{1, \ldots, k-1\}$. If the path uv exist, then u and v are *connected*. A graph is connected if all pairs of distinct vertices are connected. A graph is *discrete* if $E = \{\emptyset\}$. A graph is a *connected graph with isolated vertices* if (i) it is not discrete and (ii) the sub-graph consisting of non-isolated vertices is connected. A graph is *bipartite* if the vertices of G can be partitioned in to two subsets V_1 and V_2 in such a way that no two vertices in the same subset have an edge.

Let i, j be agents and S_i and S_j be partitions for i and j respectively. The graph $G(S_i, S_j)$ is constructed as follows:

(i) The set $V = V_1 \cup V_2$ where $V_1 = S_i$ and $V_2 = S_j$.

(ii) There are no edges between vertices in V_1 or between vertices in V_2.

(iii) There is an edge between s_i^l and s_j^k iff $s_i^l \cap s_j^k = \emptyset$ for all $s_i^l \in S_i$ and $s_j^k \in S_j$.

The graph $G(S_i, S_j)$ is bipartite. We refer to it as the *Partition Graph for i and j*. We provide several examples of such graphs below.

Example **3.2** Let $I = \{j, i\}$, $A = \{a, b, c, d, e, f, g, h\}$, $\bar{S}_j = \{ab, cd, ef, gh\}$ and $\bar{S}_i = \{ac, bd, eg, fh\}$. The induced partition Graph is shown in Figure 3.1 and it is not connected. There is no path between gh to fh. Also every vertex of it has degree 2.

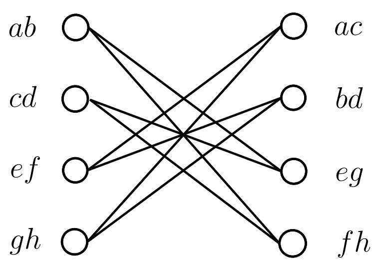

Figure 3.1: Disconnected graph

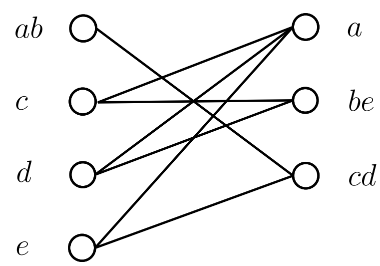

Figure 3.2: A graph with a degree one vertex

Example **3.3** Let $I = \{j, i\}$, $A = \{a, b, c, d, e\}$, $\hat{S}_j = \{ab, c, d, e\}$ and $\hat{S}_i = \{a, be, cd\}$. Figure 3.2 shows the induced partition graph which is connected. Note that degree of the vertex ab is 1.

Example **3.4** Let $I = \{j, i\}$, $A = \{a, b, c, d, e\}$, $\tilde{S}_j = \{ab, c, d, e\}$ and $\tilde{S}_i = \{a, be, cd\}$. The induced partition graph is shown in Figure 3.3 which is not connected. Note that degree of the vertex abc is 0.

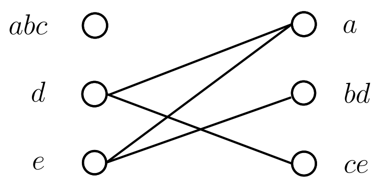

Figure 3.3: A graph with an isolated vertex

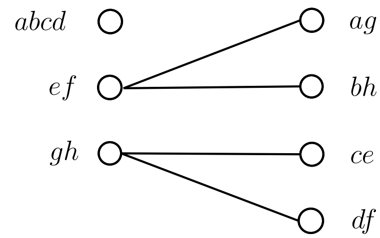

Figure 3.4: A non-connected graph with isolated vertices

Example **3.5** Let $I = \{j, i\}$, $A = \{a, b, c, d, e, f, g, h\}$, $S_j^o = \{abcd, ef, gh\}$ and $S_i^o = \{ag, bh, ce, df\}$. The induced partition graph is shown in Figure 3.4 and it is not connected. Moreover it is not a connected graph with isolated vertices.

Example **3.6** (The Allocation Problem with Selfish Preferences) There are three objects, say houses a, b and c which have to be allocated among three agents. An allocation is an ordered triple such as abc where the first, second and third components refer to the houses allocated to agents 1, 2 and 3 respectively. The set of allocations is the set $A = \{abc, acb, bac, bca, cab, cba\}$. Consider the standard model of selfish preferences. The induced partition $S^* = (S_1^*, S_2^*, S_3^*)$ is as follows:

- $S_1^* = \{\{abc, acb\}, \{bac, bca\}, \{cab, cba\}\}$.
- $S_2^* = \{\{abc, cba\}, \{bac, cab\}, \{bca, acb\}\}$.
- $S_3^* = \{\{abc, bac\}, \{cba, bca\}, \{cab, acb\}\}$.

The Partition Graph $G(S_1^*, S_2^*)$ is shown in Figure 3.5. Note that there are no common indifference classes. Clearly $G(S_1^*, S_2^*)$ is not connected, for instance there is no path between $\{cab, cba\}$ and $\{bca, acb\}$.

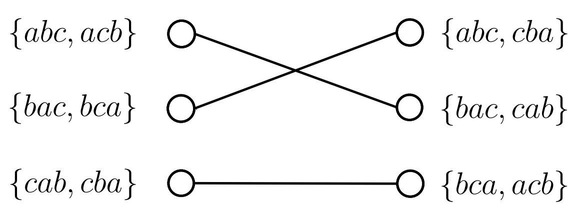

Figure 3.5: $G(S_1^*, S_2^*)$

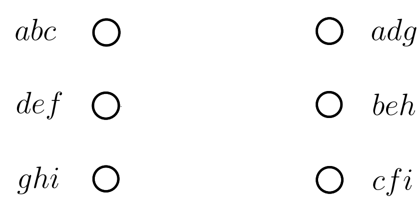

Figure 3.6: Discrete graph

Example **3.7** Let $I = \{j, i\}$, $A = \{a, b, c, d, e, f, g, h, i\}$, $S_j' = \{abc, def, ghi\}$ and $S_i' = \{adg, beh, cfi\}$. The induced partition graph in Figure 3.6, is a discrete graph.

Observation **3.2** Suppose $I = \{i, j\}$ and S is such that $G(S_i, S_j)$ is discrete (Example 3.7). Then *every* unanimous scf is dictatorial. Therefore the discrete partition graph case is trivial when there are two agents and can be excluded from consideration. In our necessary conditions (Theorem 3.1 below), we assume that the partition graph is not discrete for all pairs of agents. Note that if there are three or more agents, the pairwise discreteness of all partition graphs does not imply that every unanimous scf is dictatorial. We are unable to provide an answer for the case where partition graphs are pairwise discrete (or even some of them are pairwise discrete) but unanimity does not imply dictatorship.

Theorem **3.1** *Let $S = (S_1, \ldots, S_n)$ be such that $G(S_i, S_j)$ is not discrete for all $i, j \in I$.*

A. Suppose [$f : \mathcal{R} \to A$ is strategy-proof and unanimous $\Rightarrow f$ is dictatorial]. Then $G(S_i, S_j)$ is connected and the degree of every vertex is at least 2 for all $i, j \in I$.

B. Suppose [$f : \mathcal{R} \to A$ is strategy-proof and efficient $\Rightarrow f$ is dictatorial]. Then $G(S_i, S_j)$ is a connected graph with isolated vertices for all $i, j \in I$.

Proof: We first prove Part A. Pick an arbitrary pair $i, j \in I$. We consider three cases regarding $G(S_i, S_j)$: (I) it is not connected but has no isolated vertices (such as Example 3.2) (II) it is connected and has a vertex with degree 1 (Example 3.3) and (III) it has an isolated vertex (Example 3.4). These cases cover all possible cases in the statement of A and we show that in each case it is possible to construct a unanimous, strategy-proof and non-dictatorial scf $f : \mathcal{R} \to A$.

Case I: The vertices of $G(S_i, S_j)$ can be partitioned into subsets V' and V'' with $|V'|$, $|V''| \geq 2$ and $\{u, v\} \notin E$ for all $u \in V'$ and $v \in V''$ (In Figure 3.1, V' is the set $\{ab, cd, eg, fh\}$ and V'' is the set $\{ac, bd, ef, gh\}$). We fix two priority functions σ_1 and σ_2 as follows: $\sigma_1(1) = i$, $\sigma_1(2) = j$, $\sigma_2(1) = j$, $\sigma_2(2) = i$ and $\sigma_1(k) = \sigma_2(k)$ for all $k \neq 1, 2$. Let $\mathbb{D}$ be the set of preference profiles such that $r_1(R_i), r_1(R_j) \in V'$. The scf will be a serial dictatorship with priority σ_1 in the domain $\mathbb{D}$ and σ_2 elsewhere, i.e.

$$f(R) = \begin{cases} \text{a serial dictatorship with respect to the priority function } \sigma_1 & \text{if } R \in \mathbb{D} \\ \text{a serial dictatorship with respect to the priority function } \sigma_2 & \text{if } R \notin \mathbb{D} \end{cases}$$

We first check the strategy-proofness of f. Observe that an agent $k \neq i, j$ cannot manipulate because of the nature of serial dictatorships and the fact that they cannot affect their priority. Consider a profile $R \in \mathbb{D}$. The outcome is a R_j-maximal element outcome in the first-ranked indifference class of i in R_i, say s_i^1. Since i is a dictator, she does not manipulate. Clearly j cannot manipulate via an ordering R'_j such that $(R_i, R'_j, \ldots) \in \mathbb{D}$ because i will remain the dictator. Suppose j attempts to manipulate via an ordering R'_j

such that $(R_i, R'_j, \ldots) \notin \mathbb{D}$. He is now dictator and the outcome belongs to $r_1(R'_j)$. However, $r_1(R'_j) \cap s_i^1 \neq \emptyset$ by properties of the sets V' and V''; the outcome is therefore belongs to s_i^1. Hence j cannot manipulate. The only remaining case to consider is a possible manipulation by i from a profile not belonging to $\mathbb{D}$ to a profile in $\mathbb{D}$. The argument to show that i cannot gain is identical to the case where j attempts to manipulate from $\mathbb{D}$ to a profile outside it.

Unanimity is satisfied because f is a serial dictatorship at every profile. There are profiles in $\mathbb{D}$ and outside it where i and j's best ranked alternatives have an empty intersection. Therefore, neither i nor j are dictators. If $l \neq i, j$ is a dictator, then $G(S_i, S_l)$ and $G(S_j, G_l)$ are discrete, contradicting our assumption- i.e. f is non-dictatorial.

Case II: Let u be a vertex with degree 1. Assume w.l.o.g that $u \in S_j$ and $u = s_j^k$. By assumption, there exists a (unique) vertex $s_i^m \in S_i$ such that $\{s_i^m, s_j^k\} \in E$ (For instance in Example 3.3, s_j^k and s_i^m are *ab* and *cd* respectively). Let $\mathbb{D}$ be the set of preference profiles such that $r_1(R_i) = s_i^m$. We fix a priority function σ such that $\sigma(1) = j$ and $\sigma(2) = i$. Let $R \in \mathbb{D}$ and define the sets $T^*_{\sigma(1)}(R), T^*_{\sigma(2)}(R), \ldots, T^*_{\sigma(n)}(R)$ inductively as follows:

$$
\begin{aligned}
T^*_{\sigma(1)}(R) &= M(R_{\sigma(1)}, s_j^k \cup s_i^m) \\
T^*_{\sigma(2)}(R) &= M(R_{\sigma(2)}, T^*_{\sigma(1)}) \\
&\vdots \\
T^*_{\sigma(n)}(R) &= M(R_{\sigma(n)}, T^*_{\sigma(n-1)}(R))
\end{aligned}
$$

We define a scf $f_1^\sigma : \mathbb{D} \to A$ as follows: $f_1^\sigma(R) \in T^*_{\sigma(n)}(R)$ for all $R \in \mathbb{D}$.

Let $\bar{\sigma}$ be a priority function such that $\bar{\sigma}(1) = i$, $\bar{\sigma}(2) = j$ and $\bar{\sigma}(k) = \sigma(k)$ for all $k \neq 1, 2$. We define the following scf $f : \mathcal{R} \to A$:

$$
f(R) = \begin{cases} f_1^\sigma(R) & \text{if } R \in \mathbb{D} \\ \text{a serial dictatorship with respect to the priority function } \bar{\sigma} & \text{if } R \notin \mathbb{D} \end{cases}
$$

In order to check the strategy-proofness of f, it suffices to check that i and j cannot manipulate. Consider $R \in \mathbb{D}$ - the outcome is a R_j-maximal element in $s_j^k \cup s_i^m$. Agent j cannot manipulate since $(R_i, R'_j \ldots) \in \mathbb{D}$ for all R'_j. If the outcome belongs to s_i^m then clearly i cannot manipulate. Otherwise, the outcome belongs to s_j^k. Since s_j^k is a vertex with degree one, it has a non-empty intersection with *every* indifference class other than s_i^m. Hence the outcome belongs to $r_2(R_i)$. By construction, i cannot obtain an alternative in $r_1(R_i) = s_i^m$. If $R \notin \mathbb{D}$, agent i is dictator and cannot manipulate. Note that j cannot manipulate at $R \notin \mathbb{D}$ because $(R'_j, R_i, \ldots) \notin \mathbb{D}$ for all R'_j.

Unanimity and non-dictatorship of f is easily verified.

Case III: Let u be a vertex with degree 0. Assume w.l.o.g that $u \in S_j$ and $u = s_j^k$. Since $G(S_i, S_j)$ is not discrete, there exists an edge $\{s_i^m, s_j^l\}$ where $s_j^k \neq s_j^l$ (In Figure 3.3, the node s_j^k and the edge $\{s_i^m, s_j^l\}$ are abc and $\{d, a\}$ respectively). Let $\mathbb{D}$ be the set of preference profiles where $r_1(R_i) = s_i^m$, $r_1(R_j) = s_j^l$ and $r_2(R_j) = s_j^k$. We fix a priority function σ such that $\sigma(1) = j$ and $\sigma(2) = i$. Let $R \in \mathbb{D}$ and define the sets $T^*_{\sigma(1)}(R), T^*_{\sigma(2)}(R), \ldots, T^*_{\sigma(n)}(R)$ inductively as follows:

$$
\begin{aligned}
T^*_{\sigma(1)}(R) &= s_j^k \\
T^*_{\sigma(2)}(R) &= r_2(R_{\sigma(2)}) \cap s_j^k \\
T^*_{\sigma(3)}(R) &= M(R_{\sigma(3)}, T^*_{\sigma(2)}(R)) \\
&\vdots \\
T^*_{\sigma(n)}(R) &= M(R_{\sigma(n)}, T^*_{\sigma(n-1)}(R))
\end{aligned}
$$

We define a scf $f_1^\sigma : \mathbb{D} \to A$ as follows: $f_1^\sigma(R) \in T^*_{\sigma(n)}(R)$ for all $R \in \mathbb{D}$.

Let $\bar{\sigma}$ be a priority function such that $\bar{\sigma}(1) = i$, $\bar{\sigma}(2) = j$ and $\bar{\sigma}(k) = \sigma(k)$ for all $k \neq 1, 2$. We define the following scf $f : \mathcal{R} \to A$:

$$
f(R) = \begin{cases} f_1^\sigma(R) & \text{if } R \in \mathbb{D} \\ \text{a serial dictatorship with respect to the priority function } \bar{\sigma} & \text{if } R \notin \mathbb{D} \end{cases}
$$

We check the strategy-proofness of f. Again, it suffices to check that i and j cannot manipulate. If $R \in \mathbb{D}$, the outcome belongs to $r_2(R_j)$ i.e. s_j^k. Agent j cannot manipulate for the following reason: for any R'_j if (i) $(R_i, R'_j \ldots) \in \mathbb{D}$, then the outcome belongs to s_j^k (ii) $(R_i, R'_j \ldots) \notin \mathbb{D}$ then the outcome does not belong to $r_1(R_j)$ because j is the dictator "after" i. Consider a possible manipulation by i from R. Since s_j^k is an isolated vertex, $s_j^k \cap r_2(R_i) \neq \emptyset$. Hence i obtains a second-ranked alternative in R_i. For any R'_i (i) if $(R_j, R'_i \ldots) \in \mathbb{D}$ the outcome belongs to $r_2(R'_i)$ and $r_1(R_i) \cap r_2(R'_i) = \emptyset$ and (ii) if $(R_j, R'_i \ldots) \notin \mathbb{D}$ the outcome belongs to $r_1(R'_i)$ and $r_1(R_i) \cap r_1(R'_i) = \emptyset$. If $R \notin \mathbb{D}$, then i is a dictator and cannot manipulate. For j, the only case to consider is the one where the manipulation R'_j is such that $(R'_j, R_i, \ldots) \in \mathbb{D}$. In this case, the outcome is in s_j^k. At R the outcome is a R_j- maximal element in s_i^m. Since s_j^k is an isolated vertex, it has elements in common with s_i^m. Hence the outcome at R is weakly preferred to alternatives in s_j^k according to R_j. Thus, j cannot manipulate.

It is again straightforward to show that f is unanimous and non-dictatorial.

We now prove Part B. Pick an arbitrary pair $i, j \in I$. Suppose $G(S_i, S_j)$ is not a connected graph with isolated vertices i.e. the sub-graph of $G(S_i, S_j)$ consisting of non-isolated vertices is not connected. The vertices of $G(S_i, S_j)$ can be partitioned into subsets V' and V'' with

- $|V'|$, $|V''| \geq 2$.
- $\{u, v\} \notin E$ for all $u \in V'$ and $v \in V''$.
- The sub-graphs with vertices V' and V'' are not discrete.

For instance in Figure 3.4, V' is the set $\{abcd, ef, ag, bh\}$ and V'' is the set $\{gh, ce, df\}$. We now construct the same scf as in Case I of Part A. This scf is efficient, strategy-proof and non-dictatorial as required. ■

Observation **3.3** The scfs constructed in Cases II and III of Part A are not efficient. For instance in Example 3.3 of Case II, the outcome at (R_1, R_2) (shown in Table 3.3) is b and in Example 3.4 of Case III, the outcome at $(\bar{R}_1, \bar{R}_2)$ (shown in Table 3.4) is c.

R_1	R_2
e	cd
ab	be
d	a
c	

Table 3.3: (R_1, R_2)

$\bar{R}_1$	$\bar{R}_2$
d	a
abc	ce
e	bd

Table 3.4: $(\bar{R}_1, \bar{R}_2)$

Unfortunately the necessary condition in Part A of Theorem 3.1 is not sufficient. This is shown by the Example 3.8. In the next section, we will show that the condition in Part B is sufficient when there are two agents (Part D of Theorem 3.2).

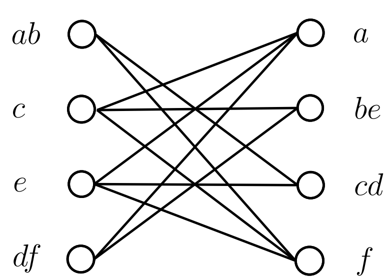

Figure 3.7: $G(S_1, S_2)$

Example **3.8** Let $I = \{1, 2\}$ and $A = \{a, b, c, d, e, f\}$. Figure 3.7 shows the induced partition graph where $S_1 = \{ab, c, e, df\}$ and $S_2 = \{a, be, cd, f\}$. Observe that this graph is connected and each vertex has degree atleast 2.

The scf is described as follows:

$$f(R)=\begin{cases} M(R_1, r_1(R_2)) & \text{if } r_1(R_2)\neq a \\ M(R_1,\{a,d\}) & \text{if } r_1(R_2)=a \text{ and } r_2(R_2)=cd \\ M(R_1,\{a,f\}) & \text{if } r_1(R_2)=a \text{ and } r_2(R_2)=f \\ a & \text{if } r_1(R_2)=a,\ r_2(R_2)=be, \\ & \quad \text{and } ab \text{ is preferred to } df \text{ according to } R_1 \\ b & \text{if } r_1(R_2)=a,\ r_2(R_2)=be, \\ & \quad \text{and } df \text{ is preferred to } ab \text{ according to } R_1 \end{cases}$$

The scf is a serial dictatorship where 2 picks first and 1 second, for all profiles *except* those where the first-ranked alternative of 2 is a. In the latter case, the outcome specified depends on 2's second-ranked alternatives and 1's preferences. In all cases, 2 gets at least a second-ranked alternative. The only profiles that are candidates for manipulation by 2 are those where her first-ranked alternative is a but the outcome is not. However, in these cases, df is preferred to ab for 1 and 2 cannot obtain a by misrepresentation. The outcome at all profiles is determined by maximizing 1's preferences over a set determined by 2's ordering; hence 1 cannot manipulate.

It is straightforward to verify that the scf is non-dictatorial and satisfies unanimity.

3.3.2 Sufficient Conditions for Dictatorship

In this section we provide various sufficient conditions for dictatorship. We introduce some definitions.

Definition **3.7** *The partition S satisfies Condition α, if (i) $v(S)\geq 2$ and (ii) there exists an agent i for whom S_i has at least three elements.*

Definition **3.8** *Let $I=\{i,j\}$. The partition S satisfies Condition β if (i) $v(S)=1$ and (ii) there exist $s_i^k, s_i^r \in S_i$, $s_i^{k'}, s_i^{r'} \in S_j$ such that $s_i^k \subset s_j^{k'}$, $s_j^{r'} \subset s_i^r$ and $s_j^{k'} \cap s_i^r = \emptyset$ (the subset relations are strict).*

Observation **3.4** Figure 3.8 shows an example of a partition graph for two players i and j where S satisfies Condition α (for instance, $S_i=\{a,bc,de,f\}$, $S_j=\{a,bc,d,ef\}$ and $V(S)=\{a,bc\}$). Observe that the vertices representing the elements of $V(S)$ have an edge with all vertices of the other agent (other than itself). This fact together with the assumption that $v(S)\geq 2$ ensures that the partition graph for any pair of agents is connected and the degree of every vertex is at least two.

Figure 3.9 illustrates Condition β (for instance, $S_i=\{a,bc,d,ef\}$, $S_j=\{a,b,de,cf\}$, $V(S)=a$ and s_i^k, s_i^r, $s_i^{k'}$, $s_i^{r'}$ are d, bc, de, b respectively). Note that the vertex representing

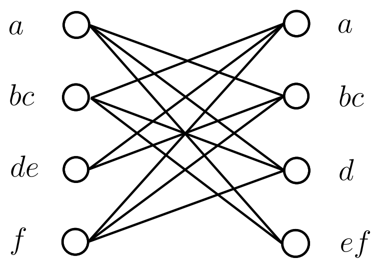

Figure 3.8: Partition graph for Condition α

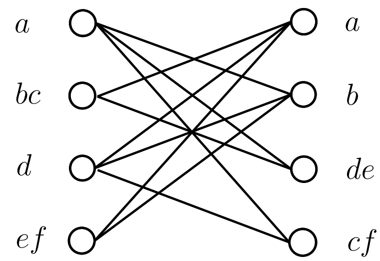

Figure 3.9: Partition graph for Condition β

the common element of the partition has an edge with all vertices of the other agent (other than itself). In addition, s_i^k has an edge with all vertices of other agent except $s_j^{k'}$. Similarly, $s_j^{r'}$ has an edge with all vertices of other agent except s_i^r. This ensures that the graph is connected and the degree of every vertex other than $s_j^{k'}$ and s_i^r is at least two. The assumption that $s_j^{k'} \cap s_i^r = \emptyset$ guarantees that the degree of these vertices is also at least two.

THEOREM **3.2** *Let $S = (S_1, \ldots, S_n)$ be a partition, $n \geq 2$.*

A. If S satisfies Condition α, then a strategy-proof and unanimous scf is dictatorial.

B. If S satisfies Condition β, then a strategy-proof and unanimous scf is dictatorial.

C. If $I = \{i, j\}$ and $G(S_i, S_j)$ is a cycle graph [2]*, then a strategy-proof and unanimous scf is dictatorial.*

D. If $I = \{i, j\}$ and $G(S_i, S_j)$ is a connected graph with isolated vertices, then a strategy-proof and efficient scf is dictatorial.

Proof **(of Part A):** Let $V(S) = \{s^1, s^2, ..., s^J\}$ where $J \geq 2$ and let $f : \mathcal{R} \to A$ be unanimous and strategy-proof scf. We prove the Theorem via the following claims.

CLAIM **3.1** *Let $R, R' \in \mathcal{R}$ be such that*

(i) $f(R) \in s^j$ where $s^j \in V(S)$ and

(ii) $\{a \in A | f(R) R_i a\} \subseteq \{a \in A | f(R) R'_i a\}$ for all i.

Then $f(R') \in s^j$.

[2] A cycle graph is a connected graph with the degree of every vertex being 2.

Proof: Pick an arbitrary agent i. We will show that $f(R'_i, R_{-i}) \in s^j$. Suppose not, i.e. $f(R'_1, R_{-1}) \notin s^j$. Clearly either $f(R)P_i f(R'_i, R_{-i})$ or $f(R'_i, R_{-i})P_i f(R)$ holds. The latter case immediately contradicts the strategy-proofness of f. If the former case holds, then condition (ii) above implies $f(R)P'_i f(R'_i, R_{-i})$. However, i will manipulate at (R'_i, R_{-i}) via R_i contradicting strategy-proofness again.

The Claim is established by repeated application of the same argument for different agents. ■

Sato (2009) proves counterparts of Claim 3.1 when the common indifference classes are singletons.

Claim **3.2** *There exists an agent $k \in I$ for whom the following holds: for all $R \in \mathcal{R}$, $f(R) \in r_1(R_k)$ whenever $r_1(R_k) \in V(S)$.*

Proof: By assumption there exists $s^j, s^k \in V(S)$. Let R^1 denote the profile where s^j and s^k are the top and bottom indifference classes respectively for all agents. By unanimity, $f(R^1) \in s^j$.

Construct new profiles by progressively making s^k and s^j the best and second-best indifference classes respectively in each agent's preferences. After changing the preferences of all agents, the outcome must belong to s^k, by unanimity. Therefore, there must exist an agent i such that (i) before agent i changes his preference, the outcome belongs to s^j and (ii) when i lifts s^k to the top of his preference, the outcome is no longer in s^j. If the outcome does not belong to either s^j or s^k, agent i will manipulate by reverting to the ordering where s^j is first and s^k is last and thereby obtaining an outcome in s^j. Therefore the outcome when i changes her ordering, must belong to s^k. Let R^2 denote the profile where which agents 1 through i have s^k first and s^j second. By the earlier argument $f(R^2) \in s^k$.

Next interchange s^j and s^k at R^2_i. Let R^3 denote the resulting preference profile. By stragegy-proofness, $f(R^3) \in s^j$.

At R^2 and R^3, lower s^j to the bottom for $1, 2, ..., i-1$ and to the second last position for $i+1, ..., n$. Let $R^{2'}$ and $R^{3'}$ denote the resulting profiles respectively. By Claim 3.1, $f(R^{2'}) \in s^k$. In order for i not to manipulate at $R^{3'}$, we must have $f(R^{3'}) \in \{s^k \bigcup s^j\}$. But if $f(R^{3'}) \in s^k$, then Claim 3.1, implies $f(R^3) \in s^k$, which is a contradiction. Therefore $f(R^{3'}) \in s^j$.

Let $d \in A \backslash \{s^j \bigcup s^k\}$ and let s^d_i denote the indifference class of agent i to which d belongs. Let R^4 be the profile shown in Table 3.5.

Agent	1	.	.	.	i-1	i	i+1	.	.	.	n
Best	.	.	.	.	.	s^j	.	.	.	.	.
	.	.	.	.	.	s_i^d	.	.	.	.	.
	.	.	.	.	.	s^k	.	.	.	.	.
	s_1^d	.	.	.	s_{i-1}^d	.	s_{i+1}^d	.	.	.	s_n^d
	s^k	.	.	.	s^k	.	s^j	.	.	.	s^j
Worst	s^j	.	.	.	s^j	.	s^k	.	.	.	s^k

Table 3.5: R^4

Since $f(R^{3'}) \in s^j$, Claim 3.1 implies that $f(R^4) \in s^j$. Let R^5 be the profile obtained by switching s^j and s^k at R^4 for agents $i+1$ through n. By strategy-proofness $f(R^5) \in \{s^j \bigcup s^k\}$. Suppose $f(R^5) \in s^k$. Let R^6 be the profile in Table 3.6. Since $f(R^5) \in s^k$, Claim 3.1 implies $f(R^6) \in s^k$. But this contradicts unanimity at R^6. Hence $f(R^5) \in s^j$. Since s^j is bottom ranked for all agents other than i and top ranked for i, it follows that the outcome belongs to s^j whenever agent i ranks s^j first.

Agent	1	.	.	.	i-1	i	i+1	.	.	.	n
Best	s_1^d	.	.	.	s_{i-1}^d	s_i^d	s_{i+1}^d	.	.	.	s_n^d
	.	.	.	.	.	s^j	.	.	.	.	.
	.	.	.	.	.	s^k	.	.	.	.	.
	.	.	.	.	.	.	.	.	.	.	.
	s^k	.	.	.	s^k	.	s^k	.	.	.	s^k
Worst	s^j	.	.	.	s^j	.	s^j	.	.	.	s^j

Table 3.6: R^6

Consider $s^k \in V(S)$ with s^k distinct from s^j. The earlier arguments can be replicated to show that there exists an agent i' such that the outcome belongs to s^k whenever i' ranks s^k first. If i and i' are distinct, the single-valuedness of f is contradicted at any profile where s^j and s^k are ranked first by i and i' respectively. This establishes the Claim. ■

We now complete the proof by showing that agent i identified in Claim 3.2 is a dictator.

Suppose this is false, i.e. there exists profile $\bar{R}$ and $f(R) \notin r_1(\bar{R}_i)$. In order not to contradict Claim 3.2 immediately, $r_1(\bar{R}_i) \notin V(S)$. Let s^j be the top-ranked indifference class in $V(S)$ in $\bar{R}_i$. By assumption there exists $s^k \in V(S)$ distinct from s^j. Construct profile R^7 as follows: (i) preferences of all agents other than i are the same as in $\bar{R}$ and (ii) $r_1(R_i^7) = r_1(\bar{R}_i)$ and $r_2(R_i^7) = s^k$. By strategy-proofness $f(R^7) \notin r_1(R_i^7)$. However, i can obtain an alternative in s^k by raising s^k to the top of her ordering (Claim 3.2). Hence $f(R^7) \in s^k$. Consider profiles R^8 and R^9 described in Tables 3.7 and 3.8 below.

Agent	1	.	.	.	i-1	i	i+1	.	.	.	n
Best	s^k	.	.	.	s^k	$r_1(\bar{R}_i)$	s^k	.	.	.	s^k
	.	.	.	.	.	s^k	.	.	.	.	.
	.	.	.	.	.	.	.	.	.	.	.
Worst	s^j	.	.	.	s^j	.	s^j	.	.	.	s^j

Table 3.7: R^8

Agent	1	.	.	.	i-1	i	i+1	.	.	.	n
Best	s^k	.	.	.	s^k	$r_1(\bar{R}_i)$	s^k	.	.	.	s^k
	.	.	.	.	.	s^j	.	.	.	.	.
	.	.	.	.	.	.	.	.	.	.	.
Worst	s^j	.	.	.	s^j	.	s^j	.	.	.	s^j

Table 3.8: R^9

By Claim 3.1 $f(R^8) \in s^k$. In addition strategy-proofness and Claim 3.2 imply $f(R^9) \in s^j$. Pick an arbitrary alternative d in $r_1(\bar{R}_i)$ and let s_l^d denote the indifference class to which d belongs for all agents $l \neq i$. Observe that s_l^d is strictly preferred to s^j for all l in R^9. Construct profile R^{10} by raising s_l^d to the top of R_l^9 for all $l \neq i$ while keeping i's preferences fixed at R_i^9. By Claim 3.1, $f(R^{10}) \in s^j$. However unanimity requires $f(R^{10}) \in \bigcap_l r_1(R_l^{10})$ where $d \in \bigcap_l r_1(R_l^{10}) \neq \emptyset$. We have a contradiction. ■

Proof **(of Part B):** Let S satisfy Condition β and suppose f is a strategy-proof and unanimous scf defined for $I = \{i, j\}$. In order to simplify the notation, we will denote the common indifference class by s, the sets s_i^k and s_i^r by y and X respectively and $s_i^{k'}$, $s_i^{r'}$ by Y and x respectively. Condition β requires $y \subset Y$, $x \subset X$ and $X \cap Y = \emptyset$.

The proof uses the following claim.

Claim **3.3** *For all profiles R such that $r_1(R_i) = s$ and $r_1(R_j) = Y$, $f(R) \in s \cup Y$.*

Proof: Suppose not, i.e. there exists R' with $r_1(R_i') = s$, $r_1(R_j') = Y$ and $f(R') \notin \{s \cup Y\}$. Let R_i'' and R_j'' be such that $r_1(R_i'') = s$, $r_2(R_i'') = y$, $r_1(R_j'') = Y$ and $r_2(R_j'') = s$. Strategy-proofness and unanimity imply $f(R_i', R_j'') \in s$ and $f(R_i'', R_j'') \in s$. However the same arguments also imply $f(R_i'', R_j') \in y$ and $f(R_i'', R_j'') \in Y$. This leads to a contradiction since f is a function. [3] ■

Let $\bar{R}$ be such that $r_1(\bar{R}_i) = s$, $r_1(\bar{R}_j) = Y$. By Claim 3.3, $f(\bar{R}_i, \bar{R}_j) \in s \cup Y$. We complete the proof by considering the following two cases.

Case 1: $f(\bar{R}) \in s$. Using the following sequence of claims we show that i is the dictator.

Claim **3.4** *For all profiles R such that $r_1(R_i) = s$, $f(R) \in s$.*

Proof: Let R^* be such that $R_i^* = \bar{R}_i$, $r_1(R_j^*) = Y$ and s is bottom-ranked according to R_j^*. If we can show $f(R^*) \in s$, then we are done. By Claim 3.3, $f(R^*) \in s \cup Y$. Note that $f(R^*) \notin Y$, otherwise j will manipulate at $\bar{R}$ via R_j^*. Therefore, $f(R^*) \in s$. ■

Claim **3.5** *For all profiles R such that $r_1(R_i) = y$, $f(R) \in y$.*

Proof: Consider the profiles R^1, R^2, R^3 and R^4 shown in Tables 3.9, 3.10, 3.11 and 3.12 respectively. The ranking of indifference classes other than the top three is the same across the four profiles, for both agents.

Agent	i	j
Best	y	x
	s	Y
	X	s
	.	.
	.	.
Worst	.	.

Table 3.9: R^1

Agent	i	j
Best	y	x
	X	Y
	s	s
	.	.
	.	.
Worst	.	.

Table 3.10: R^2

Agent	i	j
Best	y	x
	X	s
	s	Y
	.	.
	.	.
Worst	.	.

Table 3.11: R^3

Agent	i	j
Best	y	x
	s	s
	X	Y
	.	.
	.	.
Worst	.	.

Table 3.12: R^4

[3] These arguments closely follow counterparts in Sen (2001).

We begin by showing $f(R^1) \in y$. If $f(R^1) \in A \setminus \{s \cup y\}$, i will manipulate via an ordering where s is top-ranked, thereby obtaining an alternative in s (Claim 3.4). If $f(R^1) \in s$, j will manipulate via an ordering where Y is at the top obtaining an alternative in Y by unanimity. Therefore $f(R^1) \in y$.

Strategy-proofness implies $f(R^2) \in y$. Our next step is to show $f(R^3) \in y$. If $f(R^3) \in A \setminus \{y \cup X\}$, i will manipulate via an ordering where X is top-ranked, obtaining an alternative in X by unanimity. If $f(R^3) \in X \setminus x$, the outcome is ranked below Y by j (since $X \cap Y = \emptyset$) and she will manipulate via an ordering where Y is top-ranked, obtaining an alternative in Y by unanimity. If $f(R^3) \in x$, j will manipulate at R^2 via R^3_j. Therefore $f(R^3) \in y$. Again strategy-proofness implies $f(R^4) \in y$.

Finally, let R^5 be a profile such that $R^5_i = R^4_i$ and Y is bottom-ranked according to R^5_j. We complete the proof by showing that $f(R^5) \in y$. If $f(R^5) \in A \setminus \{s \cup y\}$, then i will manipulate via an ordering where s is top-ranked, obtaining an alternative in s by Claim 3.4. If $f(R^5) \in s$, j will manipulate at R^4 via R^5_j. Therefore $f(R^5) \in y$. ■

Claim **3.6** *For all profiles R such that $r_1(R_i) = X$, $f(R) \in X$.*

Proof: Suppose the Claim is false, i.e. there exists a profile $\tilde{R}$ with $r_1(\tilde{R}_i) = X$ but $f(\tilde{R}) \notin X$. Let R''_i be such that $r_1(R''_i) = X$ and $r_2(''_i) = y$. By strategy-proofness and Claim 3.5, $f(R''_i, \tilde{R}_j) \in y$. Let R''_j be such that top and bottom ranked indifference classes are Y and s respectively. Since $y \subset Y$, strategy-proofness implies that $f(R''_i, R''_j) \in Y$. Since $X \cap Y = \emptyset$, $f(R''_i, R''_j) \notin X$. Let R'''_i be such that $r_1(R'''_i) = X$ and $r_2(R''_i) = s$. By strategy-proofness and Claim 3.4, $f(R'''_i, R''_j) \in s$. If $f(R'''_i, R''_j) \in s$, then j will manipulate via an ordering whose top-ranked indifference class has a non-empty intersection with X. This proves the Claim.

■

Claim **3.7** *For all profiles R, $f(R) \in r_1(R_i)$.*

Proof: Suppose the Claim is false, i.e. there exists a profile R but $f(R) \notin r_1(R_i)$ and $r_1(R_i) \neq X, y, s$. Let $r_1(R_i) = Z$. The following cases arise.

Case 1: $Z \cap Y = \emptyset$. The arguments in Claim 3.6 with Z substituted for X can be replicated to establish $f(R) \in Z$.

Case 2: $Z \cap Y \neq \emptyset$. At R_j, lift Y to the top keeping the relative ranking of other indifference classes same and let R'_j be the resulting ordering. By unanimity, $f(R_i, R'_j) \in Z \cap Y$. Let R'_i be such that $r_1(R'_i) = Z$ and $r_2(R'_i) = s$. By unanimity once again, $f(R'_i, R'_j) \in Z \cap Y$.

We will now move Y downwards by progressive local switches in order to return to R_j while arguing that the outcome remains in Z, thereby contradicting our initial assumption.

Suppose such a local switch does not involve s and Y is above s in j's ranking. The outcome at such a profile cannot an alternative not in $Z \cup s$; in that case, i would manipulate by raising s to the top and getting s (Claim 3.4). Nor can the outcome belong to s because j would manipulate via R'_j obtaining an alternative in Y. Consequently, the outcome at such a profile must belong to Z.

Now suppose that we reach a profile where Y and s are contiguous in j's ordering and Y is above s. Let this ordering for j be denoted by $\hat{R}_j$. By the arguments of the previous paragraph, $f(R'_i, \hat{R}_j) \in Z$. If $f(R'_i, \hat{R}_j) \notin Y$, then the outcome must in fact, be above Y in $\hat{R}_j$. By switching Y and s in $\hat{R}_j$, the outcome must remain in Z (by the earlier) arguments. Continuing with the switches as required, we can conclude that $f(R'_i, R_j)$ and $f(R)$ both belong to Z contradicting our assumption.

Suppose therefore that $f(R'_i, \hat{R}_j) \in Y$, i.e. $f(R'_i, \hat{R}_j) \in Y \cap Z$. Let $\hat{R}_i$ be an ordering where Z and X are ranked first and second respectively. By strategy-proofness, $f(\hat{R}) \in Z$. Now switch Y and s in $\hat{R}_j$. The outcome at this profile must belong to $Y \cup s$. Since $X \cap (Y \cup s) = \emptyset$, the outcome does not belong to X. If the outcome is ranked below X in $\hat{R}_i$, i will manipulate by raising X to the top (Claim 3.6). Therefore the outcome belongs to Z. Now reverting back to R'_i, we observe that the outcome remains in Z. Continuing with the switches in j's ordering, we conclude that $f(R'_i, R_j)$ and $f(R)$ both belong to Z, once again contradicting our assumption. ■

We have established that i is a dictator completing Case 1.

Case 2: Suppose $f(\bar{R}) \in Y$. We will show that j is the dictator.

CLAIM **3.8** *For all profiles R, such that $r_1(R_j) = s$, $f(R) \in s$.*

Proof: Consider the profiles R^6, R^7, R^8 and R^9 shown in Tables 3.13, 3.14, 3.15 and 3.16 respectively. The ranking of indifference classes other than the top three is the same across the four profiles, for both agents.

By Claim 3.3, $f(R_i^6, \bar{R}_j) \in s \cup Y$. If $f(R_i^6, \bar{R}_j) \in s$, then i will manipulate at $\bar{R}$ via R_i^6. Therefore $f(R_i^6, \bar{R}_j) \in Y$. Strategy-proofness implies $f(R^6) \in Y$. Strategy-proofness and $X \cap Y = \emptyset$ implies $f(R^7) \notin \{s \cup X\}$; otherwise i will manipulate at R^6 via R_i^7. Unanimity implies that $f(R^7) \in y$. Since $y \subset Y$, $f(R^7) \in Y$. Note that strategy-proofness implies $f(R^8) \in s \cup Y$, because R^8 is obtained by switch s and Y for j at R^7. However, $f(R^8) \in Y$ would lead to manipulation by i via an ordering where s is top-ranked. Hence $f(R^8) \in s$. By strategy-proofness, $f(R^9) \in s$. Let R^{10} be such that $R_j^{10} = R_j^9$, $r_1(R_i^{10}) = X$ and s is at

Agent	i	j
Best	s	Y
	X	s
	y	x
	.	.
	.	.
Worst	.	.

Table 3.13: R^6

Agent	i	j
Best	X	Y
	s	s
	y	x
	.	.
	.	.
Worst	.	.

Table 3.14: R^7

Agent	i	j
Best	X	s
	s	Y
	y	x
	.	.
	.	.
Worst	.	.

Table 3.15: R^8

Agent	i	j
Best	X	s
	s	x
	y	Y
	.	.
	.	.
Worst	.	.

Table 3.16: R^9

the bottom according to R_i^{10}. Note that $f(R^{10}) \in x \cup s$, otherwise j will manipulate via an ordering where x is top-ranked, obtaining an alternative in x by unanimity. But $f(R^{10}) \notin x$, otherwise i would manipulate at R^9 via R_i^{10}. Therefore, $f(R^{10}) \in s$. This establishes the fact that $f(R) \in s$ for all R such that $r_1(R_j) = s$. ■

Claims 3.5, 3.6 and 3.7 can now be replicated with agent i replaced by j to show that j is the dictator. ■

Proof **(of Part C):** Let $G(S_i, S_j)$ be a cycle graph and suppose f satisfies strategy-proofness and unanimity on this domain. In order to simplify the notation, we will denote the elements of S_i and S_j (i.e the vertices of $G(S_i, S_j)$) by $X, Y, Z, P, Q, T, \ldots$ etc.

We identify two properties of $G(S_i, S_j)$ that will be required for the argument.

Claim **3.9** *1. $|S_i| = |S_j| \geq 3$. 2. If (X, P) is an edge where $X \in S_i$ and $P \in S_j$, there exist $Y \in S_i$, $Q \in S_j$ such that (i) $Y \cap Q = \emptyset$ (ii) $Y \cap P \neq \emptyset$ and (iii) $X \cap Q \neq \emptyset$.*

Proof: 1. $|S_i| = |S_j|$ is a standard property of bipartite cycle graphs (see page 24 of West (2001)). The only case to rule out is $|S_i| = |S_j| = 2$. Suppose this is true. Let $S_i = \{X, Y\}$ and $S_j = \{P, Q\}$. Since the degree of each vertex is 2, it must be the case that $(X, P), (X, Q), (Y, P)$ and (Y, Q) are all edges, i.e. $X \cap P, X \cap Q, Y \cap P$ and $Y \cap Q$ are all empty. This implies $(X \cup Y) \cap (P \cup Q) = \emptyset$ which is impossible because $X \cup Y = P \cup Q = A$.
2. Let (X, P) be an edge with $X \in S_i$ and $P \in S_j$. There must exist $Q \in S_j$ such that $X \cap Q \neq \emptyset$, i.e. (X, Q) is not an edge. Since Q has degree 2, there exist vertices $Y, Z \in S_i$ such that (Y, Q) and (Z, Q) are edges. Suppose $Y \cap P$ and $Z \cap P$ are both empty. Then P is degree at least three which is impossible. Hence either Y or Z (or both) has a non-empty intersection with P. ■

Claim **3.10** *For all profiles R such that $r_1(R_i) \cap r_1(R_j) = \emptyset$, $f(R) \in r_1(R_i) \cup r_1(R_j)$.*

Proof: Pick R such that $r_1(R_i) = X$ and $r_1(R_j) = P$ and $X \cap P = \emptyset$. Assume for contradiction that $f(R) \notin X \cup P$. From Claim 3.9 (Part 2), there exists $Y \in S_i$, $Q \in S_j$ such that $X \cap Q \neq \emptyset$, $Y \cap P \neq \emptyset$ and $Y \cap Q = \emptyset$.

Raise Y to the second-ranked position in R_1 keeping the ranking of other indifference classes the same and let R'_i be the resulting ordering. Similarly, raise Q to the second-ranked position keeping the relative ranking of other indifference classes the same and let R'_j be the resulting ordering. Standard strategy-proofness and unanimity arguments can be applied to show that either $f(R'_i, R'_j) \in X \cap Q$ or $f(R'_i, R'_j) \in Y \cap P$. We complete the proof of the claim by showing that $f(R_i, R_j) \in X$ if $f(R'_i, R'_j) \in X \cap Q$ (an analogous argument establishes $f(R_i, R_j) \in P$ if $f(R'_i, R'_j) \in Y \cap P$).

Strategy-proofness and unanimity imply $f(\bar{R}_i, \bar{R}_j) \in X$ where $r_1(\bar{R}_i) = X$, $r_1(\bar{R}_j) = P$ and $r_2(\bar{R}_j) = Q$. If $r_2(R_j) = r_2(\bar{R}_j) = Q$, we are done. So, let $r_2(R_j) = T \neq Q$. We consider two cases.

Case 1: $T \cap X = \emptyset$. Let R'' be a profile such that $r_1(R''_i) = X$, $r_1(R''_j) = P$, $r_2(R''_j) = Q$ and $r_3(R''_j) = T$. Note that $f(R'') \in X$. Since degree of T is 2, there exists $Z \in S_i$ such that $Z \cap T = \emptyset$. Applying Claim 3.9 (Part 1), $Z \cap P \neq \emptyset$.

Suppose $Z \cap Q = \emptyset$. Raise Z to the second position in R''_i keeping the ranking of other indifference classes the same and let R'''_i be the resulting ordering. By strategy-proofness, $f(R'''_i, R''_j) \in X$. At R''_j switch Q and T and let R'''_j be the resulting ordering. By strategy-proofness and unanimity, $f(R'''_i, R'''_j) \in X$. If $r_3(R_j) = Q$, strategy-proofness and unanimity imply $f(R'''_i, R_j) \in X$; moreover $f(R) \in X$. Let $r_3(R_j) = P' \neq Q$. By the fact that the degree of P' is 2, there exists $X' \in S_i$ such that $X' \cap P' = \emptyset$ and $X' \cap P \neq \emptyset$. Let $\tilde{R}_i$ be an ordering such that $r_1(\tilde{R}_i) = X$ and $r_2(\tilde{R}_i) = X'$. By strategy-proofness, $f(\tilde{R}_i, R'''_j) \in X$. Let $\tilde{R}_j$ be such that $r_1(\tilde{R}_j) = P$, $r_2(\tilde{R}_j) = T$ and $r_3(\tilde{R}_j) = P'$. By strategy-proofness, $f(\tilde{R}_i, \tilde{R}_j) \in X$. Again by strategy-proofness, $f(\tilde{R}_i, R_j) \in X$ and $f(R_i, R_j) \in X$.

Suppose $Z \cap Q \neq \emptyset$. Since the degree of Z is two, there exists $Q' \in S_j$ such that $Z \cap Q' = \emptyset$. Let $\hat{R}_j$ be such that $r_1(\hat{R}_j) = P$, $r_2(\hat{R}_j) = Q$, $r_3(\hat{R}_j) = T$ and $r_4(\hat{R}_j) = Q'$. Note that $f(R'''_i, \hat{R}_j) \in X$. Let R^*_j be such that $r_1(R^*_j) = P$, $r_2(R^*_j) = T$, $r_3(R^*_j) = Q'$ and $r_4(R^*_j) = Q$. By strategy-proofness and unanimity, $f(R'''_i, R^*_j) \in X$. Applying these arguments repeatedly, we conclude that $f(R) \in X$.

Case 2: $T \cap X \neq \emptyset$. Since the degree of P is 2, there exists $Y' \in S_i$ such that $Y' \cap T = \emptyset$ and $Y' \cap P \neq \emptyset$. At R''_i lift Y' to the second ranked position keeping the relative ranking of other indifference classes same and let R^*_i be the resulting ordering. By strategy-proofness and unanimity, we can infer that $f(R^*_i, R''_j) \in X$ and $f(R^*_i, R'''_j) \in X$. Applying these arguments repeatedly, $f(R) \in X$ follows. ■

The proof is completed by showing that the agent from whose first-ranked indifference class the outcome is picked in Claim 3.10 is in fact, a dictator. This can be done by replicating the arguments in Claim B in Sen (2001). The only requirement for the argument is for the graph $G(S_i, S_j)$ to be connected. ■

Proof **(of Part D):** Let $G(S_i, S_j)$ be a connected graph with isolated vertices and suppose f satisfies strategy-proofness and efficiency on this domain.

Claim **3.11** *For all R such that $r(R_i) \cap r(R_j) = \emptyset$, either $f(R) \in r_1(R_i)$ or $f(R) \in r_1(R_j)$.*

Proof: Pick R such that $r(R_1) \cap r(R_2) = \emptyset$ and denote $r_1(R_i)$ and $r_1(R_j)$ by X and P respectively. Suppose $f(R) \notin X \cup P$. There exists $Q \in S_j$ such that $X \cap Q \neq \emptyset$ and $Y \in S_i$ such that $Y \cap P \neq \emptyset$.

Let R'_i and R'_j be such that $r_1(R'_i) = X$, $r_2(R'_i) = Y$, $r_1(R'_j) = P$ and $r_2(R'_j) = Q$. By strategy-proofness and efficiency, $f(R'_i, R_j) \notin Y \cap P$. By strategy-proofness, $f(R'_i, R'_j) \in P$. Similarly, strategy-proofness and efficiency imply that $f(R_i, R'_j) \in X \cap Q$. Once again applying strategy-proofness we get $f(R'_i, R'_j) \in X$ - a contradiction. ■

Suppose $X \in S_i$ is an isolated vertex in $G(S_i, S_j)$. If $r_1(R_i) = X$ at R, we have $r_1(R_i) \cap r_1(R_j) \neq \emptyset$. The same conclusion holds if X is isolated and belongs to S_j.

Finally, since non-isolated vertices are connected in $G(S_i, S_j)$, the arguments in Claim B in Sen (2001) can be used in conjunction with Claim 3.11 to demonstrate the existence of a dictator. ■

We make several observations about our result.

Observation **3.5** Part A of Theorem 3.2 generalizes Sato (2009) in two ways. It requires only two rather three common indifference classes. Moreover, these common indifference classes need not be singletons. Our argument follows the general structure of Sato's arguments (with appropriate refinements) which in turn follows that of Reny (2001) and several other papers.

Observation **3.6** Parts B, C and D are two-person results. The generalization to an arbitrary number of agents is not straightforward. The main difficulty is that the standard "cloning" arguments for the situation where all agents have a common domain, cannot be used. It is not clear whether the pairwise conditions used in Parts B, C and D are sufficient for dictatorship generally.

Observation **3.7** Part B of Theorem 3.1 together with Part D of Theorem 3.2 provide a necessary and sufficient condition for strategy-proofness and efficiency to imply dictatorship, in the case of two agents.

3.4 Conclusion

The chapter investigates the effect of the partition structure on dictatorship results in the exogenous indifference classes model. The focus of the analysis is the pairwise partition function graph. We provide necessary conditions and stronger sufficient conditions on these graphs for strategy-proofness and unanimity (or efficiency) to imply dictatorship.

Several natural questions remain open. Some of our results apply only to the two-voter case and a gap exists between our necessary and sufficient conditions. These questions appear to be difficult but we hope to make progress in answering them in the future.

Chapter 4

Further Results on Dictatorial Domains

4.1 Introduction

The incompatibility between strategy-proofness and non-dictatorship is a major issue in social choice. The seminal result of Gibbard (1973) and Satterthwaite (1975) states that a surjective and strategy-proof social choice function (scf) with a range of at least three alternatives, defined over the complete domain, is dictatorial. Aswal et al. (2003) show that the assumption of a complete domain is far from being necessary for this result. They show that a large class of domains (including several that are "small") are *dictatorial* - i.e. domains with the property that all strategy-proof and unanimous scfs (with a range of at least three) defined over such domains, are dictatorial. A complete characterization of dictatorial domains is a natural objective but appears to difficult to provide. Our goal in this chapter is to generalize the sufficiency result of Aswal et al. (2003) and unify existing results in the area.

It will be helpful to briefly recount the result of Aswal et al. (2003). Fix an arbitrary domain. They say that two alternatives a and b are *connected* if there exists a preference in the domain where a is ranked first and b, second and another preference where the reverse is true. They consider the following graph: each alternative is a vertex and there is an edge between a pair of vertices if the two alternatives represented by the vertices, are connected. A domain is *linked* if this graph is "sufficiently dense". Specifically, there should exist an arrangement of the vertices such that the first three are mutually connected and each vertex is connected to at least two in the set of vertices that precedes it. Their main result is that every linked domain is dictatorial. They show the existence of a variety of linked domains including those that are linear in the number of alternatives. However, this result is far from a characterization - for instance, the circular domains defined in Sato (2010) and are not linked.

We generalize the linked domain result in two ways. The first way is to weaken the notion of connectedness between a pair of alternatives to *weak connectedness* while retaining the "connection structure" of the induced graph as in linkedness. The second way is to strengthen the notion of connectedness but weakening the "connection structure" on the induced graph.

Two alternatives a and b are weakly connected if there exists a (possibly empty) set of alternatives B and *four* orderings in the domain such that there is a *reversal* between B and b when a is top-ranked and there is a reversal between B and a when b is top-ranked. Reversality requires alternatives between a and b to belong to B in the case where B is better than b. Similarly, alternatives between b and a to belong to B in the case where B is better than a. A domain is called a β domain if we can arrange all the alternatives (vertices in the induced graph) in a way that the first three are mutually weakly connected and each alternative is weakly connected to at least two in the set of alternatives (vertices) that precedes it. Our first result is that β domains are dictatorial. These domains are obviously supersets of linked domains - it is also possible to find β domains that are smaller than any linked domain.

Property T between a and b requires the following "intermediateness" property in addition to weak connectedness: for any alternative c other than a and b, there exists two orderings in the domain, one where c is above b while a at the top and another where c is above a while b at the top. A domain is called a γ domain if its induced graph is connected in the usual graph-theoretic sense, i.e. there exists a path between any two alternatives(vertices). Our second result is that all γ domains whose induced graph *is not* a star-graph, are dictatorial domains. The same result holds in the star-graph case with mild additional conditions. These results generalize results on circular domains in Sato (2010) and Chatterji et al. (2013). Finally, we apply our result to a facility location problem in a restricted environment.

The chapter is organized as followed. Section 4.2 contains a description of the model. Sections 4.3 and 4.4 contain the results on β and γ domains respectively. Section 4.5 provides an application while Section 4.6 concludes.

4.2 Basic notation and definitions

Let $A = \{a_1, \ldots, a_m\}$ denote a finite set of alternatives with $m \geq 3$. Let $I = \{1, 2, \ldots, n\}$, $n \geq 2$ be a finite set of agents. Let $\mathbb{P}$ denote the set of strict orderings [1] of the elements of A. An admissible domain is a set $\mathbb{D} \subset \mathbb{P}$. A typical preference orderings will be denoted by P_i where aP_ib will signify that a is preferred (strictly) to b under P_i. A preference profile is

[1] A strict ordering is a complete, transitive and antisymmetric binary relation.

an element of the set $\mathbb{D}^n$. Preference profiles will be denoted by $P, \bar{P}, P'$ etc and their i^{th} components as $P_i, \bar{P}_i, P'_i$ respectively with $i = 1, 2, \ldots, n$. Let $(\bar{P}_i, P_{-i})$ denote the preference profile where the i^{th} component of the profile P is replaced by $\bar{P}_i$.

Given $P_i \in \mathbb{D}$, let $r_k(P_i)$ denote the k^{th} ranked alternative in P_i, $k = 1, \ldots, m$, i.e., $[r_k(P_i) = a_j] \Rightarrow [|\{a_k \in A : a_k P_i a_j\}| = k - 1]$. For an ordering $P_i \in \mathbb{D}$ and $a_j \in A$, we let $B(a_j, P_i)$ denote the set of alternatives that are strictly better than a_j according to P_i, while $W(a_j, P_i)$ denotes the set of alternatives that are strictly worse than a_j according to P_i. Let $M(a_j, a_k, P_i)$ be the set alternatives that are strictly worse than a_j and strictly better than a_k according to P_i.

Definition **4.1** *A social choice function (scf) f is a mapping $f : \mathbb{D}^n \longmapsto A$.*

Some familiar properties of scfs are stated below.

Definition **4.2** *A scf f satisfies unanimity, if for all $P \in \mathbb{D}^n$, $f(P) = a$ whenever $a = r_1(P_i)$ for all $i \in I$.*

If an alternative is top-ranked by all voters, the scf must pick that alternative.

A scf is strategy-proof if no voter can obtain a strictly better alternative by misrepresenting her preferences for any announcements of preferences of the other voters.

Definition **4.3** *A scf $f : \mathbb{P} \to A$ is manipulable by agent i at a profile $P \in \mathbb{P}$ via P'_i if*

$$f(P'_i, P_{-i}) P_i f(P).$$

A scf f is strategy-proof if it is not manipulable by any agent at any profile.

A scf is a dictatorship if a particular voter always gets her best alternative.

Definition **4.4** *A scf f is dictatorial if there is an individual $i \in I$ such that $f(P) = r_1(P_i)$ for all $P \in \mathbb{D}^n$*

The following well-known result provides a full characterization of strategy-proof scfs for the domain $\mathbb{P}$.

Theorem **4.1** *(Gibbard (1973),Satterthwaite (1975)) A scf $f : \mathbb{P}^n \to A$ is strategy-proof and satisfies unanimity if and only if it is dictatorial.*

Unfortunately, there is a large class of preference domains where strategy-proofness implies dictatorship, so that there is no escape from this unpleasant dilemma. These domains which we define formally below, are the objects of our study.

Definition **4.5** *The domain* $\mathbb{D} \subset \mathbb{P}$ *is dictatorial if, for all scfs* $f : \mathbb{D}^n \longmapsto A$ *is strategy-proof and satisfies unanimity implies* f *is dictatorial.*

Throughout the chapter, we shall restrict attention to domains that are *minimally rich.*

Definition **4.6** *A domain* $\mathbb{D}$ *is minimally rich if, for all* $a \in A$*, there exists* $P_i \in \mathbb{D}$ *such that* $r_1(P_i) = a$*.*

The minimal richness assumption guarantees that every alternative is top-ranked for some ordering in the domain. This is a standard assumption in the literature, for instance Aswal et al. (2003).

4.3 β Domains

We first introduce the notion of weak connectedness. In what follows, we fix a domain $\mathbb{D} \subset \mathbb{P}$.

Definition **4.7** *A pair of alternatives* a_j, a_k *is weakly connected, denoted by* $a_j \overset{w}{\sim} a_k$ *if there exists* $B \subset A$ *(possibly empty) and* $P_i, \bar{P}_i, P_i', P_i'' \in \mathbb{D}$ *such that*

1. $r_1(P_i) = r_1(\bar{P}_i) = a_j$ *and* $r_1(P_i') = r_1(P_i'') = a_k$*.*
2. $B = M(a_j, a_k, P_i)$ *and* $B \subset W(a_k, \bar{P}_i)$*.*
3. $B = M(a_k, a_j, P_i')$ *and* $B \subset W(a_j, P_i'')$*.*

The weak connectedness concept is illustrated below.

P_i	$\bar{P}_i$	P_i'	P_i''
a_j	a_j	a_k	a_k
B	.	B	.
a_k	a_k	a_j	a_j
.	B	.	.
.	.	.	B
.	.	.	.

Table 4.1: Weak connectedness

The idea is quite simple. There exists a set B such that there is a *reversal* between B and a_k when a_j is top-ranked and there is a reversal between B and a_j when a_k is top-ranked. Reversality requires *all* alternatives between a_j and a_k to belong to B in the case where B is better than a_k. Similarly, *all* alternatives between a_k and a_j to belong to B in the case where B is better than a_j.

Observation **4.1** In case B is the empty set, weak connectedness reduces to connectedness in the sense of Aswal et al. (2003).

A β *domain* can be defined in the same way that a linked domain was defined in Aswal et al. (2003).

Definition **4.8** *Let $B \subset A$ and let $a_j \notin B$. Then a_j is linked to B if there exists $a_k, a_r \in B$ such that $a_j \overset{w}{\sim} a_k$ and $a_j \overset{w}{\sim} a_r$.*

Definition **4.9** *The domain $\mathbb{D}$ is called a β domain if there exists a one to one function $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$ such that*

(i) $a_{\sigma(1)} \overset{w}{\sim} a_{\sigma(2)}$

(ii) a_j *is linked to* $\{a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(j-1)}\}$, $j = 3, \ldots, m$.

By virtue of Observation 4.1, linked domains are β domain. However, the converse is not true as the example below shows.

Example **4.1** Let $A = \{a_1, a_2, a_3, a_4\}$ and let $\bar{\mathbb{D}}$ be the domain in Table 4.2. It is clear that a_1 is connected to a_2 and a_3, a_2 is connected to a_3, but a_4 is not connected to any other alternatives. Therefore $\bar{\mathbb{D}}$ is not linked. But it is a β domain because $a_4 \overset{w}{\sim} a_1$ and $a_4 \overset{w}{\sim} a_2$.

P_i^1	P_i^2	P_i^3	P_i^4	P_i^5	P_i^6	P_i^7	P_i^8
a_1	a_1	a_2	a_2	a_3	a_3	a_4	a_4
a_2	a_3	a_1	a_3	a_1	a_2	a_1	a_2
a_4	a_4	a_4	a_4	a_2	a_1	a_2	a_1
a_3	a_2	a_3	a_1	a_4	a_4	a_3	a_3

Table 4.2: The domain $\bar{\mathbb{D}}$

It is helpful to interpret a β domain in terms of the graphs induced by weak connectedness. Let $\mathbb{D}$ be a domain. The graph $G(\mathbb{D})$ is defined as follows: the vertices of the graph are the alternatives and two vertices have an edge iff the alternatives represented by the vertices are weakly connected. The graph induced by the domain in Example 4.1 is shown in Figure 4.1.

Our first Theorem shows that the linked domain result in Aswal et al. (2003) can be generalized to β domains.

Theorem **4.2** *A β domain is a dictatorial domain.*

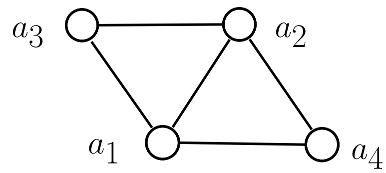

Figure 4.1: The graph $G(\bar{\mathbb{D}})$

Proof: Let $\mathbb{D}$ be a β domain and assume without loss of generality that the function σ in definition 4.9 is the identity function. For every non-empty $X \subset A$, we let $\mathbb{D}^X = \{P_i \in \mathbb{D} | r_1(P_i) \in X\}$. Similarly, for any alternative $a_i \in A$, we let $\mathbb{D}^{a_i} = \{P_i \in \mathbb{D} | r_1(P_i) = a_i\}$.

In view of Proposition 3.1 of Aswal et al. (2003) and our assumption of minimal richness, it suffices to show that if $f : \mathbb{D}^2 \to A$ is strategy-proof and unanimous, then f is dictatorial.

The following Lemma is very general.

Lemma **4.1** *Let $\mathbb{D}$ be an arbitrary domain and let a, b be arbitrary alternatives with $a \overset{w}{\sim} b$. If $f : \mathbb{D}^2 \to A$ satisfies strategy-proofness and unanimity, then $f(P) \in \{a, b\}$ for all $P \in \mathbb{D}^2$ such that $r_1(P_1), r_1(P_2) \in \{a, b\}$.*

Proof: Suppose not. Let a and b be the first ranked outcomes according to P_1 and P_2 respectively with $f(P) = c$ where $c \neq a, b$. Note that a and b must also be distinct from each other, otherwise we immediately contradict unanimity. Since $a \overset{w}{\sim} b$, there exists $B \subset A$ and $P_1', P_1'', P_2', P_2'' \in \mathbb{D}$ such that (i) $r_1(P_1') = r_1(P_1'') = a$ and $r_1(P_2') = r_1(P_2'') = b$, (ii) $B = M(a, b, P_1')$ and $B \subset W(b, P_1'')$, (iii) $B = M(b, a, P_2')$ and $B \subset W(a, P_2'')$. We consider two cases.

Case 1: $B = \emptyset$. By replicating the arguments in Claim A in Sen (2001), we can show that $f(P) \in \{a, b\}$. This leads to a contradiction.

Case 2: $B \neq \emptyset$. Observe that $f(P_1, P_2')$ cannot be b because 2 would manipulate at P via P_2'. Also note that $f(P_1, P_2') \notin W(a, P_2')$. Otherwise 2 would manipulate via an ordering where a is ranked first, thereby obtaining the outcome a (unanimity). We consider the following two cases.

Case 2.1: $f(P_1, P_2') = a$. Strategy-proofness implies that $f(P_1', P_2') = a$.

Observe that $f(P_1', P_2) \neq a$ because then 1 would manipulate at P via P_1'. Also $f(P_1', P_2) \notin W(b, P_1')$, otherwise 1 would manipulate via an ordering where b is ranked first, thereby obtaining the outcome b (unanimity). Therefore, $f(P_1', P_2) \in B \cup b$. If $f(P_1', P_2) \in B \cup b$, then 2 will manipulate at (P_1', P_2') via P_2 - a contradiction.

Case 2.2: $f(P_1, P_2') \in B$. Let $f(P_1, P_2') = d$. Then it must be the case that aP_1dP_1b. First we show that $f(P_1', P_2') \in B$. If $f(P_1', P_2') \in W(b, P_1') \bigcup \{b\}$, then 1 would manipulate at (P_1', P_2') via P_1. If $f(P_1', P_2') = a$, then 1 would manipulate at (P_1, P_2') via P_1'. Therefore $f(P_1', P_2') \in B$. Next we show that $f(P_1', P_2'') = a$. If $f(P_1', P_2'') = b$, then 2 would manipulate at (P_1', P_2') via P_2''. If $f(P_1', P_2'') \in B$, then 2 would manipulate at (P_1', P_2'') via an ordering where a is ranked first, thereby obtaining the outcome a (unanimity). If $f(P_1', P_2'') \in W(b, P_1')$, then 1 would manipulate at (P_1', P_2'') via an ordering where b is ranked first, thereby obtaining the outcome b (unanimity). Therefore $f(P_1', P_2'') = a$. At (P_1'', P_2''), $f(P_1'', P_2'') = a$, otherwise 1 would manipulate at (P_1'', P_2'') via P_1'. Finally we show that $f(P_1'', P_2') = a$. Note that $f(P_1'', P_2') \neq b$, otherwise 2 would manipulate at (P_1'', P_2'') via P_2'. Also $f(P_1'', P_2') \notin B$, otherwise 1 would manipulate at (P_1'', P_2') via an ordering where b is ranked first, thereby obtaining the outcome b (unanimity). If $f(P_1'', P_2') \in W(a, P_1')$, Then 2 would manipulate at (P_1'', P_2') via an ordering where a is ranked first, thereby obtaining the outcome a (unanimity). Therefore $f(P_1'', P_2') = a$. Note that 1 would manipulate at (P_1', P_2') via P_1'' because earlier we have shown $f(P_1', P_2') \in B$. This leads to a contradiction. ■

Our proof consists in establishing two steps.

Step 1: Let $X = \{a_1, a_2, a_3\}$. There exists $j \in \{1, 2\}$ such that $f(P) = r_1(P_j)$ for all $P \in \mathbb{D}^X \times \mathbb{D}^X$.

Step 2: Let $\bar{X} = \{a_1, a_2, \ldots, a_{l-1}\}$ and $X^* = \{a_1, a_2, \ldots, a_l\}$, $l = 4, \ldots, m$. If $f(P) = r_1(P_j)$ for all $P \in \mathbb{D}^{\bar{X}} \times \mathbb{D}^{\bar{X}}$, then $f(P) = r_1(P_j)$ for all $P \in \mathbb{D}^{X^*} \times \mathbb{D}^{X^*}$.

We proceed to establish Step 1 through a sequence of claims. First note that since $\mathbb{D}$ is a β domain and σ is the identity function, we have $a_1 \overset{w}{\sim} a_2$, $a_2 \overset{w}{\sim} a_3$ and $a_3 \overset{w}{\sim} a_1$. By Lemma 4.1, either $f(P_1, P_2) = a_1$ or $f(P_1, P_2) = a_2$ for all $P \in \mathbb{D}^2$ such that $r_1(P_1) = a_1$ and $r_1(P_2) = a_2$. Let $\bar{P}_1$ and $\bar{P}_2$ be such that $r_1(\bar{P}_1) = a_1$, $r_1(\bar{P}_2) = a_2$ and w.l.o.g. we assume that $f(\bar{P}_1, \bar{P}_2) = a_1$. We complete Step 1 by showing that agent 1 is the dictator. By Lemma 4.1 and strategy-proofness, $f(P_1, P_2) = a_1$ for all $P \in \mathbb{D}^2$ where $r_1(P_1) = a_1$ and $r_1(P_2) = a_2$. The following pair of claims are required to establish Step 1.

Claim **4.1** *For all $P \in \mathbb{D}^2$ where $r_1(P_1) = a_3$ and $r_1(P_2) = a_2$, $f(P_1, P_2) = a_3$.*

Proof: Suppose not. Then, there exists (P_1', P_2') with $r_1(P_1') = a_3$ and $r_1(P_2') = a_2$ such that $f(P_1', P_2') \neq a_3$. Lemma 4.1 implies that $f(P_1', P_2') = a_2$. By lemma 4.1 and strategy-proofness, $f(P_1, P_2) = a_2$ for all $(P_1, P_2) \in \mathbb{D}^2$ with $r_1(P_1) = a_3$ and $r_1(P_2) = a_2$. Since $a_3 \overset{w}{\sim} a_1$, there exists $B \subset A$, P_1'', and P_1^* such that $r_1(P_1'') = r_1(P_1^*) = a_3$, $B = M(a_3, a_1, P_1'')$ and $B \subset W(a_1, P_1^*)$. Note that either $a_2 \in B$ or $a_2 \notin B$. If $a_2 \in B$, then $f(P_1^*, P_2') \neq a_2$.

Otherwise, agent 1 would manipulate via an ordering where a_1 is ranked first - a contradiction. If $a_2 \notin B$, then $f(P_1'', P_2') \neq a_2$. Otherwise, agent 1 would manipulate via an ordering where a_1 is ranked first - a contradiction. ■

CLAIM **4.2** *For all $P \in \mathbb{D}^2$ where $r_1(P_1) = a_1$ and $r_1(P_2) = a_3$, $f(P_1, P_2) = a_1$.*

Proof: Suppose not. Then, there exists (P_1', P_2') with $r_1(P_1') = a_1$ and $r_1(P_2') = a_3$ such that $f(P_1', P_2') \neq a_1$. Lemma 4.1 implies that $f(P_1', P_2') = a_3$. By Lemma 4.1 and strategy-proofness, $f(P_1, P_2) = a_3$ for all $(P_1, P_2) \in \mathbb{D}^2$ with $r_1(P_1) = a_1$ and $r_1(P_2) = a_3$. Since $a_3 \overset{w}{\sim} a_2$, there exists $B \subset A$, P_2'', and P_2^* such that $r_1(P_2'') = r_1(P_2^*) = a_2$, $B = M(a_2, a_3, P_2'')$ and $B \subset W(a_3, P_2^*)$. Note that either $a_1 \in B$ or $a_1 \notin B$. If $a_1 \in B$, then $f(P_1', P_2^*) = a_2$. Otherwise if $f(P_1', P_2^*) = a_1$, agent 2 would manipulate via an ordering where a_3 is ranked first - a contradiction. If $a_1 \notin B$, then $f(P_1', P_2'') = a_2$. Otherwise, agent 2 would manipulate via an ordering where a_3 is ranked first - a contradiction. ■

Using the arguments used in the proof of Claim 4.1 and 4.2, it is straightforward to show that $f(P) = r_1(P_1)$ for all $P \in \mathbb{D}^X \times \mathbb{D}^X$. This establishes Step 1.

We now turn to Step 2. Pick an integer l in the set $\{4, \ldots, m\}$. We state our induction hypothesis below.

Induction Hypothesis (IH): $f(P) = r_1(P_1)$ for all $P \in \mathbb{D}^{\bar{X}} \times \mathbb{D}^{\bar{X}}$.

Our objective is to show that $f(P) = r_1(P_1)$ for all $P \in \mathbb{D}^{X^*} \times \mathbb{D}^{X^*}$.

Statement*: Since a_l is linked to $\{a_1, \ldots, a_{l-1}\}$, there exists $a_i, a_j \in \{a_1, \ldots, a_{l-1}\}$ such that $a_l \overset{w}{\sim} a_i$ and $a_l \overset{w}{\sim} a_j$.

CLAIM **4.3** *For all $(P_1, P_2) \in \mathbb{D}^2$ such that $P_1 \in \mathbb{D}^{a_l}$ and $P_2 \in \mathbb{D}^{\{a_i, a_j\}}$, $f(P) = r_1(P_1)$ (a_i and a_j are as specified in (*)).*

Proof: Suppose not. There exists an $(\bar{P}_1, \bar{P}_2) \in \mathbb{D}^2$ such that $\bar{P}_1 \in \mathbb{D}^{a_l}$, $\bar{P}_2 \in \mathbb{D}^{\{a_i, a_j\}}$ and $f(\bar{P}_1, \bar{P}_2) \neq a_l$. Therefore by Lemma 4.1, $f(\bar{P}_1, \bar{P}_2) = r_1(\bar{P}_2)$. Let $r_1(\bar{P}_2) = a_i$ - a similar argument holds if $r_1(\bar{P}_2) = a_j$. Since $a_l \overset{w}{\sim} a_j$, there exists an ordering P_1^* such that (i) $r_1(P_1^*) = a_l$ and (ii) $a_j P_1^* a_i$. By Lemma 4.1 and strategy-proofness $f(P_1^*, \bar{P}_2) = a_i$. Note that agent 1 would manipulate at $(P_1^*, \bar{P}_2)$ via an ordering P_1' where $r_1(P_1') = a_j$ because by induction hypothesis $f(P_1', \bar{P}_2) = a_j$ - a contradiction. ■

CLAIM **4.4** *For all $(P_1, P_2) \in \mathbb{D}^2$ such that $P_1 \in \mathbb{D}^{a_l}$ and $P_2 \in \mathbb{D}^{\bar{X}}$, $f(P) = r_1(P_1)$.*

Proof: In the view of Claim 4.3, we need to consider only the case where $P_2 \in \mathbb{D}^{a_r}$ where $a_r \in \{a_1, \ldots, a_{l-1}\}$ and $a_r \neq a_i, a_j$. Suppose there exists $(\bar{P}_1, \bar{P}_2)$ such that $r_1(\bar{P}_1) = a_l$, $r_1(\bar{P}_2) = a_r$ and $f(\bar{P}_1, \bar{P}_2) \neq a_l$. Since $a_l \overset{w}{\sim} a_i$, there exists $B \subset A$, P_1' and P_2'' such that (i) $r_1(P_1') = a_l$ and $r_1(P_2'') = a_i$, (ii) $B = M(a_l, a_i, P_1')$ and $B = M(a_i, a_l, P_2'')$. By strategy-proofness and IH, $f(P_1', \bar{P}_2) = B \cup a_i$. Claim 4.3 implies that $f(P_1', P_2'') = a_l$. Therefore, agent 2 would manipulate at (P_1', P_2'') via $\bar{P}_2$, contradicting the assumption of strategy-proofness.

■

Claim **4.5** *For all $(P_1, P_2) \in \mathbb{D}^2$ such that $P_1 \in \mathbb{D}^{\{a_i, a_j\}}$ and $P_2 \in \mathbb{D}^{a_l}$, $f(P) = r_1(P_1)$ (here too, a_i and a_j are as specified in (*)).*

Proof: Suppose not. There exists an $(\bar{P}_1, \bar{P}_2) \in \mathbb{D}^2$ such that $\bar{P}_1 \in \mathbb{D}^{\{a_i, a_j\}}$, $\bar{P}_2 \in \mathbb{D}^{a_l}$ and $f(\bar{P}_1, \bar{P}_2) \neq r_1(\bar{P}_1)$. Therefore by Lemma 4.1, $f(\bar{P}_1, \bar{P}_2) = a_l$. Let $r_1(\bar{P}_1) = a_i$ - a similar argument holds if $r_1(\bar{P}_1) = a_j$. Since $a_l \overset{w}{\sim} a_j$, there exists P_2' such that $r_1(P_2') = a_j$ and $a_l P_2' a_i$. Since, $f(\bar{P}_1, P_2') = a_i$ by IH, agent 2 would manipulate at $(\bar{P}_1, P_2')$ via $\bar{P}_2$ - a contradiction.

■

Claim **4.6** *Let $a_r \overset{w}{\sim} a_s$ and $a_r, a_s \in \{a_1, a_2, \ldots, a_{l-1}\}$. If $f(P) = a_r$ for all $(P_1, P_2) \in \mathbb{D}^2$ such that $P_1 \in \mathbb{D}^{a_r}$ and $P_2 \in \mathbb{D}^{a_l}$, then $f(P) = a_s$ for all $(P_1, P_2) \in \mathbb{D}^2$ such that $P_1 \in \mathbb{D}^{a_s}$ and $P_2 \in \mathbb{D}^{a_l}$.*

Proof: Suppose not. There exists an $(\bar{P}_1, \bar{P}_2) \in \mathbb{D}^2$ such that $\bar{P}_1 \in \mathbb{D}^{a_s}$, $\bar{P}_2 \in \mathbb{D}^{a_l}$ and $f(\bar{P}_1, \bar{P}_2) \neq a_s$. Since $a_r \overset{w}{\sim} a_s$, there exists $B \subset A$, P_1' and P_2' such that (i) $r_1(P_1') = a_s$ and $r_1(P_2') = a_r$, (ii) $B = M(a_s, a_r, P_1')$ and $B = M(a_r, a_s, P_2')$. By strategy-proofness and our assumption, $f(P_1', \bar{P}_2) \in B \cup a_r$. Since $f(P_1', P_2') = a_s$ by IH, 2 would manipulate at (P_1', P_2') via $\bar{P}_2$ - a contradiction. ■

Claim **4.7** *For all $a_r \in \{a_1, a_2, \ldots, a_{l-1}\}$, $P_1 \in \mathbb{D}^{a_r}$ and $P_2 \in \mathbb{D}^{a_l}$, $f(P_1, P_2) = a_r$.*

Proof: Pick $a_r \in \{a_1, a_2, \ldots, a_{l-1}\}$. Since $\mathbb{D}$ is a β domain, there must exist a sequence $b_0, b_1, \ldots, b_t \in \{a_1, a_2, \ldots, a_{l-1}\}$ such that $b_0 = a_j$, $b_t = a_r$ and $b_0 \overset{w}{\sim} b_1, b_1 \overset{w}{\sim} b_2, \ldots, b_{t-1} \overset{w}{\sim} b_t$. By Claim 4.5, $f(P) = a_j$ for all $P \in \mathbb{D}^2$ where $P_1 \in \mathbb{D}^{a_j}$ and $P_2 \in \mathbb{D}^{a_l}$. Applying Claim 4.6 repeatedly, it follows that $f(P) = a_r$ for all $P \in \mathbb{D}^2$ where $P_1 \in \mathbb{D}^{a_r}$ and $P_2 \in \mathbb{D}^{a_l}$. ■

Claims 4.3-4.7 establish Step 2. This completes the proof of the Theorem. ■

Observation **4.2** Aswal et al. (2003) proved that linked domains are dictatorial. Since linked domains are β domain, Theorem 4.2 clearly generalizes that of Aswal et al. (2003). We note that β domain can be much smaller than linked domains. For instance, the domain in Example 4.1 has eight orderings while the minimal linked domain with four alternatives has ten orderings. In fact, the size of a minimal dictatorial domain is $2m$, the bound that is obtained by β domains in the case where $m = 4$.

4.4 γ Domains

In this section, we consider a strengthening of the notion of weak connectedness. This generates new conditions for dictatorial domains where the induced graph on alternatives has fewer edges.

We introduce the stronger notion of weak connectedness formally below.

Definition **4.10** *A pair of alternatives a_j, a_k satisfy Property T, denoted by $a_j \approx a_k$ if $a_j \overset{w}{\sim} a_k$ and for all $a_r \neq a_j, a_k$ there exists $P_i, P'_i \in \mathbb{D}$ such that*

1. *$r_1(P_i) = a_j$ and $a_r P_i a_k$.*
2. *$r_1(P'_i) = a_k$ and $a_r P'_i a_j$.*

In addition to weak connectedness, Property T requires the following "intermediateness" property : for any alternative a_r other than a_j and a_k, there exist two orderings in the domain, one where a_r is above a_k while a_j at the top and another where a_r is above a_j while a_k at the top.

Fix a domain $\mathbb{D}$. The graph induced by Property T $\bar{G}(\mathbb{D})$ is constructed in the same way as $G(\mathbb{D})$ with weak connectedness replaced by Property T. In other words, the set of vertices in $\bar{G}(\mathbb{D})$ is A and there is an edge $\{a_j, a_k\}$ in $\bar{G}(\mathbb{D})$ if and only if $a_j \approx a_k$.

The objective of this chapter is to show that $\bar{G}(\mathbb{D})$ requires "fewer" edges than $G(\mathbb{D})$ in order to be dictatorial. In particular, we will only require $\bar{G}(\mathbb{D})$ to be *connected.* [2]

Definition **4.11** *A domain $\mathbb{D}$ is a γ domain if $\bar{G}(\mathbb{D})$ is connected.*

A γ domain may not be a β domain as the example below shows.

Example **4.2** Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ and let $\hat{\mathbb{D}}$ be the domain in Table 4.3.

The domain $\hat{\mathbb{D}}$ is a γ domain. The induced graph $\bar{G}(\hat{\mathbb{D}})$ (shown in Figure 4.2) is connected.

[2] This is the standard notion of a connected graph, i.e. a graph where there is a path between any two vertices. A complete definition can be found in West (2001).

P_i^1	P_i^2	P_i^3	P_i^4	P_i^5	P_i^6	P_i^7	P_i^8	P_i^9	P_i^{10}
a_1	a_1	a_2	a_2	a_3	a_3	a_4	a_4	a_5	a_5
a_2	a_4	a_1	a_4	a_1	a_5	a_5	a_2	a_4	a_1
a_3	a_3	a_3	a_5	a_2	a_4	a_3	a_1	a_3	a_2
a_4	a_5	a_4	a_3	a_4	a_2	a_2	a_3	a_1	a_3
a_5	a_2	a_5	a_1	a_5	a_1	a_1	a_5	a_2	a_4

Table 4.3: The domain $\hat{\mathbb{D}}$

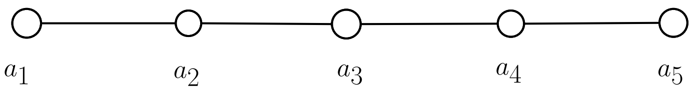

Figure 4.2: The graph $\bar{G}(\hat{\mathbb{D}})$

If a domain is a β domain, then for every $a_j \in A$, there exists a_k and a_r such that $a_j \overset{w}{\sim} a_k$ and $a_j \overset{w}{\sim} a_r$. However a_5 is not weakly connected to a_1 or a_2 or a_3. Therefore $\hat{\mathbb{D}}$ is not a β domain.

Example **4.3** Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ and let $\mathbb{D}^*$ be the domain in Table 4.4. The graph induced by $\mathbb{D}^*$ is a star graph[3] (shown in Figure 4.3). Since the star graph is connected, $\mathbb{D}^*$ is a γ domain.

P_i^1	P_i^2	P_i^3	P_i^4	P_i^5	P_i^6	P_i^7	P_i^8	P_i^9	P_i^{10}	P_i^{11}	P_i^{12}
a_1	a_1	a_2	a_2	a_3	a_3	a_3	a_4	a_4	a_4	a_5	a_5
a_4	a_3	a_4	a_5	a_1	a_2	a_2	a_1	a_3	a_3	a_4	a_3
a_2	a_5	a_1	a_3	a_2	a_4	a_5	a_5	a_2	a_5	a_2	a_1
a_5	a_2	a_3	a_1	a_5	a_1	a_1	a_2	a_1	a_1	a_1	a_4
a_3	a_4	a_5	a_4	a_4	a_5	a_4	a_3	a_5	a_2	a_3	a_2

Table 4.4: The domain $\mathbb{D}^*$

The the following we show that the *circular domain* introduced in Sato (2010), is a γ domain.

Definition **4.12** *A domain is called a circular domain ($\mathbb{D}^c$) if the elements of A can be indexed $a_1, a_2, \ldots, a_m$ so that for each $k \in \{1, 2, \ldots, m\}$, there exist two preferences P_i and P_i' in $\mathbb{D}^c$ such that*

[3]A graph $G = (N, E)$ is a *star graph* if there exists a vertex $a \in N$ (the center of the star) such that (i) for all $b \in N \setminus \{a\}$, $\{a, b\}$ is an edge in G and (ii) for all $b, c \in N \setminus \{a\}$, $\{b, c\}$ is not an edge in G.

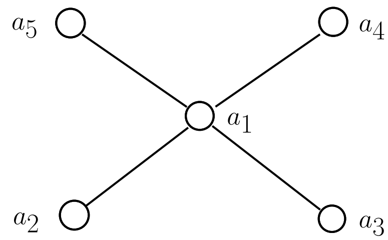

Figure 4.3: The star graph $\bar{G}(\mathbb{D}^*)$

(i) $r_1(P_i) = a_k$, $r_2(P_i) = a_{k+1}$ *and* $r_m(P_i) = a_{k-1}$.

(ii) $r_1(P_i') = a_k$, $r_2(P_i') = a_{k-1}$ *and* $r_m(P_i') = a_{k+1}$.

(Let $a_{m+1} = a_1$ *and* $a_0 = a_m$*.)*

Proposition **4.1** $\mathbb{D}^c$ *is a* γ *domain.*

Proof: First we show that for each $k \in \{1, 2, \ldots, m\}$, $\{a_k, a_{k+1}\}$ is an edge in $\bar{G}(\mathbb{D}^c)$. Since there exists $P', P'' \in \mathbb{D}^c$ where $r_1(P_i') = a_k$, $r_2(P_i') = a_{k+1}$, $r_1(P_i'') = a_{k+1}$ and $r_2(P_i') = a_k$, a_k and a_{k+1} are weakly connected. Pick an alternative $b \neq a_k, a_{k+1}$. Since there exists $P_i^1, P_i^2 \in \mathbb{D}^c$ where $r_1(P_i^1) = a_k$, $r_m(P_i^1) = a_{k+1}$, $r_1(P_i^2) = a_{k+1}$ and $r_m(P_i^2) = a_k$, b is ranked above a_{k+1} in P_i^1 and also above a_k in P_i^2. Therefore, $a_k \approx a_{k+1}$ and $\{a_k, a_{k+1}\}$ is an edge in $\bar{G}(\mathbb{D}^c)$. Since for each $k \in \{1, 2, \ldots, m\}$, $\{a_k, a_{k+1}\}$ is an edge in $\bar{G}(\mathbb{D}^c)$, it is connected. ■

Observation **4.3** $\bar{G}(\mathbb{D}^c)$ is not a star graph.

Observation **4.4** A circular domain may or may not be a β domain. Chatterji et al. (2013) introduced a more restricted class of circular domains (which they also called circular domains). These domains are β domain.

Observation **4.5** A β domain may not be a γ domain. For instance the domain in Example 4.1, is not a γ domain. The induced graph (shown in Figure 4.4) is not connected.

Figure 4.4: The graph $\bar{G}(\bar{\mathbb{D}})$

Our main result in this section shows that any $\mathbb{D}$ for which $\bar{G}(\mathbb{D})$ is connected, is dictatorial. Unfortunately, some extra conditions are needed in the very special case when $\bar{G}(\mathbb{D})$ is a star graph. We are unable to show the dictatorial result for the star graph without additional conditions but we conjecture that the additional conditions are not required. In Parts B and C of the Theorem below, we provide two independent conditions for the star-graph case that ensure dictatoriality.

THEOREM **4.3** *Let $\mathbb{D}$ be a γ domain.*

A. If $\bar{G}(\mathbb{D})$ is not a star graph, then $\mathbb{D}$ is dictatorial.

B. Let $\bar{G}(\mathbb{D})$ be a star graph and let a be the center of the star. If there exists $b, c \in A \setminus \{a\}$ such that $b \overset{w}{\sim} c$, then $\mathbb{D}$ is dictatorial.

C. Let $\bar{G}(\mathbb{D})$ be a star graph and let a be the center of the star. If there exists $P_i^1, P_i^2, P_i^3, P_i^4 \in \mathbb{D}$ such that (i) $r_1(P_i^1) = r_1(P_i^2) = b \neq a$ and $r_1(P_i^3) = r_1(P_i^4) = c \neq a$ and (ii) $M(b, a, P_i^1) = W(a, P_i^2) = M(c, a, P_i^3) = W(a, P_i^4)$, then $\mathbb{D}$ is dictatorial.

Proof: Let $\mathbb{D}$ be a γ domain and let $f : \mathbb{D}^2 \to A$ be a strategy-proof and unanimous scf [4].

Let $\bar{G}(\mathbb{D})$ be the induced connected graph. We will say a pair of alternatives $a, b \in A$ are neighbors if $\{a, b\}$ is an edge in the graph $\bar{G}(\mathbb{D})$. Agent $i \in \{1, 2\}$ is said to be decisive over $a \in A$ if for any $P \in \mathbb{D}^2$ with $r_1(P_i) = a$, $f(P) = a$. Agent $i \in \{1, 2\}$ is dictator if i is decisive over all alternatives in A.

LEMMA **4.2** *Let a and b be neighbors. For all $i, j \in \{1, 2\}$, if i is not decisive over a, then j is decisive over b.*

Proof: We assume that agent i is not decisive over a. If agent i is not decisive over a, then we argue that $f(\bar{P}_i, \bar{P}_j) \neq a$, where $r_1(\bar{P}_i) = a$ and $r_1(\bar{P}_j) = b$. If $f(\bar{P}_i, \bar{P}_j) = a$, then applying Lemma 4.1, $f(P_i, P_j) = a$ for all $(P_i, P_j) \in \mathbb{D}^2$ such that $r_1(P_i) = a$ and $r_1(P_j) = b$. In that case, we argue that agent i is decisive over a. Suppose not. Then there exists a profile $P' \in \mathbb{D}^2$, such that $r_1(P'_i) = a$ and $f(P') = c \neq a$. Since $a \approx b$, there exist P''_j with $r_1(P''_j) = b$ and $cP''_j a$. Therefore, agent j can manipulate at (P'_i, P''_j) via P'_j.

By Lemma 4.1 and our assumption, $f(\bar{P}_i, \bar{P}_j) = b$. If $f(\bar{P}_i, \bar{P}_j) = b$, then applying Lemma 4.1, $f(P_i, P_j) = b$ for all $(P_i, P_j) \in \mathbb{D}^2$ such that $r_1(P_i) = a$ and $r_1(P_j) = b$. Finally, we argue that agent j is decisive over b. Suppose not. Then there exists a profile $P' \in \mathbb{D}^2$, such that $r_1(P'_j) = b$ and $f(P') = c \neq b$. Since $a \approx b$, there exist P''_i with $r_1(P''_i) = a$ and $cP''_j b$. Therefore, agent i can manipulate at (P''_i, P'_j) via P'_i. ■

[4] In view of Proposition 3.1 of Aswal et al. (2003) and the fact that a γ domain is minimally rich, it suffices to show that if $f : \mathbb{D}^2 \to A$ is strategy-proof and unanimous, then f is dictatorial.

Lemma **4.3** *For any distinct a and b in A, it is impossible that agent 1 is decisive over a and agent 2 is decisive over b.*

Proof: Pick a profile $P \in \mathbb{D}^2$ such that $r_1(P_1) = a$ and $r_1(P_2) = b$. Since agent 1 is decisive over a, $f(P) = a$. But $f(P) = b$, because agent 2 is decisive over c. Therefore, the single-valuedness of f is contradicted. ■

Proof **(of Part A):** Suppose $\bar{G}(\mathbb{D})$ is not a star graph. We show that f is dictatorial. First we show the following claim.

Claim **4.8** *For any $a \in A$, either agent 1 is decisive over a or agent 2 is decisive over a.*

Proof: Suppose not. There exists an alternative $a \in A$ such that either both the agents are decisive over a or none of them are decisive over a. We consider the following two cases.

Case 1: Suppose both the agents are decisive over a. Since $\bar{G}(\mathbb{D})$ is connected and not a star, there exists two edges $\{a, b\}$ and $\{b, c\}$ where $a \neq c$. By Lemma 4.3, both the agents are not decisive over b. By Lemma 4.2, both the agents are decisive over c, because b and c are neighbors. Since agent 1 is decisive over a and 2 is decisive over b, Lemma 4.3 is contradicted.

Case 2: Suppose none of the agents are decisive over a. Since $\bar{G}(\mathbb{D})$ is connected, there exists an edge $\{a, b\}$ in $\bar{G}(\mathbb{D})$. By Lemma 4.2, both the agents are decisive over b. Arguments in case 1 can now be replicated with alternative a replaced by b to show a contradiction. ■

Claim **4.9** *There exists an agent who is decisive over all alternatives in A.*

Proof: Let a be any element of A. By Claim 4.8, either agent 1 is decisive over a or agent 2 is decisive over a. W.l.o.g we assume that agent 1 is decisive over a. We complete the proof by showing that 1 is decisive over all alternatives in A. Let b be any element of $A \setminus \{a\}$. We show that agent 1 is decisive over b. Since $\bar{G}(\mathbb{D})$ is connected, there exists a path $(a = a_1, a_2, \ldots, a_{k-1}, a_k = b)$ in $\bar{G}(\mathbb{D})$ from a to b. First, we show that if agent 1 is decisive over a_i then agent 1 is decisive over a_{i+1} for all $i \in \{1, 2, \ldots, k-1\}$ and applying this fact again and again we conclude that agent 1 is decisive over b. Note that a_i and a_{i+1} are neighbors in $\bar{G}(\mathbb{D})$. By Lemma 4.3, if agent 1 is decisive over a_i, then agent 2 is not decisive over a_{i+1}. By Claim 4.8, agent 1 is decisive over a_{i+1}. Therefore we conclude that agent 1 is decisive over b. Because b was arbitrary, agent 1 is decisive over all alternatives in A. ■

Claim 4.8 and Claim 4.9 establish Part A of the Theorem 4.3. ■

Proof **(of Part B):** Suppose $\bar{G}(\mathbb{D})$ is a star graph and let a be the center of the star. Let $b, c \in A \setminus \{a\}$ be such that $b \overset{w}{\sim} c$. We show that f is dictatorial. First we show the following claim.

Claim **4.10** *It is impossible for both agents* 1 *and* 2 *to be decisive over* a.

Proof: Since $b \overset{w}{\sim} c$, there exists $B \subset A$ and $P_1, P_2 \in \mathbb{D}$ such that (i) $r_1(P_1) = b$ and $r_1(P_2) = c$, (ii) $B = M(b, c, P_1)$ and (iii) $B = M(c, b, P_2)$. By Lemma 4.1, $f(P_1, P_2)$ is either b or c. Let $f(P_1, P_2) = b$ - a similar arguments works if $f(P_1, P_2) = c$. If $a \in B$, 2 would manipulate at (P_1, P_2) via an ordering where a is ranked first because 2 is decisive over a - a contradiction.

Suppose $a \notin B$. Since a and c are neighbors, $a \overset{w}{\sim} c$. Therefore, there exists $B' \subset A$ and $P_2', P_2'' \in \mathbb{D}$ such that (i) $r_1(P_2') = r_1(P_2'') = c$, (ii) $B' = M(c, a, P_2')$ and (iii) $B' \subset W(a, P_2'')$. Lemma 4.1 and strategy-proofness imply that $f(P_1, P_2') = f(P_1, P_2'') = b$. If $b \in B'$, 2 would manipulate at (P_1, P_2'') via an ordering where a is ranked first because 2 is decisive over a. Similarly, if $b \notin B'$, 2 would manipulate at (P_1, P_2') via an ordering where a is ranked first. This completes the proof of the Claim. ■

Claim **4.11** *For any* $d \in A$*, either agent* 1 *is decisive over* d *or agent* 2 *is decisive over* d.

Proof: Suppose not. Therefore, there exists an alternative $d \in A$ such that either both the agents are decisive over d or none of them are decisive over d. We consider following two cases.

Case 1: Suppose both the agents are decisive over d. Claim 4.10 implies that d is not the center of $\bar{G}(\mathbb{D})$. Since d and a are neighbors, Lemma 4.3 implies that agent 1 and 2 are not decisive over a. By Lemma 4.2, both agents are decisive over c $(\neq a, d)$, because a and c are neighbors. Since agent 1 is decisive over d and 2 is decisive over c, Lemma 4.3 is contradicted.

Case 2: In this case we consider that agent 1 and 2 are not decisive over d. First we argue that d is the center of $\bar{G}(\mathbb{D})$. If d is not the center, Lemma 4.2 implies that agent 1 and 2 are decisive over a - Claim 4.10 is contradicted. Therefore, d is the center of $\bar{G}(\mathbb{D})$. By Lemma 4.2, both agents are decisive over two distinct non-central alternatives b and c. Since agent 1 is decisive over b and 2 is decisive over c, Lemma 4.3 is contradicted. ■

Claim **4.12** *There exists an agent who is decisive over all alternatives in* A.

Proof: Replacing Claim 4.8 by Claim 4.11 in the proof of Claim 4.9 we can establish this Claim.

■

Claim 4.10-4.12 establish Part A of the Theorem 4.3. ■

Proof **(of Part C):** Suppose $\bar{G}(\mathbb{D})$ is a star graph and let a be the center of the star. Let $P_1^1, P_1^2, P_2^3, P_2^4 \in \mathbb{D}$ be such that (i) $r_1(P_1^1) = r_1(P_1^2) = b \neq a$ and $r_1(P_2^3) = r_1(P_2^4) = c \neq a$ and (ii) $M(b, a, P_1^1) = W(a, P_1^2) = M(c, a, P_2^3) = W(a, P_2^4)$. We show that f is dictatorial. First we show the following claim.

Claim **4.13** *It is impossible for both agents* 1 *and* 2 *to be decisive over* a*.*

Proof: Suppose not, i.e. agent 1 and 2 are decisive over a. Let $M(b, a, P_1^1) = W(a, P_1^2) = M(c, a, P_2^3) = W(a, P_2^4) = B$. By our assumption, $b, c \notin B$. Now consider the preference profile (P_1^1, P_2^4). We show that $f(P_1^1, P_2^4) = b$. Note that $f(P_1^1, P_2^4) \notin B$, otherwise 2 will manipulate via an ordering where a is first-ranked. Since bP_2^4a, $f(P_1^1, P_2^4) \neq a$, otherwise 2 will manipulate via an ordering where b is first-ranked. Since 1 is decisive over a, $f(P_1^1, P_2^4) \notin W(a, P_1^1)$. Therefore $f(P_1^1, P_2^4) = b$.

Strategy-proofness implies that $f(P_1^2, P_2^4) = b$. We complete the proof of the claim by showing that $f(P_1^2, P_2^3) \notin A$, because it contradicts with the fact that f is a function. Note that $f(P_1^2, P_2^3) \neq c$, otherwise 2 will manipulate at (P_1^2, P_2^4) via P_2^3. Also $f(P_1^2, P_2^3) \notin B$, otherwise 1 will manipulate via an ordering where a is first-ranked. Since cP_1^2a, $f(P_1^2, P_2^3) = a$, otherwise 1 will manipulate via an ordering where c is first-ranked. Since 2 is decisive over a, $f(P_1^2, P_2^3) \notin W(a, P_2^3)$. This completes the proof of the Claim. ■

Claim **4.14** *For any* $d \in A$*, either agent* 1 *is decisive over* d *or agent* 2 *is decisive over* d*.*

Proof: Replacing Claim 4.10 by Claim 4.12 in the proof of Claim 4.11, we can establish this Claim. ■

Claim **4.15** *There exists an agent who is decisive over all alternatives in* A*.*

Proof: Replacing Claim 4.8 by Claim 4.14 in the proof of Claim 4.9, we can establish this Claim.

■

Claim 4.13-4.15 establish Part C of Theorem 4.3. ■

This completes the proof of the Theorem. ■

OBSERVATION **4.6** Since the graph induced by a *circular domain* is connected and not a star graph (Observation 4.3), Part A of Theorem 4.3 generalizes the results in Sato (2010). Moreover, the minimal size of a γ domain is $2m$ (Example 4.2).

OBSERVATION **4.7** For the star-graph case, there are domains for which the conditions specified in Part B and Part C do not hold - the domain in Example 4.3 is such an example.

OBSERVATION **4.8** The relationship between linked domains (Aswal et al. (2003)), circular domains (Sato (2010)), circular domains (Chatterji et al. (2013)), β domains and γ domains is summarized as follows.

1. linked domains (Aswal et al. (2003)) $\subset$ β domains.

2. circular domains (Chatterji et al. (2013)) $\subset$ circular domains (Sato (2010)) $\subset$ γ domains.

3. circular domains (Chatterji et al. (2013)) $\subset$ β domains.

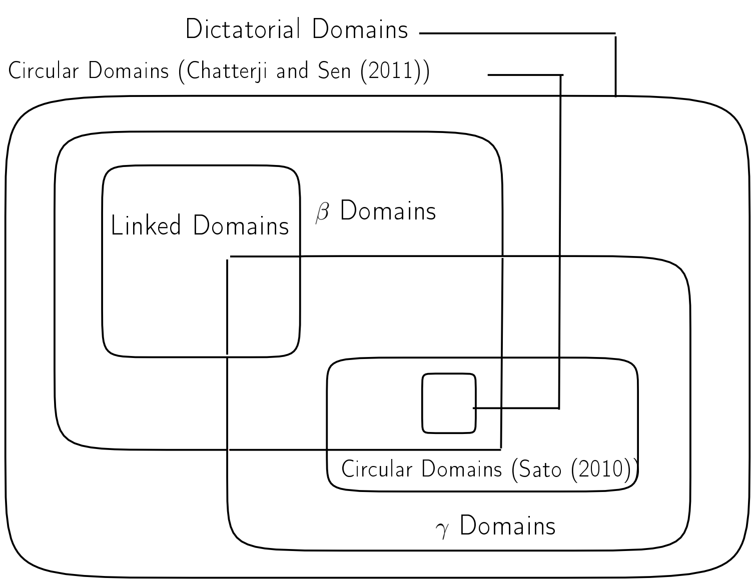

Figure 4.5: Diagrammatic representation of Observation 4.8

4.5 An Application

Consider a city with a hub. A finite number of citizens are located in the city but not at the hub. However, their locations are directly connected to the hub by a road. A public facility such as a hospital or a school has to be located in the city. The location decision is based on the preferences of citizens which are private information. What are the scfs that induce agents to report their preferences truthfully?

Let H be the hub of the city. Agents are located at a finite set of locations denoted by $a_1, \dots, a_m$, $m \geq 2$. Agents' preferences are restricted in the following manner: an agent located at a_i has one of the four orderings shown in Table 5.

P_i	$\bar{P}_i$	P_i'	P_i''
a_i	a_i	H	H
H	.	a_i	.
.	.	.	.
.	.	.	.
.	H	.	a_i

Table 4.5: Possible Preferences of an agent located at a_i

The rationale behind the preference restrictions is as follows. Some citizens want it either at their location or at the hub - these citizens prefer proximity to the facility. Thus a_i and H take first two places in their ordering and are represented either by P_i or P_i'. Some citizens want it at any residential location rather than at the hub and most prefer it when it is located near them - these preferences are represented by $\bar{P}_i$. Finally some citizens are affected by the congestion created by the facility and are strongly averse to it being located near them. They most prefer it being located at the hub for easy access. Such preferences are represented by P_i''.

A domain with the four preference orderings in Table 4.5 for each a_i, will be called a *hub domain* and denoted by $\mathbb{D}^H$.

Proposition **4.2** *A hub domain is a γ domain.*

Proof: Let $\mathbb{D}^H$ be a hub domain. First we show that for each $k \in \{1, 2, \dots, m\}$, $\{a_k, H\}$ is an edge in $\bar{G}(\mathbb{D}^H)$. Since there exists $P', P'' \in \mathbb{D}^H$ where $r_1(P_i') = a_k$, $r_2(P_i') = H$, $r_1(P_i'') = H$ and $r_2(P_i') = a_k$, a_k and a_{k+1} are weakly connected. Pick an alternative $b \neq a_k, H$. Since there exists $P_i^1, P_i^2 \in \mathbb{D}^H$ where $r_1(P_i^1) = a_k$, $r_{m+1}(P_i^1) = H$, $r_1(P_i^2) = H$ and $r_{m+1}(P_i^2) = a_k$, b is ranked above H in P_i^1 and also above a_k in P_i^2. Therefore, $a_k \approx a_{k+1}$ and $\{a_k, a_{k+1}\}$ is an edge in $\bar{G}(\mathbb{D}^c)$. Since for each $k \in \{1, 2, \dots, m\}$, $\{a_k, H\}$ is an edge in $\bar{G}(\mathbb{D}^H)$, it is connected. ■

The induced graph by a hub domain may or may not be a star-graph. In either case, it is a dictatorial domain.

THEOREM **4.4** *A hub domain is dictatorial.*

Proof: Let $\mathbb{D}^H$ be a hub domain. By Proposition 4.2, $\bar{G}(\mathbb{D}^H)$ is connected. If $\bar{G}(\mathbb{D}^H)$ is not a star-graph, we are done by Part A of Theorem 4.3.

Suppose that $\bar{G}(\mathbb{D}^H)$ is a star-graph. Note that there exists $P_i^1, P_i^2, P_i^3, P_i^4 \in \mathbb{D}^H$ such that (i) $r_1(P_i^1) = r_1(P_i^2) = a_i \neq H$ and $r_1(P_i^3) = r_1(P_i^4) = a_j \neq H$ and (ii) $M(a_i, H, P_i^1) = W(H, P_i^2) = M(a_j, H, P_i^3) = W(H, P_i^4) = \emptyset$. Therefore, by Part C of Theorem 4.3, $\mathbb{D}^H$ is dictatorial. ■

4.6 CONCLUSION

We have generalized the results of Aswal et al. (2003) in two different ways. Our results generate new examples of dictatorial domains and also unify existing results by covering some hitherto isolated cases.

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