Mechanism Design for Land Acquisition

Soumendu Sarkar

September, 2015

Thesis Supervisor : Professor Arunava Sen

Thesis submitted to the Indian Statistical Institute in partial fulfillment of the requirements for the award of the degree of Doctor of Philosophy

To Ma

Mechanism Design for Land Acquisition

Soumendu Sarkar

Contents

List of Figures

List of Tables

Acknowledgements

I am most grateful to my supervisor Prof Arunava Sen for his guidance and inspiration. In 2007, he proposed land acquisition in India as a potential application of mechanism design theory (Sen, 2007). I was delighted when he told me that I should pursue this topic as my research agenda. In this pursuit, he has been my fiercest critic and my most enthusiastic source of encouragement. If any of the components of this work stand out, it is due to his critical inputs. I am extremely fortunate to have him as my supervisor.

I am also thankful to Prof Debasis Mishra for taking interest in my work. He patiently listened to me whenever I asked for his help. His detailed comments and suggestions have been exceptionally useful.

I am indebted to Prof Avinash Dixit for his suggestion to consider the contiguity aspect of the land acquisition problem. I am also thankful to Professors Murali Agastya, Sushil Bikhchandani, Bhaskar Dutta, Parikshit Ghosh, Vijay Krishna, Vincent Merlin, Prabal Ray Chaudhuri, Rene Saran, Ram Singh and Rakesh Vohra for their audience and suggestions.

I have been blessed with great teachers throughout my life, at schools in Lumding, Guwahati and Santiniketan, at Visva Bharati, at TI, Amsterdam, and at ISI. Mr Tapan Das, who taught me Mathematics at school, has been a major inspiration.

The atmosphere at ISI has been very conducive for my research as well as personal life. It is at ISI that I earned the friendship of Abdul, Alok, Anup, Ashok, Debdatta, Dushyant, Keshav, Madhav, Mihir, Mridu and Sonal. It is very difficult to distinguish them from my family. I appreciate the support that ISI staff has offered us in the most congenial manner.

I am specially grateful to Kaustav for helping me with the mathematics.

I would like to thank colleagues at TERI University including Chuba, Manipadma-*da*, Nandan-*da*, Nidhi and Subir for their encouragement and support.

This work is a tribute to Ma, my brothers, my sisters-in-law and my beautiful niece and nephew, all of whom have been instrumental in my upbringing, education and achievements in their own ways.

Finally, a very special thanks to Vernita for being there all these years.

Chapter 1

Introduction

Conversion of land use from agriculture to industry is a typical feature of economic development in many densely populated countries. Large scale construction often requires industry or the government to acquire vast areas of land that are inhabited and often cultivated, by hundreds and even thousands of people. For some landowners, possession signifies power and status in society, while for others, it is the only means for earning a livelihood.

Adamopoulos and Restuccia (2014) use data from World Census of Agriculture to show that average farm size in the poorest 20% of countries is 1.6 hectares, while that in the richest 20% of countries is 54.1 hectares. In poor countries very small farms (less than 2 hectares) account for over 70% of total farms, whereas in rich countries they account for only 15%. In poor countries there are virtually no farms over 20 hectares, while in rich countries they account for 40% of the total number of farms. Therefore, a large number of agricultural plots have to be acquired in order to implement large projects like factories or highways. Further, these plots have to be contiguous to maintain continuity of rights of passage as well as to exploit economies of scale.

Land acquisition refers to situations where a single buyer purchases a set of land plots from multiple landowners. Often plots are required to be contiguous so that large scale construction can take place. If prices acceptable to the buyer and the landowners are publicly known, an efficient outcome is easily achieved: trade takes place with the owners of the cheapest set of contiguous plots. But in real-life situations such information is not public. The buyer then has to negotiate with individual sellers who can respond by delaying strategically - this is commonly known as *holdout*. In such exchange problems with asymmetric information, sellers can extract rent due to their private information (*information rent*) as well as due to their location.

Often, the land acquisition exercise needs assistance from a third party, like the State. The State can either negotiate with the farmers, or coerce them to trade by means of constitutional powers, or subsidize the transaction. Coercion by the State often leads to conflicts of social, political and economic significance.

Events in Singur, West Bengal (India) in 2006 highlight the dangers of ill-conceived land acquisition policies (Wikipedia, 2014b). West Bengal is a densely populated state where growth relative to other states in India has been low. In an effort to encourage industrial development, the ruling State Government led by Communist Party of India (Marxist), (CPIM), sought to acquire land in order for Tata Motors (a private company) to set up a manufacturing unit. The project by Tata Motors involved the production of the \$2500 Nano automobile. About 1000 acres of agricultural land, some of it fertile and multi-cropped, was acquired by the State Government using the provisions of the *Land Acquisition Act*, 1894. The acquired area was owned by more than 13000 farmers. Eventually more than 2000 of them refused to accept the compensation package. This led to protests by locals, activists, intellectuals and politicians. The major opposition party, the Trinamool Congress led by Mamata Banerjee staged a hunger strike demanding the release of 400 acres of disputed land that was scattered across the site. Violence erupted between supporters of the ruling and opposition parties. By October 2008, the Tatas decided to pull out of Singur. Mass violence was also witnessed in Nandigram, West Bengal, when a supposedly secret government decision to acquire land for building a Special Economic Zone became public (Wikipedia, 2014a). These events played a significant role in the defeat of the CPIM in the State Elections after more than thirty years in power.

Similar events have occurred in Orissa and Uttar Pradesh and have sparked widespread civil society protests. Several Bollywood films such as *Sarkar Raj* (2008), *Shanghai* (2012) and *Matru ki Bijli ka Mandola* (2013) have plots based on the land acquisition theme. However, several land acquisition projects have proceeded smoothly such as the DLF project in Haryana. By and large, land acquisition for railways, roads and metro rail have also been free of disputes. Land acquisition for public purposes have also created legal tangles in other countries such as the United States[1].

According to the First Fundamental Theorem of Welfare Economics, allocations achieved via competitive markets are efficient. Unfortunately, efficiency can no longer be obtained when agents have private information. Agents can then strategically withhold or misrepresent information in the expectation of private gain. An appropriate theoretical response is to design institutions or trading rules that seek to achieve desirable goals like efficiency or revenue maximization while explicitly recognizing *incentive compatibility constraints*. This is the domain of mechanism design theory. There is a relatively small literature that examines land acquisition from a mechanism design perspective (Ghatak and Ghosh, 2011; Sen, 2007; Kominers and Weyl, 2011, 2012). In this dissertation we offer such a systematic treatment.

We consider two models of land acquisition. In the LA model, no contiguity requirement is imposed on acquired land; in the LAC model, sellers are modeled as nodes on a contiguity graph and the buyer wishes to acquire a path with a predetermined number of sellers, say k. The buyer obtains a value v_0 if she gets at least k plots. There are n sellers. Each seller i holds a unit of land with valuation v_i. Both the buyer and the sellers' valuations are private information. This model is clearly a generalization of the well-known bilateral trade problem

[1] See *Kelo vs. City of New London*, 2005

studied by Chatterjee and Samuelson (1983) and Myerson and Satterthwaite (1983). We address the following questions:

(a) Is it is possible to design *successful* mechanisms for land acquisition that satisfy ex-post efficiency, Bayesian incentive compatibility, interim individual rationality and ex-post budget balance simultaneously[2]?

(b) How do the results in (a) change when contiguity requirement is imposed?

(c) How does the surplus in dominant strategy incentive compatible mechanisms behave as number of sellers becomes large?

(d) What is the nature of the *optimal* mechanism, i.e., the one that maximizes the sum of expected payoffs of agents?

We begin with a brief review of the legal provisions in India relating to land acquisition. We shall then review the general literature on mechanism design for exchange problems under asymmetric information and discuss its implications for land acquisition. Finally, we shall briefly summarize each of the chapters in this dissertation.

1.1 Eminent Domain in India

Eminent Domain[3] is a legal provision that empowers the State to acquire a citizen's private property for *public purposes* with *due compensation*. Traditionally, *public purpose* refers to provision of public goods. But eminent domain has been used to facilitate private parties in order to overcome the holdout problem. The law takes market value as basis for fair compensation but economists argue that it under-compensates the sellers relative to what they could have got. Market value based compensation may lead to over-acquisition, and may act as a subsidy to private developers who would otherwise underinvest (Miceli, 2011).

A comprehensive review of land markets in India can be found in Chakravorty (2013). Eminent Domain was legislated through the Land Acquisition Act, 1894 . Terms like *public purpose* or *market value* were vaguely defined in this legislation. Further, apart from compensation, no decision to acquire could be contested in a court of law. In 2007, after violent protests in various parts of the country over land acquisition, the Indian Government tabled a bill regarding the amendment of this Act in the parliament. The Right to Fair Compensation and Transparency in Land Acquisition, Rehabilitation and Resettlement Act or LARR, replaced the 1894 Act in 2013. Some of its highlights are listed below.

- The State can apply the law to acquire land for itself, private parties or public-private partnerships for public purposes;

[2]See Holmstrom and Myerson (1983) for the definitions of ex-ante, interim and ex-post criteria.
[3]The term is due to 17th century Dutch jurist Grotius (See Grotius (1925)).

- Public purposes are specified as strategic industries, infrastructure or housing;
- Prior consent of 80 per cent of affected families is required before acquiring land through this Act for private parties;
- Acquisition of multi-cropped land is by and large forbidden;
- The *market value* has been defined as the maximum of the legally declared floor on land prices in the area, the average registered sale price in the area and the consented amount in the case a private party is involved. The minimum compensation would be a multiple of the market value and the value of assets attached to the property;
- In addition, the affected parties benefit from various rehabilitation and resettlement allowances.

LARR became effective from January 1, 2014. The new act has been opposed by both business groups and farmers. It has also received criticism from economists for failing to elicit valuations of landowners through mechanisms such as auctions and relying instead on arbitrary compensation formulae (Ghatak and Ghosh, 2011; Ghatak et al., 2013). Subsequently, the new government at the center passed several ordinances to modify its provisions. Land acquisition thus remains a highly contested policy matter.

1.2 Related Literature

Mechanism Design provides a coherent framework for analyzing allocation problems, particularly from the aspects of incentives and private information (Royal Swedish Academy of Sciences, 2007). A mechanism consists of an allocation rule and a payment rule that summarizes the rules of trading. The simple bilateral trade problem under incomplete information was analyzed in Chatterjee and Samuelson (1983). In the double auction, trade takes place if buyer's reported valuation exceeds that of the seller's, at a price equal to an average of these two reports. When all valuations are distributed uniformly over $[0, 1]$, the double auction mechanism is *optimal*: it maximizes ex-ante welfare subject to Bayesian incentive compatibility and interim individual rationality (Myerson and Satterthwaite (1983), Section 4, p. 277-8). However, it is not efficient. The Myerson-Satterthwaite Theorem (Theorem 2, Corollary 1, p. 273) provides a general impossibility result: there does not exist a bilateral trade mechanism that satisfies ex-post efficiency, Bayesian incentive compatibility, interim individual rationality and ex-post budget balance.

The bilateral trade problem is one of several allocation models with asymmetric information. There are few allocation problems where possibility results exist. For instance, Cramton et al. (1987) consider a model of partnership in which each partner owns a share of a good and has a private valuation for the entire good. Partners report their valuations to a planner and receive a share of the good and a payment. Also, Makowski and Mezzetti

(1993) consider a model where the seller of an indivisible good wants to trade with two or more potential buyers. Each agent has a valuation for the good that is private information. McAfee (1991) has found a possibility result in a model of bilateral trade where continuous quantities are traded. These papers identify a non-empty set of priors for which successful mechanisms exist (in our sense). This naturally leads to the question whether the set of such priors can be completely characterized.

Williams (1999) (Theorem 3, p. 166) offers a characterization for such priors that is both intuitive and easy to check. He finds that every ex-post efficient and Bayesian incentive compatible mechanism is equivalent in terms of expected payoffs to an ex-post efficient and dominant strategy incentive compatible mechanism due to Groves (1973). He concludes that a *successful* mechanism can be constructed if and only if there is a Groves mechanism for the problem that results in an expected budget surplus. He provides an application of this result in a multilateral bargaining model where each of m buyers demand a unit of an item and each of n sellers sell a unit of the same item. Krishna and Perry (2000) (Theorem 2, p. 14) provide another characterization very similar to that of Williams (1999) [4]. They show that the VCG mechanism (also referred to as the pivotal mechanism in the literature), which is a member of the Groves class of mechanisms, has the highest expected budget surplus among all mechanisms that are ex-post efficient, Bayesian incentive compatible and interim individually rational. Consequently, a *successful* mechanism can be constructed if and only if the VCG mechanism for the problem results in a positive expected budget surplus [5].

Williams (1999) (Theorem 4, p. 169 and the discussion in Section 3) has applied his condition to various models where the first best can be obtained. In Chapter 3, we show that the first best can also be obtained in a different model. The Chapter also highlights the relationship between attainability of the first best and the contiguity structure of the sellers.

Following the Myerson-Satterthwaite impossibility result, the literature on bilateral trade has focussed on mechanisms like k-double auctions that depart from one or more of the Myerson-Satterthwaite requirements. Mechanisms that make a minimal departure from one of these requirements while satisfying the others are referred to as *second best*. Some of these contributions examine whether such mechanisms approximate the first best as the number of agents becomes large. See Jackson (2000) (Section 4.5) for a concise review of some of these issues and related references.

The VCG mechanism mentioned above stands out as the second best with respect to budget balance. Asymptotic properties of VCG mechanisms have been studied in the early literature in different contexts. Tideman and Tullock (1976) (p. 1149, 1155-6) conjectured that per capita VCG budget surplus converges to zero as number of individuals become large.

[4]However, there are important differences: Williams (1999) require differentiability of interim expected valuation functions of agents, but Krishna and Perry (2000) do not; the latter require the domain of valuations to be convex, but the former does not. Also, the latter is more tuned to applications in multiunit auctions, while the former focuses on multilateral bargaining problems.

[5]See Makowski and Mezzetti (1994) and Schweizer (2006) for other characterizations of possibility.

Green et al. (1976) (Section 3, p. 380-82) showed that in an economy where individuals have to make a collective decision on a public project, per capita expected VCG payments converge to zero as number of individuals become large. Rob (1982) proved similar results for an economy where individuals have to collectively choose between two public projects. Bailey (1997) and Cavallo (2006) have investigated how the VCG surplus can be redistributed optimally without violating incentive properties. Chapter 4 of this dissertation shows that the endpoints of the priors of the agents and the contiguity structure determine whether the VCG mechanism results in a redistributable surplus.

The literature on mechanism design for land acquisition is relatively sparse. Kominers and Weyl (2011) study a class of mechanisms which they call *concordance*. Sellers commit to share the aggregate proceeds of the sale in specific proportions. The buyer reports his valuation and the sellers report their shares to a planner. Trade takes place if the buyer's bid exceeds the sum of the sellers' reports. If a seller does not report the pre-determined share, he has to pay a penalty according to the externality imposed on other agents. These mechanisms satisfy truthfulness in dominant strategies or in Bayesian sense. These converge to efficiency as the number of sellers increase; these are collectively rational (i.e., the sum of payoffs over all agents is non-negative) and approximately individually rational.

Grossman et al. (2010) consider a model with two potential buyers and at least as many sellers as the number of properties demanded. In their *Strong Pareto* mechanism (SP), trade takes place with the highest bidder only if the payment of the second highest bidder is enough to pay the reservation prices of all individual owners. Proceeds of the auction are distributed back to the former owners according to fixed and exhaustive shares. Their mechanism excludes some efficient trading possibilities, but maximizes the share of the potential gains from trade among all dominant strategy incentive compatible, ex-post individually rational and budget balancing mechanisms.

Plassmann and Tideman (2010) use the concept of Clarke tax[6] to construct a mechanism for land acquisition. Let V_i be valuation of seller i, v_i be the ratio of V_i to the total of such valuations and let α_i be i's estimate of v_i. Each seller i reports $W_i = \alpha_i X - S_i$ to a planner, where X is the buyer's maximum willingness to pay. Trade takes place if $\sum_i W_i > 0$ and the buyer pays $\alpha_i X$ to each i. Each *pivotal* seller i whose announcement of W_i causes $\text{sign}(\sum_j W_j)$ to differ from $\text{sign}(\sum_{j\neq i} W_j)$ pays $|\sum_{j\neq i} W_j|$. These sellers receive at least their reservation values while non-pivotal sellers do not pay. The surplus generated is to be redistributed without distorting incentives. For instance, if $S = \sum_i S_i$ and $T_i(S)$ is the valuation tax, owner earns $V_i - T_i(S)$ is no sale occurs and $S_i - T_i(S)$ if sale occurs. Then $T_i(S)$ could be set such that each i maximizes his expected earning if and only if $S_i = V_i$. This mechanism is efficient if X coincides with the true value of the project.

Ghatak and Ghosh (2011) (p. 67-69) propose a reverse Vickrey auction with a reserve price for land sales. Trade takes place with the sellers corresponding to k lowest valuations at the price of the $k+1$-th lowest valuation, provided the latter does not exceed the reserve

[6]See Clarke (1971).

price. However, Singh (2012) (Section 3.2, p. 10-15) shows that this mechanism does not immediately extend to the situation where purchased plots are required to be contiguous .

Kominers and Weyl (2012) argue that simple mechanisms like the *posted price* mechanism may result in a low volume of trade when trade takes place with a large number of "adjacent sellers". This result is a counterpart to the result in Mailath and Postlewaite (1990) (Theorem 2, p. 357-8) which shows that the probability of efficient public decisions in large societies is low.

Apart from the literature on mechanism design, holdouts have been investigated in the framework of bargaining under complete information (Cai, 2000, 2003; Menezes and Pitchford, 2004; Roy Chowdhury and Sengupta, 2012; Xiao, 2010). These contributions primarily look at the possibility of holdout in different configurations of the Rubinstein model of bargaining with one buyer and multiple sellers.

1.3 Plan of Chapters

Apart from this introductory chapter, this dissertation comprises four chapters addressing each of the questions described earlier. We briefly summarize the content of each of these chapters.

1.3.1 Chapter 2: Possibility

The classic result by Myerson and Satterthwaite (1983) (Section 4, p. 277-8) involves a model of bilateral exchange where the buyer and seller have private information about their valuations. They show that there does not exist a mechanism that is *Bayesian Incentive Compatible* (BIC) and achieves a *first best allocation*, i.e., allows trade if and only if the seller's valuation is less than that of the buyer's (efficiency), is *ex-post budget balanced* (BB) and *interim individually rational* (IIR). Such mechanisms are referred to as *successful* in this dissertation. In this chapter, we show that the Myerson-Satterthwaite negative result does not extend to a specific multilateral allocation problem called the *Land Acquisition Problem*: there exist successful mechanisms for a wide range of specifications in this model.

There are n sellers, each holding one unit of an indivisible good. The valuation of each seller i is $v_i \in \mathbb{R}_+$. We assume that v_i's are independently and identically distributed on $[\underline{v}, \bar{v}]$ with distribution function $F(\cdot)$ and density function $f(\cdot)$. The realization of v_i is observed only by i. There is one buyer, indexed by 0, whose valuation v_0 depends on m, the number of units she gets, in the following manner:

$$v_0(m) = \begin{cases} v_0 \text{ if } m \geq k, \\ 0 \text{ otherwise.} \end{cases}$$

We assume that $v_0 \in [\underline{v_0}, \bar{v_0}]$ and $v_0 \sim G(v_0)$. We assume that F and G have continuous and positive densities $f(\cdot)$ and $g(\cdot)$ in their respective domains. The valuations of the buyer and the sellers are independently distributed . All valuations are non-negative. Own valuations are private information while the distribution functions F and G are common knowledge. In order to make the problem non-trivial , we make the following assumption.

$$\boxed{\text{ASSUMPTION } \mathbf{NT} : k\underline{v} < \bar{v}_0 \text{ and } k\bar{v} > \underline{v}_0}$$

This assumption ensures that ex-post efficiency is a non-trivial issue. If the first part of this assumption does not hold, then trade is never ex-post efficient; if the second part is violated, trade is always ex-post efficient.

A valuation profile is an $n+1$-vector $v \equiv (v_0, v_1, \ldots, v_n) \in [\underline{v_0}, \bar{v_0}] \times [\underline{v}, \bar{v}]^n$. The distribution of the random vector v is called a *prior*, denoted μ. We will refer to the model $\langle n, k, \mu \rangle$ as the *land acquisition* problem or LA. The buyer and the sellers directly report their individual valuations to a mechanism designer.

Our main result is that there exists a robust set of priors for which successful mechanisms exist in the case $n > k$. In the case where $n = k$, the standard Myerson-Satterthwaite impossibility holds. We provide a sufficient condition on the priors for the existence of a mechanism that attains the first best and also provide a weaker necessary condition. Our sufficient condition is that the lowest possible valuation of the buyer is greater than or equal to the expectation of the $k+1$-th lowest order statistic of seller valuations. Note that this condition is simple to interpret and compute since it depends only on the lower end of the buyer's support and not on the entire distribution. We provide examples to show that this condition is easy to satisfy when k is small relative to n.

1.3.2 Chapter 3: Contiguity

In the previous chapter, we showed that the Myerson-Satterthwaite negative result does not extend to the Land Acquisition problem. In this chapter, we impose the additional requirement that the buyer wants the acquired plots to be *contiguous*. A pair of plots is contiguous if they share a physical boundary. One might reasonably expect that imposing this restriction will precipitate a Myerson-Satterthwaite type impossibility result. However, we show that this is not true and results similar to the ones obtained in the previous chapter continue to hold.

There are n sellers, indexed by i, each holding one unit of an indivisible good (plot). The n indivisible items are located on a graph $\Gamma = (N, E)$ where N denotes the set of nodes (plots) and E denotes the set of edges. A pair of nodes is connected by a direct edge if they are physically adjacent to each other. A sequence of connected nodes is called a path. A path is *feasible* if it contains at least a fixed number k of nodes where $k \leq n$. A seller is *critical* if the corresponding node is in every feasible path. The assumptions of the earlier chapter on valuations and distributions are maintained. A *land acquisition problem with contiguity* or LAC is a tuple $\langle \Gamma, k, \mu \rangle$.

Our main result is that there exists a robust set of priors for which BIC mechanisms attain the first best when there are at least two distinct feasible paths. Two paths are distinct if the nodes constituting them are not identical. Similar to the analysis in Chapter 2, we provide a sufficient condition for the existence of a successful mechanism in the case with more than one feasible paths. In this case we also provide a weaker necessary condition. Furthermore, we show that the number of critical nodes has a bearing on the set of priors for which successful mechanisms exist. In particular, it becomes harder to satisfy the conditions for possibility as the number of critical nodes increases. When there is only one feasible path, it is clear that the problem reduces to the $n = k$ case in Chapter 2. Hence there does not exist a successful mechanism for any prior.

1.3.3 Chapter 4: Asymptotics

According to standard microeconomic theory, the market power of individual sellers declines as the number of sellers increases. A classic and extreme example of this is the comparison between standard monopoly and Bertrand duopoly: in the former, the market price is above the marginal cost, but in the latter, competition between two identical sellers drives market price down to marginal cost of production. In models of private information, mechanisms may fail to satisfy one or more of the Myerson-Satterthwaite criteria. A natural question is whether a second best mechanism, that minimally departs from one of these criteria, becomes successful as the number of agents become large.

In the earlier chapters we found necessary and sufficient conditions on priors for which successful mechanisms can be constructed. These mechanisms are ex post efficient, BIC, IIR and BB. However, the use of such mechanisms requires the social planner to have precise information about the underlying priors. There has been emphasis on the construction of mechanisms that are robust with respect to such information following a critique by Wilson (1987)[7]. A natural way to deal with this problem is to require mechanisms to be *dominant strategy incentive compatible*, or DSIC. The VCG mechanism is DSIC and ex-post individually rational. However, it is not BB: the sum of VCG payments can be positive, negative or zero, depending on the profile and the prior. If the sum of payments is positive, a redistribution of the surplus will improve net welfare of agents. If the sum of payments is negative, the mechanism requires an outside subsidy. The VCG mechanism therefore, becomes approximately first best in the limit if the sum of VCG payments at every profile converges to zero. In this chapter we investigate this issue.

Priors satisfy the *Trade in the Limit* or the TL condition if $\underline{v}_0 > k\underline{v}$, i.e., the lowest end of the support of the buyer's valuation is greater than k times that of the sellers' valuation. If this condition is satisfied, then trade will almost surely take place in the VCG mechanism as the number of sellers becomes large. This chapter shows that TL is a necessary and sufficient condition for almost sure positive VCG surplus in the limit in the LA model.

[7] See Bergemann and Välimäki (2006) for a survey.

There are conceptual difficulties in describing a general model for the LAC problem. This is because the underlying graph may change depending on the way new sellers are added. In this Chapter we examine some special cases where these issues can be dealt with. The first of these is a model where new sellers are added consecutively on a line. The second is a star graph where new sellers form additional edges with a fixed hub seller. We show that the TL condition can be extended to the line graph model for almost sure positive VCG surplus in the limit. We also show that a stronger condition is required in the star graph model and we identify this condition as $\underline{v}_0 > \bar{v} + \underline{v}$. It implies that buyer's lowest valuation has to be higher than any seller's highest valuation. We then generalize these conditions to sequences of graphs with special properties.

We have provided several numerical examples to illustrate these results. We generate values for the VCG sum of payments for these problems when valuations are drawn from specific uniform distributions both where TL is satisfied and where it is not.

Our results show that the asymptotic budget-balance property of the VCG mechanism does not extend in a straightforward manner to the problem of land acquisition. In our model, the relative minimum valuations of the buyer and sellers and the presence of critical sellers determine whether VCG will converge to a surplus.

1.3.4 Chapter 5: Optimality

We have seen from Chapters 2 and 3 that it is possible to design successful mechanisms only when priors satisfy certain conditions. If these conditions are not satisfied, it is natural to search for *second best* mechanisms. In this chapter, we follow the approach of Myerson and Satterthwaite (1983), who characterized the optimal mechanism for bilateral trade problem. The optimal mechanism maximizes ex-ante welfare of agents in the class of mechanisms satisfying BIC, IIR and BB. Note that for a priors for which successful mechanisms exist, it is possible to achieve the maximum sum of agents welfare at every state. Therefore, for such priors, a successful mechanism is an optimal mechanism.

The optimal mechanism by Myerson and Satterthwaite (1983) can be described as follows. For any $\alpha \geq 0$, define the virtual valuation of the buyer with valuation v_0 and that of the seller with valuation v_1 as,

$$\begin{aligned} c_0(v_0, \alpha) &= v_0 + \alpha \frac{1 - G(v_0)}{g(v_0)} \\ \text{and } c_1(v_1, \alpha) &= v_1 + \alpha \frac{F(v_1)}{f(v_1)} \text{ respectively.} \end{aligned}$$

Let trade take place when $c_0(v_0, \alpha) > c_1(v_1, \alpha)$ and no trade takes place otherwise. This is an optimal allocation rule for an $\alpha \in (0, 1]$ if in conjunction with some payment rule, it satisfies BIC and IIR and BB. These authors also provide sufficient conditions for the existence of an $\alpha \in (0, 1]$ for which this mechanism satisfies BIC, IIR and BB.

We show that the optimal mechanism for LA and LAC problems are natural extensions of the optimal mechanism for the bilateral trade model by Myerson and Satterthwaite (1983). In the bilateral trade problem the optimal mechanism allows trade whenever the virtual valuation of the buyer exceeds that of the seller. In the LA problem, it allows trade whenever the virtual valuation of the buyer exceeds the sum of the lowest k virtual valuations of sellers. In the LAC problem, it allows trade whenever the virtual valuation of the buyer exceeds the lowest sum of virtual valuations on a feasible path.

We also show that the VCG mechanism is asymptotically BB if and only if the optimal mechanism is asymptotically ex post efficient. Therefore, the results of the previous chapter exactly indicate when the optimal mechanism converges to efficiency.

Chapter 2

Possibility

2.1 Introduction

According to the First Fundamental Theorem of Welfare Economics, competitive outcomes are Pareto efficient. This result is typically not valid when one or more agents have private information regarding their preferences or valuations. An illustration of this is the classic result of Myerson and Satterthwaite (1983). They consider a model of bilateral exchange where the buyer and seller have private information about their valuations. They show that there does not exist a mechanism that is *Bayesian Incentive Compatible* and achieves a *first best allocation*, i.e., allows trade if and only if the seller's valuation is less than that of the buyer's, is *budget balanced* and *interim individually rational*. In this chapter, we show that the Myerson-Satterthwaite negative result does not extend to a specific multilateral allocation problem called the *Land Acquisition Problem*.

The land acquisition problem is one where there is a single buyer who demands k units ($k \geq 1$) (plots) of a commodity (land). There are n potential sellers ($n \geq k$) each of whom have a single unit to sell. The buyer gets a valuation v_0 if she can buy at least k units and 0 otherwise. Each seller i's valuation is v_i. All valuations are private information. This model is of great practical interest (see Chapter 1).

Our main result is that there exists a robust set of priors[1] for which Bayesian incentive compatible mechanisms attain the first best in the case $n > k$. In the case where $n = k$, the standard Myerson-Satterthwaite impossibility holds. We provide a sufficient condition on the priors for the existence of a mechanism that attains the first best and also provide a weaker necessary condition. Our sufficient condition is that the lowest possible valuation of the buyer is greater than or equal to the expectation of the $k+1$-th lowest order statistic of seller valuations. Note that this condition is simple to interpret and compute since it depends only on the lower end of the buyer's support and not on the entire distribution. We

[1] If we replace the weak inequality in (2.4) with strong inequality, the set of priors satisfying this condition is open in the Whitney $\mathscr{C}^1$-topology. Two functions are close in this topology if their values and first derivatives are close everywhere.

provide examples to show that this condition is easy to satisfy when k is small relative to n.

The literature on bilateral trade with incomplete information was initiated in Chatterjee and Samuelson (1983). They considered a specific mechanism, viz., the k-double auction. In this mechanism, trade occurs if buyer's reported valuation exceeds that of the seller's at a price equal to the average of these two reports. Myerson and Satterthwaite (1983) established a more general impossibility result and showed that the double auction is an optimal mechanism for specific valuation structures. Several papers have investigated double auctions from a second best perspective. They have shown that the double auction or variants thereof approach the first best as the number of agents become large in the limit (See Satterthwaite and Williams (1989b); Williams (1991); Gresik and Satterthwaite (1989); McAfee (1992); Rustichini et al. (1994); Yoon (2001); Reny and Perry (2003); Cripps and Swinkels (2006)).

2.2 Preliminaries

There are n sellers, indexed by i, each holding one unit of an indivisible good. The valuation of each seller i is $v_i \in \mathbb{R}_+$. We assume that v_i's are independently and identically distributed on $[\underline{v}, \bar{v}]$ with distribution function $F(\cdot)$ and density function $f(\cdot)$. The realization of v_i is observed only by i.

There is one buyer, indexed by 0, whose valuation v_0 depends on m, the number of units she gets, in the following manner:

$$v_0(m) = \begin{cases} v_0 \text{ if } m \geq k, \\ 0 \text{ otherwise.} \end{cases}$$

In other words, the buyer's valuation is $v_0 \in \mathbb{R}_+$ if she gets at least a fixed number of units k where $k \leq n$; otherwise, her value is zero. Thus, the buyer is *single-minded* in the sense of Lehmann et al. (2002) — she is only interested in those subsets of units of the good that have at least k elements. We assume that $v_0 \in [\underline{v_0}, \bar{v}_0]$ and $v_0 \sim G(v_0)$. We will assume that F and G have continuous and positive densities $f(\cdot)$ and $g(\cdot)$ in their respective domains.

The valuations of the buyer and the sellers are independently distributed . All valuations are non-negative. Own valuations are private information while the distribution functions F and G are common knowledge. In order to make the problem non-trivial , we make the following assumption.

$$\boxed{\text{ASSUMPTION } \mathbf{NT} : k\underline{v} < \bar{v}_0 \text{ and } k\bar{v} > \underline{v_0}}$$

This assumption ensures that efficiency is a non-trivial issue. If the first part of this assumption does not hold, then trade is never efficient; if the second part is violated, trade is always efficient.

A valuation profile is an $n+1$-vector $v \equiv (v_0, v_1, \ldots, v_n) \in [\underline{v_0}, \bar{v}_0] \times [\underline{v}, \bar{v}]^n$. The j-th component of v is denoted by v_j and the n-vector v_{-j} denotes the profile where the

j-th component is dropped from v. Throughout, we will use the subscripts j and $-j$ to indicate "the j-th component" and "all but the j-th component" of a vector respectively. The distribution of the random vector v is called a *prior*, denoted μ. We will refer to the model $\langle n, k, \mu \rangle$ as the *land acquisition* problem or LA.

The buyer and the sellers directly report their individual valuations to a mechanism designer. A mechanism consists of an *allocation rule* and a *transfer rule*.

A *deterministic* allocation is an $n+1$-vector x described as follows: for components $i = 1, \ldots, n$, x_i is -1 if seller i sells and 0 otherwise; $x_0 = 1$ if $\sum_{i=1}^{n} |x_i| \geq k$ and 0 otherwise. Let $\mathbb{X}$ be the set of all deterministic allocations. We provide some illustrations below.

EXAMPLE **1** Suppose $n = 1$ and $k = 1$. Then, $\mathbb{X} = \{(0,0), (1,-1)\}$.

EXAMPLE **2** If $n = 2$ and $k = 2$, $\mathbb{X} = \{(0,-1,0), (0,0,-1), (1,-1,-1), (0,0,0)\}$.

DEFINITION **1 (Allocation Rule)** *An allocation rule $P : [\underline{v_0}, \bar{v_0}] \times [\underline{v}, \bar{v}]^n \rightarrow \mathbb{X}$ maps a profile of reported values to a deterministic allocation.*

For any agent j, $P_j(v)$ is the j-th component of $P(v)$. In Example 1, suppose that at profile v, P assigns the allocation (1,-1). Then, $P_0(v) = 1$ and $P_1(v) = -1$.

DEFINITION **2 (Transfer Rule)** *A transfer rule t is a map $t : [\underline{v_0}, \bar{v_0}] \times [\underline{v}, \bar{v}]^n \rightarrow \mathbb{R}^{n+1}$.*

If $t_j(v) > 0$ (resp. $t_j(v) < 0$) then agent j pays (resp. receives) the amount $t_j(v)$.

We make the standard assumption of quasi-linear utilities.

DEFINITION **3 (Payoffs)** *Fix a mechanism (P,t). The (ex post) utility of agent j with valuation v_j reporting $\hat{v}_j$ in mechanism (P,t) is*

$$U_j^{(P,t)}(\hat{v}_j, v_{-j} | v_j) = v_j P_j(\hat{v}_j, v_{-j}) - t_j(\hat{v}_j, v_{-j}).$$

Henceforth, we shall fix the mechanism (P,t) and drop the superscript in the notation.

An important requirement for mechanisms is that they induce agents to report their valuations truthfully. Bayesian incentive compatibility ensures that truthful reporting is optimal for each agent and for each valuation in *expectation*. This expectation is computed with respect to the prior distribution of valuations of other agents and on the assumption that other agents are reporting truthfully.

DEFINITION **4 (Bayesian Incentive Compatibility)** *A mechanism is Bayesian Incentive Compatible (BIC) if for all j,*

$$E_{-j} U_j(v_j, v_{-j} | v_j) \geq E_{-j} U_j(\hat{v}_j, v_{-j} | v_j) \text{ for all } v_j \text{ and } \hat{v}_j,$$

where $E_{-j}(\cdot)$ denotes expectation taken over v_{-j}.

A mechanism is denoted BIC(μ) if it satisfies BIC with respect to prior μ.

The participation condition corresponding to expected payoffs is stated below.

Definition 5 **(Interim Individual Rationality)** *A mechanism is interim individually rational (IIR) if for all j,*

$$E_{-j}U_j(v_j, v_{-j}|v_j) \geq 0 \quad \textit{for all } v_j.$$

When truthful reporting constitutes an equilibrium, we will simplify notation and write $U_j(v)$ and $U_j(v_j)$ for the ex-post and interim utilities respectively. Henceforth, we will use E rather than $E_j E_{-j}$ to denote expectation taken over profile v.

Definition 6 **(Ex-post Efficiency)** *An allocation rule P is ex post efficient if for all v,*

$$\sum_j v_j P_j(v) \geq \sum_j v_j P'_j(v) \textit{ for any allocation rule } P'.$$

Ex-post efficient allocation rules in our problem have a straightforward characterization. Fix a valuation profile v. Let $v_{[i]}$ be i-th order statistic of the valuations of n sellers, so that $v_{[1]} \leq \ldots \leq v_{[n]}$. We will denote the seller corresponding to $v_{[i]}$ with subscript $[i]$, e.g., $t^V_{[i]}(v)$ denotes the VCG payment of the seller corresponding to $v_{[i]}$. An ex-post efficient rule P^* is described as follows: for a profile v, trade occurs if $v_0 \geq \sum_{i=1}^k v_{[i]}$, and no-trade otherwise. Note that P^* is not fully specified. These are the cases where there are more than k lowest valuation sellers and the case where the buyer's value is exactly equal to the sum of k lowest seller values. A tie-breaking rule which may involve randomization, is required to fully specify the rule. However, the subsequent analysis will not depend in any way on the choice of the tie-breaking rule. Consequently, we shall abuse notation and refer to the ex-post efficient rule as any rule satisfying the condition above and denote it by P^*.

A standard restriction on the transfer payments is that they *balance the budget*, i.e., the mechanism is self-financed and there should be no surplus.

Definition 7 **(Ex-post Budget Balance)** *A mechanism (P, t) satisfies budget balance if, for all v,*

$$\sum_{j=0}^{n} t_j(v) = 0. \tag{2.1}$$

In our model, budget balance implies that the buyer pays exactly the sum of all sellers receipts at every valuation profile.

A mechanism achieves the *first best* if it satisfies ex-post efficiency, IIR and BB. A mechanism is *successful* if (a) it is BIC with respect to the prior μ and (b) it achieves the first best.

2.2.1 The VCG Mechanism

Here we present the well-known VCG mechanism[2] and mention a related result that will be useful for our analysis. We will be closely following the notation of the textbook by Krishna (2002) who also presents these results.

The Vickery-Clarke-Groves (VCG) mechanism is a generalization[3] of mechanisms independently conceived by Vickrey (1961), Clarke (1971) and Groves (1973). The VCG payment for each agent is interpreted as the externality she imposes on other agents.

Let $SW(v)$ denote the *social welfare* or the aggregate social value realized at a profile v in an ex post efficient allocation rule, i.e., $SW(v) = \sum_{j=0}^{n} v_j P_j^*(v)$. Let $SW_{-j}(v)$ denote the social welfare at profile v aggregated over all agents other than j, i.e., $SW_{-j}(v) = \sum_{i \neq j} v_i P_i^*(v)$.

Definition 8 **(VCG Mechanism)** *The VCG mechanism is the pair (P^*, t^V) where P^* is an ex-post efficient allocation rule and t_j^V , $j = 0, \ldots, n$, is defined for all profiles v as follows:*

$$\begin{aligned} t_0^V(v) &= SW(\underline{v}_0, v_{-0}) - SW_{-0}(v) \quad (2.2) \\ t_i^V(v) &= SW(\bar{v}, v_{-i}) - SW_{-i}(v) \textit{ for } i = 1, \ldots, n. \quad (2.3) \end{aligned}$$

It is well-known that the VCG mechanism is ex-post efficient, BIC [4], IIR but not BB (see Krishna (2002)).

Example 3 In Example 1 let us suppose that the valuation for both agents lie in $[0, 1]$, so that, $\underline{v}_0 = 0$ and $\bar{v} = 1$. The ex-post efficient rule is to transfer the good to the buyer if $v_0 > v_1$, and not to transfer if $v_0 \leq v_1$. Then $SW(v) = v_0 - v_1$ if $v_0 > v_1$ and 0 otherwise. Using this we get, $t_0^V(v) = SW(0, v_1) - SW_1(v_0, v_1)$ which is v_1 if $v_0 > v_1$ and 0 otherwise. Similarly, $t_1^V(v) = SW(v_0, 1) - SW_0(v_0, v_1)$ which is $-v_0$ if $v_0 > v_1$ and 0 otherwise. Thus the sum of payments is $v_1 - v_0$ if $v_0 > v_1$ and 0 otherwise. For all profiles where $v_0 > v_1$, budget balance is violated.

The following Proposition derived by Williams (1999) and Krishna and Perry (2000) is central to the proof of our main result. See Krishna (2002) for a proof of this result.

Proposition 1 **(WKP)** *There exists a successful mechanism if and only if the VCG mechanism generates a non-negative expected surplus, i.e.,*

$$E\left(\sum_{j=0}^{n} t_j^V(v)\right) \geq 0.$$

[2]Some authors call it the pivotal mechanism.

[3]See Green and Laffont (1977)

[4]The VCG satisfies a stronger incentive compatibility condition, viz., dominant strategy incentive compatibility (DSIC), that requires every agent to be truthful for any report of other agents. It is easy to check that DSIC implies BIC, but not vice-versa.

The Proposition can be used to prove the Myerson-Satterthwaite impossibility result. To see this, observe that in Example 3, the sum of VCG payments is negative if $v_0 > v_1$ and 0 otherwise. Therefore, the expectation of this sum is negative. An application of the Proposition yields the result immediately.

2.3 Results

Our main results are presented here. First, we prove the impossibility of attaining the first-best with any BIC mechanism when the number of sellers is equal to the number of items required by the buyer to attain a positive valuation.

Theorem 1 *Suppose $n = k$. There does not exist any successful mechanism.*

Proof: We will show that the sum of VCG payments $\sum_{j=0}^{n} t_j^V$ is nonpositive at all profiles and negative in some interval of profiles. Therefore, its expectation is negative. The result follows by applying Proposition 1.

We have

$$SW(v) = \begin{cases} v_0 - \sum_{i=1}^n v_i \text{ if } v_0 > \sum_{i=1}^n v_i, \\ 0 \text{ otherwise.} \end{cases}$$

The VCG payments are given by:

$$\begin{aligned} t_0^V(v) &= SW(\underline{v}_0, v_{-0}) - SW_{-0}(v), \\ t_i^V(v) &= SW(\bar{v}, v_{-i}) - SW_{-i}(v), \quad i = 1, \ldots, n. \end{aligned}$$

The next Lemmas specify the payments for the agents.

Lemma 1 *The VCG payment of the buyer is given by*

$$t_0^V(v) = \begin{cases} \underline{v}_0 & \text{if } v_0 \geq \underline{v}_0 > \sum_{i=1}^n v_i, \\ \sum_{i=1}^n v_i & \text{if } v_0 > \sum_{i=1}^n v_i \geq \underline{v}_0, \\ 0 & \text{if } \underline{v}_0 \leq v_0 \leq \sum_{i=1}^n v_i. \end{cases}$$

Proof: If $v_0 \geq \underline{v}_0 > \sum_{i=1}^n v_i$,

$$SW(\underline{v}_0, v_{-0}) = \underline{v}_0 - \sum_{i=1}^n v_i,$$

$$\text{and } SW_{-0}(v_0, v_{-0}) = -\sum_{i=1}^n v_i.$$

$$\text{Hence } t_0^V(v) = \underline{v}_0.$$

If $v_0 > \sum_{i=1}^{n} v_i \geq \underline{v}_0$,

$$SW(\underline{v}_0, v_{-0}) = 0,$$
$$\text{and } SW_{-0}(v_0, v_{-0}) = -\sum_{i=1}^{n} v_i.$$
$$\text{Hence } t_0^V(v) = \sum_{i=1}^{n} v_i.$$

If $v_0 \leq \sum_{i=1}^{n} v_i$,

$$SW(\underline{v}_0, v_{-0}) = 0,$$
$$\text{and } SW_{-0}(v_0, v_{-0}) = 0.$$
$$\text{Hence } t_0^V(v) = 0.$$

■

Lemma 2 *When $n = k$, the VCG payment of the seller corresponding to v_i, $i = 1, \ldots, n$, is given by*

$$t_i^V(v) = \begin{cases} -(v_0 - \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j) & \text{if } \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j + \bar{v} \geq v_0 > \sum_{j=1}^{n} v_j, \\ -\bar{v} & \text{if } v_0 > \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j + \bar{v} \geq \sum_{j=1}^{n} v_j, \\ 0 & \text{if } v_0 \leq \sum_{j=1}^{n} v_j \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j + \bar{v}. \end{cases}$$

Proof: If $\sum_{\substack{j=1 \\ j \neq i}}^{n} v_j + \bar{v} \geq v_0 > \sum_{j=1}^{n} v_j$,

$$SW(\bar{v}, v_{-i}) = 0,$$
$$\text{and } SW_{-i}(v_i, v_{-i}) = v_0 - \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j.$$
$$\text{Hence } t_i^V(v) = -(v_0 - \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j).$$

If $v_0 > \sum_{\substack{j=1\\j\neq i}}^n v_j + \bar{v} \geq \sum_{j=1}^n v_j$,

$$SW(\bar{v}, v_{-i}) = v_0 - \sum_{\substack{j=1\\j\neq i}}^n v_j - \bar{v},$$
$$\text{and } SW_{-i}(v_i, v_{-i}) = v_0 - \sum_{\substack{j=1\\j\neq i}}^n v_j.$$
$$\text{Hence } t_i^V(v) = -\bar{v}.$$

If $v_0 \leq \sum_{j=1}^n v_j \leq \sum_{\substack{j=1\\j\neq i}}^n v_j + \bar{v}$,

$$SW(\bar{v}, v_{-i}) = 0,$$
$$\text{and } SW_{-i}(v_i, v_{-i}) = 0.$$
$$\text{Hence } t_i^V(v) = 0.$$

■

For any profile v, let $A(v) = \{h \in \{1, \ldots, n\} : \sum_{i=1}^n v_i < v_0 \leq \sum_{\substack{i=1\\i\neq h}}^n v_i + \bar{v}\}$. In other words, $A(v)$ is the set of sellers who influence the possibility of trade by reporting their highest valuation. Such sellers will be referred to as *trade-pivotal*.

Different mutually exclusive and exhaustive cases and corresponding sum of payments as obtained from Lemmas 1 and 2 are presented below.

Case I: $v_0 \geq \underline{v}_0 > \sum_{i=1}^n v_i$ and $A(v) \neq \emptyset$

$$\begin{aligned}
\sum_{j=0}^n t_j^V(v) &= \underline{v}_0 - \sum_{h\in A(v)} \left(v_0 - \sum_{\substack{i=1\\i\neq h}}^n v_i \right) - (n - |A(v)|)\,\bar{v} \\
&= \underline{v}_0 - \sum_{h\in A(v)} v_0 + \sum_{h\in A(v)} \sum_{i=1}^n v_i - \sum_{h\in A(v)} v_{[h]} - (n - |A(v)|)\,\bar{v} \\
&= \left(\underline{v}_0 - \sum_{h\in A(v)} v_{[h]} - (n - |A(v)|)\,\bar{v} \right) - \sum_{h\in A(v)} \left(v_0 - \sum_{i=1}^n v_i \right) \\
&\leq \left(\underline{v}_0 - \sum_{i=1}^n v_i \right) - \sum_{h\in A(v)} \left(v_0 - \sum_{i=1}^n v_i \right) \\
&\leq 0.
\end{aligned}$$

Here, $n - |A(v)|$ is the number of sellers not in $A(v)$ at profile v. The second equality is obtained by expanding $\sum_{\substack{i=1\\i\neq h}}^{n} v_i$; the third is obtained by re-arranging terms; the first inequality holds since $v_i \leq \bar{v}$ for any $i = 1, \ldots, n$ and the second inequality holds since $v_0 \geq \underline{v}_0$.

Case II: $v_0 \geq \underline{v}_0 > \sum_{i=1}^{n} v_i$ and $A(v) = \emptyset$

$$\begin{aligned}\sum_{j=0}^{n} t_j^V(v) &= \underline{v}_0 - n\bar{v} \\ &< 0.\end{aligned}$$

This holds by virtue of the assumption $\underline{v}_0 < n\bar{v}$.

Case III: $v_0 > \sum_{i=1}^{n} v_i \geq \underline{v}_0$ and $A(v) \neq \emptyset$

$$\begin{aligned}\sum_{j=0}^{n} t_j^V(v) &= \sum_{i=1}^{n} v_i - \sum_{h\in A(v)} \left(v_0 - \sum_{\substack{i=1\\i\neq h}}^{n} v_i \right) - (n - |A(v)|)\,\bar{v} \\ &= \sum_{h\in A(v)} v_h + \sum_{\substack{i=1\\i\notin A(v)}}^{n} v_i - \sum_{h\in A(v)} \left(v_0 - \sum_{\substack{i=1\\i\neq h}}^{n} v_i \right) - (n - |A(v)|)\,\bar{v} \\ &= -\sum_{h\in A(v)} \left(v_0 - \sum_{i=1}^{n} v_i \right) + \sum_{\substack{i=1\\i\notin A(v)}}^{n} (v_i - \bar{v}) \\ &< 0.\end{aligned}$$

The second equality is obtained by expanding $\sum_{i=1}^{n} v_i$; the third adds the first and the third term of the left hand side to obtain the first term, and the other two terms constitute the second term. The inequality follows because $v_0 > \sum_{i=1}^{n} v_i$ and $v_i \leq \bar{v}$ for $i = 1, \ldots, n$.

Case IV: $v_0 > \sum_{i=1}^{n} v_i \geq \underline{v}_0$ and $A(v) = \emptyset$

$$\begin{aligned}\sum_{j=0}^{n} t_j^V(v) &= \sum_{i=1}^{n} v_i - n\bar{v} \\ &= \sum_{i=1}^{n} v_i - n\bar{v} \\ &< 0.\end{aligned}$$

The inequality follows because $v_i \leq \bar{v}$ for $i = 1, \ldots, n$.

Case V: $v_0 \leq \sum_{i=1}^{n} v_i$

$$\sum_{j=0}^{n} t_j^V(v) = 0.$$

The calculations are summarized in Table 2.1. Observe that for any profile v, $\sum_{j=0}^{n} t_j^V(v) \leq 0$ and in Cases II, III and IV, $\sum_{j=0}^{n} t_j^V(v) < 0$. Therefore, as required

$$E\left(\sum_{j=0}^{n} t_j^V(v)\right) < 0.$$

■

Table 2.1: Sum of Payments when $n = k$

Case	Sum of Payments	Sign
I: $v_0 \geq \underline{v}_0 > \sum_{i=1}^{n} v_i$, $A(v) \neq \emptyset$	$\underline{v}_0 - \sum_{h\in A(v)}\left(v_0 - \sum_{\substack{i=1\\ i\neq h}}^{n} v_i\right) - (n - \lvert A(v)\rvert)\,\bar{v}$	≤ 0
II: $v_0 \geq \underline{v}_0 > \sum_{i=1}^{n} v_i$, $A(v) = \emptyset$	$\underline{v}_0 - n\bar{v}$	< 0
III: $v_0 > \sum_{i=1}^{n} v_i \geq \underline{v}_0$, $A(v) \neq \emptyset$	$\sum_{i=1}^{n} v_i - \sum_{h\in A(v)}\left(v_0 - \sum_{\substack{i=1\\ i\neq h}}^{n} v_i\right) - (n - \lvert A(v)\rvert)\,\bar{v}$	< 0
IV: $v_0 > \sum_{i=1}^{n} v_i \geq \underline{v}_0$, $A(v) = \emptyset$	$\sum_{i=1}^{n} v_i - n\bar{v}$	< 0
V: $v_0 \leq \sum_{i=1}^{n} v_i$	0	0

This result extends the Myerson-Satterthwaite result from the simple bilateral trade model, i.e., $n = k = 1$, to all LA problems where $n = k$. The next result shows that the situation changes dramatically when $n > k$.

THEOREM **2** *Assume $n > k$.*

I. Suppose μ satisfies the following condition:

$$\underline{v}_0 \geq kE\left(v_{[k+1]}\right). \tag{2.4}$$

Then there exists a successful mechanism.

II. If there exists a successful mechanism then μ satisfies the following condition:

$$\underline{v}_0 > kE\left(v_{[k+1]} \mid (\underline{v}_0 > \sum_{j=1}^{k} v_{[j]}) \cap (v_0 > \sum_{j=2}^{k+1} v_{[j]})\right). \tag{2.5}$$

Proof: For Part I, we will show that at each profile v, the sum of payments $\sum_{j=0}^{n} t_j^V$ is bounded below by $\underline{v}_0 - kv_{[k+1]}$. It follows that the expectation of the former is bounded below by the expectation of the latter expression. Therefore, if (2.4) holds, then the expected sum of payments is positive and the claim follows by Proposition 1.

The VCG payment of the buyer is given by Lemma 1. For the VCG payment of the sellers, we refer to the next two Lemmas.

LEMMA **3** *VCG payment of the seller corresponding to $v_{[i]}$, $i > k$, is 0.*

Proof: Let $v_0 > \sum_{i=1}^{k} v_{[i]}$. For any $i > k$

$$SW(\bar{v}, v_{-[i]}) = v_0 - \sum_{i=1}^{k} v_{[i]},$$

$$\text{and } SW_{-[i]}(v_{[i]}, v_{-[i]}) = v_0 - \sum_{i=1}^{k} v_{[i]}.$$

$$\text{Hence } t_{[i]}^V(v) = 0.$$

If $v_0 \le \sum_{i=1}^{k} v_{[i]}$,

$$SW(\bar{v}, v_{-[i]}) = 0,$$

$$\text{and } SW_{-[i]}(v_{[i]}, v_{-[i]}) = 0.$$

$$\text{Hence } t_{[i]}^V(v) = 0.$$

∎

LEMMA **4** *The VCG payment of the seller with valuation $v_{[i]}$, $i \le k$, is given by*

$$t_{[i]}^V(v) = \begin{cases} -(v_0 - \sum_{\substack{j=1 \\ j \ne i}}^{k} v_{[j]}) & \text{if } \sum_{j=1}^{k} v_{[j]} < v_0 \le \sum_{\substack{j=1 \\ j \ne i}}^{k+1} v_{[j]} \\ -v_{k+1} & \text{if } \sum_{j=1}^{k} v_{[j]} \le \sum_{\substack{j=1 \\ j \ne i}}^{k+1} v_{[j]} < v_0 \\ 0 & \text{if } v_0 \le \sum_{j=1}^{k} v_{[j]} \le \sum_{\substack{j=1 \\ j \ne i}}^{k+1} v_{[j]} \end{cases}$$

Proof: If $\sum_{j=1}^{k} v_{[j]} < v_0 \le \sum_{\substack{j=1 \\ j \ne i}}^{k+1} v_{[j]}$,

$$SW(\bar{v}, v_{-[i]}) = 0,$$

$$\text{and } SW_{-[i]}(v_{[i]}, v_{-[i]}) = v_0 - \sum_{\substack{j=1 \\ j \ne i}}^{k} v_{[j]}.$$

$$\text{Hence } t_{[i]}^V(v) = -(v_0 - \sum_{\substack{j=1 \\ j \ne i}}^{k} v_{[j]}).$$

If $\sum_{j=1}^{k} v_{[j]} \leq \sum_{\substack{j=1 \\ j \neq i}}^{k+1} v_{[j]} < v_0$,

$$SW(\bar{v}, v_{-[i]}) = v_0 - \sum_{\substack{j=1 \\ j \neq i}}^{k+1} v_{[j]},$$

$$\text{and } SW_{-[i]}(v_{[i]}, v_{-[i]}) = v_0 - \sum_{\substack{j=1 \\ j \neq i}}^{k} v_{[j]}.$$

$$\text{Hence } t_{[i]}^{V}(v) = -v_{[k+1]}.$$

If $v_0 \leq \sum_{j=1}^{k} v_{[j]} \leq \sum_{\substack{j=1 \\ j \neq i}}^{k+1} v_{[j]}$,

$$SW(\bar{v}, v_{-[i]}) = 0,$$

$$\text{and } SW_{-[i]}(v_{[i]}, v_{-[i]}) = 0.$$

$$\text{Hence } t_{[i]}^{V}(v) = 0.$$

■

When $n > k$, let $A(v) = \{h \in \{1, \ldots, k\} : \sum_{j=1}^{k} v_{[j]} < v_0 \leq \sum_{\substack{j=1 \\ j \neq h}}^{k+1} v_{[j]}\}$. As before, $A(v)$ represents the set of *trade-pivotal* sellers at profile v: they influence the possibility of trade by reporting their highest valuation.

Then we have the following cases for $\sum_{j=0}^{n} t_j^V(v)$.

Case I: $v_0 \geq \underline{v}_0 > \sum_{j=1}^{k} v_{[j]}$ and $A(v) \neq \emptyset$

$$\begin{aligned}
\sum_{j=0}^{n} t_j^V(v) &= \underline{v}_0 - \sum_{h \in A(v)} \left(v_0 - \sum_{\substack{j=1 \\ j \neq h}}^{k} v_{[j]} \right) - (k - |A(v)|)\, v_{[k+1]} \\
&= \underline{v}_0 - \sum_{h \in A(v)} v_{[h]} - (k - |A(v)|)\, v_{[k+1]} - |A(v)| \left(v_0 - \sum_{j=1}^{k} v_{[j]} \right) \\
&\leq \underline{v}_0 - \sum_{j=1}^{k} v_{[j]} - |A(v)| \left(v_0 - \sum_{j=1}^{k} v_{[j]} \right) \\
&\leq 0.
\end{aligned}$$

The first equality follows from Lemmas 1, 3 and 4. The second follows by subtracting and then adding $\sum_{h\in A(v)} v_{[h]}$. The first inequality follows since $\sum_{h\in A(v)} v_{[h]} + (k - |A(v)|)\, v_{[k+1]} \geq \sum_{j=1}^{k} v_{[j]}$. The final inequality follows since $\underline{v}_0 \leq v_0$ and $A(v)$ is nonempty.

Case II: $v_0 \geq \underline{v}_0 > \sum_{j=1}^{k} v_{[j]}$ and $A(v) = \emptyset$

$$\sum_{j=0}^{n} t_j^V(v) \quad = \quad \underline{v}_0 - k v_{[k+1]}.$$

The sign of $\underline{v}_0 - k v_{[k+1]}$ can be positive or negative, as examples will show.

Case III: $v_0 > \sum_{j=1}^{k} v_{[j]} \geq \underline{v}_0$ and $A(v) \neq \emptyset$

$$\begin{aligned}
\sum_{j=0}^{n} t_j^V(v) &= \sum_{j=1}^{k} v_{[j]} - \sum_{h\in A(v)} \left(v_0 - \sum_{\substack{j=1\\ j\neq h}}^{k} v_{[j]} \right) - (k - |A(v)|)\, v_{[k+1]} \\
&= \sum_{h\in A(v)} v_{[h]} + \sum_{\substack{j=1\\ j\notin A(v)}}^{k} v_{[j]} - \sum_{h\in A(v)} \left(v_0 - \sum_{\substack{j=1\\ j\neq h}}^{k} v_{[j]} \right) - (k - |A(v)|)\, v_{[k+1]} \\
&= -\sum_{h\in A(v)} \left(v_0 - \sum_{\substack{j=1\\ j\notin A(v)}}^{k} v_{[j]} \right) + \sum_{\substack{j=1\\ j\notin A(v)}}^{k} \left(v_{[j]} - v_{[k+1]} \right) \\
&< 0.
\end{aligned}$$

The first equality is by Lemmas 1, 3 and 4. The second is obtained by expanding $\sum_{j=1}^{k} v_{[j]}$; we then add the first and third terms of this expression to obtain the first term on the right hand side of the next equality sign, and the other two terms constitute the second term. The inequality follows because $v_0 > \sum_{j=1}^{k} v_{[j]}$ and $v_{[j]} \leq v_{[k+1]}$ for all $j = 1, \ldots, k$.

Case IV: $v_0 > \sum_{j=1}^{k} v_{[j]} \geq \underline{v}_0$and $A(v) = \emptyset$

$$\begin{aligned}
\sum_{j=0}^{n} t_j^V(v) &= \sum_{j=1}^{k} v_{[j]} - k v_{[k+1]} \\
&\leq 0.
\end{aligned}$$

The inequality follows because $v_0 > \sum_{j=1}^{k} v_{[j]}$ and $v_{[j]} \leq v_{[k+1]}$ for all $j = 1, \ldots, k$.

Case V: $v_0 \leq \sum_{j=1}^{k} v_{[j]}$

$$\sum_{j=0}^{n} t_j^V(v) = 0.$$

These cases are summarized in the table below.

Table 2.2: Sum of Payments when $n > k$

Case	**Sum of Payments**	**Sign**
I: $v_0 \geq \underline{v}_0 > \sum_{j=1}^{k} v_{[j]}, A(v) \neq \emptyset$	$\underline{v}_0 - \sum_{h\in A(v)} \left(v_0 - \sum_{\substack{j=1\\ j\neq h}}^{k} v_{[j]} \right) - (k - \lvert A(v)\rvert)\, v_{[k+1]}$	≤ 0
II: $v_0 \geq \underline{v}_0 > \sum_{j=1}^{k} v_{[j]}, A(v) = \emptyset$	$\underline{v}_0 - kv_{[k+1]}$	$\lesseqgtr 0$
III: $v_0 > \sum_{j=1}^{k} v_{[j]} \geq \underline{v}_0, A(v) \neq \emptyset$	$\sum_{j=1}^{k} v_{[j]} - \sum_{h\in A(v)} \left(v_0 - \sum_{\substack{j=1\\ j\neq h}}^{k} v_{[j]} \right) - (k - \lvert A(v)\rvert)\, v_{[k+1]}$	< 0
IV: $v_0 > \sum_{j=1}^{k} v_{[j]} \geq \underline{v}_0, A(v) = \emptyset$	$\sum_{j=1}^{k} v_{[j]} - kv_{k+1}$	≤ 0
V: $v_0 \leq \sum_{j=1}^{k} v_{[j]}$	0	0

The following Lemma will be used for proving Part I.

Lemma **5** *For all v,*

$$\sum_{j=0}^{n} t_j^V(v) \geq \underline{v}_0 - kv_{[k+1]}.$$

Proof: For $h \in A(v)$, $v_0 \leq \sum_{\substack{j=1\\ j\neq h}}^{k+1} v_{[j]}$. Therefore, $v_0 - \sum_{\substack{j=1\\ j\neq h}}^{k} v_{[j]} \leq v_{[k+1]}$. Therefore, in Case I of Table 2.2,

$$\begin{aligned}
\sum_{j=0}^{n} t_j^V(v) &= \underline{v}_0 - \sum_{h\in A(v)} \left(v_0 - \sum_{\substack{j=1\\ j\neq h}}^{k} v_{[j]} \right) - (k - \lvert A(v)\rvert)\, v_{[k+1]} \\
&\geq \underline{v}_0 - \sum_{h\in A(v)} v_{[k+1]} - (k - \lvert A(v)\rvert)\, v_{[k+1]} \\
&= \underline{v}_0 - kv_{[k+1]}.
\end{aligned}$$

In Case II, $\sum_{j=0}^{n} t_j^V(v) = \underline{v}_0 - kv_{[k+1]}$.

In Case III,

$$\begin{aligned}
\sum_{j=0}^{n} t_j^V(v) &= \sum_{j=1}^{k} v_{[j]} - \sum_{h\in A(v)} \left(v_0 - \sum_{\substack{j=1\\ j\neq h}}^{k} v_{[j]} \right) - (k - \lvert A(v)\rvert)\, v_{[k+1]} \\
&\geq \sum_{j=1}^{k} v_{[j]} - \sum_{h\in A(v)} v_{[k+1]} - (k - \lvert A(v)\rvert)\, v_{[k+1]} \\
&\geq \underline{v}_0 - kv_{[k+1]}.
\end{aligned}$$

In Case IV,

$$\sum_{j=0}^{n} t_j^V(v) = \sum_{j=1}^{k} v_{[j]} - kv_{[k+1]} \geq \underline{v}_0 - kv_{[k+1]}.$$

In Case V,

$$\sum_{j=0}^{n} t_j^V(v) = 0 \geq v_0 - \sum_{j=1}^{k} v_{[j]} \geq \underline{v}_0 - kv_{[k+1]}.$$

$\blacksquare$

If (2.4) holds, Lemma 5 implies

$$E\left(\sum_{j=0}^{n} t_j^V(v)\right) \geq \underline{v}_0 - kE\left(v_{[k+1]}\right) \geq 0.$$

Part I now follows by Proposition 1.

For Part II, note that $v_0 > \sum_{j=2}^{k+1} v_{[j]}$ implies

$$\begin{aligned}
& v_0 > \sum_{j=2}^{k+1} v_{[j]} = \sum_{j=1}^{k+1} v_{[j]} - v_{[1]} \geq \sum_{j=1}^{k+1} v_{[j]} - v_{[2]} \geq \cdots \geq \sum_{j=1}^{k+1} v_{[j]} - v_{[k]} \\
& \Rightarrow v_0 > \sum_{\substack{j=1 \\ j \neq 1}}^{k+1} v_{[j]} \geq \sum_{\substack{j=1 \\ j \neq 2}}^{k+1} v_{[j]} \geq \cdots \geq \sum_{\substack{j=1 \\ j \neq k}}^{k+1} v_{[j]} \\
& \Rightarrow A(v) = \emptyset.
\end{aligned}$$

Furthermore,

$$v_0 \geq \underline{v}_0 > \sum_{j=1}^{k} v_{[j]}.$$

Therefore, $\left((\underline{v}_0 > \sum_{j=1}^{k} v_{[j]}) \cap (v_0 > \sum_{j=2}^{k+1} v_{[j]})\right)$ refers to profiles in Case II of table 2.2, i.e., profiles that satisfy $v_0 \geq \underline{v}_0 > \sum_{j=1}^{k} v_{[j]}, A(v) = \emptyset$.

If mechanism M is BIC(μ) and achieves the first best, then by Proposition 1,

$$0 \leq E\left(\sum_{j=0}^{n} t_j^V(v)\right)$$

$$\text{or, } 0 \leq E\left(\sum_{j=0}^{n} t_j^V(v) \mid (\underline{v}_0 > \sum_{j=1}^{k} v_{[j]}) \cap (v_0 > \sum_{j=2}^{k+1} v_{[j]})\right) \Pr\left((\underline{v}_0 > \sum_{j=1}^{k} v_{[j]}) \cap (v_0 > \sum_{j=2}^{k+1} v_{[j]})\right)$$
$$+ E\left(\sum_{j=0}^{n} t_j^V(v) \mid (\underline{v}_0 > \sum_{j=1}^{k} v_{[j]})' \cup (v_0 > \sum_{j=2}^{k+1} v_{[j]})'\right) \Pr\left((\underline{v}_0 > \sum_{j=1}^{k} v_{[j]})' \cup (v_0 > \sum_{j=2}^{k+1} v_{[j]})'\right)$$

But by cases I and III-V of table 2.2, the second component of this sum of products is negative since $\sum_{j=0}^{n} t_j^V(v)$ takes negative or zero value at $(\underline{v}_0 > \sum_{j=1}^{k} v_{[j]})' \cup (v_0 > \sum_{j=2}^{k+1} v_{[j]})'$. Therefore,

$$E\left(\sum_{j=0}^{n} t_j^V(v) \mid (\underline{v}_0 > \sum_{j=1}^{k} v_{[j]}) \cap (v_0 > \sum_{j=2}^{k+1} v_{[j]})\right) > 0.$$

Since $\sum_{j=0}^{n} t_j^V(v) = \underline{v}_0 - kv_{[k+1]}$ when $\underline{v}_0 > \sum_{j=2}^{k+1} v_{[j]}$, the claim follows. ■

A corollary relating to the case $k = 1$ follows. This is a special case of a result by Williams (1999) (Theorem 4, p.169). He considers a multilateral bargaining problem with m buyers demanding a unit each and n sellers selling a unit each. In his result, the existence of a successful mechanism depends on m, n and the prior distribution.

Corollary 1 *Suppose $n > 1$ and $k = 1$.*

I. *Suppose μ satisfies the following condition:*

$$\underline{v}_0 \geq E\left(v_{[2]}\right). \tag{2.6}$$

Then there exists a successful mechanism.

II. *If there exists a successful mechanism then μ satisfies the following condition:*

$$\underline{v}_0 > E\left(v_{[2]} \mid (\underline{v}_0 > v_{[1]}) \cap (v_0 > v_{[2]})\right). \tag{2.7}$$

2.3.1 Examples

We provide examples of priors where BIC mechanisms can achieve the first best.

Example 4 Seller valuations are distributed uniformly in $[0, 1]$. Assumption NT requires $\bar{v}_0 > 0$ and $\underline{v}_0 < k$. In this case, $E\left(v_{[k+1]}\right) = \frac{k+1}{n+1}$. According to Part I of Theorem 2, $\underline{v}_0 \geq \frac{k(k+1)}{n+1}$ guarantees the existence of BIC mechanisms that achieve the first best. For instance, if $n = 2$ and $k = 1$, $\underline{v}_0 \geq \frac{2}{3}$ is the required condition. Since $\frac{k(k+1)}{n+1} \to 0$ as $n \to \infty$, it becomes easier to satisfy the sufficient condition as the number of sellers increase.

Example 5 Let $n = 2$, $k = 1$. Assume v_i's are uniformly distributed in $[0, 100]$ and v_0 uniformly distributed in $[\underline{v}_0, 100]$, where $\underline{v}_0 \geq 0$. Condition (2.4) is satisfied if $\underline{v}_0 \geq E(v_{[2]}) = \int_0^{100} \frac{2x^2}{10000} dx = \frac{200}{3}$.

Further calculations show that

$$E\left(\sum_{j=0}^{n} t_j^V(v) \mid (\underline{v}_0 > \sum_{j=1}^{k} v_{[j]}) \cap (v_0 > \sum_{j=2}^{k+1} v_{[j]})\right)$$
$$= \frac{1}{150} \times \frac{\underline{v}_0(\underline{v}_0 - 50)}{\Pr((\underline{v}_0 > \sum_{j=1}^{k} v_{[j]}) \cap (v_0 > \sum_{j=2}^{k+1} v_{[j]}))}.$$

Therefore, (2.5) is satisfied for all $\underline{v}_0 \in (50, 100)$.

From Table 2.2, we get

$$\sum_{j=0}^{2} t_j^V(v) = \begin{cases} \underline{v}_0 - v_0 & \text{if } v_2 \geq v_0 \geq \underline{v}_0 > v_1, \\ \underline{v}_0 - v_2 & \text{if } v_0 \geq v_2 \geq \underline{v}_0 > v_1, \\ \underline{v}_0 - v_2 & \text{if } v_0 \geq \underline{v}_0 \geq v_2 > v_1, \\ v_1 - v_0 & \text{if } v_2 \geq v_0 \geq v_1 \geq \underline{v}_0, \\ v_1 - v_2 & \text{if } v_0 > v_2 \geq v_1 \geq \underline{v}_0, \\ 0 & \text{if otherwise.} \end{cases}$$

The expectation of this sum is

$$\frac{1}{5000} \times \left(-\frac{1}{12}(100 - \underline{v}_0)^3 - \frac{1}{3}(100 - \underline{v}_0)^2 + \frac{1}{6}\underline{v}_0^3\right).$$

This expression is zero when $\underline{v}_0 \approx 60.41$ and is strictly increasing in $[50, 100]$. Therefore, there is a region of $\underline{v}_0$ where the sufficient condition is not necessary (the approximate region $[60.41, 66.67]$). There is also a region where the necessary condition is not sufficient (the approximate region $[50, 60.41]$).

2.4 Discussion regarding the Possibility Result

Recall that a seller i is *trade-pivotal* at a profile $(v_{[i]}, v_{-[i]})$ if trade takes place at this profile but not when he reports his highest valuation, i.e., at $(\bar{v}, v_{-[i]})$. He is *non-pivotal* if trade takes place at both $(v_{[i]}, v_{-[i]})$ and $(\bar{v}, v_{-[i]})$. When $n > k$, non-pivotal sellers receive $v_{[k+1]}$ while pivotal sellers get less. Thus, the receipt of any seller is at most $v_{[k+1]}$. Consequently the receipt of all sellers is at most $kv_{[k+1]}$. The sufficient condition for achieving the first best can be interpreted as follows: *the lowest value buyer can compensate all successful sellers in expectation*. If $n = k$, all trade-pivotal sellers receive $\bar{v}$. Then the lowest value buyer cannot compensate all the sellers for any realization of v.

Condition (2.5) requires the expectation of the sum of payments to be positive in the critical region corresponding to Case II of Table 2.2. If this condition does not hold, the expected sum of VCG payments is definitely negative.

Note that having more sellers than the number of items required by the buyer in the LA model introduces seller competition. Our conditions suggest that if this competition is intense enough, the first best can be achieved.

There are isolated examples in the mechanism design literature where Myerson-Satterthwaite impossibility does not hold. Two prominent examples are (a) partnership dissolution(Cramton et al., 1987) and (b) the trading of an indivisible object among one seller and two potential buyers (Makowski and Mezzetti, 1993).

2.5 Conclusion

The results in this chapter show that Myerson-Satterthwaite impossibility does not extend to the LA model in an unqualified manner. We provide a simple sufficient condition on priors which ensures the attainment of the first best outcome by BIC mechanisms. The next chapter extends the analysis to a model where the contiguity of acquired plots is important.

Chapter 3

Contiguity

3.1 Introduction

Myerson and Satterthwaite (1983) showed that in the model of bilateral trade with asymmetric information, no Bayesian Incentive Compatible (BIC) mechanism can achieve the first best. In the previous chapter, we showed that this negative result does not extend to the Land Acquisition problem. In this chapter, we impose the additional requirement that the buyer wants the acquired plots to be *contiguous*. One might reasonably expect that imposing this restriction will precipitate a Myerson-Satterthwaite type impossibility result. However, we show that this is not true and results similar to the ones obtained in the previous chapter continue to hold. We also show that the *degree* of contiguity, i.e., the number of *critical plots*, affects attainability of the first best in a direct way.

A pair of plots is contiguous if they share a physical boundary. For example, consider the figures below that show the physical location of four rectangular plots. In the first figure, all the plots are contiguous (we are assuming for simplicity that plots sharing a vertex are also contiguous. In the middle figure, none of the plots are contiguous. In the figure on the right, only the lower plots are contiguous.

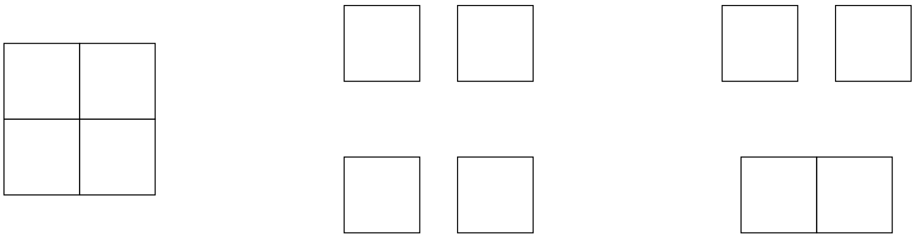

Figure 3.1: Sets of plots with different contiguity structures

Each of these situation can be represented by a graph. The nodes in the graph represent sellers. Two nodes are connected by an edge if the corresponding plots are contiguous. The graphs representing the situation in Figure 3.1 are shown in Figure 3.2.

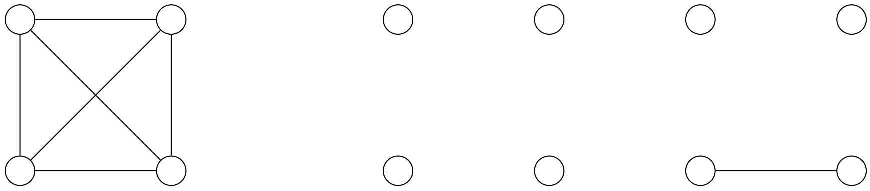

Figure 3.2: Graphs representing the contiguity structure in Figure 3.1

Figure 3.3 shows five different contiguity structures.

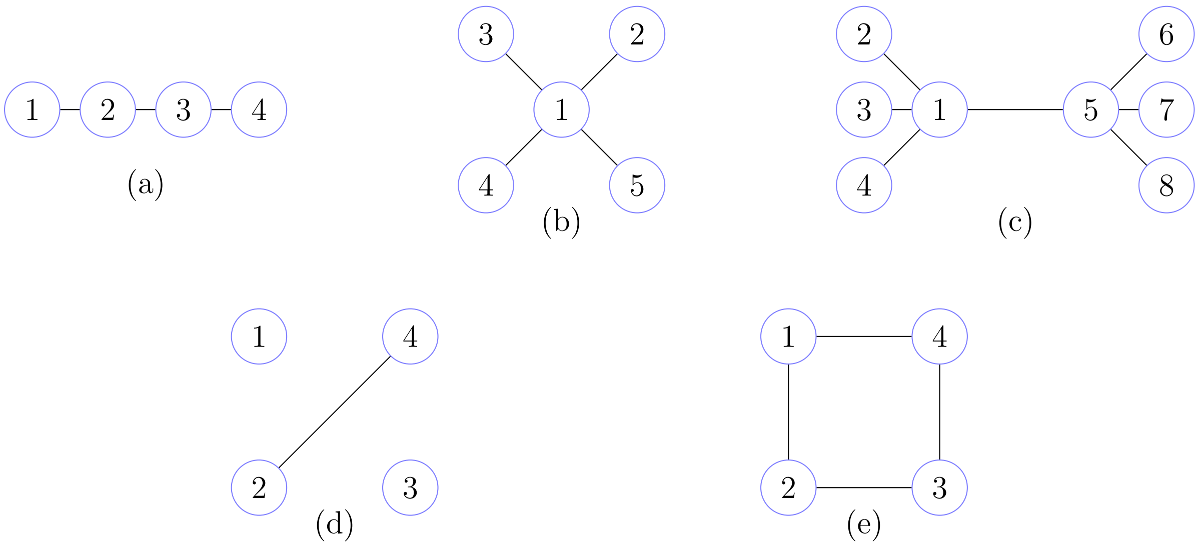

Figure 3.3: Different graphs

A path is a sequence of connected nodes. There is a single buyer who requires k contiguous plots ($k \geq 2$), i.e., a collection of nodes that constitute a path of length k. The buyer gets valuation v_0 in this case and zero otherwise. Each seller i's valuation is v_i and all valuations are private information.

We draw attention to the fact that certain nodes in these graphs have special significance. For instance in Figure 3.3 (b), if at least 2 nodes are required then all feasible paths must contain node 1. Similarly, if at least 4 nodes are required in (c) then all feasible paths must contain 1 and 5. We shall call such nodes in a LA problem *critical nodes*.

Our main result is that there exists a robust set of priors for which BIC mechanisms attain the first best when there are at least two feasible distinct paths. Two paths are distinct if the nodes constituting them are not identical. For instance, if the buyer wants two contiguous plots, there exist successful mechanisms for all but one graph in Figure 3.3, i.e., (d). Similar to the analysis in Chapter 2, we provide a sufficient condition for the existence of a successful mechanism in the case with more than one feasible paths. In this case we also provide a weaker necessary condition. Furthermore, we show that the number of critical nodes has a bearing on the set of priors for which successful mechanisms exist. In particular, it becomes harder to satisfy the conditions for possibility as the number of critical nodes increases.

The importance of contiguity in the LA problem has been generally recognized. Kominers and Weyl (2012) discuss the importance and difficulties of acquiring contiguous plots. Singh (2012) provide an example to show that a solution proposed by Ghatak and Ghosh (2011) in a model without contiguity runs into difficulty if contiguity is required. However, none of these papers formally model the contiguity issue. We believe that this chapter is novel in this respect.

3.2 Preliminaries

There are n sellers, indexed by i, each holding one unit of an indivisible good (plot). The n indivisible items are located on a graph $\Gamma = (N, E)$ where N denotes the set of nodes (plots) and E denotes the set of edges. A pair of nodes is connected by a direct edge if they are physically adjacent to each other. A sequence of connected nodes is called a path. A path is feasible if it contains at least a fixed number k of nodes where $k \leq n$.

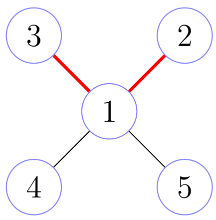

Figure 3.4: A feasible path in the star graph when $k = 3$

The valuation of each seller i is $v_i \in [\underline{v}, \bar{v}]$. We assume that v_i's are independently and identically distributed random variables with distribution function $F(\cdot)$ and density function $f(\cdot)$. The realization of v_i is observed only by i.

There is one buyer, indexed by 0. Her valuation is $v_0 \in \mathbb{R}_+$ if she acquires a feasible path. We assume that $v_0 \in [\underline{v_0}, \bar{v_0}]$ and $v_0 \sim G(v_0)$. We will assume that F and G have continuous and positive densities $f(\cdot)$ and $g(\cdot)$ in their respective domains.

As before, the valuations of the buyer and the sellers are independently distributed . All valuations are non-negative. Own valuations are private information while the distribution functions F and G are common knowledge. In order to make the problem non-trivial , we make the same assumption as in Chapter 2:

$$\boxed{\text{ASSUMPTION } \textbf{NT} : k\underline{v} < \bar{v}_0 \text{ and } k\bar{v} > \underline{v}_0}$$

As before, this assumption ensures that ex-post efficiency is a non-trivial issue. If the first part does not hold, then the buyer's valuation for any feasible path will always be less than the sum of valuations of the sellers constituting it. Consequently, trade will never be ex-post efficient. If the second part is violated, then the buyer's valuation will always exceed this sum of valuations. Then trade is ex-post efficient for any feasible path.

As before, a valuation profile is an $n+1$-vector $v \equiv (v_0, v_1, \ldots, v_n) \in [\underline{v_0}, \bar{v_0}] \times [\underline{v}, \bar{v}]^n$. The j-th component of v is denoted by v_j and the n-vector v_{-j} denotes the profile where the j-th component is dropped from v. Throughout, we will use the subscripts j and $-j$ to indicate "the j-th component" and "all but the j-th component" of a vector respectively. The distribution of the random vector v is called a *prior*, denoted μ. A *land acquisition problem with contiguity* or LAC is a tuple $\langle \Gamma, k, \mu \rangle$.

The definitions of allocation rules, transfer rules, payoffs, BIC and IR in Chapter 2 for the LA problem extend in a straightforward manner to the LAC problem.

Ex-post efficient allocations in LAC are defined as follows. Let the feasible paths in Γ be denoted by $\mathcal{P}_1, \ldots, \mathcal{P}_q$ with $q \geq 1$. Consider a valuation profile v. The sum of valuations in path $\mathcal{P}_i$ will be denoted by $S_i(v)$, $i = 1, \ldots, q$. These sums are ordered as follows: $S_{[1]}(v) \leq \ldots \leq S_{[q]}(v)$. The paths corresponding to these sums are denoted by $\mathcal{P}_{[1]}(v), \ldots, P_{[q]}(v)$ respectively. Efficiency requires trade to take place with sellers in $\mathcal{P}_{[1]}(v)$ if $v_0 > S_{[1]}(v)$; if $v_0 \leq S_{[1]}(v)$ then trade does not occur. This is illustrated in the following example.

EXAMPLE **6** Consider the graph in Figure 3.5. Suppose $k = 3$, i.e., there are two feasible paths $\{123\}$ and $\{234\}$. Consider the following valuations : $v_1 = 1$, $v_2 = 9$, $v_3 = 9$ and $v_4 = 8$.

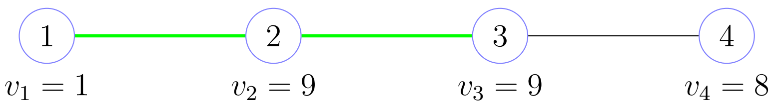

Figure 3.5: $\mathcal{P}_{[1]}(v)$

Here $\mathcal{P}_{[1]}(v) = \{123\}$ and $S_{[1]}(v) = 19$. Efficiency requires trade with sellers 1, 2 and 3 if $v_0 > 19$.

3.3 Main Result

In the case $q = 1$, i.e., there is only one feasible path, it is clear that the problem reduces to the $n = k$ case in Chapter 2. Hence Theorem 1 applies: there does not exist a successful mechanism for any prior. According to Theorem 3 below, the impossibility result breaks down when $q > 1$. Discussion and interpretation of the conditions in the Theorem is postponed till Subsection 3.3.1.

THEOREM **3** *Let $\langle \Gamma, k, \mu \rangle$ be an LAC with $q > 1$.*

I. Suppose μ satisfies the following condition:

$$\underline{v}_0 \geq E\left(\sum_{i\in\mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i})\right) - (k-1)E\left(S_{[1]}(v)\right). \tag{3.1}$$

Then there exists a successful mechanism with respect to μ.

II. Suppose there exists a successful mechanism with respect to μ. Then the following holds:

$$\underline{v}_0 > E\left(\sum_{i\in\mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i}) - (k-1)S_{[1]}(v) \middle| v \in \tilde{V}\right), \tag{3.2}$$

where

$$\tilde{V} = \{v \in [\underline{v}_0, \bar{v}_0] \times [\underline{v}, \bar{v}]^n : \underline{v}_0 > S_{[1]}(v) \textit{ and } v_0 > S_{[1]}(\bar{v}, v_{-i}) \textit{ for all } i \in \mathcal{P}_{[1]}(v)\}.$$

Proof: We rely again on Proposition 1 and compute the expected value of the sum of VCG payments.

In Part I, we will show that for all v,

$$\sum_{j=0}^{n} t_j^V \geq \underline{v}_0 - \sum_{i\in\mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i}) + (k-1)S_{[1]}(v).$$

If (3.1) holds, then the expected sum of payments is non-negative and the result follows by an application of Proposition 1.

The VCG payment of the buyer is given by Lemma 6.

Lemma **6** *The VCG payment of the buyer is given by*

$$t_0^V(v) = \begin{cases} \underline{v}_0 & \textit{if } v_0 \geq \underline{v}_0 > S_{[1]}(v), \\ S_{[1]}(v) & \textit{if } v_0 > S_{[1]}(v) \geq \underline{v}_0, \\ 0 & \textit{if } \underline{v}_0 \leq v_0 \leq S_{[1]}(v). \end{cases}$$

Proof: If $v_0 \geq \underline{v}_0 > S_{[1]}(v)$,

$$SW(\underline{v}_0, v_{-0}) = \underline{v}_0 - S_{[1]}(v),$$
$$\text{and } SW_{-0}(v_0, v_{-0}) = -S_{[1]}(v).$$
$$\text{Hence } t_0^V(v) = \underline{v}_0.$$

If $v_0 > S_{[1]}(v) \geq \underline{v}_0$,

$$\begin{aligned} SW(\underline{v}_0, v_{-0}) &= 0, \\ \text{and } SW_{-0}(v_0, v_{-0}) &= -S_{[1]}(v). \\ \text{Hence } t_0^V(v) &= S_{[1]}(v). \end{aligned}$$

If $v_0 \leq S_{[1]}(v)$,

$$\begin{aligned} SW(\underline{v}_0, v_{-0}) &= 0, \\ \text{and } SW_{-0}(v_0, v_{-0}) &= 0. \\ \text{Hence } t_0^V(v) &= 0. \end{aligned}$$

■

For the VCG payment of the sellers, we refer to the next two Lemmas.

LEMMA **7** *The VCG payment of any seller $i \in \mathcal{P}_{[j]}(v), j > 1$, is 0.*

Proof: Let $v_0 > S_{[1]}(v)$. For any seller $i \in \mathcal{P}_{[j]}(v), j > 1$,

$$\begin{aligned} SW(\bar{v}, v_{-i}) &= v_0 - S_{[1]}(v), \\ \text{and } SW_{-i}(v_i, v_{-i}) &= v_0 - S_{[1]}(v). \\ \text{Hence } t_i^V(v) &= 0. \end{aligned}$$

If $v_0 \leq S_{[1]}(v)$,

$$\begin{aligned} SW(\bar{v}, v_{-i}) &= 0, \\ \text{and } SW_{-i}(v_i, v_{-i}) &= 0. \\ \text{Hence } t_i^V(v) &= 0. \end{aligned}$$

■

LEMMA **8** *The VCG payment of any seller $i \in \mathcal{P}_{[1]}(v)$, is given by*

$$t_i^V(v) = \begin{cases} -(v_0 - S_{[1]}(v) + v_i) & \text{if } S_{[1]}(v) < v_0 \leq S_{[1]}(\bar{v}, v_{-i}), \\ -(S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v) + v_i) & \text{if } S_{[1]}(v) \leq S_{[1]}(\bar{v}, v_{-i}) < v_0, \\ 0 & \text{if } v_0 \leq S_{[1]}(v) \leq S_{[1]}(\bar{v}, v_{-i}). \end{cases}$$

Proof: If $S_{[1]}(v) < v_0 \leq S_{[1]}(\bar{v}, v_{-i})$,

$$\begin{aligned} SW(\bar{v}, v_{-i}) &= 0, \\ \text{and } SW_{-i}(v_i, v_{-i}) &= v_0 - S_{[1]}(v) + v_i. \\ \text{Hence } t_i^V(v) &= -(v_0 - S_{[1]}(v) + v_i). \end{aligned}$$

If $S_{[1]}(v) \leq S_{[1]}(\bar{v}, v_{-i}) < v_0$,

$$\begin{aligned} SW(\bar{v}, v_{-i}) &= v_0 - S_{[1]}(\bar{v}, v_{-i}), \\ \text{and } SW_{-i}(v_i, v_{-i}) &= v_0 - S_{[1]}(v) + v_i. \\ \text{Hence } t_i^V(v) &= -(S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v) + v_i). \end{aligned}$$

If $v_0 \leq S_{[1]}(v) \leq S_{[1]}(\bar{v}, v_{-i})$,

$$\begin{aligned} SW(\bar{v}, v_{-i}) &= 0, \\ \text{and } SW_{-i}(v_i, v_{-i}) &= 0. \\ \text{Hence } t_i^V(v) &= 0. \end{aligned}$$

■

When $q > 1$, let $A(v) = \{h \in \mathcal{P}_{[1]} : S_{[1]}(v) < v_0 \leq S_{[1]}(\bar{v}, v_{-i})\}$. As before, $A(v)$ represents the set of *trade-pivotal* sellers at profile v: they influence the possibility of trade by reporting their highest valuation.

Then we have the following cases for $\sum_{j=0}^{n} t_j^V(v)$.

Case I: $v_0 \geq \underline{v}_0 > S_{[1]}(v)$ and $A(v) \neq \emptyset$

$$\begin{aligned} \sum_{j=0}^{n} t_j^V(v) &= \underline{v}_0 - \sum_{i \in A(v)} \big(v_0 - S_{[1]}(v) + v_i\big) - \sum_{\substack{i \in \mathcal{P}_{[1]}(v) \\ i \notin A(v)}} \big(S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v) + v_i\big) \\ &= \underline{v}_0 - |A(v)| \big(v_0 - S_{[1]}(v)\big) - \sum_{i \in A(v)} v_i - \sum_{\substack{i \in \mathcal{P}_{[1]}(v) \\ i \notin A(v)}} \big(S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v)\big) - \sum_{\substack{i \in \mathcal{P}_{[1]}(v) \\ i \notin A(v)}} v_i \\ &= \underline{v}_0 - S_{[1]}(v) - |A(v)| \big(v_0 - S_{[1]}(v)\big) - \sum_{\substack{i \in \mathcal{P}_{[1]}(v) \\ i \notin A(v)}} \big(S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v)\big) \\ &\leq 0. \end{aligned}$$

The first equality follows from Lemmas 6, 7 and 8. The second follows by taking the v_i terms out from the parentheses. The next equality sums these v_i terms up to $S_{[1]}(v)$. The inequality follows because $\underline{v}_0 \leq v_0$ and $A(v)$ is nonempty.

Case II: $v_0 \geq \underline{v}_0 > S_{[1]}(v)$ and $A(v) = \emptyset$

$$\begin{aligned}\sum_{j=0}^{n} t_j^V(v) &= \underline{v}_0 - \sum_{i \in \mathcal{P}_{[1]}(v)} \big(S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v) + v_i\big) \\ &= \underline{v}_0 - \sum_{i \in \mathcal{P}_{[1]}(v)} \big(S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v)\big) - S_{[1]}(v) \\ &= \underline{v}_0 + (k-1)S_{[1]}(v) - \sum_{i \in \mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i})\end{aligned}$$

The first equality follows from Lemmas 6, 7 and 8. The second follows by taking out the v_i terms out of the parentheses. The third follows by taking out the $S_{[1]}(v)$ terms out of the parentheses and collecting them together. The sign of the resulting expression can be positive or negative, as examples will show.

Case III: $v_0 > S_{[1]}(v) \geq \underline{v}_0$ and $A(v) \neq \emptyset$

$$\begin{aligned}\sum_{j=0}^{n} t_j^V(v) &= S_{[1]}(v) - \sum_{i \in A(v)} \big(v_0 - S_{[1]}(v) + v_i\big) - \sum_{\substack{i \in \mathcal{P}_{[1]}(v) \\ i \notin A(v)}} \big(S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v) + v_i\big) \\ &= S_{[1]}(v) - |A(v)| \big(v_0 - S_{[1]}(v)\big) - \sum_{i \in A(v)} v_i - \sum_{\substack{i \in \mathcal{P}_{[1]}(v) \\ i \notin A(v)}} \big(S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v)\big) - \sum_{\substack{i \in \mathcal{P}_{[1]}(v) \\ i \notin A(v)}} v_i \\ &= -|A(v)| \big(v_0 - S_{[1]}(v)\big) - \sum_{\substack{i \in \mathcal{P}_{[1]}(v) \\ i \notin A(v)}} \big(S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v)\big) \\ &< 0.\end{aligned}$$

The first equality follows from Lemmas 6, 7 and 8. The second follows by taking out the v_i terms out of the parentheses. The third follows after canceling out the $S_{[1]}(v)$'s. The resulting expression is nonpositive since $v_0 > S_{[1]}(v)$ and $S_{[1]}(\bar{v}, v_{-i}) \geq S_{[1]}(v)$.

Case IV: $v_0 > S_{[1]}(v) \geq \underline{v}_0$ and $A(v) = \emptyset$

$$\begin{aligned}\sum_{j=0}^{n} t_j^V(v) &= S_{[1]}(v) - \sum_{i \in \mathcal{P}_{[1]}(v)} \big(S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v)\big) - S_{[1]}(v) \\ &= - \sum_{i \in \mathcal{P}_{[1]}(v)} \big(S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v)\big) \\ &\leq 0.\end{aligned}$$

The first equality follows from Lemmas 6, 7 and 8. The second follows by taking out the v_i terms out of the parentheses. The third follows after canceling out the $S_{[1]}(v)$'s. The resulting expression is nonpositive since $S_{[1]}(\bar{v}, v_{-i}) \geq S_{[1]}(v)$.

Case V: $v_0 \leq S_{[1]}(v)$

$$\sum_{j=0}^{n} t_j^V(v) = 0.$$

These cases are summarized in Table 3.1 .

Table 3.1: Sum of Payments when $n > k$

Case	**Sum of Payments**	**Sign**
I: $v_0 \geq \underline{v}_0 > S_{[1]}(v), A(v) \neq \emptyset$	$\underline{v}_0 - S_{[1]}(v) - \lvert A(v)\rvert(v_0 - S_{[1]}(v)) - \sum_{i \in \mathcal{P}_{[1]}(v), i \notin A(v)} (S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v))$	≤ 0
II: $v_0 \geq \underline{v}_0 > S_{[1]}(v), A(v) = \emptyset$	$\underline{v}_0 + (k-1)S_{[1]}(v) - \sum_{i \in \mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i})$	$\lesseqgtr 0$
III: $v_0 > S_{[1]}(v) \geq \underline{v}_0, A(v) \neq \emptyset$	$-\lvert A(v)\rvert(v_0 - S_{[1]}(v)) - \sum_{i \in \mathcal{P}_{[1]}(v), i \notin A(v)} (S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v))$	< 0
IV: $v_0 > S_{[1]}(v) \geq \underline{v}_0, A(v) = \emptyset$	$-\sum_{i \in \mathcal{P}_{[1]}(v)} (S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v))$	≤ 0
V: $v_0 \leq S_{[1]}(v)$	0	0

Lemma **9** *For all v,*

$$\sum_{j=0}^{n} t_j^V(v) \geq \underline{v}_0 + (k-1)S_{[1]}(v) - \sum_{i \in \mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i}).$$

Proof: For $i \in A(v)$, $v_0 \leq S_{[1]}(\bar{v}, v_{-i})$. Therefore, in Case I of Table 3.1,

$$\begin{aligned}\sum_{j=0}^{n} t_j^V(v) &= \underline{v}_0 - S_{[1]}(v) - \lvert A(v)\rvert(v_0 - S_{[1]}(v)) - \sum_{\substack{i \in \mathcal{P}_{[1]}(v) \\ i \notin A(v)}} (S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v)) \\ &\geq \underline{v}_0 - S_{[1]}(v) - \sum_{i \in \mathcal{P}_{[1]}(v)} (S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v)) \\ &= \underline{v}_0 + (k-1)S_{[1]}(v) - \sum_{i \in \mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i}).\end{aligned}$$

In Case II, $\sum_{j=0}^{n} t_j^V(v) = \underline{v}_0 + (k-1)S_{[1]}(v) - \sum_{i\in\mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i})$.

In Case III,

$$\begin{aligned}
\sum_{j=0}^{n} t_j^V(v) &= -|A(v)|(v_0 - S_{[1]}(v)) - \sum_{\substack{i\in\mathcal{P}_{[1]}(v)\\ i\notin A(v)}} (S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v)) \\
&= S_{[1]}(v) - \sum_{i\in A(v)} \big(v_0 - S_{[1]}(v) + v_i\big) - \sum_{\substack{i\in\mathcal{P}_{[1]}(v)\\ i\notin A(v)}} \big(S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v) + v_i\big) \\
&\geq S_{[1]}(v) - \sum_{i\in\mathcal{P}_{[1]}(v)} \big(S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v) + v_i\big) \\
&\geq \underline{v}_0 + (k-1)S_{[1]}(v) - \sum_{i\in\mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i}).
\end{aligned}$$

In Case IV,

$$\begin{aligned}
\sum_{j=0}^{n} t_j^V(v) &= - \sum_{i\in\mathcal{P}_{[1]}(v)} (S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v)) \\
&= S_{[1]}(v) - \sum_{i\in\mathcal{P}_{[1]}(v)} \big(S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v)\big) - S_{[1]}(v) \\
&\geq \underline{v}_0 + (k-1)S_{[1]}(v) - \sum_{i\in\mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i}).
\end{aligned}$$

In Case V,

$$\begin{aligned}
\sum_{j=0}^{n} t_j^V(v) = 0 &\geq \underline{v}_0 - S_{[1]}(v) \\
&\geq \underline{v}_0 - S_{[1]}(v) - \sum_{i\in\mathcal{P}_{[1]}(v)} \big(S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v)\big) \\
&= \underline{v}_0 + (k-1)S_{[1]}(v) - \sum_{i\in\mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i}).
\end{aligned}$$

■

If (3.1) holds, Lemma 9 implies

$$E\left(\sum_{j=0}^{n} t_j^V(v)\right) \geq \underline{v}_0 + (k-1)E\big(S_{[1]}(v)\big) - E\left(\sum_{i\in\mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i})\right) \geq 0.$$

Part I now follows by Proposition 1.

For part II, note that the profiles in $\tilde{V}$ correspond to Case II of Table 3.1 and all other profiles correspond to the other Cases. If a successful mechanism exists then by Proposition 1,

$$\begin{aligned} 0 &\leq E\left(\sum_{j=0}^{n} t_j^V(v)\right) \\ &= E\left(\sum_{j=0}^{n} t_j^V(v)\middle| v \in \tilde{V}\right) \times \Pr\left(v \in \tilde{V}\right) \\ &+ E\left(\sum_{j=0}^{n} t_j^V(v)\middle| v \notin \tilde{V}\right) \times \Pr\left(v \notin \tilde{V}\right) \end{aligned}$$

The second component of this sum of products is negative since $\sum_{j=0}^{n} t_j^V(v)$ takes negative or zero value when $v \notin \tilde{V}$. Therefore,

$$E\left(\sum_{j=0}^{n} t_j^V(v)\middle| v \in \tilde{V}\right) > 0$$

Since $\sum_{j=0}^{n} t_j^V(v) = \underline{v}_0 + (k-1)S_{[1]}(v) - \sum_{i \in \mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i})$ when $v \in \tilde{V}$, the claim follows.
■

3.3.1 Discussion and Examples

Here we elaborate on the notion of *trade-pivotality* and use it to provide an interpretation of Theorem 3. We also provide another interpretation of this result using the notion of *rent* or *surplus* earned by agents in the VCG mechanism. Further, we offer some numerical examples to show application of the sufficient condition in Part I of Theorem 3.

Suppose efficiency requires trade to take place at a profile v. Recall that a successful seller $i \in \mathcal{P}_{[1]}(v)$ is trade-pivotal at v if trade does not take place at $(\bar{v}, v_{-i})$, i.e., when seller i reports his highest possible valuation. A successful seller is not trade-pivotal if trade continues to take place at $(\bar{v}, v_{-i})$. We illustrate trade-pivotality with the help of two examples for different valuation profiles for the contiguity structure described in Figure 3.5. As before (Figure 3.5), $n = 4$, $k = 3$ and $q = 2$. However the supports of the prior distributions are as follows: $\underline{v}_0 = 25$, $\bar{v}_0 = 35$, $\underline{v} = 0$, $\bar{v} = 10$. Let $v_0 = 26$.

Recall that sellers 1, 2 and 3 trade at v. If seller 1's valuation is 10 instead of 3, the sum of the valuations on paths $\{123\}$ and $\{234\}$ are 28 and 26 respectively. Hence trade does not take place at $(10, v_{-1})$, i.e., seller 1 is trade-pivotal at v. But sellers 2 and 3 are not trade-pivotal at v: if seller 2 has a valuation of 10, the sum of valuations on $\{123\}$ is 20 and

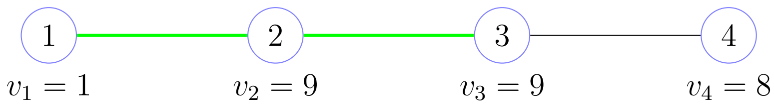

Figure 3.6: Feasible path $\mathcal{P}_{[1]}(v)$ is in green

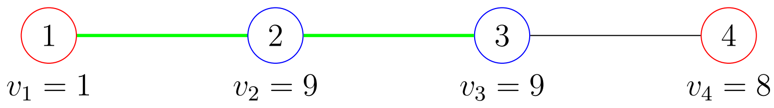

Figure 3.7: Pivotal sellers and non-pivotal sellers at v

trade can take place at $(10, v_{-2})$; same follows for seller 3. Pivotal and non-pivotal sellers and the efficient feasible path is shown in Figure 3.7.

Now consider the profile $v'_0 = 28$, , $v'_1 = 1$, $v'_2 = 2$, $v'_3 = 3$ and $v'_4 = 2$. Trade takes place at v' with sellers 1, 2 and 3. Note that trade also takes place when the buyer's valuation is the lowest possible, i.e., $\underline{v}_0 = 25$. Furthermore, no successful seller at v' is trade-pivotal. This is illustrated in Figure 3.8.

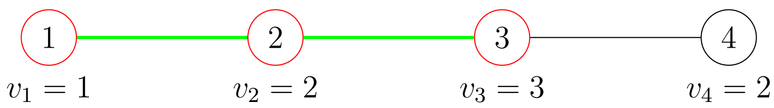

Figure 3.8: Pivotal sellers at v'

Returning to the statement of Theorem 3, the set $\tilde{V}$ is the set of profiles v such that (i) trade takes place at $(\underline{v}_0, v_{-0})$ and therefore also at v, and (ii) all successful sellers are non-pivotal at v. Hence, v' belongs to $\tilde{V}$ but $v \notin \tilde{V}$.

Pick $v \in \tilde{V}$ and a successful seller i. Suppose i's valuation changes to $\bar{v}$. Since i is not trade-pivotal, trade still takes place and the sum of valuations of the successful sellers in the profile $(\bar{v}, v_{-i})$ is $S_{[1]}(\bar{v}, v_{-i})$. The sum of valuations of all other successful sellers at v is $S_{[1]}(v) - v_i$. The difference of these two terms, summed over all successful sellers, is $\sum_{i \in \mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i}) - (k-1) S_{[1]}(v)$[1]. Part I of Theorem 3 states that there exists a successful mechanism if the expectation of this term is at most $\underline{v}_0$. Part II states that if there exists a successful mechanism, then the expectation of this term, conditional on the profile belonging to $\tilde{V}$, is less than $\underline{v}_0$.

By Proposition 1, a successful mechanism exists if and only if the expected sum of VCG payments is non-negative. In Lemma 9, we showed that the sum of VCG payments is bounded below by $\underline{v}_0 - \sum_{i \in \mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i}) + (k-1) S_{[1]}(v)$ at any profile v. Part I of Theorem 3 is

[1] In the example shown in Figure 3.8, this term is (10+2+3)+ (1+10+3)+(1+2+10)- 2(1+2+3)= 30.

a direct consequence of this Lemma. For Part II, note that the set $\tilde{V}$ corresponds to Case II in Table 3.1. For all $v \in \tilde{V}$, $\underline{v}_0 > S_{[1]}(v)$, i.e., trade takes place at $(\underline{v}_0, v_{-0})$ and therefore, also at v. Furthermore, $v_0 > S_{[1]}(\bar{v}, v_{-i})$ for all $i \in \mathcal{P}_{[1]}(v)$: trade takes place if any seller i, who is successful at v, changes his valuation to $\bar{v}$. Therefore, no successful seller at v is trade-pivotal, i.e., the set $A(v)$ is empty. According to Table 3.1, the sum of VCG payments, $\underline{v}_0 - \sum_{i \in \mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i}) + (k-1)S_{[1]}(v)$, can be positive only at such profiles. For all other profiles the sum of VCG payments is non-positive. It follows that the conditional expectation in Condition 3.2 must be positive in order for the expected sum of VCG payments to be non-negative.

Conditions (3.1) and (3.2) are counterparts of Conditions (2.4) and (2.5) in Chapter 2. In the model in Chapter 2, there is no notion of contiguity and any set of k plots is feasible. Efficiency requires trade with the k lowest valuation sellers. Consider a profile v where all successful sellers are non-pivotal. If any successful seller at v now switches to the highest valuation, then the set of sellers with the next k lowest valuations are successful. Therefore, $S_{[1]}(v)$ and $\sum_{i \in \mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i})$ are the counterparts of $\sum_{j=1}^{k} v_{[j]}$ and $\sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k+1} v_j$ respectively. Therefore, $\sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k+1} v_j - (k-1) \sum_{j=1}^{k} v_{[j]}$ or $k v_{k+1}$ is the counterpart of $\sum_{i \in \mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i}) - (k-1)S_{[1]}(v)$ in this chapter.

Lemma 8 shows the VCG payments received by successful sellers. Fix a profile where trade takes place, i.e., $v_0 > S_{[1]}(v)$. Recall that a seller is trade-pivotal if he can change the trade decision by reporting his highest possible valuation, i.e., $v_0 \leq S_{[1]}(\bar{v}, v_{-i})$. Such a seller receives a VCG rent of $v_0 - S_{[1]}(v)$. In contrast, a seller who is not trade-pivotal, receives a VCG rent of $S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v)$. Since for a seller who is not trade-pivotal, $S_{[1]}(v) \leq S_{[1]}(\bar{v}, v_{-i}) < v_0$, his VCG rent is less than that of a trade-pivotal seller. In the illustration corresponding to Figure 3.7 above, seller 1, who is trade-pivotal receives a VCG rent of 7. Sellers 2 and 3 who are not trade-pivotal receive VCG rents of 1 each.

Similarly, a buyer can be trade-pivotal if she can change the trade decision by reporting her lowest possible valuation, i.e., $\underline{v}_0 \leq S_{[1]}(v) < v_0$. Such a buyer receives a VCG rent of $v_0 - S_{[1]}(v)$. In contrast, a buyer who is not trade-pivotal, receives a VCG rent of $v_0 - \underline{v}_0$. In the illustration above, the buyer is not trade-pivotal since trade can take place at her lowest valuation, 25.

By adding $E(v_0)$ on both sides of the sufficiency condition and re-arranging terms, we get

$$E(v_0) - \underline{v}_0 + E\left(\sum_{i \in \mathcal{P}_{[1]}(v)} \left(S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v)\right)\right) \leq E\left(v_0 - S_{[1]}(v)\right), \tag{3.3}$$

i.e., expected gains from trade must be larger than the expected sum of rents earned by $k+1$ non-trade pivotal agents. Since gains from trade can be of either sign and rents are non-negative, this condition implies that whenever there is positive gains from trade, such gains must be relatively large.

It is straightforward to find a complete characterization of the priors for which successful mechanisms exist by looking at Table 3.1. It presents the possible sum of VCG payments when $q > 1$ for partitions of the set of valuation profiles, marked I-V. By Proposition 1, the expected sum of VCG payments have to be positive for the existence of a successful mechanism. Note that only in Case II, i.e., when no successful seller is trade-pivotal, the sum of VCG payments *may* be positive. Therefore, a necessary and sufficient condition for the existence of a successful mechanism is described as follows: the integral of the sum of VCG payments weighted by probability densities over partition II must exceed the negative of that over all other partitions. We have decomposed this condition into two separate conditions for the sake of simplicity and ease of interpretation.

With increase in k, the expression on the left hand side of this inequality becomes smaller and the components of the sum on the right hand side increase in number. Although an exact characterization of this comparative static effect is not possible without fixing the prior, the surplus condition becomes harder to satisfy here as well.

The next two examples provide explicit calculations for the sufficient condition in the case of line and star contiguity structures in special cases.

Example 7 Consider the line contiguity structure as in Figure 3.8 with $n = 4$ and $k = 3$. Assume that the valuations of the sellers are uniformly and independently distributed on [0,10]. Let $v^{14}_{[1]} = \min\{v_1, v_4\}$ and $v^{14}_{[2]} = \max\{v_1, v_4\}$. Then

$$\begin{aligned}
S_{[1]}(v) &= v^{14}_{[1]} + v_2 + v_3,\\
S_{[1]}(10, v^{14}_{[2]}, v_2, v_3) &= v^{14}_{[2]} + v_2 + v_3,\\
S_{[1]}(v^{14}_{[1]}, v^{14}_{[2]}, 10, v_3) &= v^{14}_{[1]} + 10 + v_3,\\
S_{[1]}(v^{14}_{[1]}, v^{14}_{[2]}, v_2, 10) &= v^{14}_{[1]} + 10 + v_2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{i\in\mathcal{P}_{[1]}(v)} S_{[1]}(10, v_{-i}) - (k-1)S_{[1]}(v)\\
&= (v^{14}_{[2]} + v_2 + v_3) + (v^{14}_{[1]} + 10 + v_3) + (v^{14}_{[1]} + 10 + v_2) - 2(v^{14}_{[1]} + v_2 + v_3)\\
&= v^{14}_{[2]} + 20.
\end{aligned}$$

Since, $E(v^{14}_{[2]}) = \int_0^{10} \frac{x^2}{50} dx = \frac{20}{3}$, Condition (3.1) is satisfied if $\underline{v}_0 \geq 26\frac{2}{3}$. Note that Assumption NT requires $\underline{v}_0 < 30$.

Example 8 Consider the line contiguity structure as in Figure 3.9 with $n = 4$ and $k = 3$. Assume that the valuations of the sellers are uniformly and independently distributed on [0,10]. Let $v^{234}_{[i]}$, $i = 1, 2, 3$, represent the i-th lowest order statistic of v_2, v_3 and v_4. Then

$$\begin{aligned} S_{[1]}(v) &= v^{234}_{[1]} + v^{234}_{[2]} + v_1, \\ S_{[1]}(10, v^{234}_{[2]}, v^{234}_{[3]}, v_1) &= v^{234}_{[2]} + v^{234}_{[3]} + v_1, \\ S_{[1]}(v^{234}_{[1]}, 10, v^{234}_{[3]}, v_1) &= v^{234}_{[1]} + v^{234}_{[3]} + v_1, \\ S_{[1]}(v^{234}_{[1]}, v^{234}_{[2]}, v^{234}_{[3]}, 10) &= v^{234}_{[1]} + v^{234}_{[2]} + 10. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{i \in \mathcal{P}_{[1]}(v)} S_{[1]}(10, v_{-i}) - (k-1) S_{[1]}(v) \\ &= (v^{234}_{[2]} + v^{234}_{[3]} + v_1) + (v^{234}_{[1]} + v^{234}_{[3]} + v_1) + (v^{234}_{[1]} + v^{234}_{[2]} + 10) - 2(v^{234}_{[1]} + v^{234}_{[2]} + v_1) \\ &= 2v^{234}_{[3]} + 10 \end{aligned}$$

As $E(v^{234}_{[3]}) = \int_0^{10} \frac{3x^3}{1000} dx = \frac{15}{2}$, condition (3.1) is satisfied for all $\underline{v}_0 \geq 10 + 2 \times \frac{15}{2} = 25$. Therefore, there exist BIC mechanisms for $n = 4, k = 3, v_i \overset{\text{iid}}{\sim} U[0, 10]$, and 1, 2, 3, 4 aligned on a star graph, if the buyer has a valuation of at least 25.

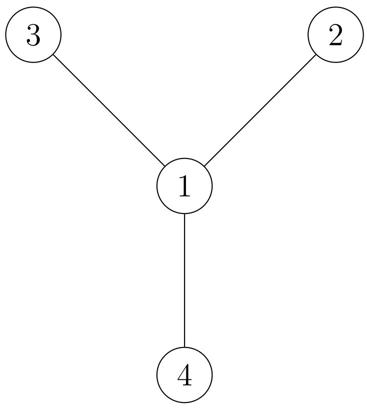

Figure 3.9: The star graph with four nodes

3.4 Critical Sellers

In this section, we show that *critical sellers* play an important role in the existence of successful mechanisms. The higher the number of critical sellers, the more difficult it is for a prior to satisfy the conditions for existence. This is explained by the fact that critical sellers extract the highest possible information rent in the VCG mechanism since they are present in all feasible paths in a graph. However, since not all sellers can possibly become critical, the set of priors for which successful mechanisms exist is non-empty.

Consider an LAC $\langle \Gamma, k, \mu \rangle$. A seller i is *critical* if $i \in \mathcal{P}_j$ for all j. Thus sellers 2 and 3 are critical in Figure 3.8 while seller 1 is critical in Figure 3.9. Let $c(\Gamma)$ denote the set of critical sellers in $\langle \Gamma, k, \mu \rangle$. Note that $c(\Gamma)$ depends only on Γ and not on valuations (unlike pivotal sellers). Also, $q > 1$ implies $|c(\Gamma)| \leq k - 1$.

Conditions (3.1) and (3.2) can be reformulated to account for critical nodes.

THEOREM 4 *Let $\langle \Gamma, k, \mu \rangle$ be an LAC with $q > 1$.*

I. Suppose μ satisfies the following condition:

$$\underline{v}_0 \geq |c(\Gamma)|\bar{v} + E\left(\sum_{i \in \mathcal{P}_{[1]}(v) \setminus c(\Gamma)} \left(S_{[1]}(\bar{v}, v_{-i}) + v_i \right) - (k - |c(\Gamma)|) S_{[1]}(v) \right). \tag{3.4}$$

Then there exists a successful mechanism with respect to μ.

II. Suppose there exists a successful mechanism with respect to μ. Then the following holds:

$$\underline{v}_0 > |c(\Gamma)|\bar{v} + E\left(\sum_{i \in \mathcal{P}_{[1]}(v) \setminus c(\Gamma)} \left(S_{[1]}(\bar{v}, v_{-i}) + v_i \right) - (k - |c(\Gamma)|) S_{[1]}(v) \middle| v \in \tilde{V} \right) \tag{3.5}$$

where

$$\tilde{V} = \{ v \in [\underline{v}_0, \bar{v}_0] \times [\underline{v}, \bar{v}]^n : \underline{v}_0 > S_{[1]}(v) \text{ and } v_0 > S_{[1]}(\bar{v}, v_{-i}) \text{ for all } i \in \mathcal{P}_{[1]}(v) \}$$

Proof: We will show that, for all v,

$$\begin{aligned} &\sum_{i \in \mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i}) - (k-1) S_{[1]}(v) \\ &= |c(\Gamma)|\bar{v} + \sum_{i \in \mathcal{P}_{[1]}(v) \setminus c(\Gamma)} \left(S_{[1]}(\bar{v}, v_{-i}) + v_i \right) - (k - |c(\Gamma)|) S_{[1]}(v). \end{aligned}$$

Since critical nodes must be present in all feasible paths, i.e., $c(\Gamma) \subset \mathcal{P}_j$ for all j, we must have $c(\Gamma) \subset \mathcal{P}_{[1]}(v)$ for all v. It follows that for all v,

$$S_{[1]}(v) = \sum_{i \in \mathcal{P}_{[1]}(v)} v_i = \sum_{i \in c(\Gamma)} v_i + \sum_{i \in \mathcal{P}_{[1]}(v) \setminus c(\Gamma)} v_i. \tag{3.6}$$

Further, since $c(\Gamma) \subset \mathcal{P}_j$ for all j, a change in the valuation of a seller corresponding to a critical node affects the sum of valuations on all feasible paths identically. Therefore, $\mathcal{P}_{[1]}(v) = \mathcal{P}_{[1]}(\bar{v}, v_{-i})$ for any critical node $i \in c(\Gamma)$. Then, for any $i \in c(\Gamma)$,

$$S_{[1]}(\bar{v}, v_{-i}) = \bar{v} + \sum_{i \in \mathcal{P}_{[1]}(\bar{v}, v_{-i})} v_i = \bar{v} + \sum_{j \in \mathcal{P}_{[1]}(v) \setminus \{i\}} v_j = \bar{v} + S_{[1]}(v) - v_i. \tag{3.7}$$

Therefore,

$$\begin{aligned}
&\sum_{i \in \mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i}) - (k-1) S_{[1]}(v) \\
&= \sum_{i \in \mathcal{P}_{[1]}(v)} \big(S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v)\big) + S_{[1]}(v) \\
&= \sum_{i \in c(\Gamma)} \big(S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v) + v_i\big) + \sum_{i \in \mathcal{P}_{[1]}(v) \setminus c(\Gamma)} \big(S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v) + v_i\big) \\
&= |c(\Gamma)| \bar{v} + \sum_{i \in \mathcal{P}_{[1]}(v) \setminus c(\Gamma)} \big(S_{[1]}(\bar{v}, v_{-i}) + v_i\big) - (k - |c(\Gamma)|) S_{[1]}(v).
\end{aligned}$$

The first equality follows from $|\mathcal{P}_{[1]}(v)| = k$. The second equality follows from $c(\Gamma) \subset \mathcal{P}_{[1]}(v)$ and (3.6). The final equality follows from (3.7) and $|\mathcal{P}_{[1]}(v) \setminus c(\Gamma)| = k - |c(\Gamma)|$. ■

Corollary 2 *Suppose there exists a successful mechanism with respect to μ. Then*

$$\underline{v}_0 > |c(\Gamma)| \bar{v}. \tag{3.8}$$

Proof: If a seller increases his valuation from v_i to $\bar{v}$, the sum of valuations on no feasible path can decrease. Therefore,

$$S_{[1]}(v) \leq S_{[1]}(\bar{v}, v_{-i}).$$

Consequently,

$$\begin{aligned}
&\sum_{i \in \mathcal{P}_{[1]}(v) \setminus c(\Gamma)} \big(S_{[1]}(\bar{v}, v_{-i}) + v_i\big) - (k - |c(\Gamma)|) S_{[1]}(v) \\
&= \sum_{i \in \mathcal{P}_{[1]}(v) \setminus c(\Gamma)} \big(S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v) + v_i\big) \\
&\geq \sum_{i \in \mathcal{P}_{[1]}(v) \setminus c(\Gamma)} v_i \\
&\geq 0, \\
\Rightarrow & E\left(\sum_{i \in \mathcal{P}_{[1]}(v) \setminus c(\Gamma)} \big(S_{[1]}(\bar{v}, v_{-i}) + v_i\big) - (k - |c(\Gamma)|) S_{[1]}(v) \Big| v \in \tilde{V} \right) \geq 0.
\end{aligned}$$

Hence, if $\underline{v} \leq |c(\Gamma)|\bar{v}$, Condition (3.5) cannot hold. ■

Theorem 4 and Corollary 1 highlight the importance of critical nodes. The count of such nodes puts a lower bound on the support of the buyer's valuation essential for the existence of a successful mechanism. We have observed that not all nodes on a feasible path can become critical nodes. Therefore, the existence of critical nodes does not preclude the attainability of the first best.

A critical seller who is not trade-pivotal earns a VCG rent of $\bar{v}$. In contrast, a critical seller who is trade-pivotal earns a VCG rent of $v_0 - S_{[1]}(v)$. For a fixed profile with sellers of both characteristics, the former gets less rent than the latter.

If a successful mechanism exists for prior μ then Corollary 2 implies that the restriction on the buyer's valuation becomes tighter as the number of critical sellers increases. The following comparative static result on the number of critical nodes may seem plausible.

> Fix n, k and μ. Let Γ and Γ' be two graphs with $|c(\Gamma)| \geq |c(\Gamma')|$. Let μ be a prior such that a successful mechanism exists for $\langle \Gamma, k, \mu \rangle$. Then a successful mechanism must exist for $\langle \Gamma', k, \mu \rangle$.

This conjecture is verified in Examples 7 and 8. At this moment we are unable to establish it for general distributions and contiguity structures.

3.5 Conclusion

In this chapter, we have modeled contiguity structures as a graph. The possibility results of the earlier chapter continue to remain valid in this model. The analysis highlights the importance of critical sellers.

Chapter 4

Asymptotics

4.1 Introduction

According to standard microeconomic theory, the *market power* of individual sellers declines as the number of sellers increases. A classic and extreme example of this is the comparison between standard monopoly and Bertrand duopoly: in the former, the market price is above the marginal cost, but in the latter, competition between two identical sellers drives market price down to marginal cost of production. In models of private information, mechanisms may fail to satisfy one or more of the Myerson-Satterthwaite criteria like ex post efficiency, IIR or BB. A natural question is whether a mechanism converges to first best (or some weakening of the first best notion) as the number of agents become large. This question has been investigated extensively in the literature[1]. Roughly speaking, in such models, intensifying competition weakens the influence of agents' private information. Consequently, incentive compatible mechanisms begin to approximate the first best. The validity of this reasoning depends on the setting. For instance, this is not true in problems involving public goods. See the review article by Jackson (2000) for elucidation.

In the earlier chapters we found necessary and sufficient conditions on priors for which successful mechanisms can be constructed. These mechanisms are efficient, BIC, IIR and BB. However, the use of such mechanisms requires the social planner to have precise information about the underlying priors. There has been emphasis on the construction of mechanisms that are robust with respect to such assumptions following a critique by Wilson (1987)[2]. A natural way to deal with such a problem is to require mechanisms to be *dominant strategy incentive compatible*, or DSIC: a mechanism (P, t) is DSIC if for all j,

$$v_j P_j(v_j, v_{-j}) - t_j(v_j, v_{-j}) \geq v_j P_j(\hat{v}_j, v_{-j}) - t_j(\hat{v}_j, v_{-j}) \text{ for all } v_j, \hat{v}_j, \text{ and } v_{-j}.$$

In other words, truthful reporting is a weakly dominant strategy for every agent. The

[1] Satterthwaite and Williams (1989a); Gresik and Satterthwaite (1989); Williams (1991); McAfee (1992); Rustichini et al. (1994); Satterthwaite and Williams (2002); Cripps and Swinkels (2006)

[2] See Bergemann and Välimäki (2006) for a survey.

corresponding participation condition is *ex-post individual rationality*, or IR: a mechanism is IR if for all j,

$$v_j P_j(v_j, v_{-j}) - t_j(v_j, v_{-j}) \geq 0 \text{ for all } v_j.$$

The VCG mechanism is DSIC and IR. However, it is not BB: the sum of VCG payments can be positive, negative or zero, depending on the profile and the prior. If the sum of payments is positive, a redistribution of the surplus will improve net welfare of agents. If the sum of payments is negative, the mechanism requires an outside subsidy. The VCG mechanism therefore, becomes approximately first best in the limit if the sum of VCG payments at every profile converges to zero. In this chapter we investigate this issue.

In our model, priors satisfy the *Trade in the Limit* or the TL condition if $\underline{v}_0 > k\underline{v}$, i.e., the lowest end of the support of the buyer's valuation is greater than k times that of the sellers' valuation. If this condition is satisfied, then trade will almost surely take place in the VCG mechanism as the number of sellers becomes large. This chapter shows that TL is a necessary and sufficient condition for almost sure positive VCG surplus in the limit in the LA model.

There are conceptual difficulties in describing a general model for the LAC problem. This is because the underlying graph may change depending on the way new sellers are added. In this chapter we examine some special cases where these issues can be dealt with. The first of these is a model where new sellers are added consecutively on a line. The second is a star graph where new sellers form additional edges with a fixed hub seller. We show that the TL condition can be extended to the line graph model for almost sure positive VCG surplus in the limit. We also show that a stronger condition is required in the star graph model and we identify this condition as $\underline{v}_0 > \bar{v} + \underline{v}$. It implies that buyer's lowest valuation has to be higher than any seller's highest valuation. We then generalize these conditions to sequences of graphs with special properties.

We have provided several numerical examples to illustrate these results. We generate values for the VCG sum of payments for these problems when valuations are drawn from specific uniform distributions both where TL is satisfied and where it is not.

Asymptotic properties of VCG mechanisms have been examined in various contexts. Tideman and Tullock (1976) conjectured that the per capita VCG budget surplus converges to zero as number of individuals become large. Green et al. (1976) demonstrated the same property in an economy where individuals have to make a collective decision on a public project. Rob (1982) extended the result to choice between two public projects and used weaker distributional assumptions. Bailey (1997) and Cavallo (2006) have investigated redistribution of VCG surplus that do not violate incentives. Our results illustrate that convergence of the per capita VCG surplus to zero is not a general phenomenon. Strong conditions on the prior and the underlying graph are required to achieve such an outcome.

Heller and Hills (2008) have made a conjecture that the holdout problem in land assembly will resolve if property rights are vested in a committee of representative sellers. This approach can be modelled as bilateral bargaining with multiple objects which has been studied recently by Jackson et al. (2015). They show that all sequential equilibria are either efficient or approximately efficient even in the presence of significant asymmetric information. This is in sharp contrast with our results, particularly when the contiguity structure of land plots is taken into account.

4.2 Results

In this Section, we present four sets of results. These are with respect to the LA model, the LAC model with line contiguity, the LAC model with star contiguity and general contiguity structures respectively.

4.2.1 Convergence in the LA Model

Refer to the sequence of LA models $\langle m, k, \mu \rangle_{m=n}^{\infty}$ in Chapter 2 where $n > k$. Let v_0 and $v_1, \ldots, v_m$ be independently distributed in $[\underline{v}_0, \bar{v}_0]$ and $[\underline{v}, \bar{v}]$ respectively. Let the corresponding distribution functions be $G(\cdot)$ and $F(\cdot)$ respectively.

The priors satisfy the *Trade in the Limit* condition, or TL if

$$\underline{v}_0 > k\underline{v}.$$

The following result shows that TL is a necessary and sufficient condition for the VCG surplus to be positive almost everywhere.

Proposition 2 *Consider the sequence of LA models $\langle m, k, \mu \rangle_{m=n}^{\infty}$ with $n > k$. Then $\Pr(\sum_{j=0}^{m} t_j^V(v) > 0) \to 1$ as $m \to \infty$ if and only if TL holds.*

Proof: **Only if part:** Suppose $\underline{v}_0 \leq k\underline{v}$. We show that $\sum_{j=0}^{n} t_j^V(v) < 0$ almost whenever trade takes place. The sum of VCG payments at different profiles are listed in the table below which is reproduced from Chapter 2.

Table 4.1: Sum of Payments when $n > k$

Case	**Sum of Payments**	**Sign**
I: $v_0 \geq \underline{v}_0 > \sum_{j=1}^{k} v_{[j]}, A(v) \neq \emptyset$	$\underline{v}_0 - \sum_{h \in A(v)} \left(v_0 - \sum_{\substack{j=1 \\ j \neq h}}^{k} v_{[j]} \right) - (k - \lvert A(v) \rvert)\, v_{[k+1]}$	≤ 0
II: $v_0 \geq \underline{v}_0 > \sum_{j=1}^{k} v_{[j]}, A(v) = \emptyset$	$\underline{v}_0 - k v_{[k+1]}$	$\lesseqgtr 0$
III: $v_0 > \sum_{j=1}^{k} v_{[j]} \geq \underline{v}_0, A(v) \neq \emptyset$	$\sum_{j=1}^{k} v_{[j]} - \sum_{h \in A(v)} \left(v_0 - \sum_{\substack{j=1 \\ j \neq h}}^{k} v_{[j]} \right) - (k - \lvert A(v) \rvert)\, v_{[k+1]}$	< 0
IV: $v_0 > \sum_{j=1}^{k} v_{[j]} \geq \underline{v}_0, A(v) = \emptyset$	$\sum_{j=1}^{k} v_{[j]} - k v_{k+1}$	≤ 0
V: $v_0 \leq \sum_{j=1}^{k} v_{[j]}$	0	0

Notice that Cases I and II do not arise when $\underline{v}_0 \leq k\underline{v}$. Further, if trade takes place, $\sum_{j=0}^{m} t_j^V(v) = 0$ only in countably many instances of Case IV. Consequently, $\sum_{j=0}^{m} t_j^V(v) < 0$ at almost every profile where trade takes place.

If part: Suppose $\underline{v}_0 > k\underline{v}$. By assumption NT (see Chapter 2) and the hypothesis, $k\underline{v} < \underline{v}_0 < k\bar{v}$. Therefore, $\underline{v} < \frac{\underline{v}_0}{k} < \bar{v}$. Let $F(\cdot)$ be the c.d.f. of $v_1, \ldots, v_m$. Since it is a monotonic increasing function in $[\underline{v}, \bar{v}]$,

$$\begin{aligned} &F(\underline{v}) < F\left(\frac{\underline{v}_0}{k}\right) < F(\bar{v}), \\ &\text{i.e., } 0 < F\left(\frac{\underline{v}_0}{k}\right) < 1. \end{aligned}$$

Recall from Lemma 5 in Chapter 2 that at all profiles v, $\sum_{j=0}^{m} t_j^V(v) \geq \underline{v}_0 - kv_{[k+1]}$. Therefore,

$$\Pr\left(\sum_{j=0}^{m} t_j^V(v) > 0\right) \geq \Pr\left(\underline{v}_0 - kv_{[k+1]} > 0\right). \tag{4.1}$$

We show that $\Pr\left(\underline{v}_0 - kv_{[k+1]} > 0\right) \to 1$ as $m \to \infty$. Notice that

$$\begin{aligned} \Pr\left(\underline{v}_0 - kv_{[k+1]} > 0\right) &= \Pr\left(v_{[k+1]} < \frac{\underline{v}_0}{k}\right) \\ &= \sum_{r=k+1}^{m} \binom{m}{r} \left\{F\left(\frac{\underline{v}_0}{k}\right)\right\}^r \left\{1 - F\left(\frac{\underline{v}_0}{k}\right)\right\}^{m-r} \\ &= 1 - \sum_{r=0}^{k} \binom{m}{r} \left\{F\left(\frac{\underline{v}_0}{k}\right)\right\}^r \left\{1 - F\left(\frac{\underline{v}_0}{k}\right)\right\}^{m-r}. \end{aligned}$$

For every $r \in \{0, \ldots, k\}$,

$$\binom{m}{r} \left\{F\left(\frac{\underline{v}_0}{k}\right)\right\}^r \left\{1 - F\left(\frac{\underline{v}_0}{k}\right)\right\}^{m-r} = \frac{F\left(\frac{\underline{v}_0}{k}\right)}{r!} \times \frac{m(m-1)\cdots(m-r+1)}{1/\left\{1 - F\left(\frac{\underline{v}_0}{k}\right)\right\}^{m-r}}.$$

The first term of this product is a constant. The second term has a polynomial of degree r in m in the numerator and an exponential term of degree $m - r$ in the denominator. The second term converges to zero as $m \to \infty$, and therefore, each of the $k + 1$ components of the sum converges to zero individually. Therefore, $\Pr\left(\underline{v}_0 - kv_{[k+1]} > 0\right) \to 1$ as $m \to \infty$. ■

Observation 1 In LA, probability of trade taking place is $\Pr(v_0 > \sum_{j=1}^{k} v_{[j]})$. Since,

$$\Pr\left(v_0 > \sum_{j=1}^{k} v_{[j]}\right) \geq \Pr\left(\underline{v}_0 > \sum_{j=1}^{k} v_{[j]}\right),$$

trade takes place almost surely in the limit if $\underline{v}_0 > k\underline{v}$. This is the reason we call it a *trade in the limit* condition.

Observation 2 Since $\Pr\left(v_{[k]} > \underline{v} + \epsilon\right) \to 0$ as $m \to \infty$, if $\underline{v}_0 = k\underline{v}$, the deficit approaches zero. This will not hold if $k\underline{v} > \underline{v}_0$.

4.2.2 Convergence in the LAC Model with Line Contiguity

A definition of a sequence of LAC problems is required in order to investigate its asymptotic properties. While adding new nodes to a graph Γ, we have to specify how new edges are constructed. Depending on the initial graph and how new nodes and edges are added, the nature of the graph may change. Initially, we will restrict ourselves to a general asymptotic feature of the VCG mechanism in the LAC model when the underlying graph is a line.

We construct a sequence of line graphs as follows. The graph $L(1)$ consists of a single path of length k. For any natural number $m \geq 1$, the graph $L(m+1)$ is constructed by adding an $m+1$-th node to $L(m)$ via an edge $(m, m+1)$. See Figure 4.1. This results in a connected acyclic graph, also known as a *tree*. Every path with k nodes is called a feasible path. For a line graph with m nodes, there are $m-k+1$ distinct feasible paths when $m > k$. Let us call these paths $\mathcal{P}_1, \ldots, \mathcal{P}_{m-k+1}$. Valuations of the buyer are drawn independently from prior μ. We assume that v_0 follows c.d.f. $G(\cdot)$ with support $[\underline{v}_0, \bar{v}_0]$ and $v_1, \ldots, v_m$ follow c.d.f. $F(\cdot)$ with support $[\underline{v}_0, \bar{v}_0]$. Let the sums of valuations of sellers on feasible path $\mathcal{P}_i$ for a profile v be denoted as $S_i(v)$. Let us order these sums as $S_{[1]}(v) \leq \ldots \leq S_{[m-k+1]}(v)$ and let the corresponding feasible paths be $\mathcal{P}_{[1]}(v), \ldots, \mathcal{P}_{[m-k+1]}(v)$. The buyer requires a feasible path. Efficiency requires that trade takes place with sellers in $\mathcal{P}_{[1]}(v)$ whenever $v_0 > S_{[1]}(v)$ and trade does not take place otherwise.

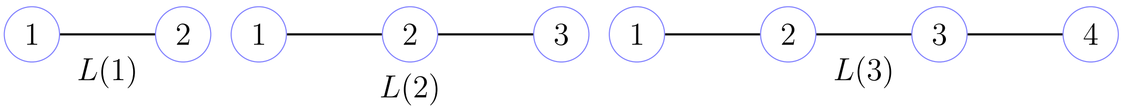

Figure 4.1: Construction of a sequence of line graphs when $k = 2$

The following result shows that TL is a necessary and sufficient condition for the VCG surplus to be positive almost everywhere.

Proposition 3 *Consider the sequence of LAC models with line contiguity* $\langle L(m), k, \mu \rangle_{m=1}^{\infty}$. *Then* $\Pr(\sum_{j=0}^{m} t_j^V(v) > 0) \to 1$ *as* $m \to \infty$ *if and only if TL holds.*

Proof: **Only if part:** Suppose $\underline{v}_0 \leq k\underline{v}$. We show that $\sum_{j=0}^{m} t_j^V(v) < 0$ almost whenever trade takes place. Table 3.1 in Chapter 3 that lists the different cases for VCG payments is reproduced below. Notice that Cases I and II do not arise when $\underline{v}_0 \leq k\underline{v}$. Further, if trade takes place, $\sum_{j=0}^{m} t_j^V(v) = 0$ only in countably many instances of Case IV. Consequently, $\sum_{j=0}^{m} t_j^V(v) < 0$ almost whenever trade takes place.

If part: Suppose $\underline{v}_0 > k\underline{v}$. By assumption NT (see Chapter 2) and the hypothesis, $k\underline{v} < \underline{v}_0 < k\bar{v}$. Therefore,

$$0 < F\left(\frac{\underline{v}_0}{k}\right) < 1,$$

Table 4.2: Sum of Payments when $n > k$

Case	Sum of Payments	Sign
I: $v_0 \geq \underline{v}_0 > S_{[1]}(v), A(v) \neq \emptyset$	$\underline{v}_0 - S_{[1]}(v) - \lvert A(v)\rvert(v_0 - S_{[1]}(v)) - \sum_{\substack{i\in\mathcal{P}_{[1]}(v) \\ i\notin A(v)}} (S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v))$	≤ 0
II: $v_0 \geq \underline{v}_0 > S_{[1]}(v), A(v) = \emptyset$	$\underline{v}_0 + (k-1)S_{[1]}(v) - \sum_{i\in\mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i})$	$\lesseqgtr 0$
III: $v_0 > S_{[1]}(v) \geq \underline{v}_0, A(v) \neq \emptyset$	$-\lvert A(v)\rvert(v_0 - S_{[1]}(v)) - \sum_{\substack{i\in\mathcal{P}_{[1]}(v) \\ i\notin A(v)}} (S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v))$	< 0
IV: $v_0 > S_{[1]}(v) \geq \underline{v}_0, A(v) = \emptyset$	$-\sum_{i\in\mathcal{P}_{[1]}(v)} (S_{[1]}(\bar{v}, v_{-i}) - S_{[1]}(v))$	≤ 0
V: $v_0 \leq S_{[1]}(v)$	0	0

as before. Recall from Lemma 9 in Chapter 3 that at all profiles v,

$$\sum_{j=0}^{m} t_j^V(v) \geq \underline{v}_0 + (k-1)S_{[1]}(v) - \sum_{i\in\mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i}).$$

We show that

$$\Pr\left(\underline{v}_0 + (k-1)S_{[1]}(v) - \sum_{i\in\mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i}) > 0\right) \to 1 \text{ as } n \to \infty.$$

A set of feasible paths $\mathcal{F}(L(m), k, \mu)$ will be called *independent* if no two feasible paths in it share a node, i.e.,

$$\mathcal{P}_i, \mathcal{P}_j \in \mathcal{F}(L(n), k, \mu) \Rightarrow V(\mathcal{P}_i) \cap V(\mathcal{P}_j) = \emptyset,$$

where $V(L)$ is the set of nodes in graph L. Note that we can always construct a non-empty independent set of feasible paths by including $\mathcal{P}_1$ and then including the feasible paths $\mathcal{P}_{k+1}$ if $n \geq 2k$, $\mathcal{P}_{2k+1}$ if $n \geq 3k$ and so on. This set will contain at most $\left[\frac{m-k+1}{k}\right]$ feasible paths where $\left[\frac{m-k+1}{k}\right]$ is the integral part of the improper fraction $\frac{m-k+1}{k}$. Denote this independent set of feasible paths by $\mathcal{F}^*$. See an illustration in Figure 4.2 below.

Figure 4.2: Construction of F^* shown with red edges when $k = 2$

At any profile v let the highest valuation of a seller on a feasible path $\mathcal{P} \in \mathcal{F}^*$ be denoted as $\tilde{v}^{\mathcal{P}}$. Each of the $\tilde{v}^{\mathcal{P}}$s are functions of a sample of size k of independent draws from $[\underline{v}, \bar{v}]$. Since all seller values are drawn independently from $F(\cdot)$, $\tilde{v}^{\mathcal{P}_1}, \tilde{v}^{\mathcal{P}_{k+1}}, \ldots, \tilde{v}^{\mathcal{P}_{m-k+1}}$ are independent random variables. Furthermore, $\Pr(\tilde{v}^{\mathcal{P}} \leq x) = F(x)^k$. Order $\tilde{v}^{\mathcal{P}}$, $\mathcal{P} \in \mathcal{F}^*$, as

$\tilde{v}_{[1]} \leq \ldots \leq \tilde{v}_{\left[\frac{m-k+1}{k}\right]}$. Let the corresponding feasible paths be $\tilde{\mathcal{P}}_{[1]}(v), \ldots, \tilde{\mathcal{P}}_{\left[\frac{m-k+1}{k}\right]}(v)$ and the corresponding sums of valuations be $\tilde{\mathcal{S}}_{[1]}(v), \ldots, \tilde{\mathcal{S}}_{\left[\frac{m-k+1}{k}\right]}(v)$. As before,

$$\text{i.e., } 0 < F\left(\frac{\underline{v}_0}{k}\right) < 1.$$

Therefore,

$$0 < F\left(\frac{\underline{v}_0}{k}\right)^k < 1.$$

Since $S_{[1]}(v) \leq \tilde{S}_{[1]}(v)$ and $\tilde{\mathcal{P}}_{[2]}(v)$ does not contain $i \in \mathcal{P}_{[1]}(v)$, $S_{[1]}(\bar{v}, v_{-i}) \leq \tilde{S}_{[2]}(v)$. The following Lemma shows that $\Pr(\underline{v}_0 > S_{[2]}(v)) \to 1$ as $m \to \infty$.

Lemma **10** *Suppose* $\underline{v}_0 > k\underline{v}$. *Then*

$$Pr(\underline{v}_0 > \tilde{S}_{[2]}(v)) \text{ as } n \to \infty.$$

Proof:

$$\begin{aligned}
&\Pr\left(\underline{v}_0 \geq \tilde{S}_{[2]}(v)\right) \geq \Pr\left(\tilde{v}_{[2]} \leq \frac{\underline{v}_0}{k}\right) \\
&= 1 - \left\{1 - F\left(\frac{\underline{v}_0}{k}\right)^k\right\}^m - mF\left(\frac{\underline{v}_0}{k}\right)^k \left\{1 - F\left(\frac{\underline{v}_0}{k}\right)^k\right\}^{m-1}.
\end{aligned}$$

Both the second and third term are fractions that converge to zero as $m \to \infty$. Hence the result. ■

Since,

$$\Pr(v_0 > S_{[1]}(v)) \geq \Pr(v_0 > S_{[1]}(\bar{v}, v_{-i})) \geq \Pr(\underline{v}_0 > S_{[1]}(\bar{v}, v_{-i})) \geq Pr(\underline{v}_0 > \tilde{S}_{[2]}(v)), \quad (4.2)$$

trade almost surely takes place as number of sellers become large. Since $\Pr(\underline{v}_0 > S_1(v)) \to 1$, Case III, IV and V are ruled out almost everywhere in the limit. Further, recall that the set of trade-pivotal sellers at a profile v is

$$A(v) = \{i \in \mathcal{P}_{[1]}(v) : v_0 > S_{[1]}(v), v_0 \leq S_{[1]}(\bar{v}, v_{-i})\}.$$

By Lemma 10, $A(v)$ is empty almost surely for large m. This rules out Case I almost everywhere in the limit. Furthermore, by (4.2), $\Pr\left(S_{[1]}(\bar{v}, v_{-i}) > S_{[1]}(v)\right) \to 0$. It follows that $\Pr\left(\underline{v}_0 + (k-1)S_{[1]}(v) - \sum_{i \in \mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i}) > 0\right) \to \Pr(\underline{v}_0 > S_{[1]}(v))$ which has been shown to approach 1 as $m \to \infty$. ■

The earlier observations on the behavior of $\sum_{j=0}^n t_j^V(v)$ when $\underline{v}_0 \leq k\underline{v}$ remain valid.

4.2.3 The Star Graph

Recall from Chapter 3 that a critical seller is a node that is contained in every feasible path. A star graph contains a non-empty set of critical sellers. Consider the following sequence of star graphs with $k = 2$: Let $\Gamma^*(1) = \langle\{1, 2\}, \{(1, 2)\}\rangle$ where the first component is the set of nodes V_1 and the second is the set of edges E_1. For any $m \geq 1$, construct $\Gamma^*(m + 1)$ as $\langle\{V_m \cup \{m + 2\}, E_m \cup \{(1, m + 2)\}\rangle$. The figure below illustrates this construction.

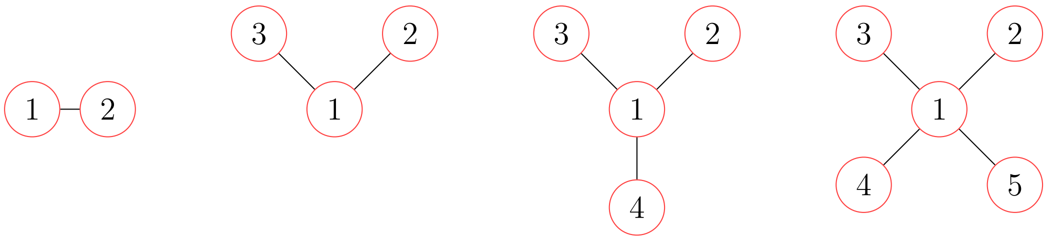

Figure 4.3: A sequence of star graphs when $k = 2$

A prior satisfies condition TLS1 if

$$\underline{v}_0 > \bar{v} + \underline{v}.$$

It can be interpreted as the counterpart of TL for star graphs with $k = 2$.

Proposition 4 *Consider the sequence:* $\langle\Gamma^*(m), 2, \mu\rangle_{m=1}^{\infty}$. *Then* $\Pr(\sum_{j=0}^{m} t_j^V(v) > 0) \to 1$ *as* $m \to \infty$ *if and only if TLS1 holds.*

Proof: **Only if part:** Suppose $\underline{v}_0 \leq \bar{v} + \underline{v}$. The sum of VCG payments in Case II of Table 4.2 is

$$\begin{aligned}
&\underline{v}_0 + (k-1)S_{[1]}(v) - \sum_{i \in \mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i}) \\
&= \underline{v}_0 + \left(v_1 + v_{[1]}^{-\{1\}}\right) - \left(\bar{v} + v_{[1]}^{-\{1\}}\right) - \left(v_1 + v_{[2]}^{-\{1\}}\right) \\
&= \underline{v}_0 - \bar{v} - v_{[2]}^{-\{1\}} \\
&\leq \underline{v} - v_{[2]}^{-\{1\}} \\
&\leq 0,
\end{aligned}$$

where $v_{[i]}^{-\{1\}}$ is the i-th order statistic of the $m-1$ valuations of all sellers other than 1. Since this is the only Case where the VCG sum of payments can be positive, hence the claim.

If part: Let $\underline{v}_0 > \bar{v} + \underline{v}$. Recall that at all profiles v,

$$\begin{aligned}
\sum_{j=0}^{m} t_j^V(v) &\geq \underline{v}_0 + (k-1)S_{[1]}(v) - \sum_{i \in \mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i}) \\
&= \underline{v}_0 - \bar{v} - v_{[2]}^{-\{1\}}.
\end{aligned}$$

Therefore,

$$\Pr\left(\sum_{j=0}^{m} t_j^V(v) > 0\right) \geq \Pr\left(\underline{v}_0 - \bar{v} - v_{[2]}^{-\{1\}} > 0\right)$$
$$= \Pr\left(v_{[2]}^{-\{1\}} < \underline{v}_0 - \bar{v}\right).$$

We will show that $\Pr\left(\sum_{j=0}^{m} t_j^V(v) > 0\right) \to 1$. According to assumption NT (see Chapter 3), $2\bar{v} > \underline{v}_0$. It follows that $\bar{v} > \underline{v}_0 - \bar{v}$. By hypothesis, $\underline{v}_0 - \bar{v} > \underline{v}$. Therefore, $\underline{v} < \underline{v}_0 - \bar{v} < \bar{v}$. This implies,

$$\text{i.e., } 0 < F\left(\underline{v}_0 - \bar{v}\right) < 1.$$

Consequently,

$$\begin{aligned}&\Pr\left(v_{[2]}^{-\{1\}} < \underline{v}_0 - \bar{v}\right)\\ &= 1 - \left\{1 - F\left(\underline{v}_0 - \bar{v}\right)\right\}^{m-1} - (m-1)F\left(\underline{v}_0 - \bar{v}\right)\left\{1 - F\left(\underline{v}_0 - \bar{v}\right)\right\}^{m-2},\end{aligned}$$

which converges to 1 as $m \to \infty$. ■

4.2.4 A Generalization

The following Propositions extend the results of the earlier subsections to a sequence of graphs under certain conditions.

Let $\Gamma(1)$ be a feasible path. For any $\Gamma(m)$, $m \geq 1$, let $\Gamma(m+1)$ be any arbitrary supergraph of $\Gamma(m)$ of order $m+1$. We say that a sequence of graphs $\Gamma(m)_{m=1}^{\infty}$ satisfies the *line inclusion* property if for any $m' \geq 1$, we can find a natural number m such that $L(m')$ is a subgraph of $\Gamma(m)$.

Proposition 5 *Consider a sequence of graphs $\Gamma(m)_{m=1}^{\infty}$ that satisfies the Line Inclusion property. Then for any sequence of LAC problems $\langle\Gamma(m), k, \mu\rangle_{m=n}^{\infty}$, the VCG mechanism almost surely results in a surplus if and only if $\underline{v}_0 > k\underline{v}$.*

Proof: This proof is almost identical to that of Proposition 3 and hence omitted. ■

Let $\Gamma(1)$ be a connected graph with a nonempty set of critical sellers $c(\Gamma(1))$. Let $|c(\Gamma)| = C$. These critical sellers form a path of length C, say $\{c_1 c_2 \cdots c_C\}$. For any $\Gamma(m)$, $m \geq 1$, let $\Gamma(m+1)$ be a supergraph of $\Gamma(m)$ such that $\Gamma(m+1) = \{V(m) \cup \{m+1\}, E(m) \cup \{x\}\}$ where $x \in \{(m+1, c_1), (c_C, m+1)\}$. In other words, the supergraph adds a new edge at the endpoints of the path $\{c_1 c_2 \cdots c_C\}$. Note that for any $m \geq 1$, $|c(\Gamma(m))| = C$. We say that such a sequence $\Gamma(m)_{m=1}^{\infty}$ satisfies the *C-preservation* property. Since a graph can have at most $k-1$ critical nodes, $C \leq k-1$. For example, $\langle\Gamma^*(m), 2, \mu\rangle_{m=1}^{\infty}$ satisfies 1-preservation property.

Proposition 6 *Let $\Gamma(1)$ be a connected graph with a nonempty set of critical sellers $c(\Gamma(1))$. Consider a sequence of graphs $\Gamma(m)_{m=1}^{\infty}$ that satisfies C-preservation property. For the sequence of LAC problems $\langle \Gamma(m), k, \mu \rangle_{m=1}^{\infty}$, the VCG mechanism almost surely results in a surplus if and only if*

$$\underline{v}_0 > C\bar{v} + (k - C)\underline{v}.$$

Proof: First note that for any Γ of order m in such a sequence,

$$\begin{aligned}
\underline{v}_0 + (k-1)S_{[1]}(v) - \sum_{i \in \mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i}) &= \underline{v}_0 + (k-1)\left[\sum_{i \in c(\Gamma)} v_i + \sum_{j=1}^{k-C} v_{[j]}^{-c(\Gamma)}\right] \\
&\quad - \sum_{i \in c(\Gamma)} \left[\bar{v} + \sum_{\substack{j \in c(\Gamma) \\ j \neq i}} v_j + \sum_{j=1}^{k-C} v_{[j]}^{-c(\Gamma)}\right] \\
&\quad - \sum_{i \notin c(\Gamma)} \left[\sum_{i \in c(\Gamma)} v_i + \sum_{\substack{j=1 \\ j \neq i}}^{k-C+1} v_{[j]}^{-c(\Gamma)}\right] \\
&= \underline{v}_0 - C\bar{v} - (k-C)v_{[k-C+1]}^{-c(\Gamma)}
\end{aligned}$$

where $v_{[i]}^{-c(\Gamma)}$ is the i-th order statistic of the $m - C$ valuations of all non-critical sellers.

Only if part: Suppose $\underline{v}_0 \leq C\bar{v} + (k - C)\underline{v}$. Then

$$\begin{aligned}
\underline{v}_0 + (k-1)S_{[1]}(v) - \sum_{i \in \mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i}) &= \underline{v}_0 - C\bar{v} - (k-C)v_{[k-C+1]}^{-c(\Gamma)} \\
&\leq (k-C)\left(\underline{v} - v_{[k-C+1]}^{-c(\Gamma)}\right) \\
&\leq 0
\end{aligned}$$

Consequently, the sum of payments in Case II of Table 4.2 cannot be positive. Since this is the only Case where the VCG sum of payments can be positive, hence the claim.

If part: Let $\underline{v}_0 > C\bar{v} + (k - C)\underline{v}$. Recall that at all profiles v,

$$\begin{aligned}
\sum_{j=0}^{m} t_j^V(v) &\geq \underline{v}_0 + (k-1)S_{[1]}(v) - \sum_{i \in \mathcal{P}_{[1]}(v)} S_{[1]}(\bar{v}, v_{-i}) \\
&= \underline{v}_0 - C\bar{v} - (k-C)v_{[k-C+1]}^{-c(\Gamma)}.
\end{aligned}$$

Therefore,

$$\Pr\left(\sum_{j=0}^{m} t_j^V(v) > 0\right) \geq \Pr\left(\underline{v}_0 - C\bar{v} - (k-C)v_{[k-C+1]}^{-c(\Gamma)} > 0\right)$$
$$= \Pr\left(v_{[k-C+1]}^{-c(\Gamma)} < \frac{\underline{v}_0 - C\bar{v}}{k-C}\right).$$

We will show that $\Pr\left(v_{[k-C+1]}^{-c(\Gamma)} < \frac{\underline{v}_0 - C\bar{v}}{k-C}\right) \to 1$. By assumption NT (see Chapter 3), $k\bar{v} > \underline{v}_0$ it follows that $(k-C)\bar{v} > \underline{v}_0 - C\bar{v}$. By hypothesis, $\underline{v}_0 - C\bar{v} > (k-C)\underline{v}$. Therefore, $\underline{v} < \frac{\underline{v}_0 - C\bar{v}}{k-C} < \bar{v}$. This implies

$$\text{i.e., } 0 < F\left(\frac{\underline{v}_0 - C\bar{v}}{k-C}\right) < 1.$$

Consequently,

$$\Pr\left(v_{[k-C+1]}^{-c(\Gamma)} < \frac{\underline{v}_0 - C\bar{v}}{k-C}\right)$$
$$= 1 - \sum_{r=0}^{k-C} \binom{m-C}{r} \left\{F\left(\frac{\underline{v}_0 - C\bar{v}}{k-C}\right)\right\}^r \left\{1 - F\left(\frac{\underline{v}_0 - C\bar{v}}{k-C}\right)\right\}^{m-C-r}.$$

Since each of the components of the sum on the right hand side converges to 0 as $m \to \infty$, $\Pr\left(v_{[k-C+1]}^{-c(\Gamma)} < \frac{\underline{v}_0 - \bar{v}}{k-C}\right)$ converges to 1 as $m \to \infty$. ■

4.3 Discussion

The TL condition can be interpreted as follows. Fix a valuation of the buyer. Then there always exists a tuple of seller valuations for which trade takes place. Note that TL does not mean that trade takes place everywhere. Assumption NT states that $k\bar{v} > \underline{v}_0$, and therefore, there exists a set of valuations for which trade does not take place.

We showed that (a) if TL holds then VCG almost always results in a surplus if the number of sellers is large, both in LA and in the LAC problem with line contiguity; (b) if the corresponding equality holds, then VCG sum of payments converges to zero from the left hand side; (c) if $\underline{v}_0 < k\underline{v}$, then VCG sum of payments never results in a surplus.

The intuition behind these results is clear. Recall that the VCG payment for each agent is interpreted as the externality she imposes on other agents. We have used the statistical fact that the k-th lowest order statistic of a sample of size $n > k$ drawn from any continuous distribution approaches the lower end of its support in probability as n becomes large. It follows that, as the number of sellers become large, the probability that the buyer with a given valuation will find an efficient set of sellers to trade with becomes very high. Further, the probability that the buyer will continue to trade if one seller reports a high valuation becomes very high as well. Therefore, the externality imposed by any successful seller on other agents at a profile becomes small. TL ensures that whenever trade takes place the externality imposed by the successful sellers at a profile is less than that by the buyer.

In Tables 4.3 and 4.4, we provide several numerical examples[3] to illustrate this result for the LA problem and the LAC problem with line contiguity when $k = 2$. We generate values for the VCG sum of payments for these problems when valuations are drawn from specific uniform distributions pertaining to cases where (a) TL holds, (b) if the corresponding equality holds and (c) if $\underline{v}_0 < k\underline{v}$. The numerical data confirms our results.

In the LAC problem with star contiguity, the hub of the star graph represents a critical seller, who by definition, is in every feasible path. Recall from Chapter 3 that a critical seller who is not trade-pivotal must receive a payment of $\bar{v}$ at a profile where trade takes place. The TLS1 condition can be interpreted as follows. Fix a valuation of the buyer. Then there always exists a tuple of seller valuations such that trade takes place where one of the sellers is critical.

Condition TLS1 requires the buyer to have a very high valuation relative to the sellers. It follows that the presence of critical sellers makes convergence to surplus less likely. We showed that in the problem with star contiguity, (a) if TLS1 holds then VCG almost always results in a surplus if the number of sellers is large; (b) if the corresponding equality holds, then VCG sum of payments converges to zero from the left hand side; (c) if $\underline{v}_0 < \bar{v} + \underline{v}$, then VCG sum of payments never results in a surplus.

Note that TLS1 implies that trade can take place at a profile where the non-critical seller reports a value that is low enough. The statistical facts mentioned above implies that probability of trade taking place becomes high as number of sellers become large. TLS1 ensure that whenever trade takes place the externality imposed by the successful sellers, accounting for the critical one, is less than that by the buyer.

In Table 4.5, we provide several numerical examples to illustrate this result for the LAC problem with star contiguity. We generate values for the VCG sum of payments when (a) TL holds, (b) TL holds with equality and (c) $\underline{v}_0 < \bar{v} + \underline{v}$.

Table 4.3: Sum of Payments in the LA model, $k = 2$

m	$v_0 \sim U[200, 300], v_i \sim U[100, 300]$	$v_0 \sim U[200, 300], v_i \sim U[50, 300]$	$v_0 \sim U[50, 300], v_i \sim U[100, 300]$
10	0	-21.266	0
100	-10.898	93.171	-2.858
1000	-0.713	95.429	0
10000	-0.073	99.802	-0.057

We have also shown that the convergence result for line contiguity can be extended to sequences of graphs satisfying the line inclusion property. This property implies that for any

[3]The numerical data presented in these Tables has been generated through programs written in GNU Octave, a high level interactive language for numerical computations.

Table 4.4: Sum of Payments in LAC with Line Contiguity, $k = 2$

m	$v_0 \sim U[200, 300], v_i \sim U[100, 300]$	$v_0 \sim U[200, 300], v_i \sim U[50, 300]$	$v_0 \sim U[50, 300], v_i \sim U[100, 300]$
10	0	-99.58	0
100	0	28.86	0
1000	-20.087	80.831	0
10000	-1.205	92.203	-2.615

Table 4.5: Sum of Payments in LAC with Star Contiguity, $k = 2$

m	$v_0 \sim U[400, 1000], v_i \sim U[100, 300]$	$v_0 \sim U[400, 1000], v_i \sim U[50, 300]$	$v_0 \sim U[300, 1000], v_i \sim U[100, 300]$
10	-126.86	13.06	-145.54
100	-5.593	47.422	-20.986
1000	-0.729	49.01	-55.136
10000	-0.05	49.931	-100.11

integer $m > k$, one can always find a graph in the sequence which has the line graph of order m embedded in it.

We also showed that the convergence result for star contiguity is extendable to sequences of graphs satisfying the C-preservation property. Note that an arbitrary sequence of graphs cannot satisfy both line inclusion and preservation together when there is at least one critical seller. In particular, the preservation property implies that one cannot find a path of length more than k.

4.4 Conclusion

In this Chapter we focussed on the behavior of VCG surplus as the number of sellers become large but the structure of the underlying graphs are preserved. We showed that under some mild conditions, VCG almost surely results in a surplus for the LA problem and the LAC problem with line contiguity. This condition requires that for any valuation of the buyer, there always exists a profile such that trade takes place. We showed that the corresponding condition changes when we allow for critical sellers like those in a star graph.

Chapter 5

Optimality

5.1 Introduction

We have seen from Chapters 2 and 3 that it is possible to design successful mechanisms only when priors satisfy certain conditions. If these conditions are not satisfied, it is natural to search for *second best* mechanisms. In this chapter, we follow the approach of Myerson and Satterthwaite (1983), who characterized the optimal mechanism for the bilateral trade problem. The optimal mechanism maximizes ex-ante welfare of agents in the class of mechanisms satisfying BIC, IIR and BB. Note that for priors for which a successful mechanism exists, it is possible to achieve the maximum sum of agents welfare in every state. Therefore, for such priors, a successful mechanism is an optimal mechanism.

The optimal mechanism characterized by Myerson and Satterthwaite (1983) can be described as follows. For any $\alpha \geq 0$, define the virtual valuation of the buyer with valuation v_0 and that of the seller with valuation v_1 by,

$$c_0(v_0, \alpha) = v_0 + \alpha \frac{1 - G(v_0)}{g(v_0)}$$
$$\text{and } c_1(v_1, \alpha) = v_1 + \alpha \frac{F(v_1)}{f(v_1)} \text{ respectively.}$$

Trade takes place if and only if $c_0(v_0, \alpha) > c_1(v_1, \alpha)$. Myerson and Satterthwaite (1983) provide a sufficient condition on the priors in order for the trading rule to be optimal for some value of α. For instance, in the case where both $F(\cdot)$ and $G(\cdot)$ are independent $U[0, 1]$ distributions, $\alpha = \frac{1}{3}$. In this case, the optimal mechanism coincides with an equilibrium of the double auction mechanism described by Chatterjee and Samuelson (1983).

We show that the optimal mechanism for LA and LAC problems are natural extensions of the optimal mechanism for the bilateral trade model. In the bilateral trade problem the optimal mechanism allows trade whenever the virtual valuation of the buyer exceeds that of the seller. In the LA problem, it allows trade whenever the virtual valuation of the buyer exceeds the sum of the lowest k virtual valuations of sellers. In the LAC problem, it allows trade whenever the virtual valuation of the buyer exceeds the lowest sum of virtual valuations on a feasible path.

For priors for which successful mechanisms do not exist, the optimal mechanism is not efficient. A natural question is whether the optimal mechanism converges to efficiency as the number of sellers becomes large. We show that the the optimal mechanism is asymptotically efficient if and only if the VCG mechanism is asymptotically BB. Therefore, the conditions for which the optimal mechanism is asymptotically efficient are precisely those for which the VCG mechanism is asymptotically BB. These have been derived in the earlier chapter.

We have explicitly calculated the α characterizing the optimal mechanism for several configurations of n and k when all valuations are independently and uniformly distributed over $[0, 1]$. We compare the expected welfare in these optimal mechanisms to that in the posted price mechanism and a mechanism suggested by Ghatak and Ghosh (2011).

5.2 Optimal Mechanisms

The optimal mechanism is defined as follows.

Definition 9 **(Optimal Mechanism)** *Let $\mathscr{M}$ be the set of mechanisms satisfying BIC, IIR and BB. An optimal mechanism is a solution to the following maximization problem.*

$$\max_{(P,t)\in\mathscr{M}} E\left(\sum_{j=0}^{n} U_j^{(P,t)}(v \mid v_j)\right). \tag{5.1}$$

Note that a successful mechanism (whenever it exists) is optimal. In Chapters 2 and 3 we have identified the necessary and sufficient conditions on priors for existence of successful mechanisms. There are priors for which these conditions are not satisfied e.g., where all valuations are distributed independently and uniformly on $[0, 1]$. In view of this observation, the results of this chapter are of interest only for such priors.

We will consider the LA and the LAC models separately. The optimal allocation rule for both these models are characterized in terms of the *virtual valuations* of the agents. For any $\alpha \geq 0$, the virtual valuations of the buyer and the sellers are functions

$$\begin{aligned} c_0(v_0, \alpha) &= v_0 - \alpha\frac{1 - G(v_0)}{g(v_0)}; \\ c_i(v_i, \alpha) &= v_i + \alpha\frac{F(v_i)}{f(v_i)}, i \in \{1, \ldots, n\}. \end{aligned} \tag{5.2}$$

5.2.1 The LA Model

The first Theorem characterizes the optimal mechanism for the LA model.

THEOREM 5 *Consider an LA model* $\langle n, k, \mu \rangle$*. For any* $\alpha \geq 0$*, let* $\hat{c}_{[j]}(v, \alpha)$ *be the* j*th lowest among* $c_i(v_i, \alpha) = v_i + \alpha \frac{F(v_i)}{f(v_i)}, i \in \{1, \dots, n\}$*. For all* j*, let* $c_j(v_j, 1)$ *be an increasing functions of* v_j*. Then there exists* $\alpha \in (0, 1]$ *and payment functions* $t_j(v)$*,* $j = 0, \dots, n$*, such that following allocation rule* P^α *together with payment rule* t *solves* (5.1)*:*

$$P_0^\alpha(v) = \begin{cases} 1 \text{ if } c_0(v_0, \alpha) > \sum_{j=1}^n \hat{c}_{[j]}(v, \alpha); \\ 0 \text{ otherwise}; \end{cases}$$

$$P_i^\alpha(v) = \begin{cases} -1 \text{ if } c_0(v_0, \alpha) > \sum_{j=1}^n \hat{c}_{[j]}(v, \alpha) \text{ and } c_i(v_i, \alpha) \leq \hat{c}_{[k]}(v, \alpha); \\ 0 \text{ otherwise}. \end{cases} \tag{5.3}$$

REMARK 1 The result above assumes that for all j, $c_j(v_j, 1)$ are increasing functions of v_j. This property is also known as the *monotone hazard rate* property. It is satisfied by a large class of distributions. For example, $U[\underline{v}, \bar{v}]$ satisfies this.

Proof: The following result is standard in the literature and known as *Revenue Equivalence*. It states that for any mechanism satisfying BIC, expected utility of every agent is determined by the allocation rule up to an additive constant.

LEMMA 11 *For any mechanism* (P, t) *satisfying BIC, for all* $j \in \{0, \dots, n\}$*,*

$$E_{-j}U_j(v_j, v_{-j} \mid v_j) = E_{-j}U_j(\alpha_j, v_{-j} \mid v_j) + \int_{\alpha_j}^{v_j} E_{-j}P_j(z, v_{-j})dz \text{ for all } v_j, \tag{5.4}$$

where $\alpha_0 = \underline{v}_0$ *and* $\alpha_i = \bar{v}$ *for* $i \in \{1, \dots, n\}$*.*

Proof: See Krishna (2002). ■

Refer to (5.1). The following Lemma characterizes $\mathscr{M}$.

LEMMA 12 *I. A mechanism* (P, t) *belongs to* $\mathscr{M}$ *only if*

$$\begin{aligned} &E_{-0}U_0(\underline{v}_0, v_{-0} \mid \underline{v}_0) + \sum_{i=1}^n E_{-i}U_i(\bar{v}, v_{-i} \mid \bar{v}) \\ &= \int_{\underline{v}_0}^{\bar{v}_0} \underbrace{\int_{\underline{v}}^{\bar{v}} \dots \int_{\underline{v}}^{\bar{v}}}_{n \text{ times}} \left[P_0(v) \left(v_0 - \frac{1 - G(v_0)}{g(v_0)} \right) + \sum_{i=1}^n P_i(v) \left(v_i + \frac{F(v_i)}{f(v_i)} \right) \right] \\ &\times f(v_1) \dots f(v_n) g(v_0) dv_0 \dots dv_n \end{aligned} \tag{5.5}$$

II. *Further, there exists payment functions* $t_j(v)$*,* $j = 0, \dots, n$*, such that* $(P, t) \in M$ *if and only if*

(i) $E_{-0}P_0(v)$ *is increasing in* v_0*,* *(ii)* $E_{-i}P_i(v)s$ *are decreasing in* v_i*, and*

(iii) the expression in (5.5) *is nonnegative.*

Proof: Let (P,t) be BIC and BB. First, we will show that (P,t) satisfies (5.5). In Lemma 11, we note that for the buyer $j=0$, $\alpha_j = \underline{v}_0$ and for each of the seller, $\alpha_j = \bar{v}$. Therefore,

$$E_{-0}U_0(v \mid v_0) = E_{-0}U_0(\underline{v}_0, v_{-0} \mid \underline{v}_0) + \int_{\underline{v}_0}^{v_0} E_{-0}P_0(z, v_{-0})dz \tag{5.6}$$

$$E_{-i}U_i(v \mid v_i) = E_{-i}U_i(\bar{v}, v_{-i} \mid \bar{v}) - \int_{v_i}^{\bar{v}} E_{-i}P_i(z, v_{-i})dz \tag{5.7}$$

.

Further, we obtain,

$$\begin{aligned}
&\int_{\underline{v}_0}^{\bar{v}_0}\int_{\underline{v}}^{\bar{v}}\dots\int_{\underline{v}}^{\bar{v}} v_0(P_0(v))f(v_1)\dots f(v_n)g(v_0)dv_0\dots dv_n \\
&+\sum_{i=1}^{n}\int_{\underline{v}_0}^{\bar{v}_0}\int_{\underline{v}}^{\bar{v}}\dots\int_{\underline{v}}^{\bar{v}} v_iP_i(v)f(v_1)\dots f(v_n)g(v_0)dv_0\dots dv_n \\
&=\int_{\underline{v}_0}^{\bar{v}_0}\int_{\underline{v}}^{\bar{v}}\dots\int_{\underline{v}}^{\bar{v}} v_0(P_0(v))f(v_1)\dots f(v_n)g(v_0)dv_0\dots dv_n \\
&-\sum_{i=1}^{n}\int_{\underline{v}_0}^{\bar{v}_0}\int_{\underline{v}}^{\bar{v}}\dots\int_{\underline{v}}^{\bar{v}} t_i(v)f(v_1)\dots f(v_n)g(v_0)dv_0\dots dv_n \\
&+\sum_{i=1}^{n}\int_{\underline{v}_0}^{\bar{v}_0}\int_{\underline{v}}^{\bar{v}}\dots\int_{\underline{v}}^{\bar{v}} t_i(v)f(v_1)\dots f(v_n)g(v_0)dv_0\dots dv_n \\
&-\sum_{i=1}^{n}\int_{\underline{v}_0}^{\bar{v}_0}\int_{\underline{v}}^{\bar{v}}\dots\int_{\underline{v}}^{\bar{v}} v_iP_i(v)f(v_1)\dots f(v_n)g(v_0)dv_0\dots dv_n \\
&=\int_{\underline{v}_0}^{\bar{v}_0}\left[\int_{\underline{v}}^{\bar{v}}\dots\int_{\underline{v}}^{\bar{v}}\left\{v_0(P_0(v))-\sum_{i=1}^{n}t_i(v)\right\}f(v_1)\dots f(v_n)dv_1\dots dv_n\right]g(v_0)dv_0 \\
&+\sum_{i=1}^{n}\int_{\underline{v}}^{\bar{v}}\left[\int_{\underline{v}_0}^{\bar{v}_0}\underbrace{\int_{\underline{v}}^{\bar{v}}\dots\int_{\underline{v}}^{\bar{v}}}_{n-1\text{ times}}\{v_iP_i(v)-t_i(v)\}\,g(v_0)\frac{f(v_1)\dots f(v_n)}{f(v_i)}\frac{dv_0\dots dv_n}{dv_i}\right]f(v_i)dv_i \\
&=\int_{\underline{v}_0}^{\bar{v}_0}E_{-0}U_0(v\mid v_0)g(v_0)dv_0+\sum_{i=1}^{n}\int_{\underline{v}}^{\bar{v}}E_{-i}U_i(v\mid v_i)f(v_i)dv_i
\end{aligned}$$

$$
\begin{aligned}
&= E_{-0}U_0(\underline{v}_0, v_{-0} \mid \underline{v}_0) + \int_{\underline{v}_0}^{\bar{v}_0}\int_{\underline{v}_0}^{v_0} E_{-0}P_0(z_0, v_{-0})dz_0 g(v_0)dv_0 \\
&+ \sum_{i=1}^{n}\left[E_{-i}U_i(\bar{v}, v_{-i} \mid \bar{v}) + \int_{\underline{v}}^{\bar{v}}\int_{v_i}^{\bar{v}} E_{-i}P_i(z_i, v_{-i})dz_i f(v_i)dv_i\right] \\
&= E_{-0}U_0(\underline{v}_0, v_{-0} \mid \underline{v}_0) + \sum_{i=1}^{n} E_{-i}U_i(\bar{v}, v_{-i} \mid \bar{v}) + \sum_{i=1}^{n}\left[\int_{\underline{v}}^{\bar{v}} E_{-i}P_i(v_i, v_{-i})F(v_i)dv_i\right] \\
&+ \int_{\underline{v}_0}^{\bar{v}_0} E_{-0}P_0(v_0, v_{-0})(1 - G(v_0))dv_0 \\
&= E_{-0}U_0(\underline{v}_0, v_{-0} \mid \underline{v}_0) + \sum_{i=1}^{n} E_{-i}U_i(\bar{v}, v_{-i} \mid \bar{v}) \\
&+ \sum_{i=1}^{n}\int_{\underline{v}_0}^{\bar{v}_0}\int_{\underline{v}}^{\bar{v}}\dots\int_{\underline{v}}^{\bar{v}} P_i(v)\frac{F(v_i)}{f(v_i)}f(v_1)\dots f(v_n)g(v_0)dv_0\dots dv_n \\
&+ \int_{\underline{v}_0}^{\bar{v}_0}\int_{\underline{v}}^{\bar{v}}\dots\int_{\underline{v}}^{\bar{v}} P_0(v)\frac{1 - G(v_0)}{g(v_0)}f(v_1)\dots f(v_n)g(v_0)dv_0\dots dv_n.
\end{aligned}
$$

The first equality is obtained by first subtracting and then adding the second term on the right hand side; the second equality uses the independence of valuations. The third equality uses the definitions of the appropriate expectations; the fourth one uses equations (5.6) and (5.7); the fifth equality uses integration by parts to remove the second integral; the sixth equality is obtained by using the expression for the expectations. The final expression is obtained by transposing and re- arranging terms on both sides. It follows that,

$$
\begin{aligned}
&E_{-0}U_0(\underline{v}_0, v_{-0} \mid \underline{v}_0) + \sum_{i=1}^{n} E_{-i}U_i(\bar{v}, v_{-i} \mid \bar{v}) \\
&= \int_{\underline{v}_0}^{\bar{v}_0}\int_{\underline{v}}^{\bar{v}}\dots\int_{\underline{v}}^{\bar{v}} P_0(v)\left[v_0 - \frac{1 - G(v_0)}{g(v_0)}\right]f(v_1)\dots f(v_n)g(v_0)dv_0\dots dv_n \\
&+ \sum_{i=1}^{n}\int_{\underline{v}_0}^{\bar{v}_0}\int_{\underline{v}}^{\bar{v}}\dots\int_{\underline{v}}^{\bar{v}} P_i(v)\left[v_i + \frac{F(v_i)}{f(v_i)}\right]f(v_1)\dots f(v_n)g(v_0)dv_0\dots dv_n
\end{aligned} \tag{5.8}
$$

which is the expression in (5.5) above.

Now we show the necessity condition in the second part of the Theorem. If a mechanism is BIC, then for any two valuations v_0 and $\hat{v}_0$ of the buyer,

$$
\begin{aligned}
E_{-0}U_0(v_0, v_{-0} \mid v_0) &= v_0E_{-0}P_0(v_0, v_{-0}) - E_{-0}t_0(v_0, v_{-0}) \\
&\geq v_0E_{-0}P_0(\hat{v}_0, v_{-0}) - E_{-0}t_0(\hat{v}_0, v_0)
\end{aligned}
$$

and

$$
\begin{aligned}
E_{-0}U_0(\hat{v}_0, v_{-0} \mid \hat{v}_0) &= \hat{v}_0E_{-0}P_0(\hat{v}_0, v_{-0}) - E_{-0}t_0(\hat{v}_0, v_{-0}) \\
&\geq \hat{v}_0E_{-0}P_0(v_0, v_{-0}) - E_{-0}t_0(v_0, v_{-0}).
\end{aligned}
$$

Therefore,

$$\begin{aligned}
E_{-0}U_0(\hat{v}_0, v_{-0} \mid \hat{v}_0) - E_{-0}U_0(v_0, v_{-0} \mid v_0) \;&\geq\; \hat{v}_0 E_{-0}P_0(v_0, v_{-0}) - E_{-0}t_0(v_0, v_{-0}) \\
&\quad\; - v_0 E_{-0}P_0(v_0, v_{-0}) + E_{-0}t_0(v_0, v_{-0}) \\
&= \; (\hat{v}_0 - v_0)E_{-0}P_0(v_0, v_{-0})
\end{aligned}$$

and,

$$\begin{aligned}
E_{-0}U_0(\hat{v}_0, v_{-0} \mid \hat{v}_0) - E_{-0}U_0(v_0, v_{-0} \mid v_0) \;&\leq\; \hat{v}_0 E_{-0}P_0(\hat{v}_0, v_{-0}) - E_{-0}t_0(\hat{v}_0, v_{-0}) \\
&\quad\; - v_0 E_{-0}P_0(\hat{v}_0, v_{-0}) + E_{-0}t_0(\hat{v}_0, v_0) \\
&= \; (\hat{v}_0 - v_0)E_{-0}P_0(\hat{v}_0, v_{-0}).
\end{aligned}$$

This gives us the following inequality:

$$(\hat{v}_0 - v_0)E_{-0}P_0(v_0, v_{-0}) \leq EU_0(\hat{v}_0) - EU_0(v_0) \leq (\hat{v}_0 - v_0)E_{-0}P_0(\hat{v}_0, v_{-0}) \tag{5.9}$$

It follows that $E_{-0}P_0(v_0, v_{-0})$ is increasing in v_0. A similar argument shows that $E_{-i}P_i(v_i, v_{-i})$ is decreasing in v_i. Since,

$$E_{-0}U_0(\underline{v}_0, v_{-0} \mid \underline{v}_0) = \min_{v_0 \in [\underline{v}_0, \bar{v}_0]} E_{-0}U_0(v_0, v_{-0} \mid v_0),$$

and

$$E_{-i}U_i(\bar{v}, v_{-i} \mid \bar{v}) = \min_{v_i \in [\underline{v}, \bar{v}]} E_{-i}U_i(v_i, v_{-i} \mid v_i),$$

by (5.8), (P, t) satisfies IIR if the expression in (5.5) is nonnegative.

To show sufficiency, suppose conditions (i)-(iii) hold; we show the existence of payment functions $t_j(v)$s such that (P, t) is BIC, IIR and BB. We may consider the following payment function among many other possibilities:

$$\begin{aligned}
t_i(v) = \frac{1}{n}\int_{z_0=\underline{v}_0}^{v_0} z_0 d(E_{-0}P_0(z_0, v_{-0})) - \int_{z_i=\underline{v}}^{v_i} z_i d(-E_{-i}P_i(z_i, v_{-i})) \\
+ \bar{v}E_{-i}(P_i(\bar{v}, v_{-i})) - \frac{1}{n}E_{-i}\left(\int_{z_0=\underline{v}_0}^{v_0} z_0 d(E_{-0}P_0(z_0, v_{-0}))\right) \\
+ \int_{z_i=\underline{v}}^{\bar{v}} z_i d(-E_{-i}P_i(z_i, v_{-i})).
\end{aligned} \tag{5.10}$$

Since $E_{-0}P_0(v)$ is increasing in v_0 and $E_{-i}P_i(v)$s are decreasing in respective v_is, the first two terms on the right hand side of (5.10) are positive; the first term is a function of v_0 alone and the second term is a function of v_i alone; the remaining part are constant terms added make $E_{-i}U_i(\bar{v}, v_{-i} \mid \bar{v}) = 0$ for all $i = 1, \ldots, n$. To check that this payment function leads to BIC, we have for the buyer,

$$
\begin{aligned}
&E_{-0}U_0(v \mid \underline{v}_0) - E_{-0}U_0(\hat{v}_0, v_{-0} \mid \underline{v}_0) \\
&= v_0 E_{-0}P_0(v) - v_0 E_{-0}P_0(\hat{v}_0, v_{-0}) - E_{-0}\left(\sum_{i=1}^{n} t_i(v) - \sum_{i=1}^{n} t_i(\hat{v}_0, v_{-0})\right) \\
&= v_0 E_{-0}P_0(v_0, v_{-0}) - v_0 E_{-0}P_0(\hat{v}_0, v_{-0}) \\
&- E_{-0}\left[\int_{z_0=\underline{v}_0}^{v_0} z_0 d(E_{-0}P_0(z_0, v_{-0})) - \sum_{i=1}^{n}\int_{z_i=\underline{v}}^{v_i} z_i d(-E_{-i}P_i(z_i, v_{-i}))\right. \\
&\left.- \int_{z_0=\underline{v}_0}^{\hat{v}_0} z_0 d(E_{-0}P_0(z_0, v_{-0})) + \sum_{i=1}^{n}\int_{z_i=\underline{v}}^{v_i} z_i d(-E_{-i}P_i(z_i, v_{-i}))\right] \\
&= v_0 \int_{z_0=\hat{v}_0}^{v_0} d(E_{-0}P_0(z_0, v_{-0})) - \int_{z_0=\hat{v}_0}^{v_0} z_0 d(E_{-0}P_0(z_0, v_{-0})) \\
&= \int_{z_0=\hat{v}_0}^{v_0} (v_0 - z_0) d(E_{-0}P_0(z_0, v_{-0})) \\
&\geq 0.
\end{aligned}
$$

The first equality follows by the definition of expected utilities and the second by substitution of (5.10); the second and the fourth term under the square braces on the right hand side of the second equality cancel out; the remaining part is a function of v_0, and hence taking expectation over valuations other than 0 does not affect it. The inequality follows from the fact that if $v_0 > \hat{v}_0$ then $v_0 - z_0$ is positive throughout the range of the integral, and the sign of the differential is positive by assumption; if on the other hand, $v_0 < \hat{v}_0$, $v_0 - z_0$ is negative, so we can interchange the limits of the integral and replace $v_0 - z_0$ with $z_0 - v_0$ to obtain a positive sign. For the sellers, we have,

$$
\begin{aligned}
&E_{-i}U_i(v \mid v_i) - U_i(\hat{v}_i, v_{-i} \mid v_i) \\
&= E_{-i}t_i(v) - v_i E_{-i}P_i(v) - E_{-i}t_i(\hat{v}_i, v_{-i}) + v_i E_{-i}P_i(\hat{v}_i, v_{-i}) \\
&= E_{-i}\left(\int_{z_i=\underline{v}}^{\hat{v}_i} z_i d(-E_{-i}P_i(z_i, v_{-i})) - \int_{z_i=\underline{v}}^{v_i} z_i d(-E_{-i}P_i(z_i, v_{-i}))\right) \\
&- v_i E_{-i}P_i(v) + v_i E_{-i}(P_i(\hat{v}_i, v_{-i})) \\
&= E_{-i}\left(\int_{z_i=v_i}^{\hat{v}_i} z_i d(-E_{-i}(P_i(z_i, v_{-i}))) - v_i \int_{z_i=v_i}^{\hat{v}_i} d(-E_{-i}P_i(z_i, v_{-i}))\right) \\
&= \int_{z_i=v_i}^{\hat{v}_i} (z_i - v_i) d(-E_{-i}P_i(z_i, v_{-i})) \\
&\geq 0,
\end{aligned}
$$

by arguments similar to the case of the buyer. Since (P, t) is BIC, (5.5) is satisfied. Since $E_{-i}U_i(\bar{v}, v_{-i} \mid \bar{v}) = 0$ for all $i = 1, \ldots, n$ and the expression in (5.5) is pre-assumed to be positive, we must have $E_{-0}U_0(\underline{v}_0, v_{-0} \mid \underline{v}_0) \geq 0$. Since $E_{-0}U_0(v \mid v_0)$ is increasing in v_0 and $E_{-i}U_i(v \mid v_i)$'s are decreasing in v_i for all $i = 1, \ldots, n$, (P, t) must be IR. By construction, (P, t) is BB. ■

Since $U_j^{(P,t)}(v \mid v_j) = v_j P_j(v) - t_j(v)$, and (P, t) is BB, (5.1) can be re-written as

$$\max_{P,t} \int_{\underline{v}_0}^{\bar{v}_0} \underbrace{\int_{\underline{v}}^{\bar{v}} \cdots \int_{\underline{v}}^{\bar{v}}}_{n \text{ times}} \sum_{j=0}^{n} v_j P_j(v) f(v_1) \ldots f(v_n) g(v_0) dv_0 \ldots dv_n \tag{5.11}$$

subject to the expression (5.5) being nonnegative.

Then, using (5.2), our problem can be written as

$$\max_{P} \int_{\underline{v}_0}^{\bar{v}_0} \underbrace{\int_{\underline{v}}^{\bar{v}} \cdots \int_{\underline{v}}^{\bar{v}}}_{n \text{ times}} \sum_{j=0}^{n} v_j P_j(v) f(v_1) \ldots f(v_n) g(v_0) dv_0 \ldots dv_n$$
$$\text{such that } \int_{\underline{v}_0}^{\bar{v}_0} \underbrace{\int_{\underline{v}}^{\bar{v}} \cdots \int_{\underline{v}}^{\bar{v}}}_{n \text{ times}} \sum_{j=0}^{n} c_j(v_j, 1) P_j(v) f(v_1) \ldots f(v_n) g(v_0) dv_0 \ldots dv_n \geq 0.$$

We construct the Lagrangian objective function as follows.

$$\begin{aligned} \Lambda &= \int_{\underline{v}_0}^{\bar{v}_0} \underbrace{\int_{\underline{v}}^{\bar{v}} \cdots \int_{\underline{v}}^{\bar{v}}}_{n \text{ times}} \sum_{j=0}^{n} \left[v_j + \lambda c_j(v_j, 1)\right] P_j(v) f(v_1) \ldots f(v_n) g(v_0) dv_0 \ldots dv_n \\ &= (1+\lambda) \int_{\underline{v}_0}^{\bar{v}_0} \underbrace{\int_{\underline{v}}^{\bar{v}} \cdots \int_{\underline{v}}^{\bar{v}}}_{n \text{ times}} \sum_{j=0}^{n} c_j\left(v_j, \frac{\lambda}{1+\lambda}\right) P_j(v) f(v_1) \ldots f(v_n) g(v_0) dv_0 \ldots dv_n \end{aligned} \tag{5.12}$$

Any P for which (5.5) is zero and maximizes the Lagrangian (5.12) for some $\lambda \geq 0$ must be a solution to our problem. Let us write $\hat{c}_{[j]}(v, \alpha)$ for the jth lowest among $c_1(v_1, \alpha), \ldots, c_n(v_n, \alpha)$. Now consider the allocation rule P^α in (5.3). Observe that P^α maximizes Λ for $\alpha = \frac{\lambda}{1+\lambda}$. Now we need to show that P^α meets the conditions of Theorem 12, and there exists an α for which (5.5) is zero.

If $c_0(\cdot, 1)$ and $c_i(\cdot, 1)$s are increasing on $[\underline{v}_0, \bar{v}_0]$ and $[\underline{v}, \bar{v}]$ respectively, then for any $\alpha \in [0, 1]$, $c_0(\cdot, \alpha)$ and $c_i(\cdot, \alpha)$s are increasing. Therefore both P_0^α and P_i^α are increasing in v_0 and decreasing in v_i. It follows that $E_{-0}P_0^\alpha$ is increasing in v_0 and $E_{-i}P_i^\alpha$ is decreasing in v_i.

Let

$$L(\alpha) = \int_{\underline{v}_0}^{\bar{v}_0} \underbrace{\int_{\underline{v}}^{\bar{v}} \cdots \int_{\underline{v}}^{\bar{v}}}_{n \text{ times}} \sum_{j=0}^{n} c_j(v_j, 1) P_j^\alpha(v) f(v_1) \ldots f(v_n) g(v_0) dv_0 \ldots dv_n \tag{5.13}$$

By construction, $L(1) \geq 0$. Further, P_0^α is increasing in α and P_i^α's are decreasing in α. If $\alpha < \beta$, $L(\alpha)$ and $L(\beta)$ will be different only in profiles v such that $P^\beta(v) \equiv 0$ but $P_0^\alpha(v) = 1$, $P_i^\alpha(v) = -1$ where i corresponds to the lowest k among $\hat{c}_i(v_i, \alpha)$. This implies $c_0(v_0, \beta) < \sum_{j\in\{1,\ldots,k\}} \hat{c}_{[j]}(v, \beta)$ and therefore, $c_0(v_0, 1) < \sum_{j\in\{1,\ldots,k\}} \hat{c}_{[j]}(v, 1)$. So, P^β eliminates the virtually inefficient trades allowed by P^α. The equation $c_0(v_0, \alpha) = \sum_{j\in\{1,\ldots,k\}} \hat{c}_{[j]}(v_j, \alpha)$ has many solutions that vary continuously in α. Therefore, $L(\alpha)$ is continuous, increasing and $L(1) \geq 0$. If a successful mechanism exists, it solves (5.1). Since we are considering priors for which no successful mechanism exists, $L(0) < 0$. Hence there exists an $\alpha \in (0, 1]$ such that $L(\alpha) = 0$. ■

5.2.2 The LAC Model

The optimal mechanism for the LAC model $\langle \Gamma, k, \mu \rangle$ is characterized below. As noted in Chapter 3, an efficient allocation rule in this model is different from that in the LA model. Recall that a path in Γ is feasible if it contains at least k nodes. The buyer realizes a valuation of v_0 if she acquires a feasible path. Let there be q distinct feasible paths in Γ, where $q \geq 1$. Let the sums of valuations of sellers on these feasible paths be ordered as $S_{[1]}(v) \leq \ldots \leq S_{[q]}(v)$. Let the corresponding feasible paths be denoted as $\mathcal{P}_{[1]}(v), \ldots, \mathcal{P}_{[k]}(v)$. Efficiency requires trade to take place with sellers in $\mathcal{P}_{[1]}(v)$ if $v_0 > S_{[1]}(v)$. For example, in a 3 seller problem where the buyer wants two items, if buyer's valuation is 20 and sellers 1, 2 and 3 aligned on a line have valuations 4, 9 and 5 respectively, trade occurs with sellers 1 and 2. In the LA model with the same profile, trade would take place with sellers 1 and 3.

Theorem 6 *Fix an LAC model $\langle \Gamma, k, \mu \rangle$. For any $\alpha \geq 0$, let $\tilde{S}_{[1]}(v, \alpha) = \sum_{i\in\mathcal{P}_{[1]}(v)} c_i(v_i, \alpha)$. For all j, let $c_j(\cdot, 1)s$ be an increasing function of v_js. Then there exists $\alpha \in (0, 1]$ and payment functions $t_j(v)$, $j = 0, \ldots, n$, such that the following allocation rule P^α together with payment rule t solves (5.1):*

$$\tilde{P}_0^\alpha(v) = \begin{cases} 1 & \text{if } c_0(v_0, \alpha) > \tilde{S}_{[1]}(v, \alpha); \\ 0 & \text{otherwise}; \end{cases}$$
$$\tilde{P}_i^\alpha(v) = \begin{cases} -1 & \text{if } c_0(v_0, \alpha) > \tilde{S}_{[1]}(v, \alpha) \text{ and } i \in \tilde{\mathcal{P}}_{[1]}(v, \alpha); \\ 0 & \text{otherwise}, \end{cases} \tag{5.14}$$

Proof: We repeat the same steps as in the proof of Theorem 5 till we construct the following Lagrangian objective function.

$$\begin{aligned}\Lambda &= \int_{\underline{v}_0}^{\bar{v}_0} \underbrace{\int_{\underline{v}}^{\bar{v}} \dots \int_{\underline{v}}^{\bar{v}}}_{n \text{ times}} \sum_{j=0}^{n} \left[v_j + \lambda c_j(v_j, 1)\right] P_j(v) f(v_1) \dots f(v_n) g(v_0) dv_0 \dots dv_n \\ &= (1+\lambda) \int_{\underline{v}_0}^{\bar{v}_0} \underbrace{\int_{\underline{v}}^{\bar{v}} \dots \int_{\underline{v}}^{\bar{v}}}_{n \text{ times}} \sum_{j=0}^{n} c_j\left(v_j, \frac{\lambda}{1+\lambda}\right) P_j(v) f(v_1) \dots f(v_n) g(v_0) dv_0 \dots dv_n \qquad (5.15)\end{aligned}$$

We observe that $\tilde{P}^{\alpha}$ maximizes Λ for $\alpha = \frac{\lambda}{1+\lambda}$. Now we need to show that $\tilde{P}^{\alpha}$ meets the conditions of Theorem 12, and there exists an α for which the expression in (5.5) is zero.

If $c_0(\cdot, 1)$ and $c_i(\cdot, 1)$'s are increasing on $[\underline{v}_0, \bar{v}_0]$ and $[\underline{v}, \bar{v}]$, then for any $\alpha \in [0, 1]$, $c_0(\cdot, \alpha)$ and $c_i(\cdot, \alpha)$ are increasing. Therefore both $\tilde{P}_0^{\alpha}$ and $\tilde{P}_i^{\alpha}$ are increasing in v_0 and decreasing in v_i and consequently $E_{-0}\tilde{P}_0^{\alpha}$ is increasing in v_0 and $E_{-i}\tilde{P}_i^{\alpha}$ is decreasing in v_i.

Let

$$L^c(\alpha) = \int_{\underline{v}_0}^{\bar{v}_0} \underbrace{\int_{\underline{v}}^{\bar{v}} \dots \int_{\underline{v}}^{\bar{v}}}_{n \text{ times}} \sum_{j=0}^{n} c_j(v_j, 1) \tilde{P}_j^{\alpha}(v) f(v_1) \dots f(v_n) g(v_0) dv_0 \dots dv_n \qquad (5.16)$$

By construction, $L^c(1) \geq 0$. Further, $\tilde{P}_0^{\alpha}$ is increasing in α and $\tilde{P}_i^{\alpha}$'s are decreasing in α. Therefore, if $\alpha < \beta$, $L^c(\alpha)$ and $L^c(\beta)$ will be different only in profiles v such that $\tilde{P}^{\beta}(v) \equiv 0$ but $\tilde{P}_0^{\alpha}(v) = 1$, $\tilde{P}_i^{\alpha}(v) = -1$ where $i \in \tilde{\mathcal{P}}_{[1]}(v, \alpha)$. This implies $c_0(v_0, \beta) \leq \tilde{S}_{[1]}(v, \beta)$ and so $c_0(v_0, 1) \leq \tilde{S}_{[1]}(v, 1)$. So, $\tilde{P}^{\beta}$ eliminates the virtually inefficient trades allowed by $\tilde{P}^{\alpha}$. The equation $c_0(v_0, \alpha) = \tilde{S}_{[1]}(v, \alpha)$ has many solutions that vary continuously in α. Therefore, $L^c(\alpha)$ is continuous, increasing and $L^c(1) \geq 0$. If a successful mechanism does not exist, $L^c(0) < 0$. Hence there exists an $\alpha \in (0, 1]$ such that $L^c(\alpha) = 0$. ■

5.2.3 Asymptotic Properties of the Optimal Mechanism

In this subsection, we establish a general relationship between the asymptotic properties of the VCG and optimal mechanism.

Proposition **7** *Suppose* $E\left(\sum_{j=0}^{n} t_j^V(v)\right) < 0$. *The expected sum of VCG payments converges to zero as* $n \to \infty$ *if and only if the expected welfare in the optimal mechanism converges to the expected welfare of the (ex-post) efficient mechanism: as* $n \to \infty$,

$$E\left(\sum_{j=0}^{n} t_j^V(v)\right) \to 0 \iff E\left(\sum_{j=0}^{n} E_{-j} U_j^{OPT}(v \mid v_j)\right) \to E\left(\sum_{j=0}^{n} v_j P_j^*(v)\right) \qquad (5.17)$$

Proof: **If part:** Let

$$E\left(\sum_{j=0}^{n} E_{-j}U_j^{OPT}(v \mid v_j)\right) \to E\left(\sum_{j=0}^{n} v_j P_j^*(v)\right)$$

for all v as $n \to \infty$. Since $U_j^V(v \mid v_j) = v_j P_j^*(v) - t_j^V(v)$, this implies

$$\sum_{j=0}^{n} E\left(U_j^{OPT}(v \mid v_j)\right) \to E\left(\sum_{j=0}^{n} U_j^V(v \mid v_j)\right) + E\left(\sum_{j=0}^{n} t_j^V(v)\right)$$
$$\text{or, } E\left(\sum_{j=0}^{n} t_j^V(v)\right) \to \sum_{j=0}^{n} E\left(U_j^{OPT}(v \mid v_j)\right) - E\left(\sum_{j=0}^{n} U_j^V(v \mid v_j)\right).$$

But by hypothesis, the left hand side of this relation is negative. Also, by definition of the optimal mechanism, the right hand side of this relation is non-negative. Therefore, $E\left(\sum_{j=0}^{n} t_j^V(v)\right) \to 0$.

Only if part: Let $E\left(\sum_{j=0}^{n} t_j^V(v)\right) \to 0$ as $n \to \infty$. Then,

$$E\left(\sum_{j=0}^{n} U_j^V(v \mid v_j)\right) = E\left(\sum_{j=0}^{n} \left(v_j P_j^*(v) - t_j^V(v)\right)\right) \to E\left(\sum_{j=0}^{n} v_j P_j^*(v)\right)$$

Since

$$E\left(\sum_{j=0}^{n} U_j^V(v)\right) \leq E\left(\sum_{j=0}^{n} U_j^{OPT}(v)\right) \leq E\left(\sum_{j=0}^{n} v_j P_j^*(v)\right),$$

it follows that,

$$E\left(\sum_{j=0}^{n} E_{-j}U_j^{OPT}(v \mid v_j)\right) \to E\left(\sum_{j=0}^{n} v_j P_j^*(v)\right).$$

■

The intuition behind Proposition 7 is as follows: recall that VCG is ex-post efficient and hence results in the highest possible expected welfare among all mechanisms that are BIC and IIR. On the other hand, the optimal mechanism results in the highest possible expected welfare among all mechanisms that are BIC, IIR and BB. If the expected budget surplus of VCG is zero in the limit as n becomes large, then it implies that VCG is optimal in the limit as n becomes large. Therefore, in the limit, expected utilities of all agents in VCG and optimal mechanism must be the same. For the converse claim, if the sum of expected utilities of all agents in VCG and optimal mechanism are the same in the limit, expected budget surplus of VCG must be zero in the limit.

Proposition 7 is useful in relating the results of the previous chapter to that of this chapter. Recall that condition TL requires $\underline{v}_0 > k\underline{v}$. Since convergence in probability implies convergence in distribution, if condition TL holds then for large n, $E\left(\sum_{j=0}^{n} t_j^V(v)\right) > 0$ for the LA model $\langle n, k, \mu\rangle$ as well as the LAC model with line contiguity, $\langle L(n), k, \mu\rangle$. This implies that if TL holds, the optimal mechanism is ex-post efficient for large n in these models. Further, we also stated that if $\underline{v}_0 = k\underline{v}$, the sum of VCG payments converges to zero as number of sellers become large. By Proposition 7, the optimal mechanism becomes asymptotically efficient. Similar results hold for contiguity structures with critical sellers when the counterpart of the TL condition for such models is substituted appropriately. We write these observations in the form of Corollaries given below.

Corollary **3** *Suppose TL holds. Then there exists an integer N such that for $n > N$, the optimal mechanism is ex-post efficient in both the LA problem $\langle n, k, \mu\rangle$ as well as in the LAC problem with line contiguity $\langle L(n), k, \mu\rangle$.*

Corollary **4** *Suppose $\underline{v}_0 = k\underline{v}$. Then for both the sequences $\langle m, k, \mu\rangle_{m=n}^{\infty}$ as well as in the LAC problem with line contiguity $\langle L(m), k, \mu\rangle_{m=n}^{\infty}$,*

$$E\left(\sum_{j=0}^{n} E_{-j}U_j^{OPT}(v \mid v_j)\right) \to E\left(\sum_{j=0}^{n} v_j P_j^*(v)\right).$$

as $n \to \infty$.

Corollary **5** *Let C be the number of critical sellers in a sequence of graphs $\langle \Gamma(n), k, \mu\rangle$ with the preservation property. Suppose $\underline{v}_0 > C\bar{v} + (k - C)\underline{v}$. Then there exists an integer N such that for $n > N$, the optimal mechanism is ex-post efficient in $\langle \Gamma(n), k, \mu\rangle$.*

Corollary **6** *Let C be the number of critical sellers in a sequence of graphs $\langle \Gamma(n), k, \mu\rangle_{m=n}^{\infty}$ with the preservation property. Suppose $\underline{v}_0 = C\bar{v} + (k - C)\underline{v}$. Then for $\langle \Gamma(n), k, \mu\rangle_{m=n}^{\infty}$,*

$$E\left(\sum_{j=0}^{n} E_{-j}U_j^{OPT}(v \mid v_j)\right) \to E\left(\sum_{j=0}^{n} v_j P_j^*(v)\right).$$

as $n \to \infty$.

5.3 Discussion and Examples

The constant α corresponding to the optimal mechanism has an interpretation. As shown in (5.12) and (5.15), α is related to the Lagrangian multiplier λ corresponding to the feasibility condition $(P, t) \in \mathscr{M}$ or equivalently, the expression in (5.5) being nonnegative. If $\lambda = 0$, then $\alpha = 0$ and efficiency is achieved by an optimal mechanism. If $\lambda > 0$ then $\alpha > 0$, and efficiency is not achievable. Therefore, α denotes the relative impact of the feasibility constraint on the solution of (5.1).

The value of the optimal α depends on the priors as well as on n and k. If it is a prior for which a successful mechanism exists, as characterized in the previous chapters, then the expression in (5.5) is strictly positive and $\alpha = 0$. Otherwise, $\alpha \in (0, 1]$.

Example **9** Two mechanisms that are DSIC, IR and BB, and therefore lie in $\mathscr{M}$, are the *posted price mechanism* (PP) and the mechanism by Ghatak and Ghosh (2011) (MGG). The mechanism PP is defined as follows: announce a constant c; trade takes place with the k sellers with lowest valuations at price c if their valuations lie below c and the valuation of the buyer is above kc. The constant c can be chosen to maximize the possibility of trade. The mechanism MGG is a reverse Vickrey auction defined for $n > k$. Trade takes place with the k sellers with lowest valuations at a price equivalent to the $k + 1$-th lowest valuation if the buyer's valuation is larger than k times this price.

The table below compares the performance of these mechanisms with that of the optimal mechanism. All valuations are distributed independently and uniformly on $[0, 1]$. The first column specifies n and k. The second column specifies the value of α in the optimal mechanism[1]. The third column specifies the posted price in the optimal posted price mechanism. The next three columns show the expected welfare in these three mechanisms and that in an unconstrained efficient mechanism.

Table 5.1: Relative Performance of Different Mechanisms

Specification	α	Price Posted	OPT Welfare	PP Welfare	MGG Welfare	Unconst. Welfare
$n = k = 1$	0.33	0.5	0.14	0.125	NA	0.167
$n = k = 2$	0.5	0.33	0.025	0.018	NA	0.042
$n = 3, k = 2$	0.48	0.317	0.021	0.013	0.015	0.03

Example **10** Consider an LAC model where $n = 3$ and $k = 2$ and the underlying graph is a line. If valuations are drawn independently from $U[0, 1]$, the optimal mechanism is given by $\alpha \approx 0.479669$. This results in an expected welfare of 0.042. The expected welfare for an unconstrained efficient mechanism is 0.0667.

5.4 Conclusion

In this chapter, we characterized the optimal mechanism for the LA and LAC problems. These extend the characterization of the optimal mechanism for the bilateral trade problem by Myerson and Satterthwaite (1983) in an intuitive way. Further, we also establish a useful connection between the asymptotic properties of the optimal mechanism and the VCG mechanism.

[1]Computing the α for general LA models $\langle n, k, \mu \rangle$ is computationally difficult.

Bibliography

Adamopoulos, T. and D. Restuccia (2014): “The size distribution of farms and international productivity differences,” *The American Economic Review*, 104, 1667–1697.

Bailey, M. J. (1997): “The demand revealing process: to distribute the surplus,” *Public Choice*, 91, 107–126.

Bergemann, D. and J. Välimäki (2006): “Information in Mechanism Design,” Centre for Economic and Policy Research.

Cai, H. (2000): “Delay in multilateral bargaining under complete information,” *Journal of Economic Theory*, 93, 260–276.

——— (2003): “Inefficient Markov Perfect Equilibria in Multilateral Bargaining,” *Economic Theory*.

Cavallo, R. (2006): “Optimal decision-making with minimal waste: Strategyproof redistribution of VCG payments,” in *International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, Hakodate, Japan, 882–889.

Chakravorty, S. (2013): *The Price of Land: Acquisition, Conflict, Consequence*, Oxford University Press, New Delhi.

Chatterjee, K. and W. Samuelson (1983): “Bargaining under Incomplete Information,” *Operations Research*, 31, 835–851.

Clarke, E. H. (1971): “Multipart Pricing of Public Goods,” *Public Choice*, 11, 17–33.

Cramton, P., R. Gibbons, and P. Klemperer (1987): “Dissolving a partnership efficiently,” *Econometrica*, 615–632.

Cripps, M. W. and J. M. Swinkels (2006): “Efficiency of Large Double Auctions,” *Econometrica*, 74, 47–92.

Fudenberg, D., M. Mobius, and A. Szeidl (2007): “Existence of equilibrium in large double auctions,” *Journal of Economic Theory*, 133, 550–567.

Ghatak, M. and P. Ghosh (2011): "The Land Acquisition Bill: A Critique and a Proposal," *Economic & Political Weekly*, 46, 65.

Ghatak, M., S. Mitra, D. Mookherjee, and A. Nath (2013): "Land Acquisition and Compensation: What Really Happened in Singur?" *Economic and Political Weekly*.

Government of India, M. o. L. and Justice (1894): "The Land Acquisition Act, 1894," http://dolr.nic.in/hyperlink/acq.htm.

——— (2013): "The Right to Fair Compensation and Transparency in Land Acquisition, Rehabilitation and Resettlement Act, 2013," http://indiacode.nic.in/acts-in-pdf/302013.pdf.

Green, J., E. Kohlberg, and J.-J. Laffont (1976): "Partial equilibrium approach to the free-rider problem," *Journal of Public Economics*, 6, 375–394.

Green, J. and J. Laffont (1977): "Characterization of satisfactory mechanisms for the revelation of preferences for public goods," *Econometrica*, 45, 427–438.

Gresik, T. A. and M. A. Satterthwaite (1989): "The Rate at Which a Simple Market Converges to Efficiency as the Number of Traders Increase: An Asymptotic Result for Optimal Trading Mechanism," *Journal of Economic Theory*, 48, 304–332.

Grossman, Z., J. Pincus, and P. Shapiro (2010): "A second-best mechanism for land assembly," Working paper.

Grotius, H. (1925): *De Jure Belli ac Pacis (On the Law of War and Peace)(1625), trans. Francis W. Kelsey*, Oxford: Clarendon Press.

Groves, T. (1973): "Incentives in Teams," *Econometrica*, 41, 617–631.

Heller, M. and R. Hills (2008): "Land assembly districts," *Harvard Law Review*, 1465–1527.

Holmstrom, B. and R. B. Myerson (1983): "Efficient and Durable Decision Rules with Incomplete Information," *Econometrica*, 51, 1799–1819.

Jackson, M. O. (2000): "Mechanism theory," *The Encyclopedia of Life Support Systems*.

Jackson, M. O., H. Sonnenschein, and Y. Xing (2015): "The Efficiency of Bargaining with Many Items," *Available at SSRN 2580894*.

Kominers, S. D. and E. G. Weyl (2011): "Concordance among holdouts," in *Proceedings of the 12th ACM conference on Electronic commerce*, ACM, 219–220.

——— (2012): "Holdout in the assembly of complements: A problem for market design," *The American Economic Review*, 102, 360–365.

Krishna, V. (2002): *Auction Theory*, Academic Press.

Krishna, V. and M. Perry (2000): "Efficient mechanism design," Working Paper.

Lehmann, D., L. Oćallaghan, and Y. Shoham (2002): "Truth revelation in approximately efficient combinatorial auctions," *Journal of the ACM (JACM)*, 49, 577–602.

Mailath, G. J. and A. Postlewaite (1990): "Asymmetric information bargaining problems with many agents," *The Review of Economic Studies*, 57.

Makowski, L. and C. Mezzetti (1993): "The possibility of efficient mechanisms for trading an indivisible object," *Journal of Economic Theory*, 59, 451–465.

——— (1994): "Bayesian and weakly robust first best mechanisms: characterizations," *Journal of Economic Theory*, 64, 500–519.

McAfee, R. P. (1991): "Efficient allocation with continuous quantities," *Journal of Economic Theory*, 53, 51–74.

——— (1992): "A Dominant Strategy Double Auction," *Journal of Economic Theory*, 56, 434–450.

Menezes, F. and R. Pitchford (2004): "A Model of Seller Holdout," *Economic Theory*.

Miceli, T. J. (2011): *The economic theory of eminent domain: private property, public use*, Cambridge University Press.

Myerson, R. and M. Satterthwaite (1983): "Efficient Mechanisms for Bilateral Trading," *Journal of Economic Theory*, 29, 265–281.

Plassmann, F. and T. Tideman (2010): "Providing Incentives for Efficient Land Assembly," Tech. rep., working paper.

Reny, P. J. and M. Perry (2003): "Toward a Strategic Foundation for Rational Expectations Equilibrium," Working Paper.

Rob, R. (1982): "Asymptotic efficiency of the demand revealing mechanism," *Journal of Economic Theory*, 28, 207–220.

Roy Chowdhury, P. and K. Sengupta (2012): "Transparency, complementarity and holdout," *Games and Economic Behavior*, 75, 598–612.

Royal Swedish Academy of Sciences (2007): "Scientific Bacground on Mechanism Design Theory," http://www.nobelprize.org/nobel_prizes/economic-sciences/laureates/2007/advanced-economicsciences2007.pdf.

Rustichini, A., M. A. Satterthwaite, and S. R. Williams (1994): "Convergence to Efficiency in a Simple Market with Incomplete Information," *Econometrica*, 62, 1041–63.

Satterthwaite, M. A. and S. R. Williams (1989a): "Bilateral Trade with the Sealed Bid k-Double Auction: Existence and Efficiency," *Journal of Economic Theory*, 48, 107–133.

——— (1989b): "The Rate of Convergence to Efficiency in the Buyer's Bid Double Auction as the Market Becomes Large," *Review of Economic Studies*, 56, 477–498.

——— (2002): "The Optimality of a Simple Market Mechanism," *Econometrica*, 70, 1841–1863.

Schweizer, U. (2006): "Universal possibility and impossibility results," *Games and Economic Behavior*, 57, 73–85.

Sen, A. (2007): "The Theory of Mechanism Design: An Overview," *Economic and Political Weekly*, 42, 8–13.

Singh, R. (2012): "Inefficiency and Abuse of Compulsory Land Acquisition: An Enquiry into the Way Forward," CDE Working Paper No. 209.

Tideman, T. N. and G. Tullock (1976): "A new and superior process for making social choices," *The Journal of Political Economy*, 1145–1159.

Vickrey, W. (1961): "Counterspeculation, Auctions and Competitive Selaed Tenders," *Journal of Finance*, 16, 8–37.

Wikipedia (2014a): "Nandigram violence — Wikipedia, The Free Encyclopedia," .

——— (2014b): "Singur Tata Nano controversy — Wikipedia, The Free Encyclopedia," .

Williams, S. R. (1991): "Existence and Convergence of Equilibria in the Buyer's Bid Double Auction," *Review of Economic Studies*, 58, 351–374.

——— (1999): "A Characterization of Efficient, Bayesian Incentive Compatible Mechanisms," *Economic Theory*, 14, 155–180.

Wilson, R. (1981): "Incentive Efficiency of Double Auctions," *Econometrica*, 53, 1101–1115.

——— (1987): "Game-Theoretic Approaches to Trading Processes," in *Advances in Economic Theory: Fifth World Congress*, ed. by T. Bewley, Cambridge University Press, chap. 2, 33–77.

Xiao, J. (2010): "Bargaining Order in a Multi-Person Bargaining Game," *Available at SSRN 1685463*.

Yoon, K. (2001): "The Modified Vickrey Double Auction," *Journal of Economic Theory*, 101, 572–584.