K-Theory of Quadratic Modules:
A Study of Roy's Elementary Orthogonal Group

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Dedicated

to

My Parents and Teachers

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Abstract

This thesis discusses the K-theory of quadratic modules by studying Roy's elementary orthogonal group of the quadratic space $Q \perp H(P)$ over a commutative ring A. We establish a set of commutator relations among the elementary generators of Roy's elementary orthogonal group and use this to prove Quillen's local-global principle for this elementary group. We also obtain a result on extendability of quadratic modules. We establish normality of the elementary orthogonal group under certain conditions and prove stability results for the K_1 group of this orthogonal group. We also prove that Roy's elementary orthogonal group and Petrov's odd hyperbolic unitary group coincides when the quadratic modules Q and P are free.

Contents

Chapter 1

Introduction

In its most familiar versions, algebraic K-theory consists of the study of groups of classes of algebraic objects. It focuses on a sequence of abelian groups $K_n(A)$ associated to each ring A which encode deep arithmetic information about the ring. The first of these is $K_0(A)$, the Grothendieck group which generalizes the construction of the ideal class group of a ring, using projective modules. It is used to create a dimension for R-modules that lack a basis. The group K_1 was defined by H. Bass, K_2 by J. Milnor and, subsequently, higher K-functors by D. Quillen and others. The group $K_1(A)$ generalizes the group of units of a ring. The group $K_2(A)$ measures the fine details of row-reduction of matrices over A.

In 1976, D. Quillen and A.A. Suslin independently proved the famous local-global principle to settle the question of J.-P. Serre as to whether projective modules over a polynomial extension of a field are free. This principle demonstrates that a finitely presented module over a polynomial ring $R[X]$ is extended from R if and only if it is locally extended from $R_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of the commutative ring R. Later, J.-P. Serre related these questions with the question of efficient generation of ideals in polynomial rings. Over the years, several new cases and versions of the local-global principle have been established. Related to these is the dilation principle which says the following: Suppose $\alpha(X) \in \mathrm{GL}(n, R[X])$ is such that $\alpha(0) = I$ and $\alpha_s(X) \in \mathrm{E}(n, R_s[X])$ for some non-nilpotent element $s \in R$. Then there exists some $\beta(X) \in \mathrm{E}(n, R[X])$ such that $\beta(0) = I$ and $\beta_s(X) = \alpha(bX) \in \mathrm{E}(n, R[X])$, where $b \in s^l R$, for some $l \gg 0$.

Localization is one of the most powerful tools in the study of structure of quadratic modules and more generally, of algebraic groups over rings. It helps to reduce many important problems over arbitrary commutative rings to similar problems for semilocal rings. There are two well-known versions of localization: *localization and patching* as proposed by D. Quillen in [43] and A.A. Suslin in [57], and *localization-completion* as proposed by A. Bak in [12].

A. Roy studied a generalization of quadratic forms and their similarity groups over projective modules in his Ph. D. thesis. In this work, we study these quadratic modules and the corresponding orthogonal groups and establish extendability results. Towards that, we establish a *dilation principle* and a *local-global principle.* We use these to deduce the *action version of local-global principle.* We also prove normality of the Roy's elementary orthogonal group in the corresponding orthogonal group and a stability theorem for the corresponding quotient group K_1. The analysis of these quadratic modules involves finding suitable commutator formulae among the elementary generators of Roy's orthogonal group. The commutator relations turn out to be rather technical and we obtain these relations by relating the elementary generators of Roy's group to a different group studied by G. Tang. We then verify them directly by hand, though a knowledge of the software GAP (see [26]) helped in discovering their form in very small dimensions. We obtain several such commutator formulae and apply them to the proofs of the above mentioned results.

To describe the results more precisely, let A be a commutative Noetherian ring in which 2 is invertible and let B be the polynomial A-algebra $A[X_1, \ldots, X_n]$ in n indeterminates. Let $Q = (Q, q)$ be a quadratic space over B and let $Q_0 = (Q_0, q_0)$ be the reduction of Q modulo the ideal of B generated by $X_1, \ldots, X_n$. In [58], A.A. Suslin and V.I. Kopeĭko proved that if Q is stably extended from A and for every maximal ideal $\mathfrak{m}$ of A, the Witt index of $A_{\mathfrak{m}} \otimes_A (Q_0, q_0)$ is larger than the Krull dimension of A, then (Q, q) is extended from A. In the doctoral thesis of R.A. Rao (see [44, 45]), it was shown that one can improve this result to Witt index at least d, when A is a local ring at a non-singular point of an affine variety of dimension d over an infinite field. Moreover, a question posed at the end of the thesis asks whether extendability can be shown for quadratic spaces with

Witt index at least d over polynomial extensions of any equicharacteristic regular local ring of dimension d. In this thesis, we give an affirmative answer to this question. The analysis of the equicharacteristic regular local ring is done by a patching argument, akin to the one developed by A. Roy in his paper [49]. This argument reduces the problem to the case of a complete equicharacteristic regular ring; which is a power series ring over a field, provided one can patch the information.

We show that the patching process is possible by establishing a local-global principle for the elementary orthogonal group of a quadratic space with a hyperbolic summand. For this, we follow the broad outline of A.A. Suslin's method in [57] which leads to a K_1 analogue of D. Quillen's local-global principle in [43]. Instead of using Suslin's 'theory of generic elementary forms', we follow the more 'hands-on' approach via the yoga of commutators. For this, we first find an appropriate generating set for Roy's group using a lemma of V. Suresh in [55].

1.1 Roy's Orthogonal Group

1.1.1 A Brief Historical Review

A. Roy defined elementary orthogonal transformations in [48] for quadratic spaces with a hyperbolic summand over a commutative ring in which 2 is invertible. These transformations (over fields) are classically known as *Siegel transformations* or *Eichler transformations* in the literature. These transformations (in matrix form) of quadratic spaces (V, q) over finite fields was defined by L.E. Dickson in p.126, p.135 of [23], which is an unaltered republication of the first edition (Teubner, Leipzig, 1901).

Later in [24], J. Dieudonné extended Dickson's results to infinite fields. These orthogonal transformations (in matrix form) over general fields, also appeared in the paper [50] of C.L. Siegel with an alternate interpretation in [51]. There he used it to define the mass for the representation of 0 by an indefinite quadratic form. M. Eichler studied these transformations of $Q \perp H(k)$, where $H(k)$ is the hyperbolic plane, in his study of the orthogonal group over fields k and made the first systematic use of them in his famous book

"Quadratische Formen und Orthogonale Gruppen", first published in 1952, and reprinted in 1974 (Eichler credits Siegel's 1935 paper for introducing these transformations in the notes in §3, p.212 of his book, and also refers to the paper [51] of Siegel in p.218).

A. Roy studied C.T.C. Wall's paper [63], who relied on Eichler's book and rewrote the transformations of Eichler appearing in Wall's paper. In his doctoral thesis (1967), A. Roy generalizes these transformations to any commutative ring R in which 2 is invertible. We shall call these the DSER (Dickson-Siegel-Eichler-Roy) elementary orthogonal transformations or just Roy's elementary orthogonal transformation group. A. Bak was aware of Roy's transformations which he mentions in the introduction of his doctoral thesis. A. Bak and L.N. Vaserstein independently defined transformations over Λ-rings in their respective doctoral theses which reminds us of Roy's transformations. However, the groups generated by these are not always comparable to the one generated by Roy's transformations.

In [39], V. Petrov introduced a new classical-like group called *odd unitary group* over odd form rings. This group generalizes and unifies all known classical groups such as the quadratic groups of A. Bak (see [11,28]), Hermitian groups (see [11,33]), classical Chevalley groups, and the group $\mathrm{U}_{2n+1}(R)$ of E. Abe (see [1]). V. Petrov established normality of the elementary subgroup of odd unitary group and surjective stability for odd unitary K_1. In [39], Petrov describes the elementary subgroup of an odd hyperbolic unitary group. We shall compare this group over commutative ring with Roy's group in Section 3.2 of Chapter 3.

We will see that Roy's elementary group coincides with Petrov's odd hyperbolic unitary group over commutative rings! Indeed, we first verified that the former is contained in the latter but realized later that the groups are the same. In other words, one may think of our study of Roy's group as a concrete realization of Petrov's group. We can now ask the question: Is the ESD group the correct generalization of Roy's group to form rings which is the concrete realization of Petrov's group?

Let G be an isotropic reductive algebraic group over a commutative ring R. In [40], V. Petrov and A. Stavrova introduced the notion of an elementary subgroup $\mathrm{E}(R)$ of the group of points $G(R)$. Let P be a parabolic subgroup of the reductive group G over R, and let U_P be its unipotent radical. There is a unique parabolic subgroup P^- in G that is opposite to P with respect to L_P. Then they define the elementary subgroup $\mathrm{E}_P(R)$ corresponding to P as the subgroup of $G(R)$ generated as an abstract group by $\mathrm{U}_P(R)$ and $\mathrm{U}_{P^-}(R)$. In [40, §7, Example 2], they state that the elementary subgroup $\mathrm{E}_P(R)$ of $\mathrm{O}^+(V,Q)$, where V is a projective module of rank $2n$ endowed with a nondegenerate quadratic form Q, coincides with the group generated by the so-called Eichler-Siegel-Dickson transvections. Here $\mathrm{O}^+(V,Q)$ denote the kernel of the Dickson map (see [33]) from the orthogonal group $\mathrm{O}(V,Q)$. As Roy's elementary transformations can be realized as Eichler-Siegel-Dickson transvections, Roy's elementary group is contained in the above mentioned elementary group.

However, we do not yet know if Roy's group coincides with the group generated by ESD transvections or not.

1.1.2 Preliminaries

Let A be a commutative ring in which 2 is invertible. A *quadratic A-module* is a pair (M,q), where M is an A-module and q is a quadratic form on M. Let M^* denote the dual of the module M. Let B_q be the symmetric bilinear form associated to q on M, which is given by $B_q(x,y) = q(x+y) - q(x) - q(y)$ and the induced map $d_{B_q} : M \to M^*$ is given by $d_{B_q}(x) = B_q(x,-)$ for $x \in M$. We say that (M,q) is a non-singular quadratic space or q is a non-singular quadratic form if d_{B_q} is an isomorphism. A *quadratic space* over A is a pair (M,q), where M is a finitely generated projective A-module and $q : M \longrightarrow A$ is a non-singular quadratic form. Given two quadratic A-modules (M_1,q_1) and (M_2,q_2), their orthogonal sum (M,q) is defined by taking $M = M_1 \oplus M_2$ and $q((x_1,x_2)) = q_1(x_1)+q_2(x_2)$ for $x_1 \in M_1, x_2 \in M_2$. Denote (M,q) by $(M_1,q_1) \perp (M_2,q_2)$ and q by $q_1 \perp q_2$.

Let P be a finitely generated projective A-module. The module $P \oplus P^*$ has a natural quadratic form given by $p((x,f)) = f(x)$ for $x \in P$ and $f \in P^*$. The corresponding bilinear form B_p is given by

$$B_p((x_1,f_1),(x_2,f_2)) = f_1(x_2) + f_2(x_1)$$

for $x_1, x_2 \in P$ and $f_1, f_2 \in P^*$.

Definition 1.1.1. The quadratic space $(P \oplus P^*, p)$, denoted by $H(P)$, is called the *hyperbolic space* of P. A quadratic space M is said to be hyperbolic, if it is isometric to $H(P)$ for some finitely generated projective module P. The quadratic space $H(A)$ is called a *hyperbolic plane*. The orthogonal sum $H(A) \perp H(A) \perp \cdots \perp H(A)$ of n hyperbolic planes is denoted by $H(A)^n$.

Definition 1.1.2. Let Q be a quadratic space.

(a) Q is said to have Witt index $\geq n$ if $Q \cong Q_0 \perp H(P)$, where $\operatorname{rank}(P) \geq n$.

(b) Q is said to have *hyperbolic* rank $\geq n$ if $Q \perp H(A)^k$ with $k \geq n$.

(c) Q is said to be *cancellative* if, for any quadratic A-spaces Q_1, Q_2 with $Q \perp Q_2 \cong Q_1 \perp Q_2$, then $Q \cong Q_1$.

If $Q \perp H(A) \cong Q_1 \perp H(A)$ implies $Q \cong Q_1$, then Q is cancellative.

Let Q be a quadratic A-space and P be a finitely generated projective A-module. Let $M = Q \perp H(P)$. This is a quadratic space with the quadratic form $q \perp p$. The associated bilinear form on M, denoted by $\langle \cdot, \cdot \rangle$, is given by

$$\langle (a, x), (b, y) \rangle = B_q(a, b) + B_p(x, y) \text{ for all } a, b \in Q \text{ and } x, y \in H(P),$$

where B_q and B_p are the bilinear forms on Q and P respectively.

Let $M = M(B, q)$ be a quadratic module over A with quadratic form q and associated symmetric bilinear form B. Then the orthogonal group of M is defined as follows:

$$\mathrm{O}_A(M) = \{\sigma \in \mathrm{Aut}_A(M) \mid q(\sigma(x)) = q(x) \text{ for all } x \in M\}, \tag{1.1.1}$$

where $\mathrm{Aut}_A(M)$ be the group of all A-linear automorphisms of M.

For A-linear maps $\alpha : Q \to P$ and $\beta : Q \to P^*$, the dual maps $\alpha^t : P^* \to Q^*$ and $\beta^t : P^{**} \simeq P \to Q^*$ are defined as $\alpha^t(\varphi) = \varphi \circ \alpha$ and $\beta^t(\varphi^*) = \varphi^* \circ \beta$ for $\varphi \in P^*$ and $\varphi^* \in P^{**}$.

We now recall from [48] that the A-linear maps $\alpha^* : P^* \to Q$ and $\beta^* : P \to Q$ are defined by $\alpha^* = d_{B_q}^{-1} \circ \alpha^t$ and $\beta^* = d_{B_q}^{-1} \circ \beta^t \circ \varepsilon$, where ε is the natural isomorphism $P \to P^{**}$. These maps are characterized by the relations

$$(f \circ \alpha)(z) = B_q(\alpha^*(f), z) \text{ for } f \in P^*, z \in Q \tag{1.1.2}$$

$$\text{and } (\beta(z))(x) = B_q(\beta^*(x), z) \text{ for } x \in P, z \in Q. \tag{1.1.3}$$

In [48], A. Roy defined the "*elementary*" transformations E_α and E^*_β of $Q \perp H(P)$ as

$$\begin{array}{ll|ll} E_\alpha(z) & = z + \alpha(z) & E^*_\beta(z) & = z + \beta(z) \\ E_\alpha(x) & = x & E^*_\beta(x) & = -\beta^*(x) + x - \frac{1}{2}\beta\beta^*(x) \\ E_\alpha(f) & = -\alpha^*(f) - \frac{1}{2}\alpha\alpha^*(f) + f & E^*_\beta(f) & = f \end{array}$$

for $z \in Q, x \in P$ and $f \in P^*$. In the same article, he also observed that these transformations are orthogonal with respect to the above quadratic form $q \perp p$.

The orthogonal group of $Q \perp H(P)$ is denoted by $\mathrm{O}_A(Q \perp H(P))$, where Q and P are finitely generated projective A-modules.

Definition 1.1.3. $\mathrm{EO}_A(Q \perp H(P))$ is defined to be the subgroup of $\mathrm{O}_A(Q \perp H(P))$ generated by E_α and E^*_β, where $\alpha \in \mathrm{Hom}_A(Q, P)$ and $\beta \in \mathrm{Hom}_A(Q, P^*)$. We call this group *Roy's elementary orthogonal group* and these transformations *Roy's elementary orthogonal transformations.*

Definition 1.1.4. For a ring R, an $R[T_1, \cdots, T_n]$-module M is *extended* from R if there exists an R-module M_0 such that $M \cong R[T_1, \cdots, T_n] \otimes_R M_0$.

More generally, if $\phi : B \to C$ is a homomorphism of rings and Q is a quadratic C-space, then we say that Q extends from B if there is a quadratic B-space Q_0 with $Q \cong Q_0 \otimes_B C$.

In [43], D. Quillen gave the following remarkable local-global criterion for a module M to be extended.

Theorem 1.1.5 (Quillen's Patching Theorem)**.** *Let A be a commutative ring. Assume M is a finitely presented module over $A[T]$ and that $M_{\mathfrak{m}}$ is an extended $A_{\mathfrak{m}}[T]$-module for each maximal ideal $\mathfrak{m}$ of A. Then M is extended.*

1.1.3 Elementary Generators in the Free Case

In this section, we assume that P and Q are free A-modules of rank m and n respectively. Then P and P^* can be identified with A^m and Q can be identified with A^n. Let $\{z_i : 1 \leq i \leq n\}$ be a basis for Q, $\{g_i : 1 \leq i \leq n\}$ be a basis for Q^*, $\{x_i : 1 \leq i \leq m\}$ be a basis for P and $\{f_i : 1 \leq i \leq m\}$ be a basis for P^*.

For a free A-module A^r of rank r, we have the projection maps $p_i : A^r \longrightarrow A$ for $1 \leq i \leq r$, which are the projections onto the i^{th} component and the inclusion maps $\eta_i : A \longrightarrow A^r$ for $1 \leq i \leq r$ which are the inclusions into the i^{th} component.

For $\alpha \in \mathrm{Hom}_A(Q, P)$ and for $1 \leq i \leq m$ and $1 \leq j \leq n$, let $\alpha_i, \alpha_{ij} \in \mathrm{Hom}_A(Q, P)$ be the maps given by

$$\alpha_i := \eta_i \circ p_i \circ \alpha \quad \text{and} \quad \alpha_{ij} := \eta_i \circ p_i \circ \alpha \circ \eta_j \circ p_j.$$

Clearly $\alpha = \Sigma_{i=1}^m \alpha_i = \Sigma_{i=1}^m \Sigma_{j=1}^n \alpha_{ij}$ Then $\alpha_i^*, \alpha_{ij}^* \in \mathrm{Hom}_A(P^*, Q)$ are the maps given by

$$\alpha_i^* := (\alpha_i)^* = \alpha^* \circ \eta_i \circ p_i \quad \text{and} \quad \alpha_{ij}^* := (\alpha_{ij})^* = \eta_j \circ p_j \circ \alpha^* \circ \eta_i \circ p_i.$$

Also, $\alpha^* = \Sigma_{i=1}^m \alpha_i^* = \Sigma_{i=1}^m \Sigma_{j=1}^n \alpha_{ij}^*$.

We can also see that these definitions of α_i^* and α_{ij}^* coincide with those obtained by using $\alpha^* = {d_{B_q}}^{-1} \circ \alpha^t \in \mathrm{Hom}_A(P^*, Q)$ for α_i and α_{ij}.

Now we shall describe how the linear transformations $E_{\alpha_{ij}}$ and $E^*_{\beta_{ij}}$ are defined in terms of the bases given above.

Let $z = \Sigma_{l=1}^n d_l z_l \in Q$ for some $d_l \in A$. Then, for $1 \leq k \leq m$ and $1 \leq l \leq n$,

$$\begin{aligned} \alpha(z_l) &= \Sigma_{k=1}^m b_{kl} x_k \text{ for some } b_{kl} \in A, \\ \alpha(z) &= \Sigma_{l=1}^n \Sigma_{k=1}^m d_j b_{kl} x_k, \\ \alpha_k(z) &= \Sigma_{l=1}^n d_l b_{kl} x_k \text{ and } \alpha_{kl}(z) = d_l b_{kl} x_k. \end{aligned}$$

For $1 \leq k \leq m$, let $w_k = \alpha^*(f_k)$. If $f = \Sigma_{k=1}^m c_k f_k$ for some $c_k \in A$, then $c_k = \langle f, x_k \rangle$ and so $\alpha^*(f) = \Sigma_{k=1}^m \langle f, x_k \rangle w_k$. If $w_k = \Sigma_{l=1}^n y_l z_l$ for some $y_l \in A$, then $w_{kl} = y_l z_l \in Q$.

For $1 \leq i, k \leq m$ and $1 \leq j \leq n$, the maps α_i^* and α_{ij}^*'s are given by

$$\alpha_i^*(f_k) = \begin{cases} w_i & \text{if} \quad k = i, \\ 0 & \text{if} \quad k \neq i. \end{cases} \qquad \alpha_{ij}^*(f_k) = \begin{cases} w_{ij} & \text{if} \quad k = i, \\ 0 & \text{if} \quad k \neq i. \end{cases}$$

Let $\beta \in \mathrm{Hom}_A(Q, P^*)$. Set $\beta^*(x_i) = v_i$ for some $v_i \in Q$. Let v_{ij} denote the element $\eta_j \circ p_j(v_i)$. Now, by defining the maps $\beta_i, \beta_{ij}, \beta_i^*$, β_{ij}^* similarly and extending these to the whole of $Q \perp H(P)$, we will get the maps as follows:
For $z \in Q$, $x \in P$, $f \in P^*$, $1 \leq i \leq m$ and $1 \leq j \leq n$;

$$\begin{aligned}
\alpha_{ij}(z, x, f) &= (0, \langle w_{ij}, z\rangle x_i, 0), & \beta_{ij}(z, x, f) &= (0, 0, \langle v_{ij}, z\rangle f_i)\,, \\
\alpha_i(z, x, f) &= (0, \langle w_i, z\rangle x_i, 0)\,, & \beta_i(z, x, f) &= (0, 0, \langle v_i, z\rangle f_i)\,, \\
\alpha(z, x, f) &= (0, \Sigma_{i=1}^m \langle w_i, z\rangle x_i, 0)\,, & \beta(z, x, f) &= (0, 0, \Sigma_{i=1}^m \langle v_i, z\rangle f_i)\,, \\
\alpha_{ij}^*(z, x, f) &= (\langle f, x_i\rangle w_{ij}, 0, 0)\,, & \beta_{ij}^*(z, x, f) &= (\langle x, f_i\rangle v_{ij}, 0, 0)\,, \\
\alpha_i^*(z, x, f) &= (\langle f, x_i\rangle w_i, 0, 0)\,, & \beta_i^*(z, x, f) &= (\langle x, f_i\rangle v_i, 0, 0)\,, \\
\alpha^*(z, x, f) &= (\Sigma_{i=1}^m \langle f, x_i\rangle w_i, 0, 0)\,, & \beta^*(z, x, f) &= (\Sigma_{i=1}^m \langle x, f_i\rangle v_i, 0, 0)\,.
\end{aligned}$$

With these notations, the orthogonal transformation $E_{\alpha_{ij}}$ of $Q \perp H(P)$ for $\alpha \in \mathrm{Hom}_A(Q, P)$ is given by the equation

$$\begin{aligned}
E_{\alpha_{ij}}(z, x, f) =& \Big(I - \alpha_{ij}^* + \alpha_{ij} - \frac{1}{2}\alpha_{ij}\alpha_{ij}^*\Big)(z, x, f) \\
=& \Big(z - \langle f, x_i\rangle w_{ij},\ x + \langle w_{ij}, z\rangle x_i - \langle f, x_i\rangle q(w_{ij}) x_i,\ f\Big).
\end{aligned}$$

The orthogonal transformation $E_{\beta_{ij}}^*$ of $Q \perp H(P)$ for $\beta \in \mathrm{Hom}_A(Q, P^*)$ is given by

$$\begin{aligned}
E_{\beta_{ij}}^*(z, x, f) =& \Big(I - \beta_{ij}^* + \beta_{ij} - \frac{1}{2}\beta_{ij}\beta_{ij}^*\Big)(z, x, f) \\
=& \Big(z - \langle f_i, x\rangle v_{ij},\ x,\ f + \langle v_{ij}, z\rangle f_i - \langle x, f_i\rangle q(v_{ij}) f_i\ \Big).
\end{aligned}$$

The inverses of the orthogonal transformations $E_{\alpha_{ij}}$ and $E_{\beta_{ij}}^*$ are given by the following:
For $1 \leq i \leq m$ and $1 \leq j \leq n$,

$$E_{\alpha_{ij}}^{-1}(z, x, f) = \Big(z + \langle f, x_i\rangle w_{ij}, x - \langle w_{ij}, z\rangle x_i - \langle f, x_i\rangle q(w_{ij}) x_i, f\Big),$$

$$E_{\beta_{ij}}^{*^{-1}}(z,x,f) = \Big(z + \langle f_i, x\rangle v_{ij},\ x,\ f - \langle v_{ij}, z\rangle f_i - \langle x, f_i\rangle q(v_{ij}) f_i\Big).$$

Since Q and P are free modules, the elements of $\mathrm{O}_A(Q \perp H(P))$ can be represented as matrices over A by choosing a basis for Q and P. i.e., we can identify $\mathrm{O}_A(Q \perp H(P))$ as a subgroup of $\mathrm{GL}_{n+2m}(A)$ and we shall denote it by $\mathrm{O}_A(Q \perp H(A)^m)$.

If Q and P are free A-modules of rank n and m respectively, then we have the elementary transformations of the type $E_{\alpha_{ij}}$ and $E^*_{\beta_{ij}}$ for $1 \le i \le m$, $1 \le j \le n$. We shall use these generators and the relations among them to prove our results. We shall denote the group $\mathrm{EO}_A(Q \perp H(P))$ by $\mathrm{EO}_A(Q \perp H(A)^m)$.

The following lemma gives a characterization of an element in the orthogonal group.

Lemma 1.1.6. *An $(n+2m) \times (n+2m)$ matrix* $T = \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{d} & \mathbf{e} & \mathbf{f} \\ \mathbf{g} & \mathbf{h} & \mathbf{j} \end{pmatrix}$ *belongs to* $\mathrm{O}_A(Q \perp H(A)^m)$ *if and only if either of the following two equations hold*:

(i) $T^t \psi T = \psi$, *for* $\psi = \begin{pmatrix} \phi & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_m & 0 \end{pmatrix}$, *where ϕ is the matrix associated to the bilinear form B_q and* $\begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$ *is the matrix of the hyperbolic form on $H(A)^m$.*

(ii) $$\begin{pmatrix} \phi^{-1}A^t\phi & \phi^{-1}H^t & \phi^{-1}D^t \\ C^t\phi & K^t & G^t \\ B^t\phi & J^t & F^t \end{pmatrix} \cdot \begin{pmatrix} A & B & C \\ D & F & G \\ H & J & K \end{pmatrix} = I_{(n+2m)\times(n+2m)}.$$

Proof. Let $\psi = \varphi \perp \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$, where φ is the matrix associated to the bilinear form B_q on Q and $\begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$ be the matrix of the hyperbolic form B_p on $H(A)^m$. By equation (1.1.1), it follows that $T \in \mathrm{O}_A(Q \perp H(A)^m)$ if and only if $T^t \psi T = \psi$. That is,

$$\begin{pmatrix} \mathbf{a} & \mathbf{d} & \mathbf{g} \\ \mathbf{b} & \mathbf{e} & \mathbf{h} \\ \mathbf{c} & \mathbf{f} & \mathbf{j} \end{pmatrix} \begin{pmatrix} \varphi & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_m & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{d} & \mathbf{e} & \mathbf{f} \\ \mathbf{g} & \mathbf{h} & \mathbf{j} \end{pmatrix} = \begin{pmatrix} \varphi & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_m & 0 \end{pmatrix}.$$

This equation is equivalent to the following set of equations:

$$\begin{aligned} &\mathbf{a}^t\varphi\mathbf{a} + \mathbf{g}^t\mathbf{c} + \mathbf{c}^t\mathbf{g} = \varphi, && \mathbf{b}^t\varphi\mathbf{a} + \mathbf{h}^t\mathbf{c} + \mathbf{e}^t\mathbf{g} = 0, && \mathbf{c}^t\varphi\mathbf{a} + \mathbf{j}^t\mathbf{c} + \mathbf{f}^t\mathbf{g} = 0, \\ &\mathbf{a}^t\varphi\mathbf{b} + \mathbf{g}^t\mathbf{e} + \mathbf{c}^t\mathbf{h} = 0, && \mathbf{b}^t\varphi\mathbf{b} + \mathbf{h}^t\mathbf{e} + \mathbf{e}^t\mathbf{h} = 0, && \mathbf{c}^t\varphi\mathbf{b} + \mathbf{j}^t\mathbf{e} + \mathbf{f}^t\mathbf{h} = I_m, \\ &\mathbf{a}^t\varphi\mathbf{c} + \mathbf{g}^t\mathbf{f} + \mathbf{c}^t\mathbf{j} = 0, && \mathbf{b}^t\varphi\mathbf{c} + \mathbf{h}^t\mathbf{f} + \mathbf{e}^t\mathbf{j} = I_m, && \mathbf{c}^t\varphi\mathbf{c} + \mathbf{j}^t\mathbf{f} + \mathbf{f}^t\mathbf{j} = 0. \end{aligned}$$

These equations are equivalent to the equation

$$T^{-1}T = I_{(n+2m)\times(n+2m)}, \text{ where } T^{-1} = \begin{pmatrix} \varphi^{-1}\mathbf{a}^t\varphi & \varphi^{-1}\mathbf{g}^t & \varphi^{-1}\mathbf{d}^t \\ \mathbf{c}^t\varphi & \mathbf{j}^t & \mathbf{f}^t \\ \mathbf{b}^t\varphi & \mathbf{h}^t & \mathbf{e}^t \end{pmatrix}. \qquad \square$$

This characterization helps us to prove normality. Also, we shall use the natural embedding $\mathrm{O}_A(Q \perp H(A)^m) \longrightarrow \mathrm{O}_A(Q \perp H(A)^{m+1})$ of groups for proving normality. Using this, define the stable orthogonal group and the stable elementary orthogonal group as follows:

$$\mathrm{O}_A = \lim_{m\to\infty} \mathrm{O}_A(Q \perp H(A)^m) \text{ and}$$

$$\mathrm{EO}_A = \lim_{m\to\infty} \mathrm{EO}_A(Q \perp H(A)^m).$$

Define $KO_{1,m}(Q \perp H(A)^m) = \mathrm{O}_A(Q \perp H(A)^m)/\mathrm{EO}_A(Q \perp H(A)^m)$, which is a coset space. The natural embedding $\mathrm{O}_A(Q \perp H(A)^m) \longrightarrow \mathrm{O}_A(Q \perp H(A)^{m+1})$ induces the stabilization map on the corresponding coset spaces.

1.2 Some more Definitions

In this section, we first recall the notion of *generalized dimension function* from [41]. Let $\mathcal{P} \subset \operatorname{Spec} A$ be a set of primes, $\mathbb{N}$ be the set of natural numbers and $d : \mathcal{P} \to \mathbb{N} \cup \{0\}$ be a function. For primes $\mathfrak{p}, \mathfrak{q}$ of $\mathcal{P}$, define a partial order $\ll$ on $\mathcal{P}$ as $\mathfrak{p} \ll \mathfrak{q}$ if and only if $\mathfrak{p} \subset \mathfrak{q}$ and $d(\mathfrak{p}) > d(\mathfrak{q})$.

Definition 1.2.1. A function $d : \mathcal{P} \to \mathbb{N} \cup \{0\}$ is a generalized dimension function if, for any ideal I of A, $V(I) \cap \mathcal{P}$ has only a finite number of minimal elements with respect to the partial ordering $\ll$.

Definition 1.2.2. We say that (P, B) is an *inner product space* (IPS) over a commutative ring R if $P \in \mathfrak{P}(R)$ (i.e., P is a finitely generated projective R-module) and $B : P \times P \to R$ is a symmetric bilinear form, satisfying the following "nonsingularity" condition:

$$\text{For any } f \in P^* = \mathrm{Hom}_R(P, R), \text{ there exists a unique } m \in P \text{ such that } f = B(-, m). \tag{1.2.1}$$

Definition 1.2.3. Let $f : R \to R'$ be a homomorphism of commutative rings. We say that an IPS (P', B') over R' is *extended from* the IPS (P, B) over R if we can write $P' = R' \otimes_R P$ and B' is given by

$$B'(r'_1 \otimes m_1, r'_2 \otimes m_2) = r'_1 r'_2 . f(B(m_1, m_2)) \; (r'_i \in R', m_i \in P).$$

An IPS (P'_1, B'_1) over R' is *stably extended* from R if there exist IPS's $(P'_2, B'_2), (P'_3, B'_3)$ extended from R such that

$$(P'_1, B'_1) \perp (P'_2, B'_2) \cong (P'_3, B'_3).$$

See [36, Chapter VII] for more details on inner product spaces.

Definition 1.2.4. Let A be an associative ring with identity. A vector $(a_1, \ldots, a_n)$ with coefficients $a_i \in A$ is called right unimodular if there are elements $b_1, \ldots, b_n \in A$ such that

$$a_1 b_1 + \ldots + a_n b_n = 1.$$

Definition 1.2.5. The ring A is said to satisfy Bass's **stable range condition** SA_l in the formulation of L.N. Vaserstein if, whenever $(a_1, \ldots a_{l+1})$ is a unimodular vector, there exist elements $b_1, \ldots, b_l \in A$ such that $(a_1 + a_{l+1} b_1, \ldots, a_l + a_{l+1} b_l)$ is unimodular.

It follows easily that $SA_l \Rightarrow SA_k$ for any $k \geq l$.

Definition 1.2.6. The **stable rank** of A, s-rank (A) is defined to be the smallest positive integer k such that A satisfies SA_l. If no such l exists, then the stable rank of A can be taken to be infinite. If A is a local ring, s-rank $(A) = 1$.

Definition 1.2.7. If V is an A-module and $v \in V$, the *order ideal* of v is defined by

$$O(v) = \{\alpha(v) | \alpha \in \mathrm{Hom}_A(V, A)\}.$$

Let R be a ring with 1 and pseudoinvolution $\sigma : R \to R, a \mapsto \overline{a}$. Let Λ be a form parameter in the sense of Bak.

Definition 1.2.8. The ring R is said to satisfy the Λ-stable rank condition Λ-SA_l if $SA_l \leq l$ and for every unimodular vector $(a_1, \ldots a_{l+1}, b_1, \ldots b_{l+1})^t \in R^{2l+2}$, there exists an $(l+1) \times (l+1)$ matrix β with $\overline{\beta} = \overline{1}\beta\overline{1}$ and $\overline{1}\beta_{ii} \in \Lambda$, such that $(a_1, \ldots a_{l+1})^t + \beta(b_1, \ldots b_{l+1})^t \in R^{l+1}$ is unimodular.

In this thesis, we shall be dealing with the case $\Lambda = 0$. i.e., when the ring is commutative.

Let $H_1, H_2, \ldots, H_r$ be subsets of a group G. Then $H_1 H_2 \cdot \ldots \cdot H_s$ denote their Minkowski product $H_1 H_2 \cdot \ldots \cdot H_r = \{h_1 h_2 \cdot \ldots \cdot h_r | h_i \in H_i\}$.

Definition 1.2.9 (Patching Technique)**.** Let **Quad** (R) denote the category of all quadratic R-spaces. Given that $\phi : B \to A$ is analytically isomorphic along a non-zero divisor s in B, we can state that the corresponding square

$$\begin{array}{ccc} \mathbf{Quad}\,(B) & \longrightarrow & \mathbf{Quad}\,(B_s) \\ \downarrow & & \downarrow \\ \mathbf{Quad}\,(A) & \longrightarrow & \mathbf{Quad}\,(A_s) \end{array}$$

is cartesian.

Given $Q_1 \in \mathbf{Quad}\,(B_s), Q_2 \in \mathbf{Quad}\,(A)$, we denote their fibre product over an isomorphism $\sigma : Q_1 \otimes A_s \xrightarrow{\cong} (Q_2)_s$ of quadratic A_s-spaces, by either $Q_1 \otimes_\sigma Q_2$ or by a triple (Q_1, σ, Q_2).

Let $Q = (Q_1, \sigma, Q_2)$ be a quadratic B-space for some $\sigma \in \mathrm{O}_{A_s}(Q \otimes A_s)$. An element $\varepsilon \in \mathrm{O}_{A_s}(Q \otimes A_s)$ is defined to be a *deeply split orthogonal transformation* if, for sufficiently large integer N, one can split ε as a product $\varepsilon = (\varepsilon_1)_s(\varepsilon_2 \otimes 1)$ with $\varepsilon_i \in \mathrm{O}(Q_i)$ for $i = 1, 2$ and $\varepsilon_2 \equiv I \bmod (s^N)$.

Definition 1.2.10. Let A be a local ring with maximal ideal $\mathfrak{m}$. We call A an *equicharacteristic* local ring if A has the same characteristic as its residue field $A/\mathfrak{m}$.

Definition 1.2.11. Let A be a local ring with maximal ideal $\mathfrak{m}$. A is said to be *complete with respect to its* $\mathfrak{m}$*-adic topology* if the natural map from A to $\varprojlim A/\mathfrak{m}^i$ is an isomorphism.

Definition 1.2.12. A regular local ring is said to be *unramified* if the characteristic of the residue field is $p \neq 0$ and p is in $\mathfrak{m}$, then p is not in $\mathfrak{m}^2$.

Notation 1.2.13. Let G be a group. For any $x, y \in G$, the commutator of x and y is denoted by $[x, y] = xyx^{-1}y^{-1}$.

1.3 Chapter-wise Summary

In Chapter 2, we state and give the explicit proofs of several commutator relations among the elementary generators for the elementary orthogonal group $\mathrm{EO}_A(Q \perp H(P))$, where A is a commutative ring, Q is a non-singular quadratic A-space of rank n and $H(P)$ is the hyperbolic space of a finitely generated projective module P of rank m with the natural quadratic form on it. We prove the commutator relations where Q and P are free modules. These proofs constitute the second chapter of this thesis and, are part of the preprint named "*Yoga of Commutators in Roy's Elementary Orthogonal Group*".

With the aid of these commutator relations, we establish a "local-global principle" of D. Quillen for the Dickson–Siegel–Eichler–Roy (DSER) elementary orthogonal transformations and a dilation principle. In this chapter, we also deduce an action version of the local-global principle. These results will appear in Chapter 3 and are used in Chapter 4 to prove certain extendability results on quadratic modules. As an interesting by-product, we realize from the yoga of commutators that the DSER group mimics G. Tang's Hermitian group (see [60]) in some features, and also the unitary transvection group of H. Bass defined in [16] in some ways. We prove that the DSER group is contained in the ESD group. We also compare the DSER group with Petrov's odd hyperbolic unitary group and show that they coincide when the projective A-modules Q and P are free and A is a commutative ring in which 2 is invertible. *In particular, the proofs of the local-global principle, normality and stability that we give for Roy's group yield proofs for the group of Petrov over a commutative ring when the projective modules Q and P are free.*

In Chapter 4, we prove that a quadratic $A[T]$-module Q with Witt index (Q/TQ) at least d, where d is the dimension of the equicharacteristic regular local ring A, is extended from A. This improves a theorem of R.A. Rao who proved it when A is the local ring at a smooth point of an affine variety over an infinite field. These results are part of an article titled "*Extendability of quadratic modules over a polynomial extension of an equicharacteristic regular local ring*" (see [5]). To establish this result, we apply the "local-global principle" established for the Dickson–Siegel–Eichler–Roy (DSER) elementary orthogonal transformations in Chapter 3.

In Chapter 5, once again we use the commutator relations of Chapter 2 to establish the we establish normality results for DSER group and stability results for DSER group under Bak's Λ-stable range condition. In particular, we establish normality when $m \geq \dim \operatorname{Max}(A) + 2$ and also when $m > l$ provided A satisfies the stable range condition 0-SA_l. This shows that the corresponding coset space K_1 is a group. We prove the surjective and injective stability of K_1 under the 0-stable range condition. We also prove the injective stability for K_1 of the orthogonal group under stable range condition. A useful tool in the proof is a decomposition theorem for the elementary subgroup that we will establish on the way under the usual stable range condition.

Chapter 2

Commutator Calculus in Roy's Elementary Orthogonal Group

For elementary groups, commutator relations are useful tools for establishing theories like local-global principle, normality etc. It involves a large body of calculations which is known as *commutator calculus*. The standard commutator formulas for GL_n was proved by L.N. Vaserstein in [62] and independently by Z.I. Borewich and N.A. Vavilov in [20]. The commutator calculus for relative elementary congruence subgroups are done in [29–31]. These commutator relations are generalized to a Chevalley group $G(R)$ over a commutative ring R by A. Stepanov in [54].

In this chapter, we establish various commutator relations among the elementary generators of Roy's elementary orthogonal group which were defined in Chapter 1. Obtaining commutator relations is the key to establish the local-global principle and the normality of the elementary subgroup in the orthogonal group we are looking at. We will use these commutator relations to prove the local-global principle over a polynomial extension in Chapter 3 and use them to prove the normality of the elementary orthogonal group in Chapter 5 .

Most of the results in this chapter are from [4].

2.1 Commutators of Elementary Transformations

In this section, we establish various commutator relations among the elementary generators of Roy's elementary orthogonal group. We will carry out the computations in two different ways - one is by choosing bases (which we call the method *using coordinates*), and the other by just using the formal definition without choosing bases (which we call the *coordinate-free method*). We need commutator relations of length up to 16. By the 'length'of a commutator, we mean the number of words in the commutator expression. We begin by recalling the definition of Roy's elementary generators by both methods which was done in the previous chapter.

The following is a coordinate-free definition of the elementary generators.

Definition 2.1.1. For $\theta \in \mathrm{Hom}_A(Q, P)$ or $\mathrm{Hom}_A(Q, P^*)$, define θ^* as $d_{B_q}^{-1} \circ \theta^t$ or $d_{B_q}^{-1} \circ \theta^t \circ \varepsilon$, where ε is the natural isomorphism $P \to P^{**}$ according to whether $\theta \in \mathrm{Hom}_A(Q, P)$ or $\mathrm{Hom}_A(Q, P^*)$ respectively. Then the elementary transformations E_θ and E_θ^{-1} are given by

$$E_\theta = I + \theta - \theta^* - \frac{1}{2}\theta\theta^*,$$
$$E_\theta^{-1} = I - \theta + \theta^* - \frac{1}{2}\theta\theta^* = E_{(-\theta)}.$$

We now recall the definition of elementary generators using coordinates from Chapter 1.

Definition 2.1.2. Let $\alpha, \delta \in \mathrm{Hom}_A(Q, P)$; $\beta, \gamma \in \mathrm{Hom}_A(Q, P^*)$ and $w_i, t_i, v_i, c_i \in Q$ for $1 \leq i \leq m$. Then, choosing bases $\{x_i\}_{i=1}^m, \{f_i\}_{i=1}^m, \{z_i\}_{i=1}^m$ respectively for P, P^*, Q, one can define the following elements in $\mathrm{Hom}_A(Q \perp H(P))$.

$$\begin{aligned}
\alpha_{ij}(z,x,f) &= (0, \langle w_{ij}, z\rangle x_i, 0), & \alpha^*_{ij}(z,x,f) &= (\langle f, x_i\rangle w_{ij}, 0, 0),\\
\delta_{kl}(z,x,f) &= (0, \langle t_{kl}, z\rangle x_k, 0), & \delta^*_{kl}(z,x,f) &= (\langle f, x_k\rangle t_{kl}, 0, 0),\\
\beta_{ij}(z,x,f) &= (0, 0, \langle v_{ij}, z\rangle f_i), & \beta^*_{ij}(z,x,f) &= (\langle x, f_i\rangle v_{ij}, 0, 0),\\
\gamma_{kl}(z,x,f) &= (0, 0, \langle c_{kl}, z\rangle f_k), & \gamma^*_{kl}(z,x,f) &= (\langle x, f_k\rangle c_{kl}, 0, 0).
\end{aligned}$$

Here w_{ij}, v_{ij} denote the elements $\eta_j \circ p_j(w_i), \eta_j \circ p_j(v_i)$ respectively and c_{kl}, t_{kl} denote the elements $\eta_l \circ p_l(c_k), \eta_l \circ p_l(t_k)$, where p_j is the j^{th} projection as defined in Section 1.1.3 of Chapter 1.

Now, for $1 \leq i,k \leq m$ and $1 \leq j,l \leq n$, the corresponding orthogonal transformations $E_{\alpha_{ij}}, E_{\delta_{kl}}, E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}$ and their inverses have the following form.

$$\begin{aligned}
E_{\alpha_{ij}}(z,x,f) &= \Big(z - \langle f, x_i\rangle w_{ij},\ x + \langle w_{ij}, z\rangle x_i - \langle f, x_i\rangle q(w_{ij}) x_i,\ f\Big),\\
E_{\delta_{kl}}(z,x,f) &= \Big(z - \langle f, x_k\rangle t_{kl},\ x + \langle t_{kl}, z\rangle x_k - \langle f, x_k\rangle q(t_{kl}) x_k,\ f\Big),\\
E^*_{\beta_{ij}}(z,x,f) &= \Big(z - \langle f_i, x\rangle v_{ij},\ x,\ f + \langle v_{ij}, z\rangle f_i - \langle x, f_i\rangle q(v_{ij}) f_i\Big),\\
E^*_{\gamma_{kl}}(z,x,f) &= \Big(z - \langle f_k, x\rangle c_{kl},\ x, f + \langle c_{kl}, z\rangle f_k - \langle x, f_k\rangle q(c_{kl}) f_k\Big),\\
E^{-1}_{\alpha_{ij}}(z,x,f) &= \Big(z + \langle f, x_i\rangle w_{ij}, x - \langle w_{ij}, z\rangle x_i - \langle f, x_i\rangle q(w_{ij}) x_i, f\Big),\\
E^{-1}_{\delta_{kl}}(z,x,f) &= \Big(\ z + \langle f, x_k\rangle t_{kl}, x - \langle t_{kl}, z\rangle x_k - \langle f, x_k\rangle q(t_{kl}) x_k, f\Big),\\
E^{*^{-1}}_{\beta_{ij}}(z,x,f) &= \Big(z + \langle f_i, x\rangle v_{ij},\ x,\ f - \langle v_{ij}, z\rangle f_i - \langle x, f_i\rangle q(v_{ij}) f_i\Big),\\
E^{*^{-1}}_{\gamma_{kl}}(z,x,f) &= \Big(z + \langle f_k, x\rangle c_{kl},\ x, f - \langle c_{kl}, z\rangle f_k - \langle x, f_k\rangle q(c_{kl}) f_k\Big).
\end{aligned}$$

The first (and the simplest) set of commutators which we compute is between elementary generators corresponding to two elements of $\mathrm{Hom}_A(Q,P)$; this is given in the following lemma.

Lemma 2.1.3. *Let $\alpha, \delta \in \mathrm{Hom}_A(Q,P)$. Then, for i,j,k,l with $1 \leq i,k \leq m$ and $1 \leq j,l \leq n$, the commutator of the type $\Big[E_{\alpha_{ij}}, E_{\delta_{kl}}\Big]$ is given by*

$$\begin{aligned}
\Big[E_{\alpha_{ij}}, E_{\delta_{kl}}\Big](z,x,f) =& \Big(I + \delta_{kl}\alpha^*_{ij} - \alpha_{ij}\delta^*_{kl}\Big)(z,x,f)\\
=& \Big(z,\ x + \langle f, x_i\rangle\langle t_{kl}, w_{ij}\rangle x_k - \langle f, x_k\rangle\langle w_{ij}, t_{kl}\rangle x_i,\ f\Big).
\end{aligned}$$

In particular, if $i = k$, then $\Big[E_{\alpha_{ij}}, E_{\delta_{kl}}\Big] = I$.

Proof. For $\alpha, \delta \in \mathrm{Hom}_A(Q,P)$ and for any i,j,k,l with $1 \leq i,k \leq m$ and $1 \leq j,l \leq n$, using the coordinate-free definition of the elementary generators, we have the commutator relation

$$
\begin{aligned}
&\Big[E_{\alpha_{ij}}, E_{\delta_{kl}}\Big](z,x,f)\\
&\qquad = E_{\alpha_{ij}} E_{\delta_{kl}} E_{\alpha_{ij}}^{-1} E_{\delta_{kl}}^{-1}(z,x,f)\\
&\qquad = E_{\alpha_{ij}} E_{\delta_{kl}} E_{\alpha_{ij}}^{-1}\Big(\Big(I-\delta_{kl}+\delta_{kl}^*-\frac{1}{2}\delta_{kl}\delta_{kl}^*\Big)(z,x,f)\Big)\\
&\qquad = E_{\alpha_{ij}} E_{\delta_{kl}}\Big(\Big(I-\delta_{kl}+\delta_{kl}^*-\frac{1}{2}\delta_{kl}\delta_{kl}^*-\alpha_{ij}+\alpha_{ij}^*-\frac{1}{2}\alpha_{ij}\alpha_{ij}^*-\alpha_{ij}\delta_{kl}^*\Big)(z,x,f)\Big)\\
&\qquad = E_{\alpha_{ij}}\Big(\Big(I-\alpha_{ij}+\alpha_{ij}^*-\frac{1}{2}\alpha_{ij}\alpha_{ij}^*-\alpha_{ij}\delta_{kl}^*+\delta_{kl}\alpha_{ij}^*\Big)(z,x,f)\Big)\\
&\qquad = \Big(I-\alpha_{ij}\delta_{kl}^*+\delta_{kl}\alpha_{ij}^*\Big)(z,x,f).
\end{aligned}
$$

Using coordinates, we may compute the above commutator as

$$
\begin{aligned}
&\Big[E_{\alpha_{ij}}, E_{\delta_{kl}}\Big](z,x,f)\\
&\qquad = E_{\alpha_{ij}} E_{\delta_{kl}} E_{\alpha_{ij}}^{-1}\Big(z+\langle f,x_k\rangle t_{kl},\ x-\langle t_{kl},z\rangle x_k-\langle f,x_k\rangle q(t_{kl})x_k,\ f\Big)\\
&\qquad = E_{\alpha_{ij}} E_{\delta_{kl}}\Big(z+\langle f,x_i\rangle w_{ij}+\langle f,x_k\rangle t_{kl},\ x-\Big\{\langle w_{ij},z\rangle+\langle f,x_i\rangle q\left(w_{ij}\right)\\
&\qquad\qquad +\langle f,x_k\rangle\langle w_{ij},t_{kl}\rangle\Big\}x_i-\Big\{\langle t_{kl},z\rangle+\langle f,x_k\rangle q(t_{kl})\Big\}x_k,\ f\Big)\\
&\qquad = E_{\alpha_{ij}}\Big(\ z+\langle f,x_i\rangle w_{ij},\ x-\Big\{\langle w_{ij},z\rangle+q(w_{ij})\langle f,x_i\rangle+\langle f,x_k\rangle\langle w_{ij},t_{kl}\rangle\Big\}x_i\\
&\qquad\qquad +\langle f,x_i\rangle\langle t_{kl},w_{ij}\rangle x_k, f\Big)\\
&\qquad = \Big(\ z,\ x+\langle f,x_i\rangle\langle t_{kl},w_{ij}\rangle x_k-\langle f,x_k\rangle\langle w_{ij},t_{kl}\rangle x_i,\ f\Big).
\end{aligned}
$$

If $i=k$, then we have

$$
\delta_{kl}\alpha_{ij}^*(z,x,f)=\Big(0,\langle f,x_i\rangle\langle t_{il},w_{ij}\rangle x_i,0\Big)=\alpha_{ij}\delta_{kl}^*(z,x,f).
$$

Hence $\Big[E_{\alpha_{ij}}, E_{\delta_{il}}\Big]=I.$ □

As a consequence of this lemma, we have the following commutator relations.

Corollary 2.1.4. *For any i,j,k,l with $1\le i,k\le m$, $1\le j,l\le n$ and for $a,b,c,d\in A$ with $ab=cd$, the following equation holds.*

$$
\Big[E_{a\alpha_{ij}}, E_{b\delta_{kl}}\Big]=\Big[E_{c\alpha_{ij}}, E_{d\delta_{kl}}\Big].
$$

Proof. For $\alpha, \delta \in \mathrm{Hom}_A(Q,P)$ and for any i,j,k,l with $1 \le i,k \le m$, $1 \le j,l \le n$ and $a,b,c,d \in A$ with $ab = cd$, we have

$$\begin{aligned}\Big[E_{a\alpha_{ij}}, E_{b\delta_{kl}}\Big] &= I - ab\alpha_{ij}\delta^*_{kl} + ab\delta_{kl}\alpha^*_{ij} \qquad \text{(by Lemma 2.1.3)}\\ &= I - cd\alpha_{ij}\delta^*_{kl} + cd\delta_{kl}\alpha^*_{ij} = \Big[E_{c\alpha_{ij}}, E_{d\delta_{kl}}\Big]. \end{aligned}$$ □

We now compute the 'mixed commutator' of elementary generators corresponding to elements of $\mathrm{Hom}_A(Q,P)$ and $\mathrm{Hom}_A(Q,P^*)$. These also yield commutator relations. The expression for the commutator as given in the proof of the lemma below may appear complicated and we need only its special case $i \neq k$. This special case can be deduced after obtaining the general expression and specializing it.

Lemma 2.1.5. *Let $\alpha \in \mathrm{Hom}_A(Q,P)$ and $\beta \in \mathrm{Hom}_A(Q,P^*)$. Then, for i,j,k,l with $1 \le i,k \le m$ and $1 \le j,l \le n$ with $i \neq k$,*

$$\begin{aligned}\Big[E_{\alpha_{ij}}, E^*_{\beta_{kl}}\Big](z,x,f) &= \Big(I - \alpha_{ij}\beta^*_{kl} + \beta_{kl}\alpha^*_{ij}\Big)(z,x,f)\\ &= \Big(\ z,\ x - \langle x, f_k\rangle\langle w_{ij}, v_{kl}\rangle x_i,\ f + \langle f, x_i\rangle\langle v_{kl}, w_{ij}\rangle f_k\ \Big).\end{aligned}$$

Remark 2.1.6. In the proof of the above lemma, we obtain an explicit expression for the commutator in the general case which specializes to the given expression when $i \neq k$.

Proof of Lemma 2.1.5. For $\alpha \in \mathrm{Hom}_A(Q,P)$, $\beta \in \mathrm{Hom}_A(Q,P^*)$ and for any i,j,k,l with $1 \le i,k \le m$ and $1 \le j,l \le n$ with $i \neq k$, we have the coordinate-free expression

$$\begin{aligned}&\Big[E_{\alpha_{ij}}, E^*_{\beta_{kl}}\Big](z,x,f)\\ &\quad= E_{\alpha_{ij}}E^*_{\beta_{kl}}E^{-1}_{\alpha_{ij}}{E^*_{\beta_{kl}}}^{-1}(z,x,f)\\ &\quad= E_{\alpha_{ij}}E^*_{\beta_{kl}}E^{-1}_{\alpha_{ij}}\Big(\Big(I - \beta_{kl} + \beta^*_{kl} - \frac{1}{2}\beta_{kl}\beta^*_{kl}\Big)(z,x,f)\Big)\\ &\quad= E_{\alpha_{ij}}E^*_{\beta_{kl}}\Big(\Big(I - \beta_{kl} + \beta^*_{kl} - \frac{1}{2}\beta_{kl}\beta^*_{kl} - \alpha_{ij} + \alpha^*_{ij} - \frac{1}{2}\alpha_{ij}\alpha^*_{ij} - \alpha_{ij}\beta^*_{kl}\\ &\qquad -\alpha^*_{ij}\beta_{kl} - \frac{1}{2}\alpha^*_{ij}\beta_{kl}\beta^*_{kl} + \frac{1}{2}\alpha_{ij}\alpha^*_{ij}\beta_{kl} + \frac{1}{4}\alpha_{ij}\alpha^*_{ij}\beta_{kl}\beta^*_{kl}\Big)(z,x,f)\Big)\\ &\quad= E_{\alpha_{ij}}\Big(\Big(I + \alpha^*_{ij} - \alpha^*_{ij}\beta_{kl} - \frac{1}{2}\alpha^*_{ij}\beta_{kl}\beta^*_{kl} + \beta^*_{kl}\alpha_{ij} + \frac{1}{2}\beta^*_{kl}\alpha_{ij}\alpha^*_{ij} + \beta^*_{kl}\alpha_{ij}\beta^*_{kl}\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\beta_{kl}^*\alpha_{ij}\alpha_{ij}^*\beta_{kl} - \frac{1}{4}\beta_{kl}^*\alpha_{ij}\alpha_{ij}^*\beta_{kl}\beta_{kl}^* - \alpha_{ij} - \frac{1}{2}\alpha_{ij}\alpha_{ij}^* - \alpha_{ij}\beta_{kl}^* \\
&+\frac{1}{2}\alpha_{ij}\alpha_{ij}^*\beta_{kl} + \frac{1}{4}\alpha_{ij}\alpha_{ij}^*\beta_{kl}\beta_{kl}^* + \beta_{kl}\alpha_{ij}^* - \beta_{kl}\alpha_{ij}^*\beta_{kl} + \frac{1}{2}\beta_{kl}\beta_{kl}^*\alpha_{ij} \\
&-\frac{1}{4}\beta_{kl}\beta_{kl}^*\alpha_{ij}\alpha_{ij}^*\beta_{kl} + \frac{1}{4}\beta_{kl}\beta_{kl}^*\alpha_{ij}\alpha_{ij}^* - \frac{1}{8}\beta_{kl}\beta_{kl}^*\alpha_{ij}\alpha_{ij}^*\beta_{kl}\beta_{kl}^*\Big)(z,x,f)\Big) \\
&= \Big(I + \beta_{kl}^*\alpha_{ij} + \frac{1}{2}\beta_{kl}^*\alpha_{ij}\alpha_{ij}^* + \beta_{kl}^*\alpha_{ij}\beta_{kl}^* - \frac{1}{2}\beta_{kl}^*\alpha_{ij}\alpha_{ij}^*\beta_{kl} - \frac{1}{4}\beta_{kl}^*\alpha_{ij}\alpha_{ij}^*\beta_{kl}\beta_{kl}^* \\
&- \alpha_{ij}^*\beta_{kl} - \frac{1}{2}\alpha_{ij}^*\beta_{kl}\beta_{kl}^* - \alpha_{ij}^*\beta_{kl}\alpha_{ij}^* + \alpha_{ij}^*\beta_{kl}\alpha_{ij}^*\beta_{kl} - \frac{1}{2}\alpha_{ij}^*\beta_{kl}\beta_{kl}^*\alpha_{ij} \\
&- \frac{1}{4}\alpha_{ij}^*\beta_{kl}\beta_{kl}^*\alpha_{ij}\alpha_{ij}^* + \frac{1}{4}\alpha_{ij}^*\beta_{kl}\beta_{kl}^*\alpha_{ij}\alpha_{ij}^*\beta_{kl} + \frac{1}{8}\alpha_{ij}^*\beta_{kl}\beta_{kl}^*\alpha_{ij}\alpha_{ij}^*\beta_{kl}\beta_{kl}^* \\
&- \alpha_{ij}\beta_{kl}^* + \frac{1}{2}\alpha_{ij}\alpha_{ij}^*\beta_{kl} + \frac{1}{4}\alpha_{ij}\alpha_{ij}^*\beta_{kl}\beta_{kl}^* - \alpha_{ij}\alpha_{ij}^*\beta_{kl} - \frac{1}{2}\alpha_{ij}\alpha_{ij}^*\beta_{kl}\beta_{kl}^* \\
&+ \alpha_{ij}\beta_{kl}^*\alpha_{ij} + \frac{1}{2}\alpha_{ij}\beta_{kl}^*\alpha_{ij}\alpha_{ij}^* + \alpha_{ij}\beta_{kl}^*\alpha_{ij}\beta_{kl}^* - \frac{1}{2}\alpha_{ij}\beta_{kl}^*\alpha_{ij}\alpha_{ij}^*\beta_{kl} \\
&- \frac{1}{4}\alpha_{ij}\beta_{kl}^*\alpha_{ij}\alpha_{ij}^*\beta_{kl}\beta_{kl}^* + \beta_{kl}\alpha_{ij}^* - \beta_{kl}\alpha_{ij}^*\beta_{kl} + \frac{1}{2}\beta_{kl}\beta_{kl}^*\alpha_{ij} + \frac{1}{4}\beta_{kl}\beta_{kl}^*\alpha_{ij}\alpha_{ij}^* \\
&-\frac{1}{4}\beta_{kl}\beta_{kl}^*\alpha_{ij}\alpha_{ij}^*\beta_{kl} - \frac{1}{8}\beta_{kl}\beta_{kl}^*\alpha_{ij}\alpha_{ij}^*\beta_{kl}\beta_{kl}^*\Big)(z,x,f).
\end{aligned}
$$

Now using coordinates, we have

$$
\begin{aligned}
\Big[E_{\alpha_{ij}}, E^*_{\beta_{kl}}\Big](z,x,f) &= E_{\alpha_{ij}}E^*_{\beta_{kl}}E^{-1}_{\alpha_{ij}}E^{*^{-1}}_{\beta_{kl}}(z,x,f) \\
&= E_{\alpha_{ij}}E^*_{\beta_{kl}}E^{-1}_{\alpha_{ij}}\Big(z + \langle x, f_k\rangle v_{kl},\ x,\ f - \Big\{\langle v_{kl}, z\rangle + \langle x, f_k\rangle q(v_{kl})\Big\}f_k\ \Big) \\
&= E_{\alpha_{ij}}E^*_{\beta_{kl}}\Big(z + \Big\{\langle f, x_i\rangle - \langle v_{kl}, z\rangle\langle f_k, x_i\rangle - \langle x, f_k\rangle\langle f_k, x_i\rangle q(v_{kl})\Big\}w_{ij} \\
&\quad +\langle x, f_k\rangle v_{kl},\ x - \Big\{\langle w_{ij}, z\rangle + \langle f, x_i\rangle q(w_{ij}) + \langle x, f_k\rangle\langle w_{ij}, v_{kl}\rangle \\
&\quad -\langle v_{kl}, z\rangle\langle f_k, x_i\rangle q(w_{ij}) - \langle f_k, x_i\rangle\langle x, f_k\rangle q(w_{ij})q(v_{kl})\Big\}x_i, \\
&\quad f - \Big\{\langle v_{kl}, z\rangle + \langle x, f_k\rangle q(v_{kl})\Big\}f_k\ \Big) \\
&= E_{\alpha_{ij}}\ \Big(\ z + \Big\{\langle f, x_i\rangle - \langle v_{kl}, z\rangle\langle f_k, x_i\rangle - \langle x, f_k\rangle\langle f_k, x_i\rangle q(v_{kl})\Big\}w_{ij} \\
&\quad + \Big\{\langle w_{ij}, z\rangle\langle f_k, x_i\rangle + \langle x, f_k\rangle\langle w_{ij}, v_{kl}\rangle\langle x_i, f_k\rangle + \langle f, x_i\rangle\langle x_i, f_k\rangle q(w_{ij}) \\
&\quad -\langle v_{kl}, z\rangle\langle f_k, x_i\rangle^2 q(w_{ij}) - \langle x, f_k\rangle\langle f_k, x_i\rangle^2 q(v_{kl})q(w_{ij})\Big\}v_{kl},\ x \\
&\quad - \Big\{\langle w_{ij}, z\rangle + \langle x, f_k\rangle\langle w_{ij}, v_{kl}\rangle + \langle f, x_i\rangle q(w_{ij}) - \langle v_{kl}, z\rangle\langle f_k, x_i\rangle q(w_{ij}) \\
&\quad -\langle x, f_k\rangle\langle f_k, x_i\rangle q(v_{kl})q(w_{ij})\Big\}\ x_i, f + \Big\{\langle w_{ij}, z\rangle\langle x_i, f_k\rangle q(v_{kl}) \\
&\quad + \langle x_i, f\rangle\langle x_i, f_k\rangle q(v_{kl})q(w_{ij}) - \langle v_{kl}, z\rangle\langle x_i, f_k\rangle^2 q(v_{kl})q(w_{ij})
\end{aligned}
$$

$$
\begin{aligned}
&+\langle f, x_i\rangle\langle v_{kl}, w_{ij}\rangle - \langle x, f_k\rangle\langle x_i, f_k\rangle^2 q(v_{kl})q(w_{ij})\\
&-\langle v_{kl}, z\rangle\langle v_{kl}, w_{ij}\rangle\langle x_i, f_k\rangle\Big\} f_k\Big)\\
=&\Big(z + \Big\{\langle w_{ij}, z\rangle + \langle x, f_k\rangle\langle w_{ij}, v_{kl}\rangle + \langle f, x_i\rangle q(w_{ij})\\
&-\langle v_{kl}, z\rangle\langle f_k, x_i\rangle q(w_{ij}) - \langle x, f_k\rangle\langle f_k, x_i\rangle q(v_{kl})q(w_{ij})\Big\}\langle x_i, f_k\rangle v_{kl}\\
&-\Big\{\langle v_{kl}, z\rangle + \langle x, f_k\rangle q(v_{kl}) + \langle x_i, f\rangle\langle v_{kl}, w_{ij}\rangle\\
&-\langle v_{kl}, z\rangle\langle v_{kl}, w_{ij}\rangle\langle x_i, f_k\rangle + \langle x_i, f\rangle\langle x_i, f_k\rangle q(v_{kl})q(w_{ij})\\
&+\langle w_{ij}, z\rangle\langle x_i, f_k\rangle q(v_{kl}) - \langle v_{kl}, z\rangle\langle x_i, f_k\rangle^2 q(v_{kl})q(w_{ij})\\
&-\langle x, f_k\rangle\langle x_i, f_k\rangle^2 q(v_{kl})q(w_{ij})\Big\}\langle x_i, f_k\rangle w_{ij},\\
&x + \Big\{-\langle x, f_k\rangle\langle w_{ij}, v_{kl}\rangle - \langle v_{kl}, z\rangle\langle x_i, f_k\rangle q(w_{ij})\\
&-\langle x, f_k\rangle\langle x_i, f_k\rangle q(w_{ij})q(v_{kl}) + \langle w_{ij}, z\rangle\langle v_{kl}, w_{ij}\rangle\langle f_k, x_i\rangle\\
&+\langle x, f_k\rangle\langle v_{kl}, w_{ij}\rangle^2\rangle\langle x_i, f_k\rangle - \langle x, f_k\rangle\langle v_{kl}, w_{ij}\rangle\langle f_k, x_i\rangle^2 q(v_{kl})q(w_{ij})\\
&-\langle w_{ij}, z\rangle\langle x_i, f_k\rangle^2 q(v_{kl})q(w_{ij}) - \langle x_i, f\rangle\langle x_i, f_k\rangle^2 q(v_{kl})q(w_{ij})^2\\
&+\langle v_{kl}, z\rangle\langle x_i, f_k\rangle^3 q(v_{kl})q(w_{ij})^2 + \langle x, f_k\rangle\langle x_i, f_k\rangle^3 q(v_{kl})q(w_{ij}))^2\Big\} x_i,\\
&f + \Big\{\langle x_i, f\rangle\langle x_i, f_k\rangle q(v_{kl})q(w_{ij}) - \langle v_{kl}, z\rangle\langle x_i, f_k\rangle^2 q(v_{kl})q(w_{ij})\\
&+\langle f, x_i\rangle\langle v_{kl}, w_{ij}\rangle + \langle w_{ij}, z\rangle\langle x_i, f_k\rangle q(v_{kl})\\
&-\langle x, f_k\rangle\langle x_i, f_k\rangle^2 q(v_{kl})q(w_{ij}) - \langle v_{kl}, z\rangle\langle v_{kl}, w_{ij}\rangle\langle x_i, f_k\rangle\Big\} f_k\Big).
\end{aligned}
$$

In the special case when $i \neq k$, using the fact that $\langle x_i, f_k\rangle = 0$, we obtain

$$
\Big[E_{\alpha_{ij}}, E^*_{\beta_{kl}}\Big](z, x, f) = \Big(z,\ x - \langle x, f_k\rangle\langle w_{ij}, v_{kl}\rangle x_i,\ f + \langle f, x_i\rangle\langle v_{kl}, w_{ij}\rangle f_k\Big).
$$

Now $\alpha_{ij}\beta^*_{kl}(z, x, f) = \Big(0, \langle x, f_k\rangle\langle w_{ij}, v_{kl}\rangle x_i, 0\Big)$, $\beta_{kl}\alpha^*_{ij}(z, x, f) = \Big(0, 0, \langle f, x_i\rangle\langle v_{kl}, w_{ij}\rangle f_k\Big)$.

Hence if $i \neq k$, then

$$
\begin{aligned}
\Big[E_{\alpha_{ij}}, E^*_{\beta_{kl}}\Big](z, x, f) &= \Big(\ z,\ x - \langle x, f_k\rangle\langle w_{ij}, v_{kl}\rangle x_i,\ f + \langle f, x_i\rangle\langle v_{kl}, w_{ij}\rangle f_k\Big)\\
&= \Big(I - \alpha_{ij}\beta^*_{kl} + \beta_{kl}\alpha^*_{ij}\Big)(z, x, f).
\end{aligned}
$$

□

The following corollary lists the resultant commutator relations from the above lemma.

Corollary 2.1.7. *For any i, j, k, l with $1 \leq i, k \leq m$, $1 \leq j, l \leq n$, $i \neq k$ and for $a, b, c, d \in A$ with $ab = cd$, the following equation holds.*

$$\Big[E_{a\alpha_{ij}}, E^*_{b\beta_{kl}}\Big] = \Big[E_{c\alpha_{ij}}, E^*_{d\beta_{kl}}\Big].$$

The lemma below computes the commutator of elementary generators corresponding to two elements of $\mathrm{Hom}_A(Q, P^*)$.

Remark 2.1.8. For any i, j, k, l with $1 \leq i, k \leq m$, $1 \leq j, l \leq n$ and $i \neq k$, the commutator $\Big[E_{\alpha_{ij}}, E^*_{\beta_{kl}}\Big]^{-1}$ is given by

$$\begin{aligned}\Big[E_{\alpha_{ij}}, E^*_{\beta_{kl}}\Big]^{-1}(z,x,f) &= \Big(z,\ x + \langle x, f_k\rangle\langle w_{ij}, v_{kl}\rangle x_i,\ f - \langle f, x_i\rangle\langle v_{kl}, w_{ij}\rangle f_k\Big)\\ &= \Big(I + \alpha_{ij}\beta^*_{kl} - \beta_{kl}\alpha^*_{ij}\Big)(z,x,f)\\ &= \Big[E^*_{\beta_{kl}}, E_{\alpha_{ij}}\Big](z,x,f).\end{aligned}$$

Lemma 2.1.9. *Let $\beta, \gamma \in \mathrm{Hom}_A(Q, P^*)$. Then, for i, j, k, l with $1 \leq i, k \leq m$ and $1 \leq j, l \leq n$, the commutator $[E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}]$ is given by*

$$\begin{aligned}\Big[E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}\Big](z,x,f) &= \Big(I + \gamma_{kl}\beta^*_{ij} - \beta_{ij}\gamma^*_{kl}\Big)(z,x,f)\\ &= \Big(z,\ x,\ f + \langle x, f_i\rangle\langle c_{kl}, v_{ij}\rangle f_k - \langle x, f_k\rangle\langle v_{ij}, c_{kl}\rangle f_i\Big).\end{aligned}$$

In particular, if $i = k$, then $[E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}] = I$.

Proof. For $\beta, \gamma \in \mathrm{Hom}_A(Q, P^*)$ and for any i, j, k, l with $1 \leq i, k \leq m$ and $1 \leq j, l \leq n$, we have the coordinate-free expression

$$\begin{aligned}\Big[E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}\Big](z,x,f) &= E^*_{\beta_{ij}}E^*_{\gamma_{kl}}E^{*\,-1}_{\beta_{ij}}E^{*\,-1}_{\gamma_{kl}}(z,x,f)\\ &= E^*_{\beta_{ij}}E^*_{\gamma_{kl}}E^{*\,-1}_{\beta_{ij}}\Big(\Big(I - \gamma_{kl} + \gamma^*_{kl} - \frac{1}{2}\gamma_{kl}\gamma^*_{kl}\Big)(z,x,f)\Big)\\ &= E^*_{\beta_{ij}}E^*_{\gamma_{kl}}\Big(\Big(I - \gamma_{kl} + \gamma^*_{kl} - \frac{1}{2}\gamma_{kl}\gamma^*_{kl} - \beta_{ij} + \beta^*_{ij} - \frac{1}{2}\beta_{ij}\beta^*_{ij} - \beta_{ij}\gamma^*_{kl}\Big)(z,x,f)\Big)\\ &= E^*_{\beta_{ij}}\Big(\Big(I - \beta_{ij} + \beta^*_{ij} - \frac{1}{2}\beta_{ij}\beta^*_{ij} - \beta_{ij}\gamma^*_{kl} + \gamma_{kl}\beta^*_{ij}\Big)(z,x,f)\Big)\end{aligned}$$

$$= \Big(I - \beta_{ij}\gamma_{kl}^* + \gamma_{kl}\beta_{ij}^*\Big)(z, x, f).$$

Using coordinates, we have

$$\begin{aligned}
\Big[E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}\Big] &= E^*_{\beta_{ij}} E^*_{\gamma_{kl}} E^{*^{-1}}_{\beta_{ij}} E^{*^{-1}}_{\gamma_{kl}}(z, x, f) \\
&= E^*_{\beta_{ij}} E^*_{\gamma_{kl}} E^{*^{-1}}_{\beta_{ij}} \Big(z + \langle x, f_k\rangle c_{kl},\ x,\ f - \langle c_{kl}, z\rangle f_k - \langle x, f_k\rangle q(c_{kl}) f_k\Big) \\
&= E^*_{\beta_{ij}} E^*_{\gamma_{kl}} \Big(z + \langle x, f_k\rangle c_{kl} + \langle x, f_i\rangle v_{ij},\ x,\ f - \Big\{\langle v_{ij}, z\rangle + \langle x, f_i\rangle q(v_{ij}) \\
&\quad + \langle x, f_k\rangle\langle v_{ij}, c_{kl}\rangle\Big\} f_i - \Big\{\langle c_{kl}, z\rangle + \langle x, f_k\rangle q(c_{kl})\Big\} f_k\Big) \\
&= E^*_{\beta_{ij}} \Big(z + \langle x, f_i\rangle v_{ij},\ x,\ f - \Big\{\langle v_{ij}, z\rangle + q(v_{ij})\langle x, f_i\rangle \\
&\quad + \langle x, f_k\rangle\langle v_{ij}, c_{kl}\rangle\Big\} f_i + \langle x, f_i\rangle\langle c_{kl}, v_{ij}\rangle f_k\Big) \\
&= \Big(z,\ x,\ f + \langle x, f_i\rangle\langle c_{kl}, v_{ij}\rangle f_k - \langle x, f_k\rangle\langle v_{ij}, c_{kl}\rangle f_i\Big).
\end{aligned}$$

If $i = k$, then

$$\gamma_{kl}\beta_{ij}^*(z, x, f) = \Big(0, 0, \langle x, f_i\rangle\langle c_{kl}, v_{ij}\rangle f_k - \langle x, f_k\rangle\langle v_{ij}, c_{kl}\rangle f_i\Big) = \beta_{ij}\gamma_{kl}^*(z, x, f).$$

Hence $\Big[E^*_{\beta_{ij}}, E^*_{\gamma_{il}}\Big] = I.$ □

Immediately, we deduce the following commutator relations.

Corollary 2.1.10. *For any i, j, k, l with $1 \le i, k \le m$, $1 \le j, l \le n$ and for $a, b, c, d \in A$ with $ab = cd$, the following equation holds.*

$$\Big[E^*_{a\beta_{ij}}, E^*_{b\gamma_{kl}}\Big] = \Big[E^*_{c\beta_{ij}}, E^*_{d\gamma_{kl}}\Big].$$

Remark 2.1.11. In the following sections, we will prove more complicated commutator relations of lengths 10 and 16; we will show how the indices may be specialized so that the commutator is non-trivial.

2.2 Triple Commutators

In this section, we prove certain triple commutator relations among the elementary generators of Roy's elementary orthogonal group. We start with a commutator of length 10 which involves a commutator of elementary generators corresponding to two elements of $\mathrm{Hom}_A(Q,P)$.

Lemma 2.2.1. *Let $\alpha, \delta \in \mathrm{Hom}_A(Q,P)$ and $\beta \in \mathrm{Hom}_A(Q,P^*)$. Then, for i,j,k,l,p,q with $1 \le i,k,p \le m$, $1 \le j,l,q \le n$ and $k \ne p$, the triple commutator $\left[E^*_{\beta_{ij}}, \left[E_{\alpha_{kl}}, E_{\delta_{pq}}\right]\right]$ is given by*

$$\left[E^*_{\beta_{ij}}, \left[E_{\alpha_{kl}}, E_{\delta_{pq}}\right]\right] = \begin{cases} E_{\lambda_{kj}} \left[E^*_{\beta_{ij}}, E_{\frac{\lambda_{kj}}{2}}\right] & \text{if} \quad i = p, \\ E_{\xi_{pj}} \left[E^*_{\beta_{ij}}, E_{\frac{\xi_{pj}}{2}}\right] & \text{if} \quad i = k, \\ I & \text{if} \quad i \ne p \text{ and } i \ne k, \end{cases}$$

where $\lambda_{kj} = \alpha_{kl}\delta^*_{pq}\beta_{ij}$ and $\xi_{pj} = -\delta_{pq}\alpha^*_{kl}\beta_{ij}$.

Proof. For $\alpha, \delta \in \mathrm{Hom}_A(Q,P)$, $\beta \in \mathrm{Hom}_A(Q,P^*)$ and for i,j,k,l,p,q with $1 \le i,k,p \le m$, $1 \le j,l,q \le n$ and $k \ne p$, we have

$$\begin{aligned} \left[E_{\alpha_{kl}}, E_{\delta_{pq}}\right](z,x,f) &= \Big(I + \delta_{pq}\alpha^*_{kl} - \alpha_{kl}\delta^*_{pq}\Big)(z,x,f) \\ &= \Big(z,\ x + \langle f, x_k\rangle\langle t_{pq}, w_{kl}\rangle x_p - \langle f, x_p\rangle\langle t_{pq}, w_{kl}\rangle x_k,\ f\Big). \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 2.1.3)} \\ \left[E_{\alpha_{kl}}, E_{\delta_{pq}}\right]^{-1}(z,x,f) &= \left[E_{\delta_{pq}}, E_{\alpha_{kl}}\right](z,x,f) \\ &= \Big(I - \delta_{pq}\alpha^*_{kl} + \alpha_{kl}\delta^*_{pq}\Big)(z,x,f) \\ &= \Big(z,\ x - \langle f, x_k\rangle\langle t_{pq}, w_{kl}\rangle x_p + \langle f, x_p\rangle\langle t_{pq}, w_{kl}\rangle x_k,\ f\Big). \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 2.1.3)} \end{aligned}$$

Hence we get the coordinate-free expression

$$\begin{aligned} \left[E^*_{\beta_{ij}}, \left[E_{\alpha_{kl}}, E_{\delta_{pq}}\right]\right](z,x,f) &= E^*_{\beta_{ij}}\left[E_{\alpha_{kl}}, E_{\delta_{pq}}\right] E^{*^{-1}}_{\beta_{ij}} \left[E_{\alpha_{kl}}, E_{\delta_{pq}}\right]^{-1}(z,x,f) \\ &= E^*_{\beta_{ij}}\left[E_{\alpha_{kl}}, E_{\delta_{pq}}\right] E^{*^{-1}}_{\beta_{ij}} \Big(\Big(I + \alpha_{kl}\delta^*_{pq} - \delta_{pq}\alpha^*_{kl}\Big)(z,x,f)\Big) \\ &= E^*_{\beta_{ij}}[E_{\alpha_{kl}}, E_{\delta_{pq}}]\Big(\Big(I + \beta^*_{ij} + \beta^*_{ij}\alpha_{kl}\delta^*_{pq} - \beta^*_{ij}\delta_{pq}\alpha^*_{kl} - \beta_{ij} - \frac{1}{2}\beta_{ij}\beta^*_{ij} \\ &\quad + \alpha_{kl}\delta^*_{pq} - \delta_{pq}\alpha^*_{kl} - \frac{1}{2}\beta_{ij}\beta^*_{ij}\alpha_{kl}\delta^*_{pq} + \frac{1}{2}\beta_{ij}\beta^*_{ij}\delta_{pq}\alpha^*_{kl}\Big)(z,x,f)\Big) \end{aligned}$$

$$
\begin{aligned}
&= E^*_{\beta_{ij}}\Big(\Big(I - \beta_{ij} + \beta^*_{ij} - \frac{1}{2}\beta_{ij}\beta^*_{ij} + \beta^*_{ij}\alpha_{kl}\delta^*_{pq} - \frac{1}{2}\beta_{ij}\beta^*_{ij}\alpha_{kl}\delta^*_{pq} \\
&+ \frac{1}{2}\beta_{ij}\beta^*_{ij}\delta_{pq}\alpha^*_{kl} + \alpha_{kl}\delta^*_{pq}\beta_{ij} + \frac{1}{2}\alpha_{kl}\delta^*_{pq}\beta^*_{ij}\beta_{ij} - \frac{1}{2}\delta_{pq}\alpha^*_{kl}\beta_{ij}\beta^*_{ij} \\
&- \delta_{pq}\alpha^*_{kl}\beta_{ij} - \frac{1}{2}\alpha_{kl}\delta^*_{pq}\beta_{ij}\beta^*_{ij}\delta_{pq}\alpha^*_{kl} - \frac{1}{2}\delta_{pq}\alpha^*_{kl}\beta_{ij}\beta^*_{ij}\alpha_{kl}\delta^*_{pq} \\
&+\frac{1}{2}\alpha_{kl}\delta^*_{pq}\beta_{ij}\beta^*_{ij}\alpha_{kl}\delta^*_{pq} + \frac{1}{2}\delta_{pq}\alpha^*_{kl}\beta_{ij}\beta^*_{ij}\delta_{pq}\alpha^*_{kl} - \beta^*_{ij}\delta_{pq}\alpha^*_{kl}\Big)(z,x,f)\Big) \\
&=\Big(I + \beta^*_{ij}\alpha_{kl}\delta^*_{pq} - \frac{1}{2}\alpha_{kl}\delta^*_{pq}\beta_{ij}\beta^*_{ij}\delta_{pq}\alpha^*_{kl} + \frac{1}{2}\beta_{ij}\beta^*_{ij}\alpha_{kl}\delta^*_{pq} \\
&+ \frac{1}{2}\alpha_{kl}\delta^*_{pq}\beta_{ij}\beta^*_{ij} - \frac{1}{2}\delta_{pq}\alpha^*_{kl}\beta_{ij}\beta^*_{ij}\alpha_{kl}\delta^*_{pq} + \alpha_{kl}\delta^*_{pq}\beta_{ij} \\
&+ \frac{1}{2}\delta_{pq}\alpha^*_{kl}\beta_{ij}\beta^*_{ij}\delta_{pq}\alpha^*_{kl} - \frac{1}{2}\beta_{ij}\beta^*_{ij}\delta_{pq}\alpha^*_{kl} - \delta_{pq}\alpha^*_{kl}\beta_{ij} \\
&- \beta^*_{ij}\delta_{pq}\alpha^*_{kl} - \frac{1}{2}\delta_{pq}\alpha^*_{kl}\beta_{ij}\beta^*_{ij} + \frac{1}{2}\alpha_{kl}\delta^*_{pq}\beta_{ij}\beta^*_{ij}\alpha_{kl}\delta^*_{pq}\Big)(z,x,f).
\end{aligned}
\tag{2.2.1}
$$

On computing using coordinates, we get

$$
\begin{aligned}
\Big[E^*_{\beta_{ij}}, \left[E_{\alpha_{kl}}, E_{\delta_{pq}}\right]\Big](z,x,f) &= E^*_{\beta_{ij}}\left[E_{\alpha_{kl}}, E_{\delta_{pq}}\right] E^{*^{-1}}_{\beta_{ij}}\left[E_{\alpha_{kl}}, E_{\delta_{pq}}\right]^{-1}(z,x,f) \\
&= E^*_{\beta_{ij}}\left[E_{\alpha_{kl}}, E_{\delta_{pq}}\right] E^{*^{-1}}_{\beta_{ij}}\Big(z,\ x - \langle f, x_k\rangle\langle t_{pq}, w_{kl}\rangle x_p \\
&\quad +\langle f, x_p\rangle\langle t_{pq}, w_{kl}\rangle x_k,\ f\Big) \\
&= E^*_{\beta_{ij}}\left[E_{\alpha_{kl}}, E_{\delta_{pq}}\right]\Big(z + \Big\{\langle f, x_p\rangle\langle t_{pq}, w_{kl}\rangle\langle x_k, f_i\rangle \\
&\quad -\langle f, x_k\rangle\langle t_{pq}, w_{kl}\rangle\langle x_p, f_i\rangle + \langle x, f_i\rangle\Big\}v_{ij}, x - \langle f, x_k\rangle\langle t_{pq}, w_{kl}\rangle x_p \\
&\quad + \langle f, x_p\rangle\langle t_{pq}, w_{kl}\rangle x_k,\ f - \Big\{\langle f, x_p\rangle\langle t_{pq}, w_{kl}\rangle\langle x_k, f_i\rangle q(v_{ij}) \\
&\quad -\langle f, x_k\rangle\langle t_{pq}, w_{kl}\rangle q(v_{ij}) + \langle x, f_i\rangle q(v_{ij}) + \langle v_{ij}, z\rangle\Big\}f_i\Big) \\
&= E^*_{\beta_{ij}}\Big(z + \Big\{\langle f, x_p\rangle\langle x_k, f_i\rangle\langle t_{pq}, w_{kl}\rangle - \langle f, x_k\rangle\langle x_p, f_i\rangle\langle t_{pq}, w_{kl}\rangle \\
&\quad + \langle x, f_i\rangle\Big\}v_{ij},\ x - \Big\{\langle x, f_i\rangle q(v_{ij}) + \langle x_k, f_i\rangle\langle f, x_p\rangle\langle t_{pq}, w_{kl}\rangle q(v_{ij}) \\
&\quad +\langle v_{ij}, z\rangle - \langle f, x_k\rangle\langle x_p, f_i\rangle\langle t_{pq}, w_{kl}\rangle q(v_{ij})\Big\}\langle f_i, x_k\rangle\langle t_{pq}, w_{kl}\rangle x_p \\
&\quad + \Big\{\langle v_{ij}, z\rangle + \langle x, f_i\rangle q(v_{ij}) + \langle f, x_p\rangle\langle x_k, f_i\rangle\langle t_{pq}, w_{kl}\rangle q(v_{ij}) \\
&\quad -\langle x_k, f\rangle\langle x_p, f_i\rangle\langle t_{pq}, w_{kl}\rangle q(v_{ij})\Big\}\langle f_i, x_p\rangle\langle t_{pq}, w_{kl}\rangle x_k, \\
&\quad f - \Big\{\langle x, f_i\rangle q(v_{ij} + \langle x_k, f_i\rangle\langle f, x_p\rangle\langle t_{pq}, w_{kl}\rangle q(v_{ij})
\end{aligned}
$$

$$
\begin{aligned}
&+\langle v_{ij}, z\rangle - \langle f, x_k\rangle\langle x_p, f_i\rangle\langle t_{pq}, w_{kl}\rangle q(v_{ij})\Big\} f_i\Big) \\
&= \Big(z + \Big\{\langle f, x_p\rangle\langle x_k, f_i\rangle - \langle f, x_k\rangle\langle x_p, f_i\rangle\Big\}\langle t_{pq}, w_{kl}\rangle v_{ij}, \\
&x - \Big\{\langle v_{ij}, z\rangle + \langle x, f_i\rangle q(v_{ij}) + \langle f_i, x_k\rangle\langle f, x_p\rangle\langle t_{pq}, w_{kl}\rangle q(v_{ij}) \\
&-\langle f_i, x_p\rangle\langle f, x_k\rangle\langle t_{pq}, w_{kl}\rangle q(v_{ij})\Big\}\langle f_i, x_k\rangle\langle t_{pq}, w_{kl}\rangle x_p \\
&+ \Big\{\langle v_{ij}, z\rangle + \langle f_i, x\rangle q(v_{ij}) + \langle f_i, x_k\rangle\langle f, x_p\rangle\langle t_{pq}, w_{kl}\rangle q(v_{ij}) \\
&-\langle f, x_k\rangle\langle f_i, x_p\rangle\langle t_{pq}, w_{kl}\rangle q(v_{ij})\Big\}\langle f_i, x_p\rangle\langle t_{pq}, w_{kl}\rangle x_k, \\
&f + \Big\{\langle f, x_p\rangle\langle f_i, x_k\rangle - \langle f, x_k\rangle\langle f_i, x_p\rangle\Big\}\langle t_{pq}, w_{kl}\rangle q(v_{ij}) f_i\Big).
\end{aligned} \tag{2.2.2}
$$

Now, for $\lambda_{kj} = \alpha_{kl}\delta^*_{pq}\beta_{ij}$ as in the statement, we can describe the maps $\lambda_{kj}, \lambda^*_{kj}, \frac{1}{2}\lambda_{kj}\lambda^*_{kj}$ and the elementary transformation $E_{\lambda_{kj}}$ as

$$
\begin{aligned}
\lambda_{kj}(z, x, f) &= \alpha_{kl}\delta^*_{pq}\beta_{ij}(z, x, f) = \Big(0, \langle v_{ij}, z\rangle\langle f_i, x_p\rangle\langle w_{kl}, t_{pq}\rangle x_k, 0\Big), \\
\lambda^*_{kj}(z, x, f) &= \beta^*_{ij}\delta_{pq}\alpha^*_{kl}(z, x, f) = \Big(\langle f, x_k\rangle\langle t_{pq}, w_{kl}\rangle\langle x_p, f_i\rangle v_{ij}, 0, 0\Big), \\
\frac{1}{2}\lambda_{kj}\lambda^*_{kj}(z, x, f) &= \Big(0, \langle f, x_k\rangle\langle f_i, x_p\rangle^2\langle w_{kl}, t_{pq}\rangle^2 q(v_{ij}) x_k, 0\Big), \\
E_{\lambda_{kj}}(z, x, f) &= \Big(I + \lambda_{kj} - \lambda^*_{kj} - \frac{1}{2}\lambda_{kj}\lambda^*_{kj}\Big)(z, x, f) \\
&= \Big(z - \langle f, x_k\rangle\langle f_i, x_p\rangle\langle w_{kl}, t_{pq}\rangle v_{ij}, x + \Big\{\langle v_{ij}, z\rangle \\
&\quad -\langle f, x_k\rangle\langle f_i, x_p\rangle\langle w_{kl}, t_{pq}\rangle q(v_{ij})\Big\} x_k, f\Big).
\end{aligned}
$$

If $i \neq k$, then, by Remark 2.1.8, we have

$$
\begin{aligned}
\left[E^*_{\beta_{ij}}, E_{\frac{\lambda_{kj}}{2}}\right](z, x, f) &= \left[E_{\frac{\lambda_{kj}}{2}}, E^*_{\beta_{ij}}\right]^{-1}(z, x, f) \\
&= \Big(I - \frac{1}{2}\beta_{ij}\lambda^*_{kj} + \frac{1}{2}\lambda_{kj}\beta^*_{ij}\Big)(z, x, f) \\
&= \Big(z,\ x + \langle x, f_i\rangle\langle f_i, x_p\rangle\langle w_{kl}, t_{pq}\rangle q(v_{ij}) x_k, \\
&\quad f - \langle f, x_k\rangle\langle f_i, x_p\rangle\langle w_{kl}, t_{pq}\rangle q(v_{ij}) f_i\Big)
\end{aligned}
$$

and hence we get

$$
E_{\lambda_{kj}}\left[E^*_{\beta_{ij}}, E_{\frac{\lambda_{kj}}{2}}\right](z, x, f) = \Big(I + \lambda_{kj} - \lambda^*_{kj} - \frac{1}{2}\lambda_{kj}\lambda^*_{kj} - \frac{1}{2}\beta_{ij}\lambda^*_{kj} + \frac{1}{2}\lambda_{kj}\beta^*_{ij}\Big)(z, x, f)
$$

$$= \Big(z - \langle f, x_k\rangle\langle x_p, f_i\rangle\langle t_{pq}, w_{kl}\rangle v_{ij}, x + \Big\{\langle f_i, x\rangle q(v_{ij}) - \langle f, x_k\rangle\langle f_i, x_p\rangle\langle t_{pq}, w_{kl}\rangle q(v_{ij}) + \langle v_{ij}, z\rangle\Big\}\langle t_{pq}, w_{kl}\rangle \langle f_i, x_p\rangle x_k, f - \langle f, x_k\rangle\langle f_i, x_p\rangle\langle t_{pq}, w_{kl}\rangle q(v_{ij}) f_i\Big). \tag{2.2.3}$$

Similarly, if $i \neq p$, we have

$$E_{\xi_{pj}}\left[E^*_{\beta_{ij}}, E_{\frac{\xi_{pj}}{2}}\right](z, x, f) = \Big(I + \xi_{pj} - \xi^*_{pj} - \frac{1}{2}\xi_{pj}\xi^*_{pj} - \frac{1}{2}\beta_{ij}\xi^*_{pj} + \frac{1}{2}\xi_{pj}\beta^*_{ij}\Big)(z, x, f)$$
$$= \Big(z - \langle f, x_k\rangle\langle x_p, f_i\rangle\langle t_{pq}, w_{kl}\rangle v_{ij}, x + \Big\{\langle f_i, x\rangle q(v_{ij}) - \langle f, x_k\rangle\langle f_i, x_p\rangle\langle t_{pq}, w_{kl}\rangle q(v_{ij}) + \langle v_{ij}, z\rangle\Big\}\langle t_{pq}, w_{kl}\rangle \langle f_i, x_p\rangle x_k, f - \langle f, x_k\rangle\langle f_i, x_p\rangle\langle t_{pq}, w_{kl}\rangle q(v_{ij}) f_i\Big). \tag{2.2.4}$$

We now consider the following possible conditions on the indices.

Case (i): $i = p$.

If $i = p$, then, by Equations (2.2.2), (2.2.1), and (2.2.3), we have

$$\left[E^*_{\beta_{ij}}, \left[E_{\alpha_{kl}}, E_{\delta_{pq}}\right]\right](z, x, f) = \Big(I - \beta^*_{pj}\delta_{pq}\alpha^*_{kl} + \alpha_{kl}\delta^*_{pq}\beta_{pj} + \frac{1}{2}\alpha_{kl}\delta^*_{pq}\beta_{pj}\beta^*_{pj} - \frac{1}{2}\beta_{pj}\beta^*_{pj}\delta_{pq}\alpha^*_{kl} - \frac{1}{2}\alpha_{kl}\delta^*_{pq}\beta_{pj}\beta^*_{pj}\delta_{pq}\alpha^*_{kl}\Big)(z, x, f)$$
$$= \Big(z - \langle f, x_k\rangle\langle t_{pq}, w_{kl}\rangle v_{pj},\ x + \Big\{\langle v_{pj}, z\rangle - \langle f, x_k\rangle\langle t_{pq}, w_{kl}\rangle q(v_{pj}) + \langle f_p, x\rangle q(v_{pj})\Big\}\langle t_{pq}, w_{kl}\rangle x_k,\ f - \langle f, x_k\rangle\langle t_{pq}, w_{kl}\rangle q(v_{pj}) f_p\Big)$$
$$= E_{\lambda_{kj}}\left[E^*_{\beta_{ij}}, E_{\frac{\lambda_{kj}}{2}}\right](z, x, f).$$

Case (ii): $i = k$.

If $i = k$, then, by Equations (2.2.2), (2.2.1), and (2.2.4), we have

$$\left[E^*_{\beta_{ij}}, \left[E_{\alpha_{kl}}, E_{\delta_{pq}}\right]\right](z, x, f) = \Big(I + \beta^*_{kj}\alpha_{kl}\delta^*_{pq} + \frac{1}{2}\beta_{kj}\beta^*_{kj}\alpha_{kl}\delta^*_{pq} - \frac{1}{2}\delta_{pq}\alpha^*_{kl}\beta_{kj}\beta^*_{kj} - \delta_{pq}\alpha^*_{kl}\beta_{kj} - \frac{1}{2}\delta_{pq}\alpha^*_{kl}\beta_{kj}\beta^*_{kj}\alpha_{kl}\delta^*_{pq}\Big)(z, x, f)$$
$$= \Big(z + \langle f, x_p\rangle\langle t_{pq}, w_{kl}\rangle v_{kj}, x - \Big\{\langle v_{kj}, z\rangle + \langle f, x_p\rangle\langle t_{pq}, w_{kl}\rangle q(v_{kj}) + \langle x, f_k\rangle q(v_{kj})\Big\}\langle t_{pq}, w_{kl}\rangle x_p, f + \langle f, x_p\rangle\langle t_{pq}, w_{kl}\rangle q(v_{kj}) f_k\Big)$$

$$= E_{\xi_{pj}} \left[E^*_{\beta_{kj}}, E_{\frac{\xi_{pj}}{2}} \right] (z, x, f).$$

Case(iii): $i \neq k$ and $i \neq p$.

If $i \neq k$ and $i \neq p$, then, by Equation (2.2.2), we have

$$\left[E^*_{\beta_{ij}}, \left[E_{\alpha_{kl}}, E_{\delta_{pq}} \right] \right] (z, x, f) = I(z, x, f). \qquad \square$$

As a consequence of the above lemma on triple commutators, we observe the following commutator relations.

Corollary 2.2.2. *For any i, j, k, l, p, q with $1 \leq i, k, p \leq m$, $1 \leq j, l, q \leq n$, $i \neq k$ and $k \neq p$ and $a, b, c, d, e, f \in A$ with $abc = def$ and $a^2bc = d^2ef$, the following equation holds.*

$$\left[E^*_{a\beta_{ij}}, \left[E_{b\alpha_{kl}}, E_{c\delta_{pq}} \right] \right] = \left[E^*_{d\beta_{ij}}, \left[E_{e\alpha_{kl}}, E_{f\delta_{pq}} \right] \right].$$

Proof. For any i, j, k, l, p, q with $1 \leq i, k, p \leq m$, $1 \leq j, l, q \leq n$, $i \neq k$ and $k \neq p$ and $a, b, c, d, e, f \in A$ with $abc = def$ and $a^2bc = d^2ef$, we have

$$\begin{aligned}
\left[E^*_{a\beta_{ij}}, \left[E_{b\alpha_{kl}}, E_{c\delta_{pq}} \right] \right] (z, x, f) &= \Big(I - a^2bc\beta^*_{ij}\delta_{pq}\alpha^*_{kl} + abc\alpha_{kl}\delta^*_{pq}\beta_{ij} + \frac{1}{2}a^2bc\alpha_{kl}\delta^*_{pq}\beta_{ij}\beta^*_{ij} \\
&\quad - \frac{1}{2}a^2bc\beta_{ij}\beta^*_{ij}\delta_{pq}\alpha^*_{kl} - \frac{1}{2}a^2b^2c^2\alpha_{kl}\delta^*_{pq}\beta_{ij}\beta^*_{ij}\delta_{pq}\alpha^*_{kl} \Big)(z, x, f) \\
&= \Big(I - d^2ef\beta^*_{ij}\delta_{pq}\alpha^*_{kl} + def\alpha_{kl}\delta^*_{pq}\beta_{ij} + \frac{1}{2}d^2ef\alpha_{kl}\delta^*_{pq}\beta_{ij}\beta^*_{ij} \\
&\quad - \frac{1}{2}d^2ef\beta_{ij}\beta^*_{ij}\delta_{pq}\alpha^*_{kl} - \frac{1}{2}d^2e^2f^2\alpha_{kl}\delta^*_{pq}\beta_{ij}\beta^*_{ij}\delta_{pq}\alpha^*_{kl} \Big)(z, x, f) \\
&= \left[E^*_{d\beta_{ij}}, \left[E_{e\alpha_{kl}}, E_{f\delta_{pq}} \right] \right] (z, x, f). \qquad \square
\end{aligned}$$

The following lemma on triple commutators involves a mixed commutator.

Lemma 2.2.3. *Let $\alpha, \delta \in \mathrm{Hom}_A(Q, P)$ and $\beta \in \mathrm{Hom}_A(Q, P^*)$. Then, for i, j, k, l, p, q with $1 \leq i, k, p \leq m$, $1 \leq j, l, q \leq n$ and $k \neq p$, the triple commutator $\left[E_{\alpha_{ij}}, \left[E_{\delta_{kl}}, E^*_{\beta_{pq}} \right] \right]$ is given by*

$$\left[E_{\alpha_{ij}}, \left[E_{\delta_{kl}}, E^*_{\beta_{pq}} \right] \right] = \begin{cases} E_{\mu_{kj}} \left[E_{\alpha_{ij}}, E_{\frac{\mu_{kj}}{2}} \right], & \text{if} \quad i = p, \\ I & \text{if} \quad i = k \quad \text{or} \quad i \neq p, \end{cases}$$

where $\mu_{kj} = \delta_{kl}\beta^*_{pq}\alpha_{ij}$.

Proof. For $\alpha, \delta \in \mathrm{Hom}_A(Q,P)$, $\beta \in \mathrm{Hom}_A(Q,P^*)$ and for i,j,k,l,p,q with $1 \leq i,k,p \leq m$, $1 \leq j,l,q \leq n$ and $k \neq p$, we have the coordinate-free expression

$$\begin{aligned}
\Big[E_{\delta_{kl}}, E^*_{\beta_{pq}}\Big](z,x,f) &= \Big(I + \beta_{pq}\delta^*_{kl} - \delta_{kl}\beta^*_{pq}\Big)(z,x,f) \\
&= \Big(z,\ x - \langle x, f_p\rangle\langle t_{kl}, v_{pq}\rangle x_k,\ f + \langle f, x_k\rangle\langle t_{kl}, v_{pq}\rangle f_p\Big). \\
&\qquad\qquad\qquad\qquad \text{(by Lemma 2.1.5)} \\
\Big[E_{\delta_{kl}}, E^*_{\beta_{pq}}\Big]^{-1}(z,x,f) &= \Big[E^*_{\beta_{pq}}, E_{\delta_{kl}}\Big](z,x,f) \\
&= \Big(z,\ x + \langle x, f_p\rangle\langle t_{kl}, v_{pq}\rangle x_k,\ f - \langle f, x_k\rangle\langle t_{kl}, v_{pq}\rangle f_p\Big). \\
&\qquad\qquad\qquad\qquad \text{(by Remark 2.1.8)}
\end{aligned}$$

Hence we get

$$\begin{aligned}
\Big[E_{\alpha_{ij}}, \Big[E_{\delta_{kl}}, E^*_{\beta_{pq}}\Big]\Big](z,x,f) &= E_{\alpha_{ij}}\Big[E_{\delta_{kl}}, E^*_{\beta_{pq}}\Big]E^{-1}_{\alpha_{ij}}\Big[E_{\delta_{kl}}, E^*_{\beta_{pq}}\Big]^{-1}(z,x,f) \\
&= E_{\alpha_{ij}}\Big[E_{\delta_{kl}}, E^*_{\beta_{pq}}\Big]E^{-1}_{\alpha_{ij}}\Big(\Big(I - \beta_{pq}\delta^*_{kl} + \delta_{kl}\beta^*_{pq}\Big)(z,x,f)\Big) \\
&= E_{\alpha_{ij}}\Big[E_{\delta_{kl}}, E^*_{\beta_{pq}}\Big]\Big(\Big(I + \alpha^*_{ij} - \alpha^*_{ij}\beta_{pq}\delta^*_{kl} + \delta_{kl}\beta^*_{pq} - \alpha_{ij} \\
&\quad -\frac{1}{2}\alpha_{ij}\alpha^*_{ij} - \beta_{pq}\delta^*_{kl} + \frac{1}{2}\alpha_{ij}\alpha^*_{ij}\beta_{pq}\delta^*_{kl}\Big)(z,x,f)\Big) \\
&= E_{\alpha_{ij}}\Big(\Big(I + \alpha^*_{ij} - \alpha^*_{ij}\beta_{pq}\delta^*_{kl} - \alpha_{ij} + \frac{1}{2}\alpha_{ij}\alpha^*_{ij}\beta_{pq}\delta^*_{kl} \\
&\quad + \delta_{kl}\beta^*_{pq}\alpha_{ij} - \frac{1}{2}\delta_{kl}\beta^*_{pq}\alpha_{ij}\alpha^*_{ij}\beta_{pq}\delta^*_{kl} - \delta_{kl}\beta^*_{pq}\delta_{kl}\beta^*_{pq} \\
&\quad +\frac{1}{2}\delta_{kl}\beta^*_{pq}\alpha_{ij}\alpha^*_{ij} - \beta_{pq}\delta^*_{kl}\beta_{pq}\delta^*_{kl} - \frac{1}{2}\alpha_{ij}\alpha^*_{ij}\Big)(z,x,f)\Big) \\
&= \Big(\Big(I - \alpha^*_{ij}\beta_{pq}\delta^*_{kl} + \alpha^*_{ij}\beta_{pq}\delta^*_{kl}\beta_{pq}\delta^*_{kl} + \frac{1}{2}\delta_{kl}\beta^*_{pq}\alpha_{ij}\alpha^*_{ij} \\
&\quad - \delta_{kl}\beta^*_{pq}\delta_{kl}\beta^*_{pq} - \frac{1}{2}\delta_{kl}\beta^*_{pq}\alpha_{ij}\alpha^*_{ij}\beta_{pq}\delta^*_{kl} \\
&\quad - \beta_{pq}\delta^*_{kl}\beta_{pq}\delta^*_{kl} + \frac{1}{2}\alpha_{ij}\alpha^*_{ij}\beta_{pq}\delta^*_{kl}\beta_{pq}\delta^*_{kl} + \delta_{kl}\beta^*_{pq}\alpha_{ij} \\
&\quad -\frac{1}{2}\alpha_{ij}\alpha^*_{ij}\beta_{pq}\delta^*_{kl}\Big)(z,x,f)\Big). \qquad (2.2.5)
\end{aligned}$$

Now if we use coordinates, we obtain

$$\Big[E_{\alpha_{ij}}, \Big[E_{\delta_{kl}}, E^*_{\beta_{pq}}\Big]\Big](z,x,f) = E_{\alpha_{ij}}\Big[E_{\delta_{kl}}, E^*_{\beta_{pq}}\Big]E^{-1}_{\alpha_{ij}}\Big[E_{\delta_{kl}}, E^*_{\beta_{pq}}\Big]^{-1}(z,x,f)$$

$$
\begin{aligned}
&= E_{\alpha_{ij}}\left[E_{\delta_{kl}}, E^*_{\beta_{pq}}\right] E^{-1}_{\alpha_{ij}}\Big(z,\ x + \langle x, f_p\rangle\langle t_{kl}, v_{pq}\rangle x_k,\ f - \langle f, x_k\rangle\langle t_{kl}, v_{pq}\rangle f_p\Big)\\
&= E_{\alpha_{ij}}\left[E_{\delta_{kl}}, E^*_{\beta_{pq}}\right]\Big(z + \langle f, x_i\rangle w_{ij} - \langle f, x_k\rangle\langle t_{kl}, v_{pq}\rangle\langle f_p, x_i\rangle w_{ij}, x\\
&\quad - \langle w_{ij}, z\rangle x_i + \langle x, f_p\rangle\langle t_{kl}, v_{pq}\rangle x_k + \langle f, x_k\rangle\langle t_{kl}, v_{pq}\rangle\langle f_p, x_i\rangle q(w_{ij})x_i\\
&\quad - \langle f, x_i\rangle q(w_{ij})x_i, f - \langle f, x_k\rangle\langle t_{kl}, v_{pq}\rangle f_p\Big)\\
&= E_{\alpha_{ij}}\Big(z + \Big\{\langle f, x_i\rangle - \langle f, x_k\rangle\langle t_{kl}, v_{pq}\rangle\langle f_p, x_i\rangle\Big\} w_{ij}, x - \Big\{\langle w_{ij}, z\rangle\\
&\quad + \langle f, x_i\rangle q(w_{ij}) + \langle f, x_k\rangle\langle t_{kl}, v_{pq}\rangle\langle f_p, x_i\rangle q(w_{ij})\Big\} x_i\\
&\quad + \Big\{\langle w_{ij}, z\rangle - \langle f, x_k\rangle\langle x_i, f_p\rangle\langle t_{kl}, v_{pq}\rangle q(w_{ij}) + \langle f, x_i\rangle q(w_{ij})\Big\}\\
&\quad \langle x_i, f_p\rangle\langle t_{kl}, v_{pq}\rangle x_k, f - \langle f, x_k\rangle\langle t_{kl}, v_{pq}\rangle^2\langle f_p, x_k\rangle f_p\Big)\\
&= \Big(z - \langle f, x_k\rangle\langle t_{kl}, v_{pq}\rangle\langle f_p, x_i\rangle w_{ij}, x + \Big\{\langle f, x_i\rangle q(w_{ij})\\
&\quad + \langle w_{ij}, z\rangle - \langle f, x_k\rangle\langle t_{kl}, v_{pq}\rangle\langle x_i, f_p\rangle q(w_{ij})\Big\}\langle t_{kl}, v_{pq}\rangle\\
&\quad \langle x_i, f_p\rangle x_k - \langle f, x_k\rangle\langle t_{kl}, v_{pq}\rangle\langle x_i, f_p\rangle q(w_{ij})x_i, f\Big). \qquad (2.2.6)
\end{aligned}
$$

The maps $\mu_{kj},\ \mu^*_{kj}, \frac{1}{2}\mu_{kj}\mu^*_{kj}$ and the elementary transformation $E^*_{\mu_{kj}}$ are given by the following expressions.

$$
\begin{aligned}
\mu_{kj}(z, x, f) &= \delta_{kl}\beta^*_{pq}\alpha_{ij}(z, x, f) = (0, \langle w_{ij}, z\rangle\langle t_{kl}, v_{pq}\rangle\langle x_i, f_p\rangle x_k, 0),\\
\mu^*_{kj}(z, x, f) &= \alpha^*_{ij}\beta_{pq}\delta^*_{kl}(z, x, f) = (\langle f, x_k\rangle\langle t_{kl}, v_{pq}\rangle\langle x_i, f_p\rangle w_{ij}, 0, 0),\\
\frac{1}{2}\mu_{kj}\mu^*_{kj}(z, x, f) &= (0, \langle f, x_k\rangle\langle t_{kl}, v_{pq}\rangle^2\langle x_i, f_p\rangle^2 q(w_{ij})x_k, 0),\\
E^*_{\mu_{kj}}(z, x, f) &= \left(I + \mu_{kj} - \mu^*_{kj} - \frac{1}{2}\mu_{kj}\mu^*_{kj}\right)(z, x, f)\\
&= (z - \langle f, x_k\rangle\langle t_{kl}, v_{pq}\rangle\langle x_i, f_p\rangle w_{ij}, x + \langle w_{ij}, z\rangle\langle t_{kl}, v_{pq}\rangle\langle x_i, f_p\rangle x_k\\
&\quad - \langle f, x_k\rangle\langle t_{kl}, v_{pq}\rangle^2\langle x_i, f_p\rangle^2 q(w_{ij})x_k,\ f\ ).
\end{aligned}
$$

If $i \neq k$, then, by Lemma 2.1.3, we have

$$
\begin{aligned}
\left[E_{\alpha_{ij}}, E_{\frac{\mu_{kj}}{2}}\right](z, x, f) &= \Big(I + \frac{1}{2}\mu_{kj}\alpha^*_{ij} - \frac{1}{2}\alpha_{ij}\mu^*_{kj}\Big)(z, x, f)\\
&= \Big(z, x + \langle f, x_i\rangle\langle t_{kl}, v_{pq}\rangle\langle x_i, f_p\rangle q(w_{ij})x_k - \langle f, x_k\rangle\langle t_{kl}, v_{pq}\rangle\langle x_i, f_p\rangle q(w_{ij})x_i, f\Big)
\end{aligned}
$$

and hence we get

$$E_{\mu_{kj}}\left[E_{\alpha_{ij}}, E_{\frac{\mu_{kj}}{2}}\right](z,x,f) = \Big(I + \mu_{kj} - \mu_{kj}^* - \frac{1}{2}\mu_{kj}\mu_{kj}^* + \frac{1}{2}\mu_{kj}\alpha_{ij}^* - \frac{1}{2}\alpha_{ij}\mu_{kj}^*\Big)(z,x,f)$$

$$= \Big(z - \langle f, x_k\rangle\langle t_{kl}, v_{pq}\rangle\langle x_i, f_p\rangle w_{ij}, x + \Big\{\langle f, x_i\rangle q(w_{ij}) + \langle w_{ij}, z\rangle - \langle f, x_k\rangle\langle t_{kl}, v_{pq}\rangle\langle x_i, f_p\rangle q(w_{ij})\Big\}\langle t_{kl}, v_{pq}\rangle \langle x_i, f_p\rangle x_k - \langle f, x_k\rangle\langle t_{kl}, v_{pq}\rangle\langle x_i, f_p\rangle q(w_{ij})x_i,\ f\Big). \quad (2.2.7)$$

We now consider the following possible conditions on the indices.

Case(i): $i = p$.

If $i = p$, then, by Equations (2.2.6), (2.2.5) and (2.2.7), we have

$$\left[E_{\alpha_{ij}}, \left[E_{\delta_{kl}}, E_{\beta_{pq}}^*\right]\right](z,x,f) = \Big(\Big(I - \alpha_{ij}^*\beta_{pq}\delta_{kl}^* + \delta_{kl}\beta_{pq}^*\alpha_{ij} + \frac{1}{2}\delta_{kl}\beta_{pq}^*\alpha_{ij}\alpha_{ij}^* - \frac{1}{2}\delta_{kl}\beta_{pq}^*\alpha_{ij}\alpha_{ij}^*\beta_{pq}\delta_{kl}^* - \frac{1}{2}\alpha_{ij}\alpha_{ij}^*\beta_{pq}\delta_{kl}^* + \alpha_{ij}^*\beta_{pq}\delta_{kl}^*\beta_{pq}\delta_{kl}^*\Big)(z,x,f)\Big)$$

$$= \Big(z - \langle f, x_k\rangle\langle t_{kl}, v_{pq}\rangle w_{ij},\ x + \langle w_{ij}, z\rangle\langle t_{kl}, v_{pq}\rangle x_k + \langle f, x_i\rangle\langle t_{kl}, v_{pq}\rangle q(w_{ij})x_k - \langle f, x_k\rangle\langle t_{kl}, v_{pq}\rangle q(w_{ij})x_i - \langle f, x_k\rangle\langle t_{kl}, v_{pq}\rangle^2 q(w_{ij})x_k, f\Big)$$

$$= E_{\mu_{kj}}\left[E_{\alpha_{ij}}, E_{\frac{\mu_{kj}}{2}}\right](z,x,f).$$

Case(ii): $i = k$ or $i \neq p$.

If $i = k$ or $i \neq p$, then, by Equation (2.2.6), we have

$$\left[E_{\alpha_{ij}}, \left[E_{\delta_{kl}}, E_{\beta_{pq}}^*\right]\right](z,x,f) = I(z,x,f). \qquad \square$$

We now deduce the commutator identities from the above lemma.

Corollary 2.2.4. *For any i,j,k,l,p,q with $1 \leq i,k,p \leq m$, $1 \leq j,l,q \leq n$, $i \neq p$ and $k \neq p$ and $a,b,c,d,e,f \in A$ with $abc = def$ and $a^2bc = d^2ef$, the following equation holds.*

$$\left[E_{a\alpha_{ij}}, \left[E_{b\delta_{kl}}, E_{c\beta_{pq}}^*\right]\right] = \left[E_{d\alpha_{ij}}, \left[E_{e\delta_{kl}}, E_{f\beta_{pq}}^*\right]\right].$$

We now compute the expression for the triple commutators which has a mixed commutator.

Lemma 2.2.5. *Let $\alpha \in \mathrm{Hom}_A(Q,P)$ and $\beta, \gamma \in \mathrm{Hom}_A(Q,P^*)$. Then, for i,j,k,l,p,q with $1 \le i,k,p \le m$, $1 \le j,l,q \le n$ and $k \ne p$, the triple commutator $\left[E^*_{\beta_{ij}}, \left[E_{\alpha_{kl}}, E^*_{\gamma_{pq}}\right]\right]$ is given by*

$$\left[E^*_{\beta_{ij}}, \left[E_{\alpha_{kl}}, E^*_{\gamma_{pq}}\right]\right] = \begin{cases} E^*_{\nu_{pj}} \left[E^*_{\beta_{ij}}, E^*_{\frac{\nu_{pj}}{2}}\right], & \text{if} \quad i = p, \\ I & \text{if} \quad i = k \quad \text{or} \quad i \ne p, \end{cases}$$

where $\nu_{pj} = -\gamma_{pq}\alpha^*_{kl}\beta_{ij}$.

Proof. For $\alpha \in \mathrm{Hom}_A(Q,P)$, $\beta, \gamma \in \mathrm{Hom}_A(Q,P^*)$ and, for i,j,k,l,p,q with $1 \le i,k,p \le m$, $1 \le j,l,q \le n$ and $k \ne p$, we have

$$\begin{aligned}\left[E_{\alpha_{kl}}, E^*_{\gamma_{pq}}\right](z,x,f) &= \Big(I + \gamma_{pq}\alpha^*_{kl} - \alpha_{kl}\gamma^*_{pq}\Big)(z,x,f) \\ &= \Big(z,\ x - \langle x, f_p\rangle\langle c_{pq}, w_{kl}\rangle x_k,\ f + \langle f, x_k\rangle\langle c_{pq}, w_{kl}\rangle f_p\Big). \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 2.1.5)} \\ [E_{\alpha_{kl}}, E^*_{\gamma_{pq}}]^{-1}(z,x,f) &= \Big(I - \gamma_{pq}\alpha^*_{kl} + \alpha_{kl}\gamma^*_{pq}\Big)(z,x,f) \\ &= \Big(z,\ x + \langle x, f_p\rangle\langle c_{pq}, w_{kl}\rangle x_k,\ f - \langle f, x_k\rangle\langle c_{pq}, w_{kl}\rangle f_p\Big). \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Remark 2.1.8)}\end{aligned}$$

Hence we get the following coordinate-free expression.

$$\begin{aligned}\left[E^*_{\beta_{ij}}, \left[E_{\alpha_{kl}}, E^*_{\gamma_{pq}}\right]\right](z,x,f) &= E^*_{\beta_{ij}}\left[E_{\alpha_{kl}}, E^*_{\gamma_{pq}}\right] E^{*^{-1}}_{\beta_{ij}}\left[E_{\alpha_{kl}}, E^*_{\gamma_{pq}}\right]^{-1}(z,x,f) \\ &= E^*_{\beta_{ij}}\left[E_{\alpha_{kl}}, E^*_{\gamma_{pq}}\right] E^{*^{-1}}_{\beta_{ij}}\Big(\Big(I - \gamma_{pq}\alpha^*_{kl} + \alpha_{kl}\gamma^*_{pq}\Big)(z,x,f)\Big) \\ &= E^*_{\beta_{ij}}\left[E_{\alpha_{kl}}, E^*_{\gamma_{pq}}\right]\Big(\Big(I - \gamma_{pq}\alpha^*_{kl} + \alpha_{kl}\gamma^*_{pq} - \beta_{ij} + \beta^*_{ij} \\ &\quad -\frac{1}{2}\beta_{ij}\beta^*_{ij} + \beta^*_{ij}\alpha_{kl}\gamma^*_{pq} - \frac{1}{2}\beta_{ij}\beta^*_{ij}\alpha_{kl}\gamma^*_{pq}\Big)(z,x,f)\Big) \\ &= E^*_{\beta_{ij}}\Big(\Big(I - \beta_{ij} + \beta^*_{ij} - \frac{1}{2}\beta_{ij}\beta^*_{ij} + \beta^*_{ij}\alpha_{kl}\gamma^*_{pq} - \frac{1}{2}\beta_{ij}\beta^*_{ij}\alpha_{kl}\gamma^*_{pq} \\ &\quad - \gamma_{pq}\alpha^*_{kl}\beta_{ij} - \alpha_{kl}\gamma^*_{pq}\alpha_{kl}\gamma^*_{pq} - \gamma_{pq}\alpha^*_{kl}\gamma_{pq}\alpha^*_{kl} - \frac{1}{2}\gamma_{pq}\alpha^*_{kl}\beta_{ij}\beta^*_{ij}\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\gamma_{pq}\alpha^*_{kl}\beta_{ij}\beta^*_{ij}\alpha_{kl}\gamma^*_{pq}\Big)(z,x,f)\Big)\\
&=\Big(\Big(I+\beta^*_{ij}\alpha_{kl}\gamma^*_{pq}-\gamma_{pq}\alpha^*_{kl}\beta_{ij}-\frac{1}{2}\gamma_{pq}\alpha^*_{kl}\beta_{ij}\beta^*_{ij}\\
&+\frac{1}{2}\beta_{ij}\beta^*_{ij}\alpha_{kl}\gamma^*_{pq}-\frac{1}{2}\gamma_{pq}\alpha^*_{kl}\beta_{ij}\beta^*_{ij}\alpha_{kl}\gamma^*_{pq}\Big)(z,x,f)\Big).
\end{aligned}
\tag{2.2.8}
$$

Now, by computing using coordinates, we have

$$
\begin{aligned}
\Big[E^*_{\beta_{ij}},\Big[E_{\alpha_{kl}},E^*_{\gamma_{pq}}\Big]\Big] &= E^*_{\beta_{ij}}\Big[E_{\alpha_{kl}},E^*_{\gamma_{pq}}\Big]E^{*^{-1}}_{\beta_{ij}}\Big[E_{\alpha_{kl}},E^*_{\gamma_{pq}}\Big]^{-1}(z,x,f)\\
&= E^*_{\beta_{ij}}\Big[E_{\alpha_{kl}},E^*_{\gamma_{pq}}\Big]E^{*^{-1}}_{\beta_{ij}}\Big(z,\ x+\langle x,f_p\rangle\langle c_{pq},w_{kl}\rangle x_k,\ f-\langle f,x_k\rangle\langle c_{pq},w_{kl}\rangle f_p\Big)\\
&= E^*_{\beta_{ij}}\Big[E_{\alpha_{kl}},E^*_{\gamma_{pq}}\Big]\Big(z+\langle f_i,x\rangle v_{ij}+\langle x,f_p\rangle\langle c_{pq},w_{kl}\rangle\langle x_k,f_i\rangle v_{ij},x\\
&\quad+\langle x,f_p\rangle\langle c_{pq},w_{kl}\rangle x_k,f-\langle v_{ij},z\rangle f_i-\langle f,x_k\rangle\langle c_{pq},w_{kl}\rangle f_p-\langle x,f_i\rangle q(v_{ij})f_i\\
&\quad-\langle x,f_p\rangle\langle c_{pq},w_{kl}\rangle\langle x_k,f_i\rangle q(v_{ij})f_i\Big)\\
&= E^*_{\beta_{ij}}\Big(z+\langle f_i,x\rangle v_{ij}+\langle x,f_p\rangle\langle c_{pq},w_{kl}\rangle\langle x_k,f_i\rangle v_{ij},\ x,\ f-\langle v_{ij},z\rangle f_i\\
&\quad-\langle x,f_i\rangle q(v_{ij})f_i-\langle x,f_p\rangle\langle c_{pq},w_{kl}\rangle\langle x_k,f_i\rangle q(v_{ij})f_i\\
&\quad-\langle v_{ij},z\rangle\langle c_{pq},w_{kl}\rangle\langle x_k,f_i\rangle f_p-\langle x,f_i\rangle\langle c_{pq},w_{kl}\rangle\langle x_k,f_i\rangle q(v_{ij})f_p\\
&\quad-\langle x,f_p\rangle\langle c_{pq},w_{kl}\rangle^2\langle x_k,f_i\rangle^2 q(v_{ij})f_p\Big)\\
&= \Big(z+\langle x,f_p\rangle\langle c_{pq},w_{kl}\rangle\langle x_k,f_i\rangle v_{ij},\ x,\ f-\Big\{\langle x,f_i\rangle q(v_{ij})\\
&\quad+\langle v_{ij},z\rangle+\langle x,f_p\rangle\langle c_{pq},w_{kl}\rangle\langle x_k,f_i\rangle q(v_{ij})\Big\}\langle c_{pq},w_{kl}\rangle\langle x_k,f_i\rangle f_p\\
&\quad+\langle x,f_p\rangle\langle c_{pq},w_{kl}\rangle\langle x_k,f_i\rangle q(v_{ij})f_i\Big).
\end{aligned}
\tag{2.2.9}
$$

The maps ν_{pj} in the statement of the lemma, as well as the other maps ν^*_{pj}, $\frac{1}{2}\nu_{pj}\nu^*_{pj}$ and the transformations $E^*_{\nu_{pj}}$ are given as

$$
\begin{aligned}
\nu_{pj}(z,x,f) &= -\gamma_{pq}\alpha^*_{kl}\beta_{ij}(z,x,f)=\Big(0,0,-\langle v_{ij},z\rangle\langle c_{pq},w_{kl}\rangle\langle f_i,x_p\rangle f_k\Big),\\
\nu^*_{pj}(z,x,f) &= -\beta^*_{ij}\alpha_{pq}\gamma^*_{kl}(z,x,f)=\Big(-\langle x,f_p\rangle\langle c_{pq},w_{kl}\rangle\langle f_i,x_k\rangle v_{ij},0,0\Big),\\
\frac{1}{2}\nu_{pj}\nu^*_{pj}(z,x,f) &= \Big(0,0,\langle x,f_p\rangle\langle c_{pq},w_{kl}\rangle^2\langle f_i,x_k\rangle^2 q(v_{ij})f_p\Big),\\
E^*_{\nu_{pj}}(z,x,f) &= \Big(I+\nu_{pj}-\nu^*_{pj}-\frac{1}{2}\nu_{pj}\nu^*_{pj}\Big)(z,x,f)\\
&= \Big(z+\langle x,f_p\rangle\langle c_{pq},w_{kl}\rangle\langle x_k,f_i\rangle v_{ij},\ x,\ f-\langle v_{ij},z\rangle\langle c_{pq},w_{kl}\rangle\langle x_k,f_i\rangle f_p
\end{aligned}
$$

$$-\langle x, f_p\rangle\langle c_{pq}, w_{kl}\rangle^2\langle x_k, f_i\rangle^2 q(v_{ij}) f_p\Big).$$

If $i \neq p$, then, by Lemma 2.1.9, we have

$$\begin{aligned}\left[E^*_{\beta_{ij}}, E^*_{\frac{\nu_{pj}}{2}}\right](z,x,f) &= \Big(I + \frac{1}{2}\nu_{pj}\beta^*_{ij} - \frac{1}{2}\beta_{ij}\nu^*_{pj}\Big)(z,x,f)\\ &= \Big(z,\ x,\ f + \langle x, f_p\rangle\langle c_{pq}, w_{kl}\rangle\langle x_k, f_i\rangle q(v_{ij}) f_i\\ &\qquad - \langle x, f_i\rangle\langle c_{pq}, w_{kl}\rangle\langle x_k, f_i\rangle q(v_{ij}) f_p\Big)\end{aligned}$$

and hence we get

$$\begin{aligned}E^*_{\nu_{pj}}\left[E^*_{\beta_{ij}}, E^*_{\frac{\nu_{pj}}{2}}\right](z,x,f) &= \Big(I + \nu_{pj} - \nu^*_{pj} - \frac{1}{2}\nu_{pj}\nu^*_{pj} + \frac{1}{2}\nu_{pj}\beta^*_{ij} - \frac{1}{2}\beta_{ij}\nu^*_{pj}\Big)(z,x,f)\\ &= \Big(z + \langle x, f_p\rangle\langle c_{pq}, w_{kl}\rangle\langle x_k, f_i\rangle v_{ij},\ x, f - \Big\{\langle x, f_i\rangle q(v_{ij}\\ &\quad + \langle v_{ij}, z\rangle + \langle x, f_p\rangle\langle c_{pq}, w_{kl}\rangle\langle x_k, f_i\rangle q(v_{ij})\Big\}\langle c_{pq}, w_{kl}\rangle\\ &\quad \langle x_k, f_i\rangle f_p + \langle x, f_p\rangle\langle c_{pq}, w_{kl}\rangle\langle x_k, f_i\rangle q(v_{ij}) f_i\Big). \end{aligned} \tag{2.2.10}$$

We now consider the following possible conditions on the indices.

Case(i): $i = k$.

If $i = k$, then, by Equations (2.2.9), (2.2.8) and (2.2.10), we have

$$\begin{aligned}\left[E^*_{\beta_{ij}}, \left[E_{\alpha_{kl}}, E^*_{\gamma_{pq}}\right]\right](z,x,f) &= \Big(\Big(I + \beta^*_{ij}\alpha_{kl}\gamma^*_{pq} - \gamma_{pq}\alpha^*_{kl}\beta_{ij} - \frac{1}{2}\gamma_{pq}\alpha^*_{kl}\beta_{ij}\beta^*_{ij}\\ &\quad + \frac{1}{2}\beta_{ij}\beta^*_{ij}\alpha_{kl}\gamma^*_{pq} - \frac{1}{2}\gamma_{pq}\alpha^*_{kl}\beta_{ij}\beta^*_{ij}\alpha_{kl}\gamma^*_{pq}\Big)(z,x,f)\Big)\\ &= \Big(z + \langle x, f_p\rangle\langle c_{pq}, w_{kl}\rangle v_{ij},\ x,\ f - \langle v_{ij}, z\rangle\langle c_{pq}, w_{kl}\rangle f_p\\ &\quad + \langle x, f_p\rangle\langle c_{pq}, w_{kl}\rangle q(v_{ij}) f_i - \langle x, f_p\rangle\langle c_{pq}, w_{kl}\rangle^2 q(v_{ij}) f_p\\ &\quad - \langle x, f_i\rangle\langle c_{pq}, w_{kl}\rangle q(v_{ij}) f_p\Big)\\ &= E^*_{\nu_{pj}}\left[E^*_{\beta_{ij}}, E^*_{\frac{\nu_{pj}}{2}}\right](z,x,f).\end{aligned}$$

Case(ii): $i = p$ or $i \neq k$.

If $i = k$ or $i \neq p$, then, by Equation (2.2.6), we have

$$\left[E^*_{\beta_{ij}}, \left[E_{\alpha_{kl}}, E^*_{\gamma_{pq}}\right]\right](z,x,f) = I(z,x,f). \qquad \square$$

The set of commutator relations we deduce from the above lemma is given in the corollary below.

Corollary 2.2.6. *For any given i, j, k, l, p, q, where $1 \leq i, k, p \leq m$, $1 \leq j, l, q \leq n$ such that $i \neq k$ and $k \neq p$ and $a, b, c, d, e, f \in A$, $\left[E^*_{a\beta_{ij}}, \left[E^*_{b\gamma_{kl}}, E_{c\alpha_{pq}}\right]\right] = \left[E^*_{d\beta_{ij}}, \left[E^*_{e\gamma_{kl}}, E_{f\alpha_{pq}}\right]\right]$ if $abc = def$ and $a^2bc = d^2ef$.*

Finally, another triple commutator is computed in the following lemma and the commutator relations which follow from this are stated in the corollary below this lemma.

Lemma 2.2.7. *Let $\alpha \in \mathrm{Hom}_A(Q, P)$ and $\beta, \gamma \in \mathrm{Hom}_A(Q, P^*)$. Then, for i, j, k, l, p, q with $1 \leq i, k, p \leq m$, $1 \leq j, l, q \leq n$ and $k \neq p$, the triple commutator $\left[E_{\alpha_{ij}}, \left[E^*_{\beta_{kl}}, E^*_{\gamma_{pq}}\right]\right]$ is given by*

$$\left[E_{\alpha_{ij}}, \left[E^*_{\beta_{kl}}, E^*_{\gamma_{pq}}\right]\right] = \begin{cases} E^*_{\eta_{kj}} \left[E_{\alpha_{ij}}, E^*_{\frac{\eta_{kj}}{2}}\right] & \text{if} \quad i = p, \\ E^*_{\vartheta_{pj}} \left[E_{\alpha_{ij}}, E^*_{\frac{\vartheta_{pj}}{2}}\right] & \text{if} \quad i = k, \\ I & \text{if} \quad i \neq p \text{ and } i \neq k, \end{cases}$$

where $\eta_{kj} = \beta_{kl}\gamma^*_{pq}\alpha_{ij}$ and $\vartheta_{pj} = \gamma_{pq}\beta^*_{kl}\alpha_{ij}$.

Proof. For $\alpha \in \mathrm{Hom}_A(Q, P)$, $\beta, \gamma \in \mathrm{Hom}_A(Q, P^*)$ and for i, j, k, l, p, q with $1 \leq i, k, p \leq m$, $1 \leq j, l, q \leq n$ and $k \neq p$, we have

$$\begin{aligned} \left[E^*_{\beta_{kl}}, E^*_{\gamma_{pq}}\right](z, x, f) &= \Big(I + \gamma_{pq}\beta^*_{kl} - \beta_{kl}\gamma^*_{kl}\Big)(z, x, f) \\ &= \Big(z,\ x,\ f + \langle x, f_k\rangle\langle c_{pq}, v_{kl}\rangle f_p - \langle x, f_p\rangle\langle v_{kl}, c_{pq}\rangle f_k\Big). \\ &\qquad\qquad\qquad\qquad \text{(by Lemma 2.1.9)} \\ \left[E^*_{\beta_{kl}}, E^*_{\gamma_{pq}}\right]^{-1}(z, x, f) &= \left[E^*_{\gamma_{pq}}, E^*_{\beta_{kl}}\right](z, x, f) \\ &= \Big(I - \gamma_{pq}\beta^*_{kl} + \beta_{kl}\gamma^*_{pq}\Big)(z, x, f) \\ &= \Big(z,\ x,\ f - \langle x, f_k\rangle\langle c_{pq}, v_{kl}\rangle f_p + \langle x, f_p\rangle\langle v_{kl}, c_{pq}\rangle f_k\Big). \\ &\qquad\qquad\qquad\qquad \text{(by Lemma 2.1.9)} \end{aligned}$$

Hence we get

$$\begin{aligned}
\left[E_{\alpha_{ij}},\left[E^*_{\beta_{kl}},E^*_{\gamma_{pq}}\right]\right](z,x,f) &= E_{\alpha_{ij}}\left[E^*_{\beta_{kl}},E^*_{\gamma_{pq}}\right]E^{-1}_{\alpha_{ij}}\left[E^*_{\beta_{kl}},E^*_{\gamma_{pq}}\right]^{-1}(z,x,f)\\
&= E_{\alpha_{ij}}\left[E^*_{\beta_{kl}},E^*_{\gamma_{pq}}\right]E^{-1}_{\alpha_{ij}}\Big(\Big(I-\gamma_{pq}\beta^*_{kl}+\beta_{kl}\gamma^*_{pq}\Big)(z,x,f)\Big)\\
&= E_{\alpha_{ij}}\left[E^*_{\beta_{kl}},E^*_{\gamma_{pq}}\right]\Big(\Big(I-\gamma_{pq}\beta^*_{kl}+\beta_{kl}\gamma^*_{pq}-\alpha_{ij}+\alpha^*_{ij}\\
&\quad -\alpha^*_{ij}\gamma_{pq}\beta^*_{kl}+\alpha^*_{ij}\beta_{kl}\gamma^*_{pq}+\frac{1}{2}\alpha_{ij}\alpha^*_{ij}\gamma_{pq}\beta^*_{kl}-\frac{1}{2}\alpha_{ij}\alpha^*_{ij}\\
&\quad -\frac{1}{2}\alpha_{ij}\alpha^*_{ij}\beta_{kl}\gamma^*_{pq}\Big)(z,x,f)\Big)\\
&= E_{\alpha_{ij}}\Big(\Big(I-\alpha_{ij}+\alpha^*_{ij}-\frac{1}{2}\alpha_{ij}\alpha^*_{ij}-\alpha^*_{ij}\gamma_{pq}\beta^*_{kl}+\alpha^*_{ij}\beta_{kl}\gamma^*_{pq}\\
&\quad +\frac{1}{2}\alpha_{ij}\alpha^*_{ij}\gamma_{pq}\beta^*_{kl}-\frac{1}{2}\alpha_{ij}\alpha^*_{ij}\beta_{kl}\gamma^*_{pq}-\frac{1}{2}\gamma_{pq}\beta^*_{kl}\alpha_{ij}\alpha^*_{ij}\\
&\quad +\frac{1}{2}\gamma_{pq}\beta^*_{kl}\alpha_{ij}\alpha^*_{ij}\gamma_{pq}\beta^*_{kl}-\frac{1}{2}\gamma_{pq}\beta^*_{kl}\alpha_{ij}\alpha^*_{ij}\beta_{kl}\gamma^*_{pq}+\beta_{kl}\gamma^*_{pq}\alpha_{ij}\\
&\quad +\frac{1}{2}\beta_{kl}\gamma^*_{pq}\alpha_{ij}\alpha^*_{ij}-\frac{1}{2}\beta_{kl}\gamma^*_{pq}\alpha_{ij}\alpha^*_{ij}\gamma_{pq}\beta^*_{kl}-\gamma_{pq}\beta^*_{kl}\alpha_{ij}\\
&\quad +\frac{1}{2}\beta_{kl}\gamma^*_{pq}\alpha_{ij}\alpha^*_{ij}\beta_{kl}\gamma^*_{pq}\Big)(z,x,f)\Big)\\
&= \Big(\Big(I+\alpha^*_{ij}\beta_{kl}\gamma^*_{pq}-\alpha^*_{ij}\gamma_{pq}\beta^*_{kl}+\frac{1}{2}\gamma_{pq}\beta^*_{kl}\alpha_{ij}\alpha^*_{ij}\gamma_{pq}\beta^*_{kl}\\
&\quad +\frac{1}{2}\alpha_{ij}\alpha^*_{ij}\beta_{kl}\gamma^*_{pq}-\frac{1}{2}\gamma_{pq}\beta^*_{kl}\alpha_{ij}\alpha^*_{ij}\beta_{kl}\gamma^*_{pq}\\
&\quad -\frac{1}{2}\alpha_{ij}\alpha^*_{ij}\gamma_{pq}\beta^*_{kl}+\beta_{kl}\gamma^*_{pq}\alpha_{ij}-\frac{1}{2}\gamma_{pq}\beta^*_{kl}\alpha_{ij}\alpha^*_{ij}\\
&\quad -\frac{1}{2}\beta_{kl}\gamma^*_{pq}\alpha_{ij}\alpha^*_{ij}\gamma_{pq}\beta^*_{kl}+\frac{1}{2}\beta_{kl}\gamma^*_{pq}\alpha_{ij}\alpha^*_{ij}\\
&\quad -\gamma_{pq}\beta^*_{kl}\alpha_{ij}+\frac{1}{2}\beta_{kl}\gamma^*_{pq}\alpha_{ij}\alpha^*_{ij}\beta_{kl}\gamma^*_{pq}\Big)(z,x,f)\Big).
\end{aligned} \tag{2.2.11}$$

Computing with coordinates, we get

$$\begin{aligned}
\left[E_{\alpha_{ij}},\left[E^*_{\beta_{kl}},E^*_{\gamma_{pq}}\right]\right](z,x,f) &= E_{\alpha_{ij}}\left[E^*_{\beta_{kl}},E^*_{\gamma_{pq}}\right]E^{-1}_{\alpha_{ij}}\left[E^*_{\beta_{kl}},E^*_{\gamma_{pq}}\right]^{-1}(z,x,f)\\
&= E_{\alpha_{ij}}\left[E^*_{\beta_{kl}},E^*_{\gamma_{pq}}\right]E^{-1}_{\alpha_{ij}}\Big(z,\ x,\ f-\langle x,f_k\rangle\langle c_{pq},v_{kl}\rangle f_p\\
&\quad +\langle x,f_p\rangle\langle v_{kl},c_{pq}\rangle f_k\Big)\\
&= E_{\alpha_{ij}}\left[E^*_{\beta_{kl}},E^*_{\gamma_{pq}}\right]\Big(z+\Big\{\langle f,x_i\rangle-\langle x,f_k\rangle\langle c_{pq},v_{kl}\rangle\langle x_i,f_p\rangle\\
&\quad +\langle x,f_p\rangle\langle c_{pq},v_{kl}\rangle\langle x_i,f_k\rangle\Big\}w_{ij},\ x-\Big\{\langle w_{ij},z\rangle+\langle f,x_i\rangle q(w_{ij})
\end{aligned}$$

$$
\begin{aligned}
&- \langle x, f_k\rangle\langle c_{pq}, v_{kl}\rangle\langle x_i, f_p\rangle q(w_{ij}) + \langle x, f_p\rangle\langle c_{pq}, v_{kl}\rangle\langle x_i, f_k\rangle \\
&q(w_{ij})\Big\}x_i, f - \langle x, f_k\rangle\langle c_{pq}, v_{kl}\rangle f_p + \langle x, f_p\rangle\langle v_{kl}, c_{pq}\rangle f_k\Big) \\
&= E_{\alpha_{ij}}\Big(z + \Big\{\langle x, f_p\rangle\langle c_{pq}, v_{kl}\rangle\langle x_i, f_k\rangle - \langle x, f_k\rangle\langle c_{pq}, v_{kl}\rangle\langle x_i, f_p\rangle \\
&+\langle f, x_i\rangle\Big\}w_{ij},\ x + \Big\{\langle x, f_k\rangle c_{pq}, v_{kl}\rangle\langle x_i, f_p\rangle q(w_{ij}) - \langle w_{ij}, z\rangle \\
&-\langle f, x_i\rangle q(w_{ij}) - \langle x, f_p\rangle\langle c_{pq}, v_{kl}\rangle\langle x_i, f_k\rangle q(w_{ij})\Big\}x_i, \\
&f + \Big\{\langle x, f_p\rangle\langle c_{pq}, v_{kl}\rangle\langle x_i, f_k\rangle q(w_{ij}) + \langle w_{ij}, z\rangle + \langle f, x_i\rangle q(w_{ij}) \\
&-\langle x, f_k\rangle\langle c_{pq}, v_{kl}\rangle\langle x_i, f_p\rangle q(w_{ij}) f_k\Big\}\langle x_i, f_p\rangle\langle c_{pq}, v_{kl}\rangle f_k \\
&+ \Big\{\langle x, f_k\rangle\langle c_{pq}, v_{kl}\rangle\langle x_i, f_p\rangle q(w_{ij}) - \langle w_{ij}, z\rangle - \langle f, x_i\rangle q(w_{ij}) \\
&-\langle x, f_p\rangle\langle c_{pq}, v_{kl}\rangle\langle x_i, f_k\rangle q(w_{ij})\Big\}\langle x_i, f_k\rangle\langle c_{pq}, v_{kl}\rangle f_p\Big) \\
&= \Big(z + \Big\{\langle x, f_p\rangle\langle x_i, f_k\rangle - \langle x, f_k\rangle\langle x_i, f_p\rangle\Big\}\langle c_{pq}, v_{kl}\rangle w_{ij}, \\
&\quad x + \Big\{\langle x, f_p\rangle\langle x_i, f_k\rangle - \langle x, f_k\rangle\langle x_i, f_p\rangle\Big\}\langle c_{pq}, v_{kl}\rangle \\
&\quad q(w_{ij})x_i,\ f + \Big\{\langle w_{ij}, z\rangle - \langle x, f_k\rangle\langle c_{pq}, v_{kl}\rangle\langle x_i, f_p\rangle \\
&\quad q(w_{ij}) + \langle f, x_i\rangle q(w_{ij}) - \langle x, f_p\rangle\langle c_{pq}, v_{kl}\rangle\langle x_i, f_k\rangle \\
&\quad q(w_{ij})\Big\}\langle x_i, f_p\rangle\langle c_{pq}, v_{kl}\rangle f_k + \Big\{\langle x, f_k\rangle\langle c_{pq}, v_{kl}\rangle \\
&\quad \langle x_i, f_p\rangle q(w_{ij}) - \langle f, x_i\rangle q(w_{ij}) - \langle w_{ij}, z\rangle \\
&\quad -\langle x, f_p\rangle\langle c_{pq}, v_{kl}\rangle\langle x_i, f_k\rangle q(w_{ij})\Big\}\langle x_i, f_k\rangle\langle c_{pq}, v_{kl}\rangle f_p\Big).
\end{aligned}
\tag{2.2.12}
$$

The transformations $\eta_{kj}, \eta^*_{kj}, \frac{1}{2}\eta_{kj}\eta^*_{kj}$ and $E^*_{\eta_{kj}}$ are given by

$$
\begin{aligned}
\eta_{kj}(z, x, f) &= \beta_{kl}\gamma^*_{pq}\alpha_{ij}(z, x, f) = \Big(0, 0, \langle w_{ij}, z\rangle\langle c_{pq}, v_{kl}\rangle\langle f_p, x_i\rangle f_k\Big), \\
\eta^*_{kj}(z, x, f) &= \alpha^*_{ij}\gamma_{pq}\beta^*_{kl}(z, x, f) = \Big(\langle x, f_k\rangle\langle c_{pq}, v_{kl}\rangle\langle f_p, x_i\rangle w_{ij}, 0, 0\Big), \\
\frac{1}{2}\eta_{kj}\eta^*_{kj}(z, x, f) &= \Big(0, 0, \langle x, f_k\rangle\langle c_{kl}, w_{pq}\rangle^2\langle f_p, x_i\rangle^2 q(w_{ij}) f_k\Big), \\
E^*_{\eta_{kj}}(z, x, f) &= \Big(z - \langle x, f_k\rangle\langle c_{pq}, v_{kl}\rangle\langle x_i, f_p\rangle w_{ij}, x,\ f + \langle w_{ij}, z\rangle\langle c_{pq}, v_{kl}\rangle\langle x_i, f_p\rangle f_k \\
&\quad -\langle x, f_k\rangle\langle c_{pq}, v_{kl}\rangle^2\langle x_i, f_p\rangle^2 q(w_{ij}) f_k\Big).
\end{aligned}
$$

If $i \neq k$, then, by Lemma 2.1.5, we have

$$[E_{\alpha_{ij}}, E^*_{\frac{\eta_{kj}}{2}}](z,x,f) = \Big(I + \frac{1}{2}\eta_{kj}\alpha^*_{ij} - \frac{1}{2}\alpha_{ij}\eta^*_{kj}\Big)(z,x,f)$$
$$= \Big(z,\ x - \langle x, f_k\rangle\langle c_{pq}, v_{kl}\rangle\langle x_i, f_p\rangle q(w_{ij})x_i,$$
$$f + \langle f, x_i\rangle\langle c_{pq}, v_{kl}\rangle\langle x_i, f_p\rangle q(w_{ij})f_k\Big)$$

and hence we get

$$E^*_{\eta_{kj}}[E_{\alpha_{ij}}, E^*_{\frac{\eta_{kj}}{2}}](z,x,f) = \Big(I + \eta_{kj} - \eta^*_{kj} - \frac{1}{2}\eta_{kj}\eta^*_{kj} + \frac{1}{2}\eta_{kj}\alpha^*_{ij} - \frac{1}{2}\alpha_{ij}\eta^*_{kj}\Big)(z,x,f)$$
$$= \Big(\ z - \langle x, f_k\rangle\langle c_{pq}, v_{kl}\rangle\langle x_i, f_p\rangle w_{ij},\ x - \langle x, f_k\rangle\langle c_{pq}, v_{kl}\rangle$$
$$\langle x_i, f_p\rangle q(w_{ij})x_i,\ f + \Big\{\langle w_{ij}, z\rangle + \langle f, x_i\rangle q(w_{ij}) - \langle x, f_k\rangle$$
$$\langle c_{pq}, v_{kl}\rangle\langle x_i, f_p\rangle q(w_{ij})\Big\}\langle c_{pq}, v_{kl}\rangle\langle x_i, f_p\rangle f_k\Big). \tag{2.2.13}$$

Similarly, if $i \neq p$, then we have

$$E^*_{\vartheta_{pj}}\left[E_{\alpha_{ij}}, E^*_{\frac{\vartheta_{pj}}{2}}\right](z,x,f) = \Big(I + \vartheta_{pj} - \vartheta^*_{pj} - \frac{1}{2}\vartheta_{pj}\vartheta^*_{pj} - \frac{1}{2}\alpha_{ij}\vartheta^*_{pj} + \frac{1}{2}\vartheta_{pj}\alpha^*_{ij}\Big)(z,x,f)$$
$$= \Big(z + \langle x, f_p\rangle\langle c_{pq}, v_{kl}\rangle\langle x_i, f_k\rangle w_{ij},\ x - \langle x, f_p\rangle\langle c_{pq}, v_{kl}\rangle$$
$$\langle x_i, f_k\rangle q(w_{ij})x_i,\ f - \Big\{\langle w_{ij}, z\rangle - \langle f, x_i\rangle q(w_{ij}) - \langle x, f_p\rangle$$
$$\langle c_{pq}, v_{kl}\rangle\langle x_i, f_k\rangle q(w_{ij})\Big\}\langle x_i, f_k\rangle\langle c_{pq}, v_{kl}\rangle f_p\Big). \tag{2.2.14}$$

We now consider the following possible conditions on the indices.

Case(i): $i = p$.

If $i = p$, then, by Equations (2.2.12), (2.2.11), and (2.2.13), we have

$$\left[E_{\alpha_{ij}}, \left[E^*_{\beta_{kl}}, E^*_{\gamma_{pq}}\right]\right](z,x,f) = \Big(\Big(I - \alpha^*_{pj}\gamma_{pq}\beta^*_{kl} - \frac{1}{2}\alpha_{pj}\alpha^*_{pj}\gamma_{pq}\beta^*_{kl} + \beta_{kl}\gamma^*_{pq}\alpha_{pj}$$
$$+ \frac{1}{2}\beta_{kl}\gamma^*_{pq}\alpha_{pj}\alpha^*_{pj} - \frac{1}{2}\beta_{kl}\gamma^*_{pq}\alpha_{pj}\alpha^*_{pj}\gamma_{pq}\beta^*_{kl}\Big)(z,x,f)\Big)$$
$$= \Big(\ z - \langle x, f_k\rangle\langle c_{pq}, v_{kl}\rangle w_{pj},\ x - \langle x, f_k\rangle\langle c_{pq}, v_{kl}\rangle q(w_{pj})x_p,$$
$$f + \langle w_{pj}, z\rangle\langle c_{pq}, v_{kl}\rangle f_k + \langle f, x_p\rangle\langle c_{pq}, v_{kl}\rangle q(w_{pj})f_k$$
$$-\langle x, f_k\rangle\langle c_{pq}, v_{kl}\rangle^2 q(w_{pj})f_k\Big)$$

$$= E^*_{\eta_{kj}} \left[E_{\alpha_{ij}}, E^*_{\frac{\eta_{kj}}{2}} \right] (z, x, f).$$

Case(ii): $i = k$.

If $i = k$, then, by Equations (2.2.12), (2.2.11), and (2.2.14), we have

$$\begin{aligned}
\left[E_{\alpha_{ij}}, \left[E^*_{\beta_{kl}}, E^*_{\gamma_{pq}} \right] \right] (z, x, f) &= \Big(\Big(I - \gamma_{pq}\beta^*_{kl}\alpha_{kj} + \alpha^*_{kj}\beta_{kl}\gamma^*_{pq} + \frac{1}{2}\alpha_{kj}\alpha^*_{kj}\beta_{kl}\gamma^*_{pq} \\
&\quad - \frac{1}{2}\gamma_{pq}\beta^*_{kl}\alpha_{kj}\alpha^*_{kj} - \frac{1}{2}\gamma_{pq}\beta^*_{kl}\alpha_{kj}\alpha^*_{kj}\beta_{kl}\gamma^*_{pq} \Big)(z, x, f) \Big) \\
&= \Big(z + \langle x, f_p \rangle \langle c_{pq}, v_{kl} \rangle w_{kj},\ x + \langle x, f_p \rangle \langle c_{pq}, v_{kl} \rangle q(w_{kj}) x_k, \\
&\quad f - \Big\{ \langle w_{kj}, z \rangle + \langle f, x_k \rangle q(w_{kj}) + \langle x, f_p \rangle \langle c_{pq}, v_{kl} \rangle q(w_{kj}) \Big\} \\
&\quad \langle c_{pq}, v_{kl} \rangle f_p \Big).
\end{aligned}$$

Case(iii): $i \neq p$ and $i \neq k$.

If $i \neq p$, then, by Equation (2.2.12), we have

$$\left[E_{\alpha_{ij}}, \left[E^*_{\beta_{kl}}, E^*_{\gamma_{pq}} \right] \right] (z, x, f) = I(z, x, f). \qquad \square$$

Corollary 2.2.8. *For any i, j, k, l, p, q with $1 \leq i, k, p \leq m$, $1 \leq j, l, q \leq n$, $i \neq k$ and $k \neq p$ and $a, b, c, d, e, f \in A$ with $abc = def$ and $a^2bc = d^2ef$, the following equation holds.*

$$\left[E_{a\alpha_{ij}}, \left[E^*_{b\beta_{kl}}, E^*_{c\gamma_{pq}} \right] \right] = \left[E_{d\alpha_{ij}}, \left[E^*_{e\beta_{kl}}, E^*_{f\gamma_{pq}} \right] \right].$$

2.3 Multiple Commutators

In this section, we establish some four-fold commutator formulae. These will be needed while proving the normality of the elementary orthogonal group. *In this section, the computations will be done without using coordinates, since the computation using coordinates is too involved.*

Lemma 2.3.1. *Let $\alpha \in \mathrm{Hom}_A(Q, P)$ and $\beta, \gamma, \mu \in \mathrm{Hom}_A(Q, P^*)$. Then, for i, j, k, l, r, s, p, q with $1 \leq i, k, r, p \leq m$, $1 \leq j, l, s, q \leq n$, $i \neq k$ and $r \neq p$, the four-fold commutator*

$\Big[[E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}], [E_{\alpha_{rs}}, E^*_{\mu_{pq}}]\Big]$ *is given by*

$$\Big[[E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}], [E_{\alpha_{rs}}, E^*_{\mu_{pq}}]\Big] = \begin{cases} \Big[E^*_{\mu_{pq}\alpha^*_{rs}}, E^*_{\beta_{ij}\gamma^*_{kl}}\Big] & \text{if} \quad k = r, \\ \Big[E^*_{\gamma_{kl}\beta^*_{ij}}, E^*_{\mu_{pq}\alpha^*_{rs}}\Big] & \text{if} \quad i = r, \\ I & \quad \text{otherwise} . \end{cases}$$

Proof. If $i \neq k$, then, by Lemma 2.1.9, we have

$$[E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}](z, x, f) = (I + \gamma_{kl}\beta^*_{ij} - \beta_{ij}\gamma^*_{kl})(z, x, f).$$

If $r \neq p$, then, by Lemma 2.1.5, we have

$$[E_{\alpha_{rs}}, E^*_{\mu_{pq}}](z, x, f) = (I + \mu_{pq}\alpha^*_{rs} - \alpha_{rs}\mu^*_{pq})(z, x, f).$$

Now if $i \neq k$ and $r \neq p$, then we get

$$\begin{aligned} \Big[[E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}], [E_{\alpha_{rs}}, E^*_{\mu_{pq}}]\Big](z, x, f) &= [E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}][E_{\alpha_{rs}}, E^*_{\mu_{pq}}][E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}]^{-1}[E_{\alpha_{rs}}, E^*_{\mu_{pq}}]^{-1}(z, x, f) \\ &= [E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}][E_{\alpha_{rs}}, E^*_{\mu_{pq}}][E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}]^{-1}\big((I - \mu_{pq}\alpha^*_{rs} \\ &\quad +\alpha_{rs}\mu^*_{pq})(z, x, f)\big) \\ &= [E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}][E_{\alpha_{rs}}, E^*_{\mu_{pq}}]\big((I - \mu_{pq}\alpha^*_{rs} + \alpha_{rs}\mu^*_{pq} - \gamma_{kl}\beta^*_{ij} \\ &\quad +\beta_{ij}\gamma^*_{kl} - \gamma_{kl}\beta^*_{ij}\alpha_{rs}\mu^*_{pq} + \beta_{ij}\gamma^*_{kl}\alpha_{rs}\mu^*_{pq})(z, x, f)\big) \\ &= [E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}]\big((I - \gamma_{kl}\beta^*_{ij} + \beta_{ij}\gamma^*_{kl} - \gamma_{kl}\beta^*_{ij}\alpha_{rs}\mu^*_{pq} + \beta_{ij}\gamma^*_{kl}\alpha_{rs}\mu^*_{pq} \\ &\quad - \mu_{pq}\alpha^*_{rs}\mu_{pq}\alpha^*_{rs} - \mu_{pq}\alpha^*_{rs}\gamma_{kl}\beta^*_{ij} + \mu_{pq}\alpha^*_{rs}\beta_{ij}\gamma^*_{kl} - \alpha_{rs}\mu^*_{pq}\alpha_{rs}\mu^*_{pq} \\ &\quad -\mu_{pq}\alpha^*_{rs}\gamma_{kl}\beta^*_{ij}\alpha_{rs}\mu^*_{pq} + \mu_{pq}\alpha^*_{rs}\beta_{ij}\gamma^*_{kl}\alpha_{rs}\mu^*_{pq})(z, x, f)\big) \\ &= (I - \gamma_{kl}\beta^*_{ij}\alpha_{rs}\mu^*_{pq} - \mu_{pq}\alpha^*_{rs}\gamma_{kl}\beta^*_{ij} + \beta_{ij}\gamma^*_{kl}\alpha_{rs}\mu^*_{pq} \\ &\quad + \mu_{pq}\alpha^*_{rs}\beta_{ij}\gamma^*_{kl})(z, x, f) \\ &= \Big[E^*_{\mu_{pq}\alpha^*_{rs}}, E^*_{\beta_{ij}\gamma^*_{kl}}\Big]\Big[E^*_{\gamma_{kl}\beta^*_{ij}}, E^*_{\mu_{pq}\alpha^*_{rs}}\Big](z, x, f). \qquad (2.3.1) \end{aligned}$$

Now if $k = r$, then Equation (2.3.1) becomes $\Big[E^*_{\mu_{pq}\alpha^*_{rs}}, E^*_{\beta_{ij}\gamma^*_{kl}}\Big](z, x, f)$ and in particular

$$\Big[E^*_{\mu_{pq}\alpha^*_{rs}}, E^*_{\beta_{ij}\gamma^*_{kl}}\Big](z, x, f) = I(z, x, f) \quad \text{if} \quad i = p,$$

and if $i = r$, then Equation (2.3.1) becomes $\left[E^*_{\gamma_{kl}\beta^*_{ij}}, E^*_{\mu_{pq}\alpha^*_{rs}}\right](z,x,f)$ and in particular

$$\left[E^*_{\gamma_{kl}\beta^*_{ij}}, E^*_{\mu_{pq}\alpha^*_{rs}}\right](z,x,f) = I(z,x,f) \quad \text{if} \quad k = p. \qquad \square$$

Lemma 2.3.2. *Let* $\alpha, \delta, \xi \in \mathrm{Hom}_A(Q,P)$ *and* $\beta \in \mathrm{Hom}_A(Q,P^*)$. *Then, for* i,j,k,l,r,s,p,q *with* $1 \le i,k,r,p \le m$, $1 \le j,l,s,q \le n$, $i \ne k$ *and* $s \ne p$, *the four-fold commutator* $\left[[E_{\alpha_{ij}}, E_{\delta_{kl}}], [E_{\xi_{rs}}, E^*_{\beta_{pq}}]\right]$ *is given by*

$$\left[[E_{\alpha_{ij}}, E_{\delta_{kl}}], [E_{\xi_{rs}}, E^*_{\beta_{pq}}]\right] = \begin{cases} \left[E_{\delta_{kl}\alpha^*_{ij}}, E_{\xi_{rs}\beta^*_{pq}}\right] & \text{if} \quad i = p, \\ \left[E_{\alpha_{ij}\delta^*_{kl}}, E_{\xi_{rs}\beta^*_{pq}}\right] & \text{if} \quad k = p, \\ I & \text{otherwise} . \end{cases}$$

Proof. If $i \ne k$, then, by Lemma 2.1.3, we have

$$[E_{\alpha_{ij}}, E_{\delta_{kl}}](z,x,f) = (I + \delta_{kl}\alpha^*_{ij} - \alpha_{ij}\delta^*_{kl})(z,x,f).$$

If $r \ne p$, then, by Lemma 2.1.5, we have

$$[E_{\xi_{rs}}, E^*_{\beta_{pq}}](z,x,f) = (I + \beta_{pq}\xi^*_{rs} - \xi_{rs}\beta^*_{pq})(z,x,f).$$

Now if $i \ne k$ and $r \ne p$, then we get

$$\begin{aligned} \left[[E_{\alpha_{ij}}, E_{\delta_{kl}}], [E_{\xi_{rs}}, E^*_{\beta_{pq}}]\right](z,x,f) &= [E_{\alpha_{ij}}, E_{\delta_{kl}}][E_{\xi_{rs}}, E^*_{\beta_{pq}}][E_{\alpha_{ij}}, E_{\delta_{kl}}]^{-1}[E_{\xi_{rs}}, E^*_{\beta_{pq}}]^{-1}(z,x,f) \\ &= [E_{\alpha_{ij}}, E_{\delta_{kl}}][E_{\xi_{rs}}, E^*_{\beta_{pq}}][E_{\alpha_{ij}}, E_{\delta_{kl}}]^{-1}\left((I - \beta_{pq}\xi^*_{rs}\right. \\ &\quad \left.\xi_{rs}\beta^*_{pq})(z,x,f)\right) \\ &= [E_{\alpha_{ij}}, E_{\delta_{kl}}][E_{\xi_{rs}}, E^*_{\beta_{pq}}]\left((I - \beta_{pq}\xi^*_{rs} + \xi_{rs}\beta^*_{pq} - \delta_{kl}\alpha^*_{ij}\right. \\ &\quad \left. + \alpha_{ij}\delta^*_{kl} + \delta_{kl}\alpha^*_{ij}\beta_{pq}\xi rs^* - \alpha_{ij}\delta^*_{kl}\beta_{pq}\xi^*_{rs})(z,x,f)\right) \\ &= [E_{\alpha_{ij}}, E_{\delta_{kl}}]\left((I - \delta_{kl}\alpha^*_{ij} + \alpha_{ij}\delta^*_{kl} + \delta_{kl}\alpha^*_{ij}\beta_{pq}\xi rs^* - \alpha_{ij}\delta^*_{kl}\beta_{pq}\xi^*_{rs}\right. \\ &\quad - \beta_{pq}\xi^*_{rs}\beta_{pq}\xi^*_{rs} - \xi_{rs}\beta^*_{pq}\xi_{rs}\beta^*_{pq} + \xi_{rs}\beta^*_{pq}\delta_{kl}\alpha^*_{ij} - \xi_{rs}\beta^*_{pq}\alpha_{ij}\delta^*_{kl} \\ &\quad \left. - \xi_{rs}\beta^*_{pq}\delta_{kl}\alpha^*_{ij}\beta_{pq}\xi^*_{rs} + \xi_{rs}\beta^*_{pq}\alpha_{ij}\delta^*_{kl}\beta_{pq}\xi^*_{rs})(z,x,f)\right) \\ &= (I + \delta_{kl}\alpha^*_{ij}\beta_{pq}\xi rs^* - \alpha_{ij}\delta^*_{kl}\beta_{pq}\xi^*_{rs} + \xi_{rs}\beta^*_{pq}\delta_{kl}\alpha^*_{ij} \end{aligned}$$

$$
\begin{aligned}
&- \xi_{rs}\beta^*_{pq}\alpha_{ij}\delta^*_{kl} - \xi_{rs}\beta^*_{pq}\delta_{kl}\alpha^*_{ij}\beta_{pq}\xi^*_{rs} \\
&+ \xi_{rs}\beta^*_{pq}\alpha_{ij}\delta^*_{kl}\beta_{pq}\xi^*_{rs})(z,x,f) \\
= &\left[E_{\alpha_{ij}\delta^*_{kl}}, E_{\xi_{rs}\beta^*_{pq}}\right]\left[E_{\delta_{kl}\alpha^*_{ij}}, E_{\xi_{rs}\beta^*_{pq}}\right](z,x,f). \qquad (2.3.2)
\end{aligned}
$$

Now if $k = p$, then Equation (2.3.2) becomes $\left[E_{\alpha_{ij}\delta^*_{kl}}, E_{\xi_{rs}\beta^*_{pq}}\right](z,x,f)$ and in particular

$$\left[E_{\alpha_{ij}\delta^*_{kl}}, E_{\xi_{rs}\beta^*_{pq}}\right](z,x,f) = I(z,x,f) \quad \text{if} \quad i = r,$$

and if $i = p$, then Equation (2.3.2) becomes $\left[E_{\delta_{kl}\alpha^*_{ij}}, E_{\xi_{rs}\beta^*_{pq}}\right](z,x,f)$ and in particular

$$\left[E_{\delta_{kl}\alpha^*_{ij}}, E_{\xi_{rs}\beta^*_{pq}}\right](z,x,f) = I(z,x,f) \quad \text{if} \quad k = r. \qquad \square$$

Lemma 2.3.3. *Let $\alpha, \delta \in \mathrm{Hom}_A(Q,P)$ and $\beta, \gamma \in \mathrm{Hom}_A(Q,P^*)$. Then, for any i,j,k,l,r,s,p,q with $1 \le i,k,r,p \le m$, $1 \le j,l,s,q \le n$, $i \ne k$ and $r \ne p$, the four-fold commutator $\left[[E_{\alpha_{ij}}, E^*_{\beta_{kl}}], [E_{\delta_{rs}}, E^*_{\gamma_{pq}}]\right]$ is given by*

$$
\left[[E_{\alpha_{ij}}, E^*_{\beta_{kl}}], [E_{\delta_{rs}}, E^*_{\gamma_{pq}}]\right] =
\begin{cases}
\left[E_{\alpha_{ij}\beta^*_{kl}}, E^*_{\gamma_{pq}\delta^*_{rs}}\right]^{-1} & \text{if} \quad k = r \text{ and } i \ne p, \\
\left[E_{\delta_{rs}\gamma^*_{pq}}, E^*_{\beta_{kl}\alpha^*_{ij}}\right] & \text{if} \quad i = p \text{ and } k \ne r, \\
I & \text{if} \quad k \ne r \text{ and } i \ne p.
\end{cases}
$$

Proof. If $i \ne k$, then, by Lemma 2.1.5, we have

$$[E_{\alpha_{ij}}, E^*_{\beta_{kl}}](z,x,f) = (I + \beta_{kl}\alpha^*_{ij} - \alpha_{ij}\beta^*_{kl})(z,x,f).$$

If $r \ne p$, then, by Lemma 2.1.5, we have

$$[E_{\delta_{rs}}, E^*_{\gamma_{pq}}](z,x,f) = (I + \gamma_{pq}\delta^*_{rs} - \delta_{rs}\gamma^*_{pq})(z,x,f).$$

Now if $i \ne k$ and $r \ne p$, then we get

$$
\begin{aligned}
\left[[E_{\alpha_{ij}}, E^*_{\beta_{kl}}], [E_{\delta_{rs}}, E^*_{\gamma_{pq}}]\right](z,x,f) &= [E_{\alpha_{ij}}, E^*_{\beta_{kl}}][E_{\delta_{rs}}, E^*_{\gamma_{pq}}][E_{\alpha_{ij}}, E^*_{\beta_{kl}}]^{-1}[E_{\delta_{rs}}, E^*_{\gamma_{pq}}]^{-1}(z,x,f) \\
&= [E_{\alpha_{ij}}, E^*_{\beta_{kl}}][E_{\delta_{rs}}, E^*_{\gamma_{pq}}][E_{\alpha_{ij}}, E^*_{\beta_{kl}}]^{-1}\big((I - \gamma_{pq}\delta^*_{rs} \\
&\quad +\delta_{rs}\gamma^*_{pq})(z,x,f)\big)
\end{aligned}
$$

$$
\begin{aligned}
&= [E_{\alpha_{ij}}, E^*_{\beta_{kl}}][E_{\delta_{rs}}, E^*_{\gamma_{pq}}] \big((I - \gamma_{pq}\delta^*_{rs} + \delta_{rs}\gamma^*_{pq} - \beta_{kl}\alpha^*_{ij} \\
&\quad +\alpha_{ij}\beta^*_{kl} + \alpha_{ij}\beta^*_{kl}\delta_{rs}\gamma^*_{pq} + \beta_{kl}\alpha^*_{ij}\gamma_{pq}\delta^*_{rs})(z,x,f)\big) \\
&= [E_{\alpha_{ij}}, E^*_{\beta_{kl}}] \big((I - \beta_{kl}\alpha^*_{ij} + \alpha_{ij}\beta^*_{kl} + \alpha_{ij}\beta^*_{kl}\delta_{rs}\gamma^*_{pq} + \beta_{kl}\alpha^*_{ij}\gamma_{pq}\delta^*_{rs} \\
&\quad - \gamma_{pq}\delta^*_{rs}\beta_{kl}\alpha^*_{ij} - \gamma_{pq}\delta^*_{rs}\gamma_{pq}\delta^*_{rs} - \delta_{rs}\gamma^*_{pq}\delta_{rs}\gamma^*_{pq} - \delta_{rs}\gamma^*_{pq}\alpha_{ij}\beta^*_{kl} \\
&\quad + \gamma_{pq}\delta^*_{rs}\beta_{kl}\alpha^*_{ij}\gamma_{pq}\delta^*_{rs} - \delta_{rs}\gamma^*_{pq}\alpha_{ij}\beta^*_{kl}\delta_{rs}\gamma^*_{pq})(z,x,f)\big) \\
&= (I + \beta_{kl}\alpha^*_{ij}\gamma_{pq}\delta^*_{rs} + \alpha_{ij}\beta^*_{kl}\delta_{rs}\gamma^*_{pq} - \gamma_{pq}\delta^*_{rs}\beta_{kl}\alpha^*_{ij} - \delta_{rs}\gamma^*_{pq}\alpha_{ij}\beta^*_{kl} \\
&\quad + \gamma_{pq}\delta^*_{rs}\beta_{kl}\alpha^*_{ij}\gamma_{pq}\delta^*_{rs} - \delta_{rs}\gamma^*_{pq}\alpha_{ij}\beta^*_{kl}\delta_{rs}\gamma^*_{pq} - \beta_{kl}\alpha^*_{ij}\beta_{kl}\alpha^*_{ij} \\
&\quad + \beta_{kl}\alpha^*_{ij}\beta_{kl}\alpha^*_{ij}\gamma_{pq}\delta^*_{rs} - \alpha_{ij}\beta^*_{kl}\alpha_{ij}\beta^*_{kl} - \beta_{kl}\alpha^*_{ij}\gamma_{pq}\delta^*_{rs}\beta_{kl}\alpha^*_{ij} \\
&\quad - \alpha_{ij}\beta^*_{kl}\alpha_{ij}\beta^*_{kl}\delta_{rs}\gamma^*_{pq} + \beta_{kl}\alpha^*_{ij}\gamma_{pq}\delta^*_{rs}\beta_{kl}\alpha^*_{ij}\gamma_{pq}\delta^*_{rs} \\
&\quad + \alpha_{ij}\beta^*_{kl}\delta_{rs}\gamma^*_{pq}\alpha_{ij}\beta^*_{kl} + \alpha_{ij}\beta^*_{kl}\delta_{rs}\gamma^*_{pq}\alpha_{ij}\beta^*_{kl}\delta_{rs}\gamma^*_{pq})(z,x,f) \\
&= (I + \beta_{kl}\alpha^*_{ij}\gamma_{pq}\delta^*_{rs} + \alpha_{ij}\beta^*_{kl}\delta_{rs}\gamma^*_{pq} - \gamma_{pq}\delta^*_{rs}\beta_{kl}\alpha^*_{ij} \\
&\qquad + \alpha_{ij}\beta^*_{kl}\delta_{rs}\gamma^*_{pq}\alpha_{ij}\beta^*_{kl}\delta_{rs}\gamma^*_{pq} - \delta_{rs}\gamma^*_{pq}\alpha_{ij}\beta^*_{kl} \\
&\qquad - \delta_{rs}\gamma^*_{pq}\alpha_{ij}\beta^*_{kl}\delta_{rs}\gamma^*_{pq} - \beta_{kl}\alpha^*_{ij}\gamma_{pq}\delta^*_{rs}\beta_{kl}\alpha^*_{ij} \\
&\qquad + \gamma_{pq}\delta^*_{rs}\beta_{kl}\alpha^*_{ij}\gamma_{pq}\delta^*_{rs} + \alpha_{ij}\beta^*_{kl}\delta_{rs}\gamma^*_{pq}\alpha_{ij}\beta^*_{kl} \\
&\qquad + \beta_{kl}\alpha^*_{ij}\gamma_{pq}\delta^*_{rs}\beta_{kl}\alpha^*_{ij}\gamma_{pq}\delta^*_{rs})(z,x,f). \qquad (2.3.3)
\end{aligned}
$$

Now if $k = r$ and $i \neq p$, then, by Equation (2.3.3), we have

$$\Big[[E_{\alpha_{ij}}, E^*_{\beta_{kl}}], [E_{\delta_{rs}}, E^*_{\gamma_{pq}}]\Big](z,x,f) = \Big[E_{\alpha_{ij}\beta^*_{kl}}, E^*_{\gamma_{pq}\delta^*_{rs}}\Big]^{-1}(z,x,f)$$

and if $i = p$ and $k \neq r$, then, by Equation (2.3.3), we have

$$\Big[[E_{\alpha_{ij}}, E^*_{\beta_{kl}}], [E_{\delta_{rs}}, E^*_{\gamma_{pq}}]\Big](z,x,f) = \Big[E_{\delta_{rs}\gamma^*_{pq}}, E^*_{\beta_{kl}\alpha^*_{ij}}\Big](z,x,f).$$

Now if $i \neq p$ and $k \neq r$, then, by Equation (2.3.3), we get

$$\Big[[E_{\alpha_{ij}}, E^*_{\beta_{kl}}], [E_{\delta_{rs}}, E^*_{\gamma_{pq}}]\Big](z,x,f) = I(z,x,f). \qquad \square$$

Lemma 2.3.4. *Let $\alpha, \delta, \xi, \mu \in \mathrm{Hom}_A(Q,P)$. Then, for i,j,k,l,r,s,p,q with $1 \leq i,k,r,p \leq m$, $1 \leq j,l,s,q \leq n$, $i \neq k$ and $r \neq p$, the four-fold commutator $\big[[E_{\alpha_{ij}}, E_{\delta_{kl}}], [E_{\xi_{rs}}, E_{\mu_{pq}}]\big]$ is given by*

$$\big[[E_{\alpha_{ij}}, E_{\delta_{kl}}], [E_{\xi_{rs}}, E_{\mu_{pq}}]\big] = I.$$

Proof. If $i \neq k$, then, by Lemma 2.1.3, we have

$$[E_{\alpha_{ij}}, E_{\delta_{kl}}](z,x,f) = (I + \delta_{kl}\alpha_{ij}^* - \alpha_{ij}\delta_{kl}^*)(z,x,f).$$

If $r \neq p$, then, by Lemma 2.1.3, we have

$$[E_{\xi_{rs}}, E_{\mu_{pq}}](z,x,f) = (I + \mu_{pq}\xi_{rs}^* - \xi_{rs}\mu_{pq}^*)(z,x,f).$$

Now if $i \neq k$ and $r \neq p$, then we get

$$\begin{aligned}
\Big[[E_{\alpha_{ij}}, E_{\delta_{kl}}], [E_{\xi_{rs}}, E_{\mu_{pq}}]\Big](z,x,f) &= [E_{\alpha_{ij}}, E_{\delta_{kl}}][E_{\xi_{rs}}, E_{\mu_{pq}}][E_{\alpha_{ij}}, E_{\delta_{kl}}]^{-1}[E_{\xi_{rs}}, E_{\mu_{pq}}]^{-1}(z,x,f) \\
&= [E_{\alpha_{ij}}, E_{\delta_{kl}}][E_{\xi_{rs}}, E_{\mu_{pq}}][E_{\alpha_{ij}}, E_{\delta_{kl}}]^{-1}\Big(\Big(I - \mu_{pq}\xi_{rs}^* \\
&\quad + \xi_{rs}\mu_{pq}^*\Big)(z,x,f)\Big) \\
&= [E_{\alpha_{ij}}, E_{\delta_{kl}}][E_{\xi_{rs}}, E_{\mu_{pq}}]\Big(\Big(I - \mu_{pq}\xi_{rs}^* + \xi_{rs}\mu_{pq}^* + \delta_{kl}\alpha_{ij}^* \\
&\quad - \alpha_{ij}\delta_{kl}^*\Big)(z,x,f)\Big) \\
&= [E_{\alpha_{ij}}, E_{\delta_{kl}}]\Big(\Big(I + \delta_{kl}\alpha_{ij}^* - \alpha_{ij}\delta_{kl}^*\Big)(z,x,f)\Big) \\
&= I(z,x,f).
\end{aligned}$$

□

Lemma 2.3.5. *Let $\beta, \gamma, \eta, \nu \in \mathrm{Hom}_A(Q,P)$. Then, for i,j,k,l,r,s,p,q with $1 \leq i,k,r,p \leq m$, $1 \leq j,l,s,q \leq n$, $i \neq k$ and $r \neq p$, the four-fold commutator $\Big[[E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}], [E^*_{\eta_{rs}}, E^*_{\nu_{pq}}]\Big]$ is given by*

$$\Big[[E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}], [E^*_{\eta_{rs}}, E^*_{\nu_{pq}}]\Big] = I.$$

Proof. If $i \neq k$, then, by Lemma 2.1.9, we have

$$[E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}](z,x,f) = (I + \gamma_{kl}\beta_{ij}^* - \beta_{ij}\gamma_{kl}^*)(z,x,f).$$

If $r \neq p$, then, by Lemma 2.1.9, we have

$$[E^*_{\eta_{rs}}, E^*_{\nu_{pq}}](z,x,f) = (I + \nu_{pq}\eta_{rs}^* - \eta_{rs}\nu_{pq}^*)(z,x,f).$$

Now if $i \neq k$ and $r \neq p$, then we get

$$\Big[[E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}], [E^*_{\eta_{rs}}, E^*_{\nu_{pq}}]\Big](z,x,f) = [E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}][E^*_{\eta_{rs}}, E^*_{\nu_{pq}}][E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}]^{-1}[E^*_{\eta_{rs}}, E^*_{\nu_{pq}}]^{-1}(z,x,f)$$

$$
\begin{aligned}
&= [E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}][E^*_{\eta_{rs}}, E^*_{\nu_{pq}}][E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}]^{-1}\Big(\Big(I - \nu_{pq}\eta^*_{rs} \\
&\quad + \eta_{rs}\nu^*_{pq}\Big)(z, x, f)\Big) \\
&= [E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}][E^*_{\eta_{rs}}, E^*_{\nu_{pq}}]\Big(\Big(I - \nu_{pq}\eta^*_{rs} - \eta_{rs}\nu^*_{pq} - \gamma_{kl}\beta^*_{ij} \\
&\quad + \beta_{ij}\gamma^*_{kl}\Big)(z, x, f)\Big) \\
&= [E^*_{\beta_{ij}}, E^*_{\gamma_{kl}}]\Big(\Big(I - \gamma_{kl}\beta^*_{ij} + \beta_{ij}\gamma^*_{kl}\Big)(z, x, f)\Big) \\
&= I(z, x, f).
\end{aligned}
$$

□

Lemma 2.3.6. *Let $\alpha, \delta \in \mathrm{Hom}_A(Q, P)$ and $\beta, \gamma \in \mathrm{Hom}_A(Q, P^*)$. Then, for any i, j, k, l, r, s, p, q with $1 \leq i, k, r, p \leq m$, $1 \leq j, l, s, q \leq n$, $i \neq k$ and $r \neq p$, the four-fold commutator $\Big[[E_{\alpha_{ij}}, E_{\delta_{kl}}], [E^*_{\beta_{rs}}, E^*_{\gamma_{pq}}]\Big]$ is given by*

$$
\Big[[E_{\alpha_{ij}}, E_{\delta_{kl}}], [E^*_{\beta_{rs}}, E^*_{\gamma_{pq}}]\Big] = \begin{cases} \Big[E_{\alpha_{ij}\beta^*_{kl}}, E^*_{\gamma_{pq}\delta^*_{rs}}\Big]^{-1} & \text{if} \quad k = r \text{ and } i \neq p, \\ \Big[E_{\delta_{rs}\gamma^*_{pq}}, E^*_{\beta_{kl}\alpha^*_{ij}}\Big] & \text{if} \quad i = p \text{ and } k \neq r, \\ I & \text{if} \quad k \neq r \text{ and } i \neq p. \end{cases}
$$

Proof. If $i \neq k$, then, by Lemma 2.1.3, we have

$$[E_{\alpha_{ij}}, E_{\delta_{kl}}](z, x, f) = (I + \delta_{kl}\alpha^*_{ij} - \alpha_{ij}\delta^*_{kl})(z, x, f).$$

If $r \neq p$, then, by Lemma 2.1.9, we have

$$[E^*_{\beta_{rs}}, E^*_{\gamma_{pq}}](z, x, f) = (I + \gamma_{pq}\beta^*_{rs} - \beta_{rs}\gamma^*_{pq})(z, x, f).$$

Now if $i \neq k$ and $r \neq p$, then, by the coordinate-free method, we get

$$
\begin{aligned}
\Big[[E_{\alpha_{ij}}, E_{\delta_{kl}}], [E^*_{\beta_{rs}}, E^*_{\gamma_{pq}}]\Big](z, x, f) &= [E_{\alpha_{ij}}, E_{\delta_{kl}}][E^*_{\beta_{rs}}, E^*_{\gamma_{pq}}][E_{\alpha_{ij}}, E_{\delta_{kl}}]^{-1}[E^*_{\beta_{rs}}, E^*_{\gamma_{pq}}]^{-1}(z, x, f) \\
&= [E_{\alpha_{ij}}, E_{\delta_{kl}}][E^*_{\beta_{rs}}, E^*_{\gamma_{pq}}][E_{\alpha_{ij}}, E_{\delta_{kl}}]^{-1}\Big(\Big(I - \gamma_{pq}\beta^*_{rs} \\
&\quad + \beta_{rs}\gamma^*_{pq}\Big)(z, x, f)\Big) \\
&= [E_{\alpha_{ij}}, E_{\delta_{kl}}][E^*_{\beta_{rs}}, E^*_{\gamma_{pq}}]\Big(\Big(I - \gamma_{pq}\beta^*_{rs} + \beta_{rs}\gamma^*_{pq} - \delta_{kl}\alpha^*_{ij} \\
&\quad + \alpha_{ij}\delta^*_{kl} + \delta_{kl}\alpha^*_{ij}\gamma_{pq}\beta^*_{rs} - \delta_{kl}\alpha^*_{ij}\beta_{rs}\gamma^*_{pq} - \alpha_{ij}\delta^*_{kl}\gamma_{pq}\beta^*_{rs}
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha_{ij}\delta^*_{kl}\beta_{rs}\gamma^*_{pq}\Big)(z,x,f)\Big)\\
&=[E_{\alpha_{ij}},E_{\delta_{kl}}]\Big(\Big(I-\delta_{kl}\alpha^*_{ij}+\alpha_{ij}\delta^*_{kl}+\delta_{kl}\alpha^*_{ij}\gamma_{pq}\beta^*_{rs}\\
&\quad-\delta_{kl}\alpha^*_{ij}\beta_{rs}\gamma^*_{pq}-\alpha_{ij}\delta^*_{kl}\gamma_{pq}\beta^*_{rs}+\alpha_{ij}\delta^*_{kl}\beta_{rs}\gamma^*_{pq}\\
&\quad-\gamma_{pq}\beta^*_{rs}\delta_{kl}\alpha^*_{ij}+\gamma_{pq}\beta^*_{rs}\alpha_{ij}\delta^*_{kl}+\beta_{rs}\gamma^*_{pq}\delta_{kl}\alpha^*_{ij}\\
&\quad-\beta_{rs}\gamma^*_{pq}\alpha_{ij}\delta^*_{kl}+\gamma_{pq}\beta^*_{rs}\delta_{kl}\alpha^*_{ij}\gamma_{pq}\beta^*_{rs}\\
&\quad-\gamma_{pq}\beta^*_{rs}\delta_{kl}\alpha^*_{ij}\beta_{rs}\gamma^*_{pq}-\gamma_{pq}\beta^*_{rs}\alpha_{ij}\delta^*_{kl}\gamma_{pq}\beta^*_{rs}\\
&\quad+\gamma_{pq}\beta^*_{rs}\alpha_{ij}\delta^*_{kl}\beta_{rs}\gamma^*_{pq}-\beta_{rs}\gamma^*_{pq}\delta_{kl}\alpha^*_{ij}\gamma_{pq}\beta^*_{rs}\\
&\quad+\beta_{rs}\gamma^*_{pq}\delta_{kl}\alpha^*_{ij}\beta_{rs}\gamma^*_{pq}+\beta_{rs}\gamma^*_{pq}\alpha_{ij}\delta^*_{kl}\gamma_{pq}\beta^*_{rs}\\
&\quad-\beta_{rs}\gamma^*_{pq}\alpha_{ij}\delta^*_{kl}\beta_{rs}\gamma^*_{pq}\Big)(z,x,f)\Big)\\
&=\Big(I+\delta_{kl}\alpha^*_{ij}\gamma_{pq}\beta^*_{rs}-\delta_{kl}\alpha^*_{ij}\beta_{rs}\gamma^*_{pq}-\alpha_{ij}\delta^*_{kl}\gamma_{pq}\beta^*_{rs}\\
&\quad+\alpha_{ij}\delta^*_{kl}\beta_{rs}\gamma^*_{pq}-\gamma_{pq}\beta^*_{rs}\delta_{kl}\alpha^*_{ij}+\gamma_{pq}\beta^*_{rs}\alpha_{ij}\delta^*_{kl}\\
&\quad+\beta_{rs}\gamma^*_{pq}\delta_{kl}\alpha^*_{ij}-\beta_{rs}\gamma^*_{pq}\alpha_{ij}\delta^*_{kl}-\beta_{rs}\gamma^*_{pq}\alpha_{ij}\delta^*_{kl}\beta_{rs}\gamma^*_{pq}\\
&\quad+\gamma_{pq}\beta^*_{rs}\delta_{kl}\alpha^*_{ij}\gamma_{pq}\beta^*_{rs}-\gamma_{pq}\beta^*_{rs}\delta_{kl}\alpha^*_{ij}\beta_{rs}\gamma^*_{pq}\\
&\quad-\gamma_{pq}\beta^*_{rs}\alpha_{ij}\delta^*_{kl}\gamma_{pq}\beta^*_{rs}+\gamma_{pq}\beta^*_{rs}\alpha_{ij}\delta^*_{kl}\beta_{rs}\gamma^*_{pq}\\
&\quad-\beta_{rs}\gamma^*_{pq}\delta_{kl}\alpha^*_{ij}\gamma_{pq}\beta^*_{rs}+\beta_{rs}\gamma^*_{pq}\delta_{kl}\alpha^*_{ij}\beta_{rs}\gamma^*_{pq}\\
&\quad+\beta_{rs}\gamma^*_{pq}\alpha_{ij}\delta^*_{kl}\gamma_{pq}\beta^*_{rs}-\delta_{kl}\alpha^*_{ij}\gamma_{pq}\beta^*_{rs}\delta_{kl}\alpha^*_{ij}\\
&\quad+\delta_{kl}\alpha^*_{ij}\gamma_{pq}\beta^*_{rs}\alpha_{ij}\delta^*_{kl}+\delta_{kl}\alpha^*_{ij}\beta_{rs}\gamma^*_{pq}\delta_{kl}\alpha^*_{ij}\\
&\quad-\delta_{kl}\alpha^*_{ij}\beta_{rs}\gamma^*_{pq}\alpha_{ij}\delta^*_{kl}+\alpha_{ij}\delta^*_{kl}\beta_{rs}\gamma^*_{pq}\alpha_{ij}\delta^*_{kl}\\
&\quad-\alpha_{ij}\delta^*_{kl}\gamma_{pq}\beta^*_{rs}\alpha_{ij}\delta^*_{kl}-\alpha_{ij}\delta^*_{kl}\beta_{rs}\gamma^*_{pq}\delta_{kl}\alpha^*_{ij}\\
&\quad+\alpha_{ij}\delta^*_{kl}\gamma_{pq}\beta^*_{rs}\delta_{kl}\alpha^*_{ij}+\delta_{kl}\alpha^*_{ij}\beta_{rs}\gamma^*_{pq}\alpha_{ij}\delta^*_{kl}\gamma_{pq}\beta^*_{rs}\\
&\quad-\delta_{kl}\alpha^*_{ij}\beta_{rs}\gamma^*_{pq}\alpha_{ij}\delta^*_{kl}\beta_{rs}\gamma^*_{pq}+\alpha_{ij}\delta^*_{kl}\beta_{rs}\gamma^*_{pq}\alpha_{ij}\delta^*_{kl}\beta_{rs}\gamma^*_{pq}\\
&\quad+\delta_{kl}\alpha^*_{ij}\gamma_{pq}\beta^*_{rs}\delta_{kl}\alpha^*_{ij}\gamma_{pq}\beta^*_{rs}-\delta_{kl}\alpha^*_{ij}\gamma_{pq}\beta^*_{rs}\delta_{kl}\alpha^*_{ij}\beta_{rs}\gamma^*_{pq}\\
&\quad-\delta_{kl}\alpha^*_{ij}\gamma_{pq}\beta^*_{rs}\alpha_{ij}\delta^*_{kl}\gamma_{pq}\beta^*_{rs}+\delta_{kl}\alpha^*_{ij}\gamma_{pq}\beta^*_{rs}\alpha_{ij}\delta^*_{kl}\beta_{rs}\gamma^*_{pq}\\
&\quad-\delta_{kl}\alpha^*_{ij}\beta_{rs}\gamma^*_{pq}\delta_{kl}\alpha^*_{ij}\gamma_{pq}\beta^*_{rs}+\delta_{kl}\alpha^*_{ij}\beta_{rs}\gamma^*_{pq}\delta_{kl}\alpha^*_{ij}\beta_{rs}\gamma^*_{pq}\\
&\quad-\alpha_{ij}\delta^*_{kl}\gamma_{pq}\beta^*_{rs}\delta_{kl}\alpha^*_{ij}\gamma_{pq}\beta^*_{rs}+\alpha_{ij}\delta^*_{kl}\gamma_{pq}\beta^*_{rs}\delta_{kl}\alpha^*_{ij}\beta_{rs}\gamma^*_{pq}
\end{aligned}
$$

$$+\alpha_{ij}\delta^*_{kl}\gamma_{pq}\beta^*_{rs}\alpha_{ij}\delta^*_{kl}\gamma_{pq}\beta^*_{rs} - \alpha_{ij}\delta^*_{kl}\gamma_{pq}\beta^*_{rs}\alpha_{ij}\delta^*_{kl}\beta_{rs}\gamma^*_{pq}$$
$$+\alpha_{ij}\delta^*_{kl}\beta_{rs}\gamma^*_{pq}\delta_{kl}\alpha^*_{ij}\gamma_{pq}\beta^*_{rs} - \alpha_{ij}\delta^*_{kl}\beta_{rs}\gamma^*_{pq}\delta_{kl}\alpha^*_{ij}\beta_{rs}\gamma^*_{pq}$$
$$-\alpha_{ij}\delta^*_{kl}\beta_{rs}\gamma^*_{pq}\alpha_{ij}\delta^*_{kl}\gamma_{pq}\beta^*_{rs})(z,x,f). \tag{2.3.4}$$

If $i = p$ or $k = r$ or $i \neq r$ and $k \neq p$, then the Equation (2.3.4) becomes

$$\begin{aligned}\Big[[E_{\alpha_{ij}}, E_{\delta_{kl}}], [E^*_{\beta_{rs}}, E^*_{\gamma_{pq}}]\Big](z,x,f) &= I + \delta_{kl}\alpha^*_{ij}\gamma_{pq}\beta^*_{rs} + \alpha_{ij}\delta^*_{kl}\beta_{rs}\gamma^*_{pq} - \gamma_{pq}\beta^*_{rs}\delta_{kl}\alpha^*_{ij}\\ &- \beta_{rs}\gamma^*_{pq}\alpha_{ij}\delta^*_{kl} - \beta_{rs}\gamma^*_{pq}\alpha_{ij}\delta^*_{kl}\beta_{rs}\gamma^*_{pq}\\ &+ \gamma_{pq}\beta^*_{rs}\delta_{kl}\alpha^*_{ij}\gamma_{pq}\beta^*_{rs} - \delta_{kl}\alpha^*_{ij}\gamma_{pq}\beta^*_{rs}\delta_{kl}\alpha^*_{ij}\\ &+ \alpha_{ij}\delta^*_{kl}\beta_{rs}\gamma^*_{pq}\alpha_{ij}\delta^*_{kl} + \alpha_{ij}\delta^*_{kl}\beta_{rs}\gamma^*_{pq}\alpha_{ij}\delta^*_{kl}\beta_{rs}\gamma^*_{pq}\\ &+ \delta_{kl}\alpha^*_{ij}\gamma_{pq}\beta^*_{rs}\delta_{kl}\alpha^*_{ij}\gamma_{pq}\beta^*_{rs} + \alpha_{ij}\delta^*_{kl}\beta_{rs}\gamma^*_{pq}\delta_{kl}\alpha^*_{ij}\gamma_{pq}\beta^*_{rs}.\end{aligned} \tag{2.3.5}$$

If (i) $i = p$ and $k \neq r$ or (ii) $i \neq r, k \neq p$ and $k \neq r$, then the Equation (2.3.5) reduces to

$$I + \delta_{kl}\alpha^*_{ij}\gamma_{pq}\beta^*_{rs} - \beta_{rs}\gamma^*_{pq}\alpha_{ij}\delta^*_{kl} = \Big[E_{\delta_{kl}\alpha^*_{ij}}, E^*_{\beta_{rs}\gamma^*_{pq}}\Big]^{-1}.$$

If (i) $k = r$ and $i \neq p$ or (ii) $i \neq r$, $k \neq p$ and $i \neq p$, then the Equation (2.3.5) reduces to

$$I + \alpha_{ij}\delta^*_{kl}\beta_{rs}\gamma^*_{pq} - \gamma_{pq}\beta^*_{rs}\delta_{kl}\alpha^*_{ij} = \Big[E_{\alpha_{ij}\delta^*_{kl}}, E^*_{\gamma_{pq}\beta^*_{rs}}\Big]^{-1}.$$

If $i = r$ or $k = p$ or if $i \neq p$ and $k \neq r$, then the Equation (2.3.4) becomes

$$\begin{aligned}\Big[[E_{\alpha_{ij}}, E_{\delta_{kl}}], [E^*_{\beta_{rs}}, E^*_{\gamma_{pq}}]\Big](z,x,f) &= I - \delta_{kl}\alpha^*_{ij}\beta_{rs}\gamma^*_{pq} - \alpha_{ij}\delta^*_{kl}\gamma_{pq}\beta^*_{rs} + \gamma_{pq}\beta^*_{rs}\alpha_{ij}\delta^*_{kl}\\ &+ \beta_{rs}\gamma^*_{pq}\delta_{kl}\alpha^*_{ij} - \gamma_{pq}\beta^*_{rs}\alpha_{ij}\delta^*_{kl}\gamma_{pq}\beta^*_{rs}\\ &+ \beta_{rs}\gamma^*_{pq}\delta_{kl}\alpha^*_{ij}\beta_{rs}\gamma^*_{pq} + \delta_{kl}\alpha^*_{ij}\beta_{rs}\gamma^*_{pq}\delta_{kl}\alpha^*_{ij}\\ &- \alpha_{ij}\delta^*_{kl}\gamma_{pq}\beta^*_{rs}\alpha_{ij}\delta^*_{kl} + \delta_{kl}\alpha^*_{ij}\beta_{rs}\gamma^*_{pq}\alpha_{ij}\delta^*_{kl}\gamma_{pq}\beta^*_{rs}\\ &+ \delta_{kl}\alpha^*_{ij}\beta_{rs}\gamma^*_{pq}\delta_{kl}\alpha^*_{ij}\beta_{rs}\gamma^*_{pq} + \alpha_{ij}\delta^*_{kl}\gamma_{pq}\beta^*_{rs}\alpha_{ij}\delta^*_{kl}\gamma_{pq}\beta^*_{rs}.\end{aligned} \tag{2.3.6}$$

If (i) $i = r$ and $k \neq p$ or (ii) $i \neq p, k \neq r$ and $k \neq p$, then the Equation (2.3.6) reduces to

$$I - \delta_{kl}\alpha^*_{ij}\beta_{rs}\gamma^*_{pq} + \gamma_{pq}\beta^*_{rs}\alpha_{ij}\delta^*_{kl} = \Big[E_{\delta_{kl}\alpha^*_{ij}}, E^*_{\gamma_{pq}\beta^*_{rs}}\Big].$$

If (i) $k = p$ and $i \neq r$ or (ii) $i \neq p, k \neq r$ and $i \neq r$, then the Equation (2.3.6) reduces to

$$I - \alpha_{ij}\delta^*_{kl}\gamma_{pq}\beta^*_{rs} + \beta_{rs}\gamma^*_{pq}\delta_{kl}\alpha^*_{ij} = \left[E_{\alpha_{ij}\delta^*_{kl}}, E^*_{\beta_{rs}\gamma^*_{pq}}\right]. \qquad \square$$

Chapter 3

Local-Global Principle for Roy's Orthogonal Group

J.-P. Serre, in his 1955 paper "Faisceaux algébriques cohérents", conjectured that a finitely generated projective module over a polynomial ring in n variables over a field is free. In 1976, this was proved independently by D. Quillen (see [43]) and A.A. Suslin (see [56]). Soon after, in [57], A.A. Suslin proved the K_1-theoretic analogue of this conjecture, which says that if k is a field and $r \geq 3$, then $\mathrm{SL}_r(k[X_1, \ldots, X_n])$ is generated by elementary matrices. An exposition of this can be found in [27]. Later, A.A. Suslin and V.I. Kopeĭko established an analogue of the above theorem for the symplectic and the orthogonal groups (see [35, 58]). They also proved the normality of the elementary subgroup in the linear, symplectic and orthogonal cases.

D. Quillen's famous local-global principle says that *a finitely presented module over a polynomial ring $R[X]$ over a commutative ring R is extended from R if and only if the localized module over $R_{\mathfrak{m}}[X]$ is extended from $R_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R.* He raised an analogous question for quadratic modules.

In [10], A. Bak et al. gave a uniform proof of local-global principle for classical groups (linear, symplectic and orthogonal) over a commutative ring with identity, and relates normality of elementary group to local-global principle. Local-global principle for transvections of a projective module with a unimodular element is proved in [18]. Also, local-global principle for general quadratic group and general Hermitian group are done in [17].

In this chapter, we use the commutator relations which we proved in Chapter 2 to prove a local-global principle for the group of Dickson–Siegel–Eichler–Roy (DSER) elementary orthogonal transformations. These results are used in Chapter 4 to prove certain extendability results on quadratic modules. Also, we can realize from the yoga of commutators that some features of Roy's group mimic Tang's well-known Hermitian group defined in [60], as well as Bass's unitary transvection group defined in [16].

Most of the results in this chapter are from [5].

3.1 Splitting Property

In this section, we state a splitting property and extend Lemma 1.4 of [55] regarding Roy's transformations.

Notation 3.1.1. $E(\alpha)$ denotes either E_α or E^*_α, where $\alpha \in \mathrm{Hom}_A(Q,P)$ or $\mathrm{Hom}_A(Q,P^*)$ respectively.

Combining Lemma 1.2 and Lemma 1.3 of [55], we have the following lemma.

Lemma 3.1.2 (Splitting Property)**.** *For $\alpha_1, \alpha_2 \in \mathrm{Hom}_A(Q,P)$ or $\mathrm{Hom}_A(Q,P^*)$, we have*

$$E(\alpha_1+\alpha_2) = E\left(\frac{\alpha_1}{2}\right) E(\alpha_2)\, E\left(\frac{\alpha_1}{2}\right) = E\left(\frac{\alpha_2}{2}\right) E(\alpha_1)\, E\left(\frac{\alpha_2}{2}\right).$$

When P is a free A-module of rank m, we write $\mathrm{O}_A(Q \perp H(A)^m)$ in place of $\mathrm{O}_A(Q \perp H(P))$. Let $\eta_i : A \to A^m$ be the inclusion map into the i^{th} component. Then η_i induces an inclusion $\eta_i : \mathrm{O}_A(Q\perp H(A)) \to \mathrm{O}_A(Q \perp H(A)^m)$ which takes $\mathrm{EO}_A(Q\perp H(A))$ into $\mathrm{EO}_A(Q\perp H(A)^m)$. For $\alpha \in \mathrm{Hom}_A(Q,A)$, let $E_i(\alpha) \in \mathrm{EO}_A(Q\perp H(A)^m)$ be the image of $E(\alpha)$ under η_i .

Lemma 3.1.3 ([55, Lemma 1.4])**.** *The group $\mathrm{EO}_A(Q\perp H(A)^m)$ is generated by $E_i(\alpha)$ $(1 \le i \le m)$, where $\alpha \in \mathrm{Hom}_A(Q,A)$.*

Lemma 3.1.4. *Following the same notation as above, the group $\mathrm{EO}_A(Q\perp H(A)^m)$ is generated by $E(\alpha_{ij})$ $(1 \le i \le m$ and $1 \le j \le n)$ with $\alpha \in \mathrm{Hom}_A(Q,P)$ or $\mathrm{Hom}_A(Q,P^*)$.*

Proof. For $\alpha \in \mathrm{Hom}_A(Q,P)$ or $\mathrm{Hom}_A(Q,P^*)$, we have $\alpha = \sum_{i=1}^m \sum_{j=1}^n \alpha_{ij}$ from the previous section. By repeated application of the splitting property, we have

$$E(\alpha) = E\left(\frac{\alpha_{11}}{2}\right) E\left(\frac{\alpha_{21}}{2}\right) \cdots E\left(\frac{\alpha_{m1}}{2}\right) E\left(\frac{\alpha_{12}}{2}\right) \cdots E\left(\frac{\alpha_{m2}}{2}\right)$$

$$\cdots E\left(\frac{\alpha_{(m-1)n}}{2}\right) E\left(\alpha_{mn}\right) E\left(\frac{\alpha_{(m-1)n}}{2}\right)\cdots E\left(\frac{\alpha_{11}}{2}\right).$$

This proves the lemma. □

3.2 Comparison of Roy's Elementary Orthogonal Group with Other Groups

The orthogonal group of $Q \perp H(P)$ is denoted by $\mathrm{O}_A\left(Q \perp H(P)\right)$, where Q and P are free A-modules of finite rank and the elementary orthogonal group is denoted by $\mathrm{EO}_A\left(Q \perp H(P)\right)$. Here, we compare Roy's elementary transformations with the so-called Eichler transformations and also with the unitary transvections.

3.2.1 Roy's Transformations as Eichler-Siegel-Dickson Transformations

In this section, we view Roy's group of elementary orthogonal transformations in terms of Eichler-Siegel-Dickson transformations. The latter are defined as follows:

Definition 3.2.1 ([28, Chapter 5])**.** Let (M, B, q) be a non-degenerate quadratic module over A and let $\mathrm{O}_A(M)$ be its orthogonal group. Let u and v be in M with u isotropic and $B(u, v) = 0$. For $r = q(v)$, define the ESD transformation $\Sigma_{u,v,r} \in \mathrm{End}(M)$ by

$$\Sigma_{u,v,r}(x) = x + uB(v, x) - vB(u, x) - urB(u, x).$$

One can easily verify the following properties:

(a) $\Sigma_{u,v,q(v)} \in \mathrm{O}_A(M)$,

(b) $\Sigma_{u,v,q(v)}\Sigma_{u,w,q(w)} = \Sigma_{u,v+w,q(v)+q(w)+h(v,w)}$,

(c) $\Sigma^{-1}_{u,v,q(v)} = \Sigma_{u,-v,q(v)}$,

(d) $\sigma\Sigma_{u,v,q(v)}\sigma^{-1} = \Sigma_{\sigma u,\sigma v,q(v)}$ for $\sigma \in \mathrm{O}_A(M)$.

(e) $\Sigma_{0,0,0} = I$.

We may regard the elementary orthogonal transformations $E_{\alpha_{ij}}$ and $E^*_{\beta_{ij}}$ as ESD transformations. More precisely, the orthogonal transformation $E_{\alpha_{ij}}$ of $Q \perp H(P)$ given by $E_{\alpha_{ij}}(z, x, f) = (z - \langle f, x_i\rangle w_{ij},\ \ x + \langle w_{ij}, z\rangle x_i - \langle f, x_i\rangle q(w_{ij})x_i,\ \ f)$ can be written as

$\Sigma_{x_i,w_{ij},q(w_{ij})}(z,x,f)$. For,

$$\begin{aligned}\Sigma_{x_i,w_{ij},q(w_{ij})}(z,x,f) =&(z,x,f)+(0,x_i,0)\langle(w_{ij},0,0),(z,x,f)\rangle-(w_{ij},0,0)\\&\langle(0,x_i,0),(z,x,f)\rangle-(0,x_i,0)q(w_{ij})\langle(0,x_i,0),(z,x,f)\rangle\\=&(z-\langle f,x_i\rangle w_{ij},\ x+\langle w_{ij},z\rangle x_i-\langle f,x_i\rangle q(w_{ij})x_i,\ f).\end{aligned}$$

Similarly, the elementary orthogonal transformation $E^*_{\beta_{ij}}$ of $Q\perp H(P)$ given by $E^*_{\beta_{ij}}(z,x,f)=(z-\langle f_i,x\rangle v_{ij},\ x,\ f+\langle v_{ij},z\rangle f_i-\langle x,f_i\rangle q(v_{ij})f_i\ )$ can be written as the ESD transformation $\Sigma_{f_i,v_{ij},q(v_{ij})}(z,x,f)$.

These elementary orthogonal transformations also satisfy the properties listed above. From this, we can conclude that Roy's group of elementary transformations $\mathrm{EO}_A\,(Q\perp H(A)^m)$ is a subgroup of the group of ESD transformations. The reverse containment of these groups will be addressed in the upcoming article [6].

3.2.2 Comparison between Roy's Elementary Orthogonal Group and Unitary Transvection Group

In this section, we will see that Roy's transformations can also be viewed as *unitary transvections* [16, Section 5] of certain types of quadratic modules over a *unitary ring* (A,λ,Λ). See [16, Section 4] for further details about unitary rings.

Definition 3.2.2 ([16, Section 5]). Let $M=V\perp H(P)$. If $x=(v,p,q)\in M$, we have $f(x,x)=f(v,v)+\langle q,p\rangle_P$. Suppose P has a unimodular element p_0. i.e., there is a $q_0\in P^*$ such that $\langle q_0,p_0\rangle_P=1$. For any elements $p_0\in P, w_0\in V$ and $a_0\in A$ with $a_0\equiv f(w_0,w_0)$ mod Λ, assume that the following conditions hold.

$$f(p_0,p_0)\in\Lambda,\ \ \langle w_0,p_0\rangle=0,\ \ f(w_0,w_0)\equiv a_0 \mod \Lambda.$$

If $x=(v,p,q)$, then σ_{p_0,a_0,w_0} is defined as

$$\sigma_{p_0,a_0,w_0}(x)=x+p_0\langle w_0,x\rangle-w_0\overline{\lambda}\langle p_0,x\rangle-p_0\overline{\lambda}a_0\langle p_0,x\rangle.$$

Now take $\Lambda=0,\lambda=1,f(w_0,w_0)=a_0$ and $\langle w_0,w_0\rangle=2f(w_0,w_0)=2a_0$. Then we have

$$E_{\alpha_{ij}}(z,x,f)=\sigma_{x_i,\frac{\langle w_{ij},w_{ij}\rangle}{2},w_{ij}}(z,x,f),$$

$$E^*_{\beta_{ij}}(z,f,x) = \sigma_{f_i, \frac{\langle v_{ij}, v_{ij}\rangle}{2}, v_{ij}}(z,f,x)$$

for $z \in V, x \in P, f \in P^*$. In this way, Roy's group could also be viewed as unitary transvection group.

3.2.3 Comparison between Roy's and Petrov's groups

In [39], V. Petrov introduced a new classical-like group called *odd unitary group* over odd form rings. This group generalizes and unifies all known classical groups such as quadratic groups, Hermitian groups, classical Chevalley groups. In Section 6 of his paper, V. Petrov defined an elementary subgroup $\mathrm{EU}_{2l}(R, \mathfrak{L})$ of an *odd hyperbolic unitary group* $\mathrm{U}_{2l}(R, \mathfrak{L})$.

We recall Petrov's definition for the odd unitary group.

Let R be a ring with pseudo-involution and V be a right R-module with an anti-Hermitian form B. Let $\mathfrak{H}$ denote the Heisenberg group of the form B. The subgroups $\mathfrak{L}_{\min}$ and $\mathfrak{L}_{\max}$ of $\mathfrak{H}$ are defined as follows:

$$\mathfrak{L}_{\min} = \{(0, a + \overline{a}) | a \in R\},$$
$$\mathfrak{L}_{\max} = \{\xi \in \mathfrak{H} | \operatorname{tr}(\xi) = 0\}.$$

An odd form parameter $\mathfrak{L}$ is a subgroup of $\mathfrak{H}$ that lies between $\mathfrak{L}_{\min}$ and $\mathfrak{L}_{\max}$ and is stable under the action of R. The pair $(R, \mathfrak{L})$ is called an odd form ring and the pair (V, q) is called an odd quadratic space, where $q = (B, \mathfrak{L})$ is an odd quadratic form. We denote B by $(\cdot,\cdot)_q$. The even part of the form parameter is denoted by $\mathfrak{L}_{ev}$. The pair (V, q) is called an odd quadratic space.

Let $T_{uv}(a)$ be the Eichler-Siegel-Dickson transvections defined in an odd quadratic space as follows:

Let u, v be vectors of an odd quadratic space V and a be an element of R such that $(u, v)_q = 0$, $(u, 0) \in \mathfrak{L}$, and $(v, a) \in \mathfrak{L}$. Then

$$T_{uv}(a)(w) = w + u\overline{1}^{-1}((v, w)_q + a(u, w)_q) + v(u, w)_q \quad \text{for } w \in V. \tag{3.2.1}$$

Suppose V_0 is an odd quadratic space with an odd quadratic form $q_0 = (B_0, \mathfrak{L})$. Then the orthogonal sum $V = H^l \perp V_0$ is called an odd hyperbolic unitary space of rank l

corresponding to the odd form parameter $\mathfrak{L}$. The unitary group $\mathrm{U}(V, q)$ in this case is called the odd hyperbolic unitary group and is denoted by $\mathrm{U}_{2l}(R, \mathfrak{L})$. Now, the elementary hyperbolic unitary group $\mathrm{EU}_{2l}(R, \mathfrak{L})$ is given to be the group generated by

$$T_{ij}(a) = T_{e_j, -e_i a \varepsilon_j}(0),\ j \neq \pm i,\ a \in R, \tag{3.2.2}$$

$$T_i(u, a) = T_{e_i, u\varepsilon_{-i}}(-\overline{\varepsilon}_{-i}\overline{1}^{-1} a \varepsilon_{-i}),\ (u, a) \in \mathfrak{L}, \tag{3.2.3}$$

$$T_i(0, a),\ a \in \mathfrak{L}_{ev}, \tag{3.2.4}$$

where $i, j = 1, \cdots, l, -l, \cdots, -1$ and $\varepsilon_i = -1$.

Now, if we take the involution to be $a \to -a$ and for $l = m$, $R = A$, where A is a commutative ring, $V_0 = Q$ and

$$\mathfrak{L} = \mathfrak{L}_{\max} = \{(u, a) : 2a - B(u, u) = 0\},$$

then we get Roy's transformations as elements in $\mathrm{EU}_{2m}(A, \mathfrak{L})$. Since Roy's elementary transformations are of the type $T_{e_i v}(a)$ or $T_{f_i w}(b)$, where $(e_i, v)_q = 0 = (f_i, v)_q$ and $a, b \in A$ such that $(v, a), (w, b) \in \mathfrak{L}$. i.e., v and w are such that $q(v) = \frac{B(v,v)}{2} = a$ and $q(w) = \frac{B(w,w)}{2} = b$.

Precisely, we can write Roy's elementary transformations as follows:

$$\begin{aligned}
E_{\alpha_{ij}}(z, x, f) &= (z, x, f) + e_i((w_{ij}, (z, x, f))_q + q(w_{ij})(e_i, (z, x, f))_q) + w_{ij}(e_i, (z, x, f))_q \\
&= T_{e_i w_{ij}}(q(w_{ij}))(z, x, f). \\
E^*_{\beta_{ij}}(z, x, f) &= (z, x, f) + f_i((v_{ij}, (z, x, f))_q + q(v_{ij})(f_i, (z, x, f))_q) + v_{ij}(f_i, (z, x, f))_q \\
&= T_{f_i v_{ij}}(q(v_{ij}))(z, x, f).
\end{aligned}$$

We now recall the following results from [39].

Lemma 3.2.3 ([39, Lemma 2]). *Let v be a vector of V such that $(e_i, v)_q = (e_{-i}, v)_q = 0$, and a be an element of R such that $(v, a) \in \mathfrak{L}$. Then $T_{e_i v}(a)$ belongs to $\mathrm{EU}_{2l}(R, \mathfrak{L})$.*

Proposition 3.2.4 ([39, Proposition 1]). *The group $\mathrm{EU}_{2l}(R, \mathfrak{L})$ coincides with the group generated by all the elements of the form $T_{e_{\pm 1} v}(a)$, where $(e_1, v)_q = (e_{-1}, v)_q = 0$ and $(v, a) \in \mathfrak{L}$.*

Since $(w_{ij}, q(w_{ij}))$, $(v_{ij}, q(v_{ij})) \in \mathfrak{L}$ and $(e_i, w_{ij})_q = (f_i, w_{ij})_q = (e_i, v_{ij})_q = (f_i, v_{ij})_q = 0$, by Lemma 3.2.3, we can conclude that $E_{\alpha_{ij}}$ and $E^*_{\beta_{ij}}$ belong to $\mathrm{EU}_{2m}(A, \mathfrak{L})$. Thus

$$\mathrm{EO}_A\left(Q \perp H(A)^m\right) \subseteq \mathrm{EU}_{2m}(A, \mathfrak{L}).$$

Now, by Proposition 3.2.4, we have

$$\begin{aligned}\mathrm{EU}_{2m}(A, \mathfrak{L}) &= \langle T_{e_{\pm 1}v}(a) | (v, a) \in \mathfrak{L}\rangle \\ &= \langle E_{\alpha_1}, E^*_{\beta_1} \text{ for } \alpha \in \mathrm{Hom}_A(Q, P), \beta \in \mathrm{Hom}_A(Q, P^*)\rangle \\ &= \langle E_{\alpha_{1j}}, E^*_{\beta_{1j}} \text{ for } 1 \leq j \leq n, \text{ and for } \alpha \in \mathrm{Hom}_A(Q, P), \beta \in \mathrm{Hom}_A(Q, P^*)\rangle.\end{aligned}$$

Since $\langle E_{\alpha_{1j}}, E^*_{\beta_{1j}}$ for $1 \leq j \leq n$ and for $\alpha \in \mathrm{Hom}_A(Q, P), \beta \in \mathrm{Hom}_A(Q, P^*)\rangle$

$$\subseteq \langle E_{\alpha_{ij}}, E^*_{\beta_{ij}} \text{ for } 1 \leq i \leq m,\ 1 \leq j \leq n \text{ and for } \alpha \in \mathrm{Hom}_A(Q, P), \beta \in \mathrm{Hom}_A(Q, P^*)\rangle,$$

we have

$$\mathrm{EU}_{2m}(A, \mathfrak{L}) = \mathrm{EO}_A(Q \perp H(A)^m).$$

Remark 3.2.5. Bak's hyperbolic general quadratic group is a special case of Petrov's odd unitary group. It is obtained by taking $V_0 = 0$ and $\mathfrak{L} = \mathfrak{L}_{ev}$ in odd hyperbolic unitary group $V = H^l \perp V_0$. Bak's group can not be compared with Roy's elementary group since, for defining Roy's elementary transformations, one need $V_0 \neq 0$.

Let $n \geq r$. Then, for $(0, a_1), \cdots, (0, a_r) \in \mathfrak{L}_{\max}$, the general Hermitian group $\mathrm{GH}(R, a_1, \cdots, a_r)$ of Bak and Tang may be regarded as a special case of $\mathrm{U}_{2(l-r)}(R, \mathfrak{L}_{\max})$ by taking $V_0 = \langle f_1, \cdots, f_r, f_{-r}, \cdots, f_{-1}\rangle$ with anti-Hermitian form B_0 given by

$$B_0\left(\sum_i f_i b_i, \sum_j f_j c_j\right) = \sum_{j=1}^{r} \overline{b_j} \overline{1}^{-1} a_j c_j + \sum_i \overline{b_i} \varepsilon_{-i} c_{-i}. \tag{3.2.5}$$

Thus in particular, if we take Q to be of rank $2r$ and $a_1 = \cdots = a_r = 0$, $R = A$; then we get $\mathrm{O}_A\left(Q \perp H(A)^m\right) = \mathrm{GH}(A, 0, \cdots, 0) = \mathrm{O}_A(H(A)^{r+m})$ which is the classical orthogonal group. But in general case, we can see that the elementary generators and the commutator relations among them mimics that of the general Hermitian group. At this point, we do not explicitly compare the elementary generators of the DSER group with that of the elementary Hermitian group.

3.3 $\mathrm{EO}_A\,(Q \perp H(A)^m)$ is perfect

In this section, we observe that the elementary orthogonal group $\mathrm{EO}_A(Q \perp H(A)^m)$ is perfect.

Theorem 3.3.1. *If $m \geq 2$, then $\mathrm{EO}_A\,(Q \perp H(A)^m)$ is perfect.*

Proof. To prove $[\mathrm{EO}_A\,(Q \perp H(A)^m)\,, \mathrm{EO}_A\,(Q \perp H(A)^m)] = \mathrm{EO}_A\,(Q \perp H(A)^m)$, we need to prove that any element in $\mathrm{EO}_A\,(Q \perp H(A)^m)$ can be written as a commutator. This follows from the commutator relation proved in Chapter 2.

Since $\mathrm{EO}_A\,(Q \perp H(A)^m)$ is generated by elementary transformations of the type $E_{\alpha_{ij}}$ and $E^*_{\beta_{ij}}$ by Lemma 3.1.4, it is enough to show that these transformations can be written as commutators of elements of $\mathrm{EO}_A\,(Q \perp H(A)^m)$. By triple commutator relations in Section 2.2 of Chapter 2, we can write the transformations $E_{\alpha_{ij}}$ and $E^*_{\beta_{ij}}$ as products of commutators of elements of the group $\mathrm{EO}_A\,(Q \perp H(A)^m)$. Thus the elements $E_{\alpha_{ij}}$ and $E^*_{\beta_{ij}}$ belong to the commutator subgroup $[\,\mathrm{EO}_A\,(Q \perp H(A)^m)\,, \mathrm{EO}_A\,(Q \perp H(A)^m)\,]$. Hence $\mathrm{EO}_A\,(Q \perp H(A)^m)$ is perfect. □

Remark 3.3.2. The condition $m \geq 2$ in the above theorem is necessary in order to have non-trivial commutator relations.

3.4 Local-Global Principle for Roy's Elementary Orthogonal Group

In this section, we establish that $\mathrm{EO}_{A[X]}(M[X])$, where $M = Q \perp H(P)$ such that Q and P are free modules of rank n and m respectively, satisfies a local-global principle.

Theorem 3.4.1 (Local-Global Principle). *Let $\theta(X) \in \mathrm{O}_{A[X]}(M[X])$. If, for all maximal ideals $\mathfrak{m}$ of A, $\theta(X)_{\mathfrak{m}} \in \mathrm{O}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) \cdot \mathrm{EO}_{A_{\mathfrak{m}}[X]}(M_{\mathfrak{m}}[X])$, then $\theta(X) \in \mathrm{O}_A(M) \cdot \mathrm{EO}_{A[X]}(M[X])$.*

Before beginning the proof, it is worthwhile to observe:

Remark 3.4.2. Replacing $\theta(X)$ by $\theta(0)^{-1}\theta(X)$, we may assume that $\theta(0) = 1$. Further, for $\theta(X) \in \mathrm{O}_A(M)\mathrm{EO}_{A[X]}(M[X])$ and $\theta(0) = I$ implies that $\theta(X) \in \mathrm{EO}_{A[X]}(M[X])$. Indeed, if $\theta(X) = \gamma\varepsilon(X)$ with $\gamma \in \mathrm{O}_A(M)$ and $\varepsilon(X) \in \mathrm{EO}_{A[X]}(M[X])$, then $\gamma = \theta(0)\varepsilon(0)^{-1} = \varepsilon(0)^{-1}$.

In view of this remark, we can rewrite Theorem 3.4.1 as follows:

Theorem 3.4.3 (**Local-Global Principle**). *Let $\theta(X) \in \mathrm{O}_{A[X]}(M[X])$ be such that $\theta(0) = I$. If $\theta(X)_{\mathfrak{m}} \in \mathrm{EO}_{A_{\mathfrak{m}}[X]}(M_{\mathfrak{m}}[X])$ for all maximal ideals $\mathfrak{m}$ of A, then $\theta(X) \in \mathrm{EO}_{A[X]}(M[X])$.*

We begin with some lemmas of which the first one is an elementary observation in group theory.

Lemma 3.4.4. *Let G be a group and $a_i, b_i \in G$ for $i = 1, ..., n$. Then*

$$\prod_{i=1}^n a_i b_i = \prod_{i=1}^n r_i b_i r_i^{-1} \prod_{i=1}^n a_i,$$

where $r_i = \prod_{j=1}^i a_j$.

Lemma 3.4.5. *The group $\mathrm{EO}_{A[X]}(M[X])$ is generated by elements of the form $\gamma E\left(X\alpha_{ij}(X)\right)\gamma^{-1}$, where $\gamma \in \mathrm{EO}_A(M)$ and $\alpha_{ij}(X) \in \mathrm{Hom}_A(Q[X], P[X])$ or $\mathrm{Hom}_A(Q[X], P^*[X])$.*

Proof. Let $\theta(X)$ be an element of $\mathrm{EO}_{A[X]}(M[X])$ such that $\theta(0) = I$. Then

$$\begin{aligned}\theta(X) &= \prod_{k=1}^r E\left(\alpha_{i_k j_k}(X)\right) = \prod_{k=1}^r E\left(\alpha_{i_k j_k}(0) + X{\alpha'}_{i_k j_k}(X)\right)\\ &= \prod_{k=1}^r E\left(\frac{\alpha_{i_k j_k}(0)}{2}\right) E\left(X{\alpha'}_{i_k j_k}(X)\right) E\left(\frac{\alpha_{i_k j_k}(0)}{2}\right) \text{ (by Splitting property)}\\ &= \prod_{k=1}^{r+1} a_k b_k,\end{aligned}$$

$$\begin{array}{llll} \text{where} & a_1 = E\left(\frac{\alpha_{i_1 j_1}(0)}{2}\right), & & b_k = E\left(X{\alpha'}_{i_k j_k}(X)\right) \text{ for } k = 1, ..., r,\\ & a_k = E\left(\frac{\alpha_{i_{k-1} j_{k-1}}(0)}{2}\right) E\left(\frac{\alpha_{i_k j_k}(0)}{2}\right) & & \text{for } k = 2, ..., r,\\ & a_{r+1} = E\left(\frac{\alpha_{i_r j_r}(0)}{2}\right), & & b_{r+1} = 1.\end{array}$$

By Lemma 3.4.4, we have

$$\theta(X) = \prod_{k=1}^{r+1} \gamma_k E\left(X{\alpha'}_{i_k j_k}(X)\right) {\gamma_k}^{-1} \prod_{k=1}^{r+1} a_k,$$

where $\gamma_k = \prod_{j=1}^{k} a_j \in \mathrm{EO}_A(M)$ and $\prod_{k=1}^{r+1} a_k = \prod_{k=1}^{r} E(\alpha_{i_k j_k}(0)) = \theta(0) = I$.
Therefore

$$\theta(X) = \prod_{k=1}^{r+1} \gamma_k E\left(X{\alpha'}_{i_k j_k}(X)\right){\gamma_k}^{-1}. \qquad \square$$

Lemma 3.4.6. *Let $\alpha, \delta \in \mathrm{Hom}(Q, P)$, $\beta, \gamma \in \mathrm{Hom}(Q, P^*)$ and s be a non-nilpotent element of A. Fix $r \in \mathbb{N}$. Let $i, k, p_t \in \{1, 2, ..., m\}$ and $j, l, q_t \in \{1, 2, ..., n\}$ for every $t \in \mathbb{N}$. Then for sufficiently large integer N, there exists a product decomposition for $E\left(\frac{a}{s^r}W_{ij}\right) E\left(s^N xY_{kl}\right) E\left(-\frac{a}{s^r}W_{ij}\right)$ in $\mathrm{EO}_{A_s}(M_s)$ given by*

$$E\left(\frac{a}{s^r}W_{ij}\right) E\left(s^N xY_{kl}\right) E\left(-\frac{a}{s^r}W_{ij}\right) = \prod_{t=1}^{\nu} E\left(s^{N_t} x_t Z_{p_t q_t}\right),$$

where $W, Y, Z \in \{\alpha, \beta, \gamma, \delta\}$, $a, x \in A$ and $x_t \in A$, $N_t \in \mathbb{N}$ for $t \in \mathbb{N}$ are chosen suitably.

Proof. To prove the lemma it is enough to consider the following cases.
Case 1: $(W, Y) \in \{(\alpha, \alpha), (\alpha, \delta), (\beta, \beta), (\beta, \gamma)\}$.

$E\left(\frac{a}{s^r}W_{ij}\right) E\left(s^N xY_{kl}\right) E\left(\frac{a}{s^r}W_{ij}\right)^{-1} = \prod_{t=1}^{\nu} E\left(s^{N_t} x_t Z_{p_t q_t}\right)$.

Subcase (a): $i \neq k$.

$$\begin{aligned} E\left(\frac{a}{s^r}W_{ij}\right) E\left(s^N xY_{kl}\right) E\left(\frac{a}{s^r}W_{ij}\right)^{-1} &= \left[E\left(\frac{a}{s^r}W_{ij}\right), E\left(s^N xY_{kl}\right)\right] E\left(s^N xY_{kl}\right) \\ &= \left[E\left(as^p W_{ij}\right), E\left(s^q xY_{kl}\right)\right] E\left(s^N xY_{kl}\right) \\ &\qquad \text{(by Corollary 2.1.4 and Corollary 2.1.10)} \\ &= \prod_{t=1}^{\nu} E\left(s^{N_t} x_t Z_{p_t q_t}\right) \quad \text{for } N_t > 0. \end{aligned}$$

This equation holds for any positive integers p, q with $p + q = N - r$.
Subcase (b): $i = k$.

$$\begin{aligned} E\left(\frac{a}{s^r}W_{ij}\right) E\left(s^N xY_{kl}\right) E\left(\frac{a}{s^r}W_{ij}\right)^{-1} &= \left[E\left(\frac{a}{s^r}W_{ij}\right), E\left(s^N xY_{kl}\right)\right] E\left(s^N xY_{kl}\right) \\ &= E\left(s^N xY_{kl}\right). \\ &\qquad \text{(by Lemma 2.1.3 and by Lemma 2.1.9)} \end{aligned}$$

Case 2: $(W, Y) \in \{(\alpha, \beta), (\beta, \alpha)\}$.

$$E\left(\frac{a}{s^r}W_{ij}\right) E\left(s^N xY_{kl}\right) E\left(\frac{a}{s^r}W_{ij}\right)^{-1} = \prod_{t=1}^{\nu} E\left(s^{N_t} x_t Z_{p_t q_t}\right).$$

Subcase (a): $i \neq k$.

For instance,

$$\begin{aligned} E\left(\frac{a}{s^r}\alpha_{ij}\right) E\left(s^N x\beta_{kl}\right) E\left(\frac{a}{s^r}\alpha_{ij}\right)^{-1} &= E_{\frac{a}{s^r}\alpha_{ij}} E^*_{s^N x\beta_{kl}} E^{-1}_{\frac{a}{s^r}\alpha_{ij}} \\ &= \left[E_{\frac{a}{s^r}\alpha_{ij}}, E^*_{s^N x\beta_{kl}}\right] E^*_{s^N x\beta_{kl}} \\ &= \left[E_{as^p\alpha_{ij}}, E^*_{s^q x\beta_{kl}}\right] E^*_{s^N x\beta_{kl}} \quad \text{(by Corollary 2.1.7)} \\ &= \textstyle\prod_{t=1}^{\nu} E(s^{N_t} x_t Z_{p_t q_t}) \text{ for } N_t > 0 \text{ and } \nu \leq 5. \end{aligned}$$

Subcase (b): $i = k$.

For instance,

$$E\left(\frac{a}{s^r}\alpha_{ij}\right) E\left(s^N x\beta_{il}\right) E\left(\frac{a}{s^r}\alpha_{ij}\right)^{-1} = E_{\frac{a}{s^r}\alpha_{ij}} E^*_{s^N x\beta_{il}} E^{-1}_{\frac{a}{s^r}\alpha_{ij}}. \tag{3.4.1}$$

Set $N = N_1 + N_2 + N_3$ such that $N_1 \geq r + 2$ and $N_2 + N_3 \geq 2r + 4$. Now, by replacing $E^*_{s^N x\beta_{il}}$ with $\left[E_{s^{N_1}\alpha_{kl}}, \left[E_{s^{N_2}x\beta^*_{il}}, E_{s^{N_3}\gamma^*_{pq}}\right]\right] \left[E_{s^N x\frac{\beta^*_{il}}{2}}, E_{s^{N_1}\alpha_{kl}}\right]$ in equation (3.4.1), and by using Lemma 2.2.7, we have

$$E_{\frac{a}{s^r}\alpha_{ij}} E^*_{s^N x\beta_{il}} E^{-1}_{\frac{a}{s^r}\alpha_{ij}} = E_{\frac{a}{s^r}\alpha_{ij}} \left[E_{s^{N_1}\alpha_{kl}}, \left[E_{s^{N_2}x\beta^*_{il}}, E_{s^{N_3}\gamma^*_{pq}}\right]\right] \left[E_{s^N x\frac{\beta^*_{il}}{2}}, E_{s^{N_1}\alpha_{kl}}\right] E^{-1}_{\frac{a}{s^r}\alpha_{ij}}.$$

Then we will see that the following are in the required product form.

(a) $E_{\frac{a}{s^r}\alpha_{ij}} E_{s^{N_1}\alpha_{kl}} E^{-1}_{\frac{a}{s^r}\alpha_{ij}}$,

(b) $E_{\frac{a}{s^r}\alpha_{ij}} \left[E_{s^{N_2}x\beta^*_{il}}, E_{s^{N_3}\gamma^*_{pq}}\right] E^{-1}_{\frac{a}{s^r}\alpha_{ij}}$,

(c) $E_{\frac{a}{s^r}\alpha_{ij}} \left[E_{s^N x\frac{\beta^*_{il}}{2}}, E_{s^{N_1}\alpha_{kl}}\right] E^{-1}_{\frac{a}{s^r}\alpha_{ij}}$.

For, (a)
$$\begin{aligned} E_{\frac{a}{s^r}\alpha_{ij}} E_{s^{N_1}\alpha_{kl}} E^{-1}_{\frac{a}{s^r}\alpha_{ij}} &= \left[E_{\frac{a}{s^r}\alpha_{ij}}, E_{s^{N_1}\alpha_{kl}}\right] E_{s^{N_1}\alpha_{kl}} \\ &= \left[E_{as^{p'}\alpha_{ij}}, E_{s^{q'}\alpha_{kl}}\right] E_{s^{N_1}\alpha_{kl}} \quad \text{(by Corollary 2.2.8(i))} \\ &= \textstyle\prod_{t=1}^{\nu} E\left(s^{N_t} x_t Z_{p_t q_t}\right) \text{ for } N_t > 0 \text{ and } \nu \leq 5. \end{aligned}$$

This equation holds for any positive integers p', q' with $p' + q' = N_1 - r$.

(b)
$$\begin{aligned} E_{\frac{a}{s^r}\alpha_{ij}} \left[E_{s^{N_2}x\beta^*_{il}}, E_{s^{N_3}\gamma^*_{pq}}\right] E^{-1}_{\frac{a}{s^r}\alpha_{ij}} &= \left[E_{\frac{a}{s^r}\alpha_{ij}} \left[E_{s^{N_2}x\beta^*_{il}}, E_{s^{N_3}\gamma^*_{pq}}\right]\right] \left[E_{s^{N_2}x\beta^*_{il}}, E_{s^{N_3}\gamma^*_{pq}}\right] \\ &= \left[E_{s^{p''}\alpha_{ij}}, \left[E_{s^{q''}x\beta^*_{il}}, E_{s^{r''}\gamma^*_{pq}}\right]\right] \left[E_{s^{N_2}x\beta^*_{il}}, E_{s^{N_3}\gamma^*_{pq}}\right] \end{aligned}$$

(by Corollary 2.2.8)

$$= \prod_{t=1}^{\nu} E\left(s^{N_t} x_t Z_{p_t q_t}\right) \text{ for } N_t > 0 \text{ and } \nu \leq 14.$$

This equation holds for any positive integers p'', q'' and r'' with $2p''+q''+r'' = N_2+N_3-2r$.

$$\text{(c) } E_{\frac{a}{s^r}\alpha_{ij}}\left[E_{s^N x\frac{\beta^*_{il}}{2}}, E_{s^{N_1}\alpha_{kl}}\right] E^{-1}_{\frac{a}{s^r}\alpha_{ij}} = \left[E_{\frac{a}{s^r}\alpha_{ij}}, \left[E_{s^N x\frac{\beta^*_{il}}{2}}, E_{s^{N_1}\alpha_{kl}}\right]\right]\left[E_{s^N x\frac{\beta^*_{il}}{2}}, E_{s^{N_1}\alpha_{kl}}\right]$$

$$= \left[E_{s^{p'''}\alpha_{ij}}, \left[E_{s^{q'''}x\frac{\beta^*_{il}}{2}}, E_{s^{r'''}\alpha_{kl}}\right]\right]\left[E_{s^N x\frac{\beta^*_{il}}{2}}, E_{s^{N_1}\alpha_{kl}}\right]$$

(by Corollary 2.2.4)

$$= \prod_{t=1}^{\nu} E\big(s^{N_t} x_t Z_{p_t q_t}\big) \text{ for } N_t > 0 \text{ and } \nu \leq 14.$$

This equation holds for any positive integers p''', q''' and r''' with $2p'''+q'''+r''' = N_1+N-2r$. Hence equation (3.4.1) is of the form $\prod_{t=1}^{\nu} E(s^{N_t} x_t Z_{p_t q_t})$ for $N_t > 0$ and $\nu \leq 52$. □

Lemma 3.4.7 (Dilation Lemma). *Let s be a non-nilpotent element of A and let $M = Q \perp H(P)$. Let $\theta(X) \in \mathrm{O}_{A[X]}(M[X])$ with $\theta(0) = I$. Let $Y, Z \in \mathrm{Hom}_A(Q, P)$ or $\mathrm{Hom}_A(Q, P^*)$. If $\theta_s(X) = (\theta(X))_s \in \mathrm{EO}_{A_s[X]}(M_s[X])$, then, for $N \gg 0$ and for all $b \in (s)^N A$, we have $\theta(bX) \in \mathrm{EO}_{A[X]}(M[X])$.*

Proof. Let $\theta_s(X) \in \mathrm{EO}_{A_s[X]}(M_s[X])$. Then $\theta_s(X) = \prod_{k=1}^{r} E\left(\alpha_{i_k j_k}(X)\right)$, where $\alpha_{i_k j_k}(X) \in \mathrm{Hom}_A(Q_s[X], P_s[X])$ or $\mathrm{Hom}_A(Q_s[X], P^*_s[X])$ for all $k \in \mathbb{N}$, $i_k \in \{1, 2, ..., m\}$ and $j_k \in \{1, 2, ..., n\}$.

Let $\alpha_{i_k j_k}(X) = \alpha_{i_k j_k}(0) + X\alpha'_{i_k j_k}(X)$. By the splitting property, we can write

$$E\left(\alpha_{i_k j_k}(X)\right) = E\left(\frac{\alpha_{i_k j_k}(0)}{2}\right) E\left(X\alpha'_{i_k j_k}(X)\right) E\left(\frac{\alpha_{i_k j_k}(0)}{2}\right).$$

Then

$$\theta_s(X) = \prod_{k=1}^{r+1} E\left(\tfrac{\alpha_{i_k j_k}(0)}{2}\right) E\left(X\alpha'_{i_k j_k}(X)\right) E\left(\tfrac{\alpha_{i_k j_k}(0)}{2}\right).$$

By Lemma 3.4.5, one has

$$\theta_s(X) = \prod_{k=1}^{r+1} \gamma_k E\left(X\alpha'_{i_k j_k}(X)\right) \gamma_k^{-1},$$

where $\gamma_k = \prod_{j=1}^{k} a_j$ with

$$\begin{aligned} a_1 &= E\left(\tfrac{\alpha_{i_1 j_1}(0)}{2}\right), \\ a_{r+1} &= E\left(\tfrac{\alpha_{i_r j_r}(0)}{2}\right), \\ a_k &= E\left(\tfrac{\alpha_{i_{k-1} j_{k-1}}(0)}{2}\right) E\left(\tfrac{\alpha_{i_k j_k}(0)}{2}\right) \quad \text{for } k = 2, ..., r. \end{aligned}$$

Hence we can write

$$\theta_s(s^N X) = \prod_{k=1}^{r+1} \gamma_k E\left(s^N X \alpha'_{i_k j_k}(s^N X)\right) \gamma_k^{-1} \quad \text{for } N \gg 0.$$

Claim : If $\xi = \prod_{j=1}^{k} E\left(c_j\right), c_j \in M_s$, then, for $\xi E\left(s^N x Z_{ij}\right) \xi^{-1}$, we have a product decomposition given by

$$\xi E\left(s^N x Z_{ij}\right) \xi^{-1} = \prod_{t=1}^{\lambda_k} E\left(s^{N_t} x_t Z_{i_t j_t}\right) \tag{3.4.2}$$

with $N_t \to \infty$ for $N \gg 0$, $x_t \in A$.

Proof of the Claim. We do this by induction on k.

Let $\xi = \xi_1 \xi_2 \ldots \xi_k$, where $\xi_i = E\left(c_i\right)$. When $k = 1$, by Lemma 3.4.6, we have a product decomposition

$$\xi_1 E\left(s^N x Z_{ij}\right) \xi_1^{-1} = \prod_{t=1}^{\lambda_1} E\left(s^{N_t} x_t Z_{i_t j_t}\right)$$

with $N_t \to \infty$ for $N \gg 0$. Now assume that the result is true for $k - 1$. i.e., we have

$$\xi_1 \xi_2 \ldots \xi_{k-1} E\left(s^N x Z_{ij}\right) \left(\xi_1 \xi_2 \ldots \xi_{k-1}\right)^{-1} = \prod_{t=1}^{\lambda_{k-1}} E\left(s^{N_t} x_t Z_{i_t j_t}\right)$$

with $N_t \to \infty$ for $N \gg 0$. Now, by Lemma 3.4.6, we can write

$$\xi_k E\left(s^N x Z_{ij}\right) \xi_k^{-1} = \prod_{t=1}^{\lambda_{k-1}} E\left(s^{N_t} x_t Z_{i_t j_t}\right) = \mu_1 \mu_2 \ldots \mu_\lambda \text{ (say)}.$$

Hence we have

$$\begin{aligned} &(\xi_1 \xi_2 \ldots \xi_{k-1} \xi_k) E\left(s^N x Z_{ij}\right) \left(\xi_1 \xi_2 \ldots \xi_{k-1}\right)^{-1} \\ &\qquad = \left(\xi_1 \xi_2 \ldots \xi_{k-1}\right) \mu_1 \mu_2 \ldots \mu_\lambda \left(\xi_1 \xi_2 \ldots \xi_{k-1}\right)^{-1} \\ &\qquad = \left(\xi_1 \xi_2 \ldots \xi_{k-1}\right) \mu_1 (\xi_1 \xi_2 \ldots \xi_{k-1})^{-1} (\xi_1 \xi_2 \ldots \xi_{k-1}) \\ &\qquad\quad \mu_2 (\xi_1 \xi_2 \ldots \xi_{k-1})^{-1} \ldots (\xi_1 \xi_2 \ldots \xi_{k-1}) \mu_\lambda \left(\xi_1 \xi_2 \ldots \xi_{k-1}\right)^{-1}. \end{aligned}$$

Now, by applying induction to each of the expressions $\xi_1\xi_2\ldots\xi_{k-1}\mu_l\left(\xi_1\xi_2\ldots\xi_{k-1}\right)^{-1}$ as l varies from 1 to λ, we have a product decomposition as in equation (3.4.2). Therefore we can write

$$\theta_s\left(s^N X\right) = \prod_{k=1}^{r+1}\prod_{t=1}^{\lambda_k} E\left(s^{N_t}x_t Z_{i_t j_t}\right) \quad \text{for } N \text{ large enough.}$$

The terms $s^{N_t}x_t$ for $1 \le t \le \lambda_k$ is contained in $M[X]$ as required. Hence

$$\theta(bX) = \prod_{k=1}^{r+1}\prod_{t=1}^{\lambda_k} E\left(s^{N_t}x_t Z_{i_t j_t}\right) \in \mathrm{EO}_{A[X]}\left(M[X]\right)$$

for all $b \in (s)^N A$. □

Proof of Theorem 3.4.3. Let $\mathfrak{m}$ be a maximal ideal of A. Choose an element $s_{\mathfrak{m}}$ from $A \setminus \mathfrak{m}$ such that

$$\theta(X)_{s_{\mathfrak{m}}} \in \mathrm{EO}_{A_{s_{\mathfrak{m}}}[X]}(M_{s_{\mathfrak{m}}}[X]).$$

Define

$$\kappa(X,Y) = \theta(X+Y)\theta(Y)^{-1}.$$

Clearly $\kappa(X,Y)_{s_{\mathfrak{m}}} \in \mathrm{EO}_{A_{s_{\mathfrak{m}}}[X,Y]}(M_{s_{\mathfrak{m}}}[X])$ and $\kappa(0,Y) = I$.

Now by applying Dilation Lemma with $A[Y]$ as the base ring, we get

$$\kappa(b_{\mathfrak{m}}X, Y) \in \mathrm{EO}_{A[X,Y]}\left(M[X,Y]\right),$$

where $b_{\mathfrak{m}} \in (s_{\mathfrak{m}}^N)$ for some $N \gg 0$.

Since A is the ideal generated by $\{s_{\mathfrak{m}}\}_{\mathfrak{m}\in \mathrm{Max}\, A}$, there exist maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ and elements $s_{\mathfrak{m}_i} \in A \setminus \mathfrak{m}_i$ such that $A = \sum_{i=1}^{r}(s_{\mathfrak{m}_i})$. Therefore

$$A = \sum_{i=1}^{r}(s_{\mathfrak{m}_i}^{N_i})$$

for any $N_i > 0$. Hence for $b_{\mathfrak{m}_i} \in (s_{\mathfrak{m}_i}^{N_i})$ with $N_i \gg 0$, we have $\sum_{i=1}^{r} b_{\mathfrak{m}_i} = 1$.

Observe that $\kappa(b_{\mathfrak{m}_i}X, Y) \in \mathrm{EO}_{A[X,Y]}(M[X,Y])$ for $1 \le i \le r$.

$$\begin{aligned}\theta(X) =&\theta(\textstyle\sum_{i=1}^{r} b_{\mathfrak{m}_i}X)\,\theta\left(\sum_{i=2}^{r} b_{\mathfrak{m}_i}X\right)^{-1}\theta\left(\sum_{i=2}^{r} b_{\mathfrak{m}_i}X\right)\ \theta\left(\sum_{i=3}^{r} b_{\mathfrak{m}_i}X\right)^{-1}\cdots\\ &\theta\left(b_{\mathfrak{m}_{r-1}}X + b_{\mathfrak{m}_r}X\right)\ \theta\left(b_{\mathfrak{m}_r}X\right)^{-1}\theta\left(b_{\mathfrak{m}_r}X\right)\\ =&\textstyle\prod_{i=1}^{r-1}\kappa(b_{\mathfrak{m}_i}X, T_i)\kappa(b_{\mathfrak{m}_r}X, 0),\end{aligned}$$

where $T_i = \sum_{k=i+1}^{r} b_{\mathfrak{m}_k}X$. Hence $\theta(X) \in \mathrm{EO}_{A[X]}(M[X])$. □

3.5 A Local-Global Principle for $\mathrm{EO}(Q \perp H(A)^m) \cdot \mathrm{O}(H(A)^m)$

In this section, we prove a local-global principle for the set $\mathrm{EO}_A(Q \perp H(A)^m) \cdot \mathrm{O}(H(A)^m)$, where Q is a free A-module of rank n. We also assume that the generalized dimension of A is at least d.

Theorem 3.5.1 ([45, Theorem 2.5]). *Let A be a ring of generalized dimension $\geq d$. Let (Q, q) be a diagonalizable quadratic A-space. Consider the quadratic A-space $Q \perp H(P)$, where* rank $(P) > d$. *Then*

$$\begin{aligned} \mathrm{O}_A(Q \perp H(P)) &= \mathrm{EO}_A(Q \perp H(P)) \cdot \mathrm{O}_A(H(P)) \\ &= \{\varepsilon\beta \mid \varepsilon \in \mathrm{EO}_A(Q \perp H(P)), \beta \in \mathrm{O}_A(H(P))\} \\ &= \{\beta\varepsilon \mid \varepsilon \in \mathrm{EO}_A(Q \perp H(P)), \beta \in \mathrm{O}_A(H(P))\} \\ &= \mathrm{O}_A(H(P)) \cdot \mathrm{EO}_A(Q \perp H(P)). \end{aligned}$$

We now prove the Dilation lemma for $\mathrm{EO}_A(Q \perp H(A)^m) \cdot \mathrm{O}(H(A)^m)$.

Lemma 3.5.2 (**Dilation Lemma**). *Let s be a non-nilpotent element of A and let $m > d$. Let $\theta(X) \in \mathrm{O}_{A[X]}(Q \otimes A[X] \perp H(A[X])^m) \cdot \mathrm{O}_{A[X]}(H(A[X])^m)$ with $\theta(0) = I$. If $\theta_s(X) = (\theta(X))_s \in \mathrm{EO}_{A_s[X]}(Q \otimes A_s[X] \perp H(A_s[X])^m) \cdot \mathrm{O}_{A_s[X]}(H(A_s[X])^m)$, then, for $d \gg 0$ and for all $b \in (s)^d A$, we have $\theta(bX) \in \mathrm{EO}_{A[X]}(Q \otimes A[X] \perp H(A[X])^m) \cdot \mathrm{O}_{A[X]}(H(A[X])^m)$.*

Proof. If $\theta_s(X) = \varepsilon(X)\beta(X)$, where $\varepsilon(X) \in \mathrm{EO}_{A_s[X]}(Q \otimes A_s[X] \perp H(A_s[X])^m)$ and $\beta(X) \in \mathrm{O}_{A_s[X]}(H(A_s[X])^m)$, then $\theta(0) = I = \varepsilon(0)\beta(0)$; whence

$$\theta_s(X) = \{\varepsilon(X)\varepsilon(0)^{-1}\}\{\beta(0)^{-1}\beta(X)\}.$$

In other words, we may assume at the onset that $\varepsilon(0) = I$ and $\beta(0) = I$. The rest of the proof follows from Lemma 3.4.7. □

We now prove the local-global principle for $\mathrm{EO}_A(Q \perp H(A)^m) \cdot \mathrm{O}(H(A)^m)$.

Theorem 3.5.3 (**Local-Global Principle**). *Let (Q, q) be a diagonalizable quadratic A-space. Let $m > d$ and let $\theta(X) \in \mathrm{O}_{A[X]}(Q \otimes A[X] \perp H(A[X])^m)$ be such that $\theta(0) = I$.*

Suppose that, for every $\mathfrak{m} \in \mathrm{Max}\,(A)$*, we have* $\alpha_{\mathfrak{m}} = \beta_{\mathfrak{m}}\gamma_{\mathfrak{m}}$*, where* $\beta_{\mathfrak{m}} \in \mathrm{EO}_{A_{\mathfrak{m}}[X]}\,((Q \otimes A_{\mathfrak{m}}[X]) \perp H(A_{\mathfrak{m}}[X])^m)$ *and* $\gamma_{\mathfrak{m}} \in \mathrm{O}_{A_{\mathfrak{m}}[X]}(H(A_{\mathfrak{m}}[X])^m)$ *with* $\beta(0) = I$*,* $\gamma(0) = I$*. Then* $\alpha = \beta\gamma$ *with* $\beta \in \mathrm{EO}_{A[X]}((Q \otimes A[X]) \perp H(A[X])^m), \gamma \in \mathrm{O}_{A[X]}(H(A[X])^m)$*.*

Proof. The proof follows in similar lines as Theorem 3.4.3 except for the following. Let $\mathfrak{m}$ be a maximal ideal of A. Choose an element $s_{\mathfrak{m}}$ from $A \setminus \mathfrak{m}$ such that

$$\theta(X)_{s_{\mathfrak{m}}} \in \mathrm{EO}_{A_{s_{\mathfrak{m}}}[X]}\,(Q \otimes A_{s_{\mathfrak{m}}}[X] \perp H(A_{s_{\mathfrak{m}}}[X])^m)\,\mathrm{O}_{A_{s_{\mathfrak{m}}}[X]}(H(A_{s_{\mathfrak{m}}}[X])^m).$$

Define

$$\kappa(X, Y) = \theta(X + Y)_{s_{\mathfrak{m}}}\theta(Y)^{-1}_{s_{\mathfrak{m}}}.$$

Then

$$\kappa(X, Y) = \varepsilon_1\eta_1\eta_2\varepsilon_2 = \varepsilon_1\eta_3\varepsilon_2 \tag{3.5.1}$$

for $\varepsilon_1, \varepsilon_2 \in \mathrm{EO}_{A_{s_{\mathfrak{m}}}[X,Y]}(Q \otimes A[X, Y] \perp h^m)$, $\eta_1, \eta_2 \in \mathrm{O}_{A_{s_{\mathfrak{m}}}[X,Y]}(h^m)$ and $\eta_3 = \eta_1\eta_2$. Since

$$\mathrm{EO}_{A_{s_{\mathfrak{m}}}[X,Y]}(Q\otimes A[X, Y] \perp h^m)\cdot\mathrm{O}_{A_{s_{\mathfrak{m}}}[X,Y]}(h^m) = \mathrm{O}_{A_{s_{\mathfrak{m}}}[X,Y]}(h^m)\cdot\mathrm{EO}_{A_{s_{\mathfrak{m}}}[X,Y]}(Q\otimes A[X, Y] \perp h^m),$$

by Theorem 3.5.1, we can write equation (3.5.1) as

$$\kappa(X, Y) = \varepsilon_1\varepsilon_2'\eta_3'$$

for some $\varepsilon_2' \in \mathrm{EO}_{A_{s_{\mathfrak{m}}}[X,Y]}(Q \otimes A[X, Y] \perp h^m)$ and $\eta_3' \in \mathrm{O}_{A_{s_{\mathfrak{m}}}[X,Y]}(h^m)$. That is,

$$\kappa(X, Y) \in \mathrm{EO}_{A_{s_{\mathfrak{m}}}[X,Y]}(Q \otimes A_{s_{\mathfrak{m}}}[X, Y] \perp h^m) \cdot \mathrm{O}_{A_{s_{\mathfrak{m}}}[X,Y]}(h^m) \text{ and } \kappa(0, Y) = I.$$

Therefore, by applying Lemma 3.5.2 with base ring $A[Y]$,

$$\kappa(b_{\mathfrak{m}}X, Y) \in \mathrm{EO}_{A[X,Y]}(Q \otimes A[X, Y] \perp h^m) \cdot \mathrm{O}_{A[X,Y]}(h^m),$$

where $b_{\mathfrak{m}} \in (s_{\mathfrak{m}}^N)$ for any sufficiently large N. □

3.6 Action Version of Local-Global Principle

In this section, we prove an "action version" of Quillen's local-global principle. We begin by recalling some known results in this direction. In a letter to H. Bass, L.N. Vaserstein proved the following action version of Quillen's well-known local-global principle.

Theorem 3.6.1 ([36, Chapter III, Theorem 2.5]). *Let $n \geq 3$ and $\nu(X) \in \mathrm{Um}_n(A[X])$. If $\nu(X) \in \mathrm{GL}_n(A_{\mathfrak{m}}[X])$, for all maximal ideals $\mathfrak{m}$ of A, then $\nu(X) \in \nu(0)\,\mathrm{GL}_n(A[X])$.*

A result similar to the one above was proved for the elementary linear group by R.A. Rao which is the following.

Theorem 3.6.2 ([46, Theorem 2.3]). *Let $\nu(X) \in \mathrm{Um}_n(A[X]), n \geq 3$. Suppose, for all maximal ideals $\mathfrak{m}$ in A, $\nu(X) \in \nu(0)\,\mathrm{E}_n(A_{\mathfrak{m}}[X])$. Then $\nu(X) \in \nu(0)\,\mathrm{E}_n(A[X])$.*

Similar results are also proved in [8, 10, 21]. More generalized results of the action version of local-global principle for Chevalley groups are established in [7, 54].

In [48], A. Roy proved the following result.

Theorem 3.6.3. *Let A be a commutative Noetherian ring and $d = \dim\ \mathrm{Max}\ A < \infty$. Let P be a finitely generated projective A-module of* rank $\geq d+1$, *and Q a quadratic A-space. Let $\mathfrak{a}$ be an ideal of A and $w \in Q \perp H(P)$ such that $Aq(w) + \mathfrak{a} = A$. Then there exist A-linear maps $\alpha_1, \cdots, \alpha_n : Q \to P$ such that*

$$o(P\text{-component of } E_{\alpha_n} \circ \cdots \circ E_{\alpha_1}(w)) + \mathfrak{a} = A$$

Then R. Parimala extended this result for generalised dimension.

Theorem 3.6.4 ([38, Theorem 3.1]). *Let A be a commutative ring and d be a generalised dimension function on* $\mathrm{Spec}\,A$. *Let (Q_0, q_0) be a quadratic A-space and let $Q = Q_0 \perp H(P)$, where P is a finitely generated projective A-module of* rank $\geq d(A)+1$. *Let $w = (z, x, f)$ be an element in $Q_0 \perp H(P)$ such that $q(w) = q_0(z) + f(x)$ is a unit in A. Then there exists $\eta = E_{\alpha_1} \circ E_{\alpha_2} \circ \cdots \circ E_{\alpha_n} \in \mathrm{EO}_A(Q_0 \perp H(P))$ such that $\eta(z, x, f) = (z', x', f')$ with x' unimodular in P.*

The above result states that elements of unit norm in a quadratic space of sufficiently large Witt index can be brought into general position by elementary orthogonal transformations. This can be considered as a quadratic analogue of a stability theorem of Eisenbud-Evans [25, Theorem A (ii)b].

R.A. Rao, in his Ph. D. thesis (1984), raised the following question.

Question 3.6.5. *Is there a "local- global" principle for the action of the elementary group* $\mathrm{EO}_{A[T]}\,(Q \otimes A[T] \perp H(A[T])^n)$ *on non-singular elements? Explicitly, let* (Q, q) *be a quadratic* A*-space and let* w *be a non-singular element in* $(Q \perp H(A)^n) \otimes A[T]$. *Assume that, for all* $\mathfrak{m} \in \mathrm{Max}\,(A)$*, there exists an element* $\sigma_{\mathfrak{m}} \in \mathrm{EO}_{A_{\mathfrak{m}}[T]}\,(Q \otimes A_{\mathfrak{m}}[T] \perp H(A_{\mathfrak{m}}[T])^n)$ *such that* $\sigma_{\mathfrak{m}} w = w\,(0)\,\mathrm{EO}_{A[T]}\,(Q \otimes A[T] \perp H(A[T])^n)$. *Does there exist an element* σ *in* $\mathrm{EO}_{A[T]}\,(Q \otimes A[T] \perp H(A[T])^n)$ *with* $\sigma w = w\,(0)$*?*

In this chapter, we give an affirmative answer to this question.

Let Q and P be free A-modules of rank n and m respectively. In the remaining part of this section, d denotes a generalized dimension function on $\mathrm{Spec}\,A$.

The main theorem of this section is:

Theorem 3.6.6. *Let* (Q, q) *be a quadratic* A*-space and let* $M = Q \perp H(A)^m$*, where* m *is at least* $d(A) + 1$. *Let* $w \in (Q \perp H(A)^m) \otimes A[T]$ *be non-singular. Suppose, for all* $\mathfrak{m} \in \mathrm{Max}\,(A)$*, there exists an element* $\sigma_{\mathfrak{m}}$ *in* $\mathrm{EO}_{A_{\mathfrak{m}}[T]}\,((Q \perp H(A)^m) \otimes A_{\mathfrak{m}}[T])$ *such that* $\sigma_{\mathfrak{m}} w = w\,(0)\,\mathrm{EO}_{A[T]}\,((Q \perp H(A)^m) \otimes A[T])$. *Then there exists an element* σ *in the elementary group* $\mathrm{EO}_{A[T]}\,(Q \otimes A[T], H(A[T])^m)$ *with* $\sigma w = w(0)$.

We begin with a lemma which uses a standard argument of L.N. Vaserstein (see [36, Chapter III, Proposition 2.3]).

Lemma 3.6.7. *Let* S *be a multiplicatively closed set in* A *and let* $n + 2m \geq 6$. *Let* $w(X) \in \mathrm{Um}_{n+2m}(A[X])$ *and let* $w(X) \in w(0)\ \mathrm{EO}\,((Q \perp H(A)^m) \otimes A[X])$. *Then there is an element* s *in* S *such that, for any* a *in* A*,*

$$w\,(X + asT) \in w(X)\ \mathrm{EO}((Q \perp H(A)^m) \otimes A[X, T]).$$

Proof. Let $\vartheta(X) \in \mathrm{EO}((Q \perp H(A)^m) \otimes A_S[X])$ such that $w(X)\vartheta(X) = w(0)$. Let

$$\theta(X, T) \ = \ \vartheta(X + T)\vartheta(X)^{-1} \ \in \ \mathrm{EO}((Q \perp H(A)^m) \otimes A_S[X, T]).$$

Then

$$w(X + T)\theta(X, T) = w(X + T)\vartheta(X + T)\vartheta(X)^{-1}$$

$$= w(0)\vartheta(X)^{-1}$$
$$= w(X) \in A_S[X,T]^{n+2m}.$$

Since $\theta(X,0) = I$, we can find $\theta^*(X,T) \in \mathrm{EO}((Q \perp H(A)^m) \otimes A[X,T])$ which localizes to $\theta(X, sT)$ for some $s \in S$ with $\theta^*(X,0) = I$ (by applying Dilation Lemma to the base ring $A[X]$). Then in $A[X,T]^n$, we have

$$w(X + sT)\theta^*(X,T) - w(X) = Tv(X,T)$$

for some $v(X,T)$ which localizes to 0. Thus, for some $s^* \in S$ and for all $a \in A$, we get

$$w(X + ass^*T)\theta^*(X, as^*T) - w(X) = Tas^*v(X, as^*T) = 0. \qquad \square$$

Proof of Theorem 3.6.6. Let w be a non-singular element in $(Q \perp H(A)^m) \otimes A[T]$. By Theorem 3.6.4, there exists an element $\eta \in \mathrm{EO}\,(Q, H(A)^m)$ such that $\eta\,(w)$ has its P-component unimodular in P. This implies that the order ideal

$$o\,(P\text{- component } (\eta\,(w))) = A.$$

which in turn implies that $o\,(\eta\,(w)) = A$. Hence $\eta\,(w)$ is unimodular in $Q \perp H(A)^m$.

Let $n + 2m \geq 6$. Let $w(X) \in \mathrm{Um}_{n+2m}(A[X])$. If, for all maximal ideals $\mathfrak{m}$ of A, $w(X)_\mathfrak{m} \in w(0)_\mathfrak{m}\,\mathrm{EO}(Q \perp H(A)^m \otimes A_\mathfrak{m}[X])$. Using Lemma 3.6.7 it follows that, for each maximal ideal $\mathfrak{m}$ of A, there exists $s_k \in A \backslash \mathfrak{m}$ such that, for all $a \in A$,

$$w(X + as_kT) \in w(X)\ \mathrm{EO}(Q \perp H(A)^m \otimes A[X,T]). \qquad (3.6.1)$$

We note that the ideal generated by $s_k's$ is the whole ring A. Therefore there exist elements $s_{k_1}, \cdots, s_{k_r}$ in $A \backslash \mathfrak{m}$ such that $a_1 s_{k_1} + \cdots + a_r s_{k_r} = 1$, where $a_i \in A$ for $1 \leq i \leq r$. In equation (3.6.1), replacing X by $a_2 s_{k_2} X + \cdots + a_r s_{k_r} X$ and $a_{s_k} T$ by $a_1 s_{k_1} X$, we get

$$\begin{aligned} w(X) &= w(a_1 s_{k_1} X + a_2 s_{k_2} X + \cdots + a_r s_{k_r} X) \\ &\in w(a_2 s_{k_2} X + \cdots + a_r s_{k_r} X)\ \mathrm{EO}((Q \perp H(A)^m) \otimes A[X]). \end{aligned}$$

Again in equation (3.6.1), replacing X by $a_3 s_{k_3} X + \cdots + a_r s_{k_r} X$ and $a_{s_k} T$ by $a_2 s_{k_2} X$, we get

$$w(a_2 s_{k_2} X + \cdots + a_r s_{k_r} X) \in w(a_3 s_{k_3} X + \cdots + a_r s_{k_r} X)\ \mathrm{EO}((Q \perp H(A)^m) \otimes A[X]).$$

Continuing in this way, we have

$$w(a_r s_{k_r} X + 0) \in w(0)\mathrm{EO}((Q \perp H(A)^m) \otimes A[X]).$$

Combining all of these, we get

$$w(X) \in w(0)\mathrm{EO}((Q \perp H(A)^m) \otimes A[X])$$

and hence the result is proved. □

CHAPTER 4

Extendability of Quadratic Modules over a Polynomial Extension of an Equicharacteristic Regular Local Ring

In this chapter, we obtain an extendability theorem for quadratic modules over polynomial rings. If A is an equicharacteristic regular local ring of dimension d, we prove that a quadratic $A[T]$-module Q for which the Witt index of Q/TQ is at least d, is extended from A. This improves a theorem of R.A. Rao which proves the above theorem when A is a local ring at a smooth point of an affine variety over an infinite field. To establish our result, we use a local-global principle for Roy's elementary orthogonal group that was proved in Chapter 3.

The results in this chapter are contained in [5].

4.1 Some Known Results

Let A be a commutative Noetherian ring in which 2 is invertible and let B be the polynomial A-algebra $A[X_1, \dots, X_n]$ in n indeterminates. Let $Q = (Q, q)$ be a quadratic space over B and let $Q_0 = (Q_0, q_0)$ be the reduction of Q modulo the ideal of B generated by $X_1, \dots, X_n$. In [58], A.A. Suslin and V.I. Kopeĭko proved that if Q is stably extended from A and if, for every maximal ideal $\mathfrak{m}$ of A, the Witt index of $A_{\mathfrak{m}} \otimes_A (Q_0, q_0)$ is larger than the Krull dimension of A, then (Q, q) is extended from A. In [19], I. Bertuccioni gave a short proof of this and another proof is in the Ph. D. thesis of R.A. Rao. In that thesis (see [44, 45]), it was shown that one can improve this result to quadratic spaces with Witt index at least d, when A is a local ring at a non-singular point of an affine variety of dimension d over an infinite field. Moreover, a question was posed at the end of the thesis whether extendability can be shown for quadratic spaces with Witt index at least d over polynomial extensions of any equicharacteristic regular local ring of dimension d. In the next section, we answer this question affirmatively.

As before, we consider the orthogonal group of $Q \perp H(P)$, denoted by $\mathrm{O}_A(Q \perp H(P))$, where Q and P are free A-modules of finite rank. Also, recall that Roy's elementary group $\mathrm{EO}_A(Q \perp H(P))$ is the subgroup of $\mathrm{O}_A(Q \perp H(P))$ generated by E_α and E^*_β, as $\alpha \in \mathrm{Hom}_A(Q, P)$ and $\beta \in \mathrm{Hom}_A(Q, P^*)$ vary.

The following cancellation theorem for quadratic spaces over semilocal rings was proved by A. Roy.

Theorem 4.1.1 ([48, Theorem 8.1])**.** *Let A be a semilocal ring and let R, R_1 and R_2 be quadratic spaces over A such that $R \perp R_1 \cong R \perp R_2$. Then $R_1 \cong R_2$.*

We now recall the following theorem of A.A. Suslin and V.I. Kopeĭko.

Theorem 4.1.2 ([58, Theorem 7.13])**.** *Let R be a commutative ring in which 2 is invertible. Any stably extended quadratic $R[T_1, \cdots, T_n]$-space Q with Witt index of $Q/(T_1, \cdots, T_n)Q$ at least $\max(2, \dim R + 1)$, is extended from R.*

In his Ph. D. thesis, R.A. Rao improved the above theorem when R is a regular ring as follows:

Theorem 4.1.3 (Extendability in the complete case)**.** *If R is a complete unramified regular local ring and Q is a quadratic $R[T_1, \cdots, T_n]$-space with Witt index of $Q/(T_1, \cdots, T_n)Q$ at least 1, then Q is extended from R.*

Definition 4.1.4. Let k be a field. A ring R is said to be of essentially finite type over k if $R = S^{-1}C$, where C is a finitely generated k-algebra and S is a multiplicatively closed subset of C.

We say R is a *regular k-spot* if R is the localisation of a finitely generated k-algebra C at a regular prime $\mathfrak{p} \in \operatorname{Spec}(C)$.

R.A. Rao, in his Ph. D. thesis, proved the following proposition.

Proposition 4.1.5 ([44, Proposition 1.3]). *Let R be a regular k-spot. Let Q be a quadratic $R[T_1, \cdots, T_n]$-space. Assume*

(i) Witt index $(\overline{Q}) > 1$*, where "bar" denotes "modulo $(T_1, \cdots, T_n)$".*

(ii) *$\overline{Q}$ is extended from k.*

Then Q is extended from R. In particular, if $\overline{Q}$ is hyperbolic, then Q itself is hyperbolic.

He also proved the following theorems.

Theorem 4.1.6 ([44, Theorem 1.1]). *Let A be a complete equicharacteristic regular local ring. Then every quadratic space Q over $A[T_1, \cdots, T_n]$ with* Witt index $(\overline{Q}) > 1$*, where "bar" denotes "modulo $(T_1, \cdots, T_n)$", is extended from A.*

Theorem 4.1.7 ([45, Theorem 3.3]). *Let $B = R[X]$, where R has dimension d. Let Q be a quadratic $R[X]$-space with hyperbolic rank $\geq d+1$. Then Q is cancellative.*

In the following proposition, the symbol $[a, b]$ denotes the quadratic space with quadratic form having its value matrix

$$\begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}.$$

Proposition 4.1.8 ([9, Proposition 3.4]). *Let A be a semilocal ring and (E, q) be a free quadratic space over A. Then E has an orthogonal decomposition*

$$E = [a_1, b_1] \perp \ldots \perp [a_n, b_n] \quad \textit{or}$$

$$E = [a_1, b_1] \perp \ldots \perp [a_n, b_n] \perp [c]$$

with $a_i, b_i \in A$ and $c, 1 - 4a_ib_i \in A^*(1 \leq i \leq n)$ according as $\dim E = 2n$ *or* $2n+1$*. If $2 \in A^*$, then E has an orthogonal basis. i.e.,*

$$E = [c_1] \perp \ldots \perp [c_m]$$

with $c_i \in A^*$ *for* $1 \leq i \leq m$*.*

The next theorem is a famous result due to M. Karoubi.

Theorem 4.1.9 ([36, Chapter VII, Theorem 2.1])**.** *Let* R *be a commutative ring in which* 2 *is invertible, and let* (P, B) *be an inner product space over* $R[T_1, \cdots, T_d]$*. If* P *is stably extended from* R*, then* (P, B) *is also stably extended from* R*.*

We now recall the famous Cohen's structure theorem.

Theorem 4.1.10 ([22, Theorem 15])**.** *A commutative regular local ring* $(R, \mathfrak{m})$ *of Krull dimension* d *is isomorphic to a formal power series ring* $k[[X]]$ *over a field if and only if* R *is equicharacteristic and is complete with respect to its* $\mathfrak{m}$*-adic topology.*

4.2 Extendability of Quadratic Modules

In this section, the principal result [Theorem 4.2.2] on the extendability of quadratic $A[T]$-spaces of Witt index $\geq d$ over an equicharacteristic regular local ring of dimension d is deduced from the local-global principle which we proved in Chapter 3.

The analysis of the equicharacteristic regular local ring is done by a patching argument, akin to the one developed by A. Roy in his article [49]. This argument reduces the problem to the case of a complete equicharacteristic regular ring; which is a power series ring over a field, provided one can patch the information. We show that the patching process is possible because of the local-global principle established for Roy's elementary group in Chapter 3.

We begin with the following crucial observation.

Lemma 4.2.1 ([42])**.** *Let* A *be a regular local ring containing a field. Let* $(Q, q) \perp H(A)$ *be a quadratic* $A[T]$*-space. If* $(Q/TQ) \perp H(A)$ *is hyperbolic, then* $(Q, q) \perp H(A)$ *is hyperbolic.*

Proof. In [42], D. Popescu showed that if A is a geometrically regular local ring (over a field k), or when the characteristic of the residue field is a regular parameter in A, then it is a filtered inductive limit of regular local rings essentially of finite type over the integers (or over k).

In view of this, we may regard $(Q, q) \perp H(A)$ to be a quadratic $B[T]$-space over some regular local ring B essentially of finite type over k with $(Q/TQ, q/(T)) \perp H(A)$ hyperbolic. In view of Proposition 4.1.5, $(Q, q) \perp H(A)$ is hyperbolic over $B[T]$, whence over $A[T]$. □

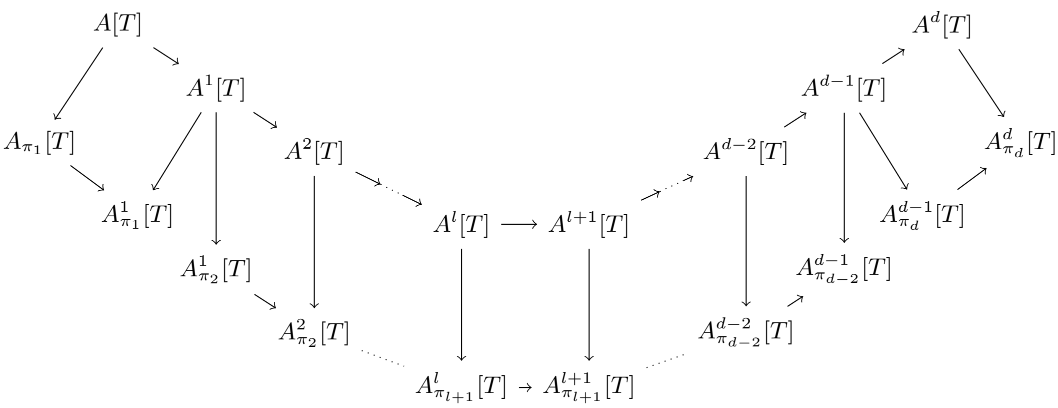
We now prove the extendability theorem.

Theorem 4.2.2. *Let* $(A, \mathfrak{m})$ *be an equicharacteristic regular local ring of dimension* d *and* $2 \in A^*$. *Then every quadratic* $A[T]$*-space* $(Q, q) \perp H(A)^n$ *with* $n \geq d$ *is extended from* A.

Proof. Let $\{\pi_1, \pi_2, \ldots, \pi_d\}$ be a regular system of parameters generating the maximal ideal $\mathfrak{m}$ of A.

Let A^l denote the $(\pi_1, \ldots, \pi_l)$-adic completion of A. We observe that A^d is isomorphic to the power series ring $k[[X_1, \ldots, X_d]]$ by Theorem 4.1.10, where k is the residue field $A/\mathfrak{m}$ of A. We also observe that A^l is the (π_l)-adic completion of A^{l-1}.

We now recall the following A. Roy's garland of patching diagrams in [49].



We now focus on the following patching square $\mathcal{P}_l(A)[T]$.

$$\begin{array}{ccc} A^l[T] & \longrightarrow & A^{l+1}[T] \\ \big\downarrow & & \big\downarrow \\ (A^l)_{\pi_{l+1}}[T] & \longrightarrow & (A^{l+1})_{\pi_{l+1}}[T] \end{array}$$

For all l, this is a cartesian square as rings. Moreover, by [37], it is also a cartesian square of quadratic spaces. This will enable us to analyze the quadratic A-space.

We prove the result by induction on $d-l$, starting with $l=0$. In this case A is a complete equicharacteristic regular local ring, whence a power series ring over its residue field. We now appeal to Theorem 4.1.6.

Now assume the result for $d-l=m$. For $d-(l+1)=m-1$, consider the patching square $\mathcal{P}_{m-1}(A)[T]$.

We fix some notations as follows:

For a regular parameter π of A, let $Q^l = Q\otimes A^l[T]$, $Q^0=Q$, ${Q^l}_\pi = Q\otimes {A^l}_\pi[T]$ and for a quadratic A-space Q_1, we denote $Q_1\otimes A^l$ by ${Q_1}^l$.

Let $(Q\perp H(A)^n)/(T) = Q_1\perp H(A)^n$, where Q_1 is the quadratic A-space $Q/(T)$. Since A^{m-1} is local, by Proposition 4.1.8, ${Q_1}^{m-1}$ is diagonalizable. Since A^{m-1} is regular, by Theorem 4.1.9, $(Q\perp H(A)^n)^{m-1}$ is stably extended from A^{m-1}. Let

$$(Q\perp H(A)^n)^{m-1}\perp H(A)^r \xrightarrow{\simeq} A^{m-1}[T]\otimes\left({Q_1}^{m-1}\perp H(A)^{n+r}\right) \quad \text{for } n\geq d.$$

Then

$$\left((Q\perp H(A)^n)^{m-1}\perp H(A)^r\right)_{\pi_m} \xrightarrow{\simeq} \left(\left(A^{m-1}\right)_{\pi_m}[T]\otimes\left(\left({Q_1}^{m-1}\right)_{\pi_m}\perp H(A)^{n+r}\right)\right) \quad \text{for } n\geq d.$$

By Theorem 4.1.7, we get the isomorphism

$$\left((Q\perp H(A)^n)^{m-1}\right)_{\pi_m} \xrightarrow{\sigma} \left(\left(A^{m-1}\right)_{\pi_m}[T]\otimes\left(\left({Q_1}^{m-1}\right)_{\pi_m}\perp H(A)^{n}\right)\right).$$

Using the extendability for quadratic spaces over $A^m[T]$ via induction hypothesis, we have

$$\tau : (Q\perp H(A)^n)^m \xrightarrow{\simeq} A^m[T]\otimes({Q_1}^m\perp H(A)^n).$$

Now, by identifying the quadratic spaces $\left(\left((Q\perp H(A)^n)^{m-1}\right)_{\pi_m}\bigotimes_{(A^{m-1})_{\pi_m}[T]}({A^m}_{\pi_m}[T])\right)$ and $\left((Q\perp H(A)^n)^{m-1}\bigotimes_{A^{m-1}[T]}A^m[T]\right)_{\pi_m}$ with $\left((Q\perp H(A)^n)^{m-1}\bigotimes_{A^{m-1}[T]}\left((A^m)_{\pi_m}[T]\right)\right)$, via the patching technique for quadratic spaces from [37], we have maps $\widetilde{\sigma}$, $\widetilde{\tau}$ corresponding to σ, τ and

$$\widetilde{\sigma}\widetilde{\tau}^{-1}\in \mathrm{O}_{(A^m)_{\pi_m}[T]}\left(((Q_1\perp H(A)^n)^m)_{\pi_m}\right).$$

Since $((A^m)_{\pi_m})_{\mathfrak{m}}$ is local, $\left({(Q_1)^m}_{\pi_m}\right)_{\mathfrak{m}}$ is diagonalizable and hence, by Theorem 3.5.1,

$$\mathrm{O}\left(\left((Q_1{}^m)_{\pi_m}\right)_{\mathfrak{m}} \perp H(A)^n\right) = \mathrm{EO}\left(\left((Q_1{}^m)_{\pi_m}\right)_{\mathfrak{m}} \perp H(A)^n\right) \cdot \mathrm{O}\left(H(A)^n\right).$$

Therefore we can write

$$\left(\widetilde{\sigma}\widetilde{\tau}^{-1}\right)_{\mathfrak{m}} = \alpha_{\mathfrak{m}}\beta_{\mathfrak{m}},$$

where $\alpha_{\mathfrak{m}} \in \mathrm{EO}_{((A^m)_{\pi_m})_{\mathfrak{m}}[T]}\left(\left((Q_1{}^m)_{\pi_m}\right)_{\mathfrak{m}} \perp H(A)^n\right)$ for some $\alpha \in \mathrm{O}_{(A^m)_{\pi_m}[T]}\left(\left((Q_1{}^m)_{\pi_m}\right) \perp H(A)^n\right)$ with $\alpha(0) = I$ and $\beta_{\mathfrak{m}} \in \mathrm{O}_{((A^m)_{\pi_m})_{\mathfrak{m}}[T]}\left(H(A)^n\right)$ for some $\beta \in \mathrm{O}_{(A^m)_{\pi_m}[T]}\left(H(A)^n\right)$ with $\beta(0) = I$, via the same argument as in Lemma 3.5.2.

Then, by Theorem 3.5.3, we have

$$\widetilde{\sigma}\widetilde{\tau}^{-1} = \alpha\beta$$

with $\alpha \in \mathrm{O}\left(\left((Q_1{}^m)_{\pi_m}\right) \perp H(A)^n\right)$, $\alpha(0) = I$, $\beta \in \mathrm{O}_{(A^m)_{\pi_m}[T]}\left(H(A)^n\right)$ and $\beta(0) = I$. Now via the 'deep splitting' technique introduced in [44] which we have described in Chapter 1, we can write $\widetilde{\sigma}\widetilde{\tau}^{-1} = \beta \in \mathrm{O}(H(A)^n)$.

We now have

$$\begin{aligned}
(Q \perp H(A)^n)^{m-1} &\simeq \left(\left((Q \perp H(A)^n)^{m-1}\right)_{\pi_m}, I, (Q \perp H(A)^n)^m\right) \\
&\simeq \left(\left(A^{m-1}\right)_{\pi_m}[T] \otimes \left(\left(Q_1{}^{m-1}\right)_{\pi_m} \perp H(A)^n\right), \alpha\beta, A^m[T] \otimes (Q_1{}^m \perp H(A)^n)\right) \\
&\simeq \left(Q_1{}^{m-1}{}_{\pi_m}[T] \perp H(A)^n, \beta, Q_1{}^m[T] \perp H(A)^n\right) \\
&\simeq Q_1{}^{m-1}[T] \perp (H(A)^n, \beta, H(A)^n) = Q_1{}^{m-1}[T] \perp Q_2,
\end{aligned}$$

where Q_2 is the quadratic $A^{m-1}[T]$-space defined by the patching technique. Now

$$Q_1{}^{m-1}[T] \perp Q_2 \perp H(A)^r \simeq Q^{m-1} \perp H(A)^r \simeq Q_1{}^{m-1}[T] \perp H(A)^{n+r}.$$

By cancellation of quadratic spaces over local rings (see Theorem 4.1.1), we have $Q_2 \perp H(A) \simeq H(A)^{n+1}$. Since $\beta(0) = I$, $Q_2/(T) \simeq H(A)^n$. Thus, by Lemma 4.2.1, Q_2 is extended from A^{m-1}, whence so is $(Q \perp H(A)^n)^{m-1}$. Hence the result is true for $l+1$. Then the theorem follows by induction. □

CHAPTER 5

Normality and Injective Stability

In 1960's, H. Bass initiated the study of the normal subgroup structure of linear groups. He introduced a new notion of dimension of rings, called stable rank, and proved that the principal structure theorems hold for groups whose degrees are large with respect to the stable rank. Later, J.S. Wilson, I.Z. Golubchik and A.A. Suslin made many other important contributions in this direction. In 1977, A.A. Suslin proved that over any commutative ring A, the group $\mathrm{E}_n(A)$ is normal in $\mathrm{GL}_n(A)$ when $n \geq 3$.

The normal subgroup structure of symplectic and classical unitary groups over rings were studied by V.I. Kopeĭko in [35], G. Taddei in [59] and by Suslin-Kopeĭko in [58]. Similar results were obtained for general quadratic groups by A. Bak, V. Petrov, and G. Tang in [14], for general Hermitian groups by G. Tang in [60] and A. Bak and G. Tang in [13], and for odd unitary groups by V. Petrov in [39] and W. Yu in [64].

The stability problem for K_1 of quadratic forms was studied in 1960's and in early 1970's by H. Bass, A. Bak, A. Roy, M. Kolster and L.N. Vaserstein. The stability theorems relate unitary groups and their elementary subgroups in different ranges. The stability results for quadratic K_1 are due to A. Bak, V. Petrov and G. Tang (see [14]), and for Hermitian K_1 are due to A. Bak and G. Tang (see [13]). Recently, in [64], W. Yu proved the K_1-stability for odd unitary groups which were introduced by V. Petrov. Stronger results for spaces over semilocal rings are due to A. Roy and M. Knebusch for quadratic spaces (see [32, 48]) and H. Reiter for Hermitian spaces (see [47]). In [52], S. Sinchuk proved injective stability for unitary K_1 under stable range condition. We adapt the method used by him for proving injective stability.

In this chapter, we establish normality results for DSER group and stability results for DSER group under Bak's Λ-stable range condition. We also prove the injective stability for K_1 of the orthogonal group under stable range condition. A useful tool in the proof is a decomposition theorem for the elementary subgroup that we will establish on the way. For proving stability, we adapt the method used in [13,14,52]. We also need some commutator relations which are proved in Chapter 2.

Let A be a commutative ring with identity in which 2 is invertible. Let $E_{\alpha_{ij}}$ and $E^*_{\beta_{ij}}$ be defined as in Chapter 1. Also, let $\mathrm{O}_A(Q \perp H(P))$ and $\mathrm{EO}_A(Q \perp H(P))$ be as defined in Chapter 1. Throughout this chapter, we assume that Q and P are free A-modules of rank n and m respectively.

Most of the results in this chapter are from [3].

5.1 Main Theorems

In this chapter, we prove the following **normality theorems**.

(i) $\mathrm{O}_A(Q \perp H(A)^{m-1})$ *normalizes* $\mathrm{EO}_A(Q \perp H(A)^m)$. *In particular,* EO_A *is a normal subgroup of* O_A.

(ii) *If* $m \geq \dim \mathrm{Max}\,(A) + 2$, *then* $\mathrm{O}_A(Q \perp H(A)^m)$ *normalizes* $\mathrm{EO}_A(Q \perp H(A)^m)$.

(iii) *If* $m > l$, *then* $\mathrm{O}_A(Q \perp H(A)^m)$ *normalizes* $\mathrm{EO}_A(Q \perp H(A)^m)$ *provided* A *satisfies the stable range condition* 0-SA_l.

Using normality theorem and a decomposition theorem, we establish the following stability theorem for KO_1.

Suppose A satisfies the stable range condition 0-SA_l. Then, for all $m \geq l+1$, the coset space $KO_{1,m}(Q \perp H(A)^m)$ is a group. Further, the canonical map

$$KO_{1,r}(Q \perp H(A)^r) \longrightarrow KO_{1,m}(Q \perp H(A)^m)$$

is surjective for $l \leq r < m$, and when $m \geq l+2$, the canonical homomorphism

$$KO_{1,m-1}(Q \perp H(A)^{m-1}) \longrightarrow KO_{1,m}(Q \perp H(A)^m)$$

is an isomorphism.

Using the decomposition theorem, we prove the following injective stability theorem for KO_1 under the usual stable range condition.

Let A be a commutative ring of stable rank l in which 2 is invertible and let $m \geq l+2$. Then the canonical map

$$KO_{1,m-1}(Q \perp H(A)^{m-1}) \longrightarrow KO_{1,m}(Q \perp H(A)^m)$$

is injective.

A key tool used in the proofs of the above theorems is a decomposition theorem for the elementary orthogonal group $\mathrm{EO}_A(Q \perp H(A)^m)$. This decomposition involves the following subgroups.

$$\begin{aligned}
C_m &= \left\langle [E_{\alpha_{mj}}, E^*_{\beta_{ik}}], [E_{\alpha_{ij}}, E_{\delta_{mk}}], E_{\alpha_{mj}} : 1 \leq i < m, 1 \leq j,k \leq n \right\rangle, \\
F_m &= \left\{ \eta\eta_1 : \eta \in \mathrm{EO}_A(Q \perp H(A)^{m-1}) \text{ and } \eta_1 \in C_m \right\}, \\
G_m &= \left\langle E_{\alpha_{jk}}, [E_{\alpha_{ir}}, E^*_{\beta_{jk}}], [E_{\alpha_{ir}}, E_{\delta_{jk}}] : 1 \leq i,j \leq m, 1 \leq r,k \leq n \right\rangle, \\
G^-_m &= \left\langle E^*_{\beta_{jk}}, [E^*_{\beta_{jk}}, E_{\alpha_{ir}}], [E^*_{\beta_{ir}}, E^*_{\gamma_{jk}}] : 1 \leq i,j \leq m, 1 \leq r,k \leq n \right\rangle, \\
L_m &= G_m \cap G^-_m = \left\langle [E_{\alpha_{ir}}, E^*_{\beta_{jk}}] : 1 \leq i,j \leq m, 1 \leq r,k \leq n \right\rangle, \\
U_m &= \left\langle E_{\alpha_{ir}}, [E_{\alpha_{ir}}, E_{\delta_{jk}}] : 1 \leq i,j \leq m, 1 \leq r,k \leq n \right\rangle, \\
U^-_m &= \left\langle E^*_{\beta_{ir}}, [E^*_{\beta_{ir}}, E^*_{\gamma_{jk}}] : 1 \leq i,j \leq m, 1 \leq r,k \leq n \right\rangle, \\
Y^+ &= \left\langle E_{\alpha_{mk}}, E_{\alpha_{m-1,k}}, [E_{\alpha_{m-1,k}}, E_{\delta_{m,j}}] : 1 \leq j,k \leq n \right\rangle \leq U_m, \\
Y^- &= \left\langle E^*_{\beta_{mk}}, E^*_{\beta_{m-1,k}}, [E^*_{\beta_{m-1,k}}, E^*_{\gamma_{m,j}}] : 1 \leq j,k \leq n \right\rangle \leq U^-_m, \\
V^+ &= \left\langle [E_{\alpha_{ik}}, E^*_{\beta_{mj}}], [E_{\alpha_{ik}}, E^*_{\beta_{m-1,j}}] : 1 \leq i \leq m-2, 1 \leq j,k \leq n \right\rangle, \\
V^- &= \left\langle [E_{\alpha_{mk}}, E^*_{\beta_{ij}}], [E_{\alpha_{m-1,k}}, E^*_{\beta_{ij}}] : 1 \leq i \leq m-2, 1 \leq j,k \leq n \right\rangle, \\
U^+ &= U_m \rtimes V^+, \quad U^- = U^-_m \rtimes V^-, \quad G_m = U_m \rtimes L_m.
\end{aligned}$$

Definition 5.1.1. Let $\theta \in \mathrm{EO}_A(Q \perp H(A)^m)$, where Q has rank n. An $G_m U^-_m F_m$-decomposition of θ is a product decomposition $\theta = \eta\xi\mu$, where $\eta \in G_m, \xi \in U^-_m$ and $\mu \in F_m$.

The **decomposition theorem** for $\mathrm{EO}_A(Q \perp H(A)^m)$ that we prove in this chapter is: *Let A satisfies the stable range condition SA_l and let $m \geq l+2$. Then every element of* $\mathrm{EO}_A(Q \perp H(A)^m)$ *has a $G_m U_m^- F_m$-decomposition.*

5.2 Roy's Elementary Group is Normalized by a Smaller Orthogonal Group

In this section, we prove that the orthogonal group $\mathrm{O}_A(Q \perp H(A)^{m-1})$ normalizes the elementary orthogonal group $\mathrm{EO}_A(Q \perp H(A)^m)$.

Now, by 3.1.4, each E_α, E_β^* for $\alpha \in \mathrm{Hom}_A(Q,P)$ and $\beta \in \mathrm{Hom}_A(Q,P^*)$ can be written as a product of $E_{\alpha_{ij}}, E^*_{\beta_{ij}}, 1 \leq i \leq m, 1 \leq j \leq n$. Hence we can consider $\mathrm{EO}_A(Q \perp H(P))$ as the group generated by $E_{\alpha_{ij}}$'s and $E^*_{\beta_{ij}}$'s for $\alpha \in \mathrm{Hom}(Q,P)$ and $\beta \in \mathrm{Hom}_A(Q,P^*)$.

Now, by the commutator relations which we proved in Chapter 2, we note the following useful interpretation.

Lemma 5.2.1. *The elementary orthogonal group $\mathrm{EO}_A(Q \perp H(A)^m)$ is generated by the elements of the type $E_{\alpha_{ij}}, E^*_{\beta_{kl}}, \left[E_{\alpha_{ij}}, E_{\delta_{kl}}\right], \left[E_{\alpha_{ij}}, E^*_{\beta_{kl}}\right], \left[E^*_{\gamma_{ij}}, E^*_{\beta_{kl}}\right]$ for $\alpha \in \mathrm{Hom}_A(Q,P)$, $\beta \in \mathrm{Hom}_A(Q,P^*)$ and for i,j,k,l with $1 \leq i,k \leq m, 1 \leq j,l \leq n$ and $i \neq k$.*

Towards the proof of the normality theorem, we first recall some of the commutator relations that we proved in Chapter 2 (Lemma 2.2.5, 2.2.1, 2.3.1, 2.3.2, 2.3.3).

Lemma 5.2.2. *Let $\alpha, \delta, \xi \in \mathrm{Hom}_A(Q,P)$ and $\beta, \gamma, \mu \in \mathrm{Hom}_A(Q,P^*)$. Then, for any given i,j,k,l such that $1 \leq i,j,t \leq m$ and $1 \leq k,l,r,s \leq n$, we have the following commutator relations.*

(i) $\left[E^*_{\beta_{ik}}, \left[E_{\alpha_{ir}}, E^*_{\gamma_{jl}}\right]\right] = E^*_{\eta_{jk}} \left[E^*_{\nu_{jk}}, E^*_{\zeta_{ik}}\right]$, *where* $\eta_{jk} = -\gamma_{jl}\alpha_{ir}\beta^*_{ik}$, $\nu_{jk} = -\frac{1}{2}\gamma_{jl}\alpha_{ir}\beta^*_{ik}$, $\zeta_{ik} = -\beta_{ik}$ *and* $i \neq j$.

(ii) $\left[E^*_{\beta_{ik}}, \left[E_{\alpha_{ir}}, E_{\delta_{jl}}\right]\right] = E_{\lambda_{jk}} \left[E_{\xi_{jk}}, E^*_{\zeta_{ik}}\right]$, *where* $\lambda_{jk} = \delta_{jl}\alpha^*_{ir}\beta_{ik}, \xi_{jk} = \frac{1}{2}\delta_{jl}\alpha^*_{ir}\beta_{ik}$, $\zeta_{ik} = \beta_{ik}$ *and* $i \neq j$.

(iii) $\left[\left[E^*_{\beta_{ir}}, E^*_{\gamma_{jl}}\right], \left[E_{\alpha_{js}}, E^*_{\mu_{tk}}\right]\right] = \left[E^*_{\zeta_{il}}, E^*_{\nu_{ts}}\right]$, *where* $\zeta_{il} = -\beta_{ir}{\gamma_{jl}}^*$, $\nu_{ts} = \mu_{tk}{\alpha_{js}}^*$ *and for* i,j,t *distinct.*

(iv) $\Big[[E_{\alpha_{ir}}, E_{\delta_{jl}}], \Big[E_{\xi_{tk}}, E^*_{\beta_{js}}\Big]\Big] = [E_{\lambda_{il}}, E_{\eta_{ts}}]$, *where* $\lambda_{il} = \alpha_{ir}{\delta_{jl}}^*$, $\eta_{ts} = \xi_{tk}{\beta_{js}}^*$ *and for* i, j, t *distinct.*

(v) $\Big[\Big[E_{\alpha_{ir}}, E^*_{\beta_{jl}}\Big], [E_{\delta_{js}}, E^*_{\gamma_{tk}}]\Big] = \Big[E_{\eta_{il}}, E^*_{\mu_{ts}}\Big]$, *where* $\eta_{il} = -\alpha_{ir}{\beta_{jl}}^*$, $\mu_{ts} = \gamma_{tk}{\delta_{js}}^*$ *and for* i, j, t *distinct.*

In particular, we have the following commutator relations.

(i) $E^*_{\mu_{kj}} = \Big[E^*_{\beta_{mj}}, [E_{\alpha_{mr}}, E^*_{\gamma_{kl}}]\Big]\Big[E^*_{\nu_{kj}}, E^*_{\zeta_{mj}}\Big]^{-1}$,

(ii) $E_{\lambda_{kj}} = \Big[E^*_{\beta_{mj}}, [E_{\alpha_{mr}}, E_{\delta_{kl}}]\Big]\Big[E_{\xi_{kj}}, E^*_{\zeta_{mj}}\Big]^{-1}$,

(iii) $[E^*_{\zeta_{il}}, E^*_{\nu_{ks}}] = \big[[E^*_{\beta_{ir}}, E^*_{\gamma_{ml}}], [E_{\alpha_{ms}}, E^*_{\mu_{kt}}]\big]$,

(iv) $[E_{\lambda_{il}}, E_{\eta_{ks}}] = \Big[[E_{\alpha_{ir}}, E_{\delta_{ml}}], \Big[E_{\xi_{kt}}, E^*_{\beta_{js}}\Big]\Big]$,

(v) $[E_{\eta_{il}}, E^*_{\mu_{ks}}] = \big[[E_{\alpha_{ir}}, E^*_{\beta_{ml}}], [E_{\delta_{ms}}, E^*_{\gamma_{kt}}]\big]$.

Lemma 5.2.3. *The elementary orthogonal group* $\mathrm{EO}_A(Q \perp H(A)^m)$ *is generated by those elementary generators having* m *as one of the subscripts.*

Proof. The commutator relations in Lemma 5.2.2 show that the group $\mathrm{EO}_A(Q \perp H(A)^m)$ is generated by the elements of type $E_{\alpha_{mj}}, E^*_{\beta_{mk}}, [E_{\alpha_{ij}}, E^*_{\beta_{mk}}]$, $[E_{\alpha_{mj}}, E^*_{\beta_{ik}}], [E_{\alpha_{mj}}, E_{\delta_{il}}]$ and $[E^*_{\beta_{ij}}, E^*_{\gamma_{mk}}]$ when Q. □

As a consequence of Lemma 5.2.3, it follows that the groups U_m^- and C_m generate the elementary group $\mathrm{EO}_A(Q \perp H(A)^m)$.

We now state the main normality result of this section.

Theorem 5.2.4. $\mathrm{O}_A(Q \perp H(A)^{m-1})$ *normalizes* $\mathrm{EO}_A(Q \perp H(A)^m)$.

Proof. For proving this, it is sufficient to prove that U_m^- and C_m are normalized by $\mathrm{O}_A(Q \perp H(A)^{m-1})$, and we do this by direct matrix calculation.

We consider the matrix representation of elements of $\mathrm{O}_A(Q \perp H(A)^m)$.

Let $T = \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{d} & \mathbf{e} & \mathbf{f} \\ \mathbf{g} & \mathbf{h} & \mathbf{j} \end{pmatrix} \in \mathrm{O}_A(Q \perp H(A)^m)$. Then

$$T^t \Psi T = \Psi, \tag{5.2.1}$$

where $\Psi = \varphi \perp \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$ is the matrix of the bilinear form associated to the quadratic form on $Q \perp H(A)^m$. Here, φ denotes the matrix corresponding to the nondegenerate bilinear form on Q and $\begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$ is the matrix of the bilinear form on the hyperbolic space. This equation is equivalent to the following set of equations.

$$\begin{array}{lll} \mathbf{a}^t\varphi\mathbf{a} + \mathbf{g}^t\mathbf{d} + \mathbf{d}^t\mathbf{g} = \varphi & \mathbf{b}^t\varphi\mathbf{a} + \mathbf{h}^t\mathbf{d} + \mathbf{e}^t\mathbf{g} = 0 & \mathbf{c}^t\varphi\mathbf{a} + \mathbf{j}^t\mathbf{d} + \mathbf{f}^t\mathbf{g} = 0 \\ \mathbf{a}^t\varphi\mathbf{b} + \mathbf{g}^t\mathbf{e} + \mathbf{d}^t\mathbf{h} = 0 & \mathbf{b}^t\varphi\mathbf{b} + \mathbf{h}^t\mathbf{e} + \mathbf{e}^t\mathbf{h} = 0 & \mathbf{c}^t\varphi\mathbf{b} + \mathbf{j}^t\mathbf{e} + \mathbf{f}^t\mathbf{h} = I_m \\ \mathbf{a}^t\varphi\mathbf{c} + \mathbf{g}^t\mathbf{f} + \mathbf{d}^t\mathbf{j} = 0 & \mathbf{b}^t\varphi\mathbf{c} + \mathbf{h}^t\mathbf{f} + \mathbf{e}^t\mathbf{j} = I_m & \mathbf{c}^t\varphi\mathbf{c} + \mathbf{j}^t\mathbf{f} + \mathbf{f}^t\mathbf{j} = 0 \end{array}$$

These equations are equivalent to the equation

$$T^{-1} = \begin{pmatrix} \varphi^{-1}\mathbf{a}^t\varphi & \varphi^{-1}\mathbf{g}^t & \varphi^{-1}\mathbf{d}^t \\ \mathbf{c}^t\varphi & \mathbf{j}^t & \mathbf{f}^t \\ \mathbf{b}^t\varphi & \mathbf{h}^t & \mathbf{e}^t \end{pmatrix}.$$

The stabilization homomorphism $\mathrm{O}_A(Q \perp H(A)^{(m-1)}) \to O_A(Q \perp H(A)^m)$ is given by

$$\begin{pmatrix} \mathbf{a}' & \mathbf{b}' & \mathbf{c}' \\ \mathbf{d}' & \mathbf{e}' & \mathbf{f}' \\ \mathbf{g}' & \mathbf{h}' & \mathbf{j}' \end{pmatrix} \mapsto \left(\begin{array}{c|cc|cc} \mathbf{a}' & \mathbf{b}' & 0 & \mathbf{c}' & 0 \\ \hline \mathbf{d}' & \mathbf{e}' & 0 & \mathbf{f}' & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline \mathbf{g}' & \mathbf{h}' & 0 & \mathbf{j}' & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) = \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{d} & \mathbf{e} & \mathbf{f} \\ \mathbf{g} & \mathbf{h} & \mathbf{j} \end{pmatrix} = T. \tag{5.2.2}$$

We now consider the generators for the subgroups U_m^- and C_m of $\mathrm{EO}_A(Q \perp H(A)^m)$ and prove that they are normalized by an element in $\mathrm{O}_A(Q \perp H(A)^{m-1})$.

Consider $T \in \mathrm{O}_A(Q \perp H(A)^{m-1})$ as an element in $\mathrm{O}_A(Q \perp H(A)^m)$ by the stabilization homomorphism. Then we conjugate the elementary generators of $\mathrm{EO}_A(Q \perp H(A)^m)$ and write the conjugated element as a product of elementary generators. Corresponding to the elementary generator $E_{\alpha_{mj}}$, we have

$$T^{-1}E_{\alpha_{mj}}T = \begin{pmatrix} I_n & 0 & -\phi^{-1}\mathbf{a}^t{\alpha_{mj}}^t\mathbf{j} \\ \mathbf{j}^t\alpha_{mj}\mathbf{a} & I_m + \mathbf{j}^t\alpha_{mj}\mathbf{b} & \mathbf{j}^t\alpha_{mj}\mathbf{c} - \mathbf{c}^t{\alpha_{mj}}^t\mathbf{j} - \frac{1}{2}\mathbf{j}^t\alpha_{mj}{\alpha_{mj}}^*\mathbf{j} \\ 0 & 0 & I_m - \mathbf{b}^t{\alpha_{mj}}^t\mathbf{j} \end{pmatrix}$$

$$= \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_m & \frac{1}{2}\mathbf{j}^t\alpha_{mj}\mathbf{c}\mathbf{b}^t{\alpha_{mj}}^t\mathbf{j} - \frac{1}{2}\mathbf{j}^t\alpha_{mj}\mathbf{b}\mathbf{c}^t{\alpha_{mj}}^t\mathbf{j} \\ 0 & 0 & I_m \end{pmatrix}$$

$$\begin{pmatrix} I_n & 0 & 0 \\ 0 & I_m & \mathbf{j}^t\alpha_{mj}\mathbf{c} - \mathbf{c}^t{\alpha_{mj}}^t\mathbf{j} \\ 0 & 0 & I_m \end{pmatrix} \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_m + \mathbf{j}^t\alpha_{mj}\mathbf{b} & 0 \\ 0 & 0 & I_m - \mathbf{b}^t{\alpha_{mj}}^t\mathbf{j} \end{pmatrix}$$

$$\begin{pmatrix} I_n & 0 & -\phi^{-1}\mathbf{a}^t{\alpha_{mj}}^t\mathbf{j} \\ \mathbf{j}^t\alpha_{mj}\mathbf{a} & I_m & -\frac{1}{2}\mathbf{j}^t\alpha_{mj}\mathbf{a}\phi^{-1}\mathbf{a}^t{\alpha_{mj}}^t\mathbf{j} \\ 0 & 0 & I_m \end{pmatrix}$$

$$= [E_{\mathbf{j}^t\alpha_{mj}\mathbf{b}\mathbf{c}^t\phi}, E_{\frac{\mathbf{j}^t\alpha_{mj}}{2}}][E_{\mathbf{c}^t\phi}, E_{\mathbf{j}^t\alpha_{mj}}][E^*_{\mathbf{b}^t\phi}, E_{\mathbf{j}^t\alpha_{mj}}]E_{\mathbf{j}^t\alpha_{mj}\mathbf{a}}.$$

Corresponding to the elementary generator $E^*_{\beta_{mj}}$, we have

$$T^{-1}E^*_{\beta_{mj}}T = \begin{pmatrix} I_n & -\phi^{-1}\mathbf{a}^t\beta^t_{mj}\mathbf{e} & 0 \\ 0 & I_m - \mathbf{c}^t\beta^t_{mj}\mathbf{e} & 0 \\ \mathbf{e}^t\beta_{mj}\mathbf{a} & \mathbf{e}^t\beta_{mj}\mathbf{b} - \mathbf{b}^t\beta^t_{mj}\mathbf{e} - \frac{1}{2}\mathbf{e}^t\beta_{mj}\beta^*_{mj}\mathbf{e} & I_m + \mathbf{e}^t\beta_{mj}\mathbf{c} \end{pmatrix}$$

$$= \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_m & 0 \\ 0 & \frac{1}{2}\mathbf{e}^t\beta_{mj}\mathbf{c}\mathbf{b}^t{\beta_{mj}}^t\mathbf{e} - \frac{1}{2}\mathbf{e}^t\beta_{mj}\mathbf{b}\mathbf{c}^t\beta^t_{mj}\mathbf{e} & I_m \end{pmatrix} \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_m & 0 \\ 0 & \mathbf{e}^t\beta_{mj}\mathbf{b} - \mathbf{b}^t\beta^t_{mj}\mathbf{e} & I_m \end{pmatrix}$$

$$\begin{pmatrix} I_n & 0 & 0 \\ 0 & I_m - \mathbf{c}^t\beta^t_{mj}\mathbf{e} & 0 \\ 0 & 0 & I_m + \mathbf{e}^t\beta_{mj}\mathbf{c} \end{pmatrix} \begin{pmatrix} I_n & -\phi^{-1}\mathbf{a}^t\beta^t_{mj}\mathbf{e} & 0 \\ 0 & I_m & 0 \\ \mathbf{e}^t\beta_{mj}\mathbf{a} & -\frac{1}{2}\mathbf{e}^t\beta_{mj}\beta^*_{mj}\mathbf{e} & I_m \end{pmatrix}$$

$$= \left[E^*_{\mathbf{e}^t\beta_{mj}\mathbf{c}\mathbf{b}^t\phi}, E^*_{\frac{\mathbf{e}^t\beta_{mj}}{2}}\right]\left[E^*_{\mathbf{b}^t\phi}, E^*_{\mathbf{e}^t\beta_{mj}}\right]\left[E_{\mathbf{c}^t\phi}, E^*_{\mathbf{e}^t\beta_{mj}}\right]E_{(\mathbf{e}^t\beta_{mj}\mathbf{a})}.$$

Corresponding to the elementary generator $[E_{\alpha_{mj}}, E^*_{\beta_{kl}}]$, we have

$$T^{-1}[E_{\alpha_{mj}}, E^*_{\beta_{kl}}]T = \begin{pmatrix} I_n & 0 & \phi^{-1}(\mathbf{d}^t\beta_{kl}\alpha^*_{mj}\mathbf{j})^t \\ -\mathbf{j}^t\alpha_{mj}\phi^{-1}\beta^t_{kl}\mathbf{d} & I_m - \mathbf{j}^t\alpha_{mj}\beta^*_{kl}\mathbf{e} & \mathbf{f}^t\beta_{kl}\phi^{-1}\alpha^t_{mj}\mathbf{j} - \mathbf{j}^t\alpha_{mj}\phi^{-1}\beta^t_{kl}\mathbf{f} \\ 0 & 0 & I_m + \mathbf{e}^t\beta_{kl}\alpha^*_{mj}\mathbf{j} \end{pmatrix}$$

$$= \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_m & \mathbf{j}^t\alpha_{mj}\phi^{-1}{\beta_{kl}}^t\left(\frac{\mathbf{f}\mathbf{e}^t - \mathbf{e}\mathbf{f}^t}{2}\right)\beta_{kl}\phi^1\alpha^t_{mj}\mathbf{j} \\ 0 & 0 & I_m \end{pmatrix}$$

$$\begin{pmatrix} I_n & 0 & 0 \\ 0 & I_m & \mathbf{f}^t\beta_{kl}\phi^{-1}\alpha^t_{mj}\mathbf{j} - \mathbf{j}^t\alpha_{mj}\phi^{-1}\beta^t_{kl}\mathbf{f} \\ 0 & 0 & I_m \end{pmatrix}$$

$$\begin{pmatrix} I_n & 0 & 0 \\ 0 & I_m - \mathbf{j}^t\alpha_{mj}\beta^*_{kl}\mathbf{e} & 0 \\ 0 & 0 & I_m + \mathbf{e}^t\beta_{kl}\alpha^*_{mj}\mathbf{j} \end{pmatrix}$$

$$\begin{pmatrix} I_n & 0 & \phi^{-1}(\mathbf{d}^t\beta_{kl}\alpha^*_{mj}\mathbf{j})^t \\ -\mathbf{j}^t\alpha_{mj}\phi^{-1}\beta^t_{kl}\mathbf{d} & I_m & -\frac{1}{2}\mathbf{j}^t\alpha_{mj}\phi^{-1}\beta^t_{kl}\mathbf{d}\phi^{-1}(\mathbf{d}^t\beta_{kl}\alpha^*_{mj}\mathbf{j})^t \\ 0 & 0 & I_m \end{pmatrix}$$

$$= \left[E_{(\frac{\mathbf{j}^t\alpha_{mj}}{2})}, E_{(\mathbf{j}^t\alpha_{mj}\phi^{-1}{\beta_{kl}}^t\mathbf{f}\mathbf{e}^t\beta_{kl})}\right]\left[E_{(\mathbf{j}^t\alpha_{mj})}, E_{(\mathbf{f}^t\beta_{kl})}\right]$$

$$\left[E_{(\mathbf{j}^t\alpha_{mj})}, E^*_{(\mathbf{e}^t\beta_{kl})}\right]E_{-(\mathbf{j}^t\alpha_{mj}\phi^{-1}\beta^t_{kl}\mathbf{d})}.$$

Corresponding to the elementary generator $[E_{\alpha_{ij}}, E^*_{\beta_{mk}}]$, we have

$$T^{-1}[E_{\alpha_{ij}}, E^*_{\beta_{mk}}]T = \begin{pmatrix} I_n & -\phi^{-1}(\mathbf{e}^t\beta_{mk}\alpha^*_{ij}\mathbf{g})^t & 0 \\ 0 & I_m - \mathbf{j}^t\alpha_{ij}\beta^*_{mk}\mathbf{e} & 0 \\ \mathbf{e}^t\beta_{mk}\alpha^*_{ij}\mathbf{g} & \mathbf{e}^t\beta_{mk}\alpha^*_{ij}\mathbf{h} - \mathbf{h}^t\alpha_{ij}\beta^*_{mk}\mathbf{e} & I_m + \mathbf{e}^t\beta_{mk}\alpha^*_{ij}\mathbf{j} \end{pmatrix}$$

$$= \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_m & 0 \\ 0 & \frac{1}{2}\mathbf{e}^t\beta_{mk}\phi^{-1}\alpha_{ij}^t\mathbf{h}\mathbf{j}^t\alpha_{ij}\phi^{-1}\beta_{mk}^t\mathbf{e} - \frac{1}{2}\mathbf{e}^t\beta_{mk}\phi^{-1}\alpha_{ij}^t\mathbf{j}\mathbf{h}^t\alpha_{ij}\phi^{-1}\beta_{mk}^t\mathbf{e} & I_m \end{pmatrix}$$

$$\begin{pmatrix} I_n & 0 & 0 \\ 0 & I_m & 0 \\ 0 & \mathbf{e}^t\beta_{mk}\alpha_{ij}^*\mathbf{h} - \mathbf{h}^t\alpha_{ij}\beta_{mk}^*\mathbf{e} & I_m \end{pmatrix} \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_m - \mathbf{j}^t\alpha_{ij}\beta_{mk}^*\mathbf{e} & 0 \\ 0 & 0 & I_m + \mathbf{e}^t\beta_{mk}\alpha_{ij}^*\mathbf{j} \end{pmatrix}$$

$$\begin{pmatrix} I_n & -\phi^{-1}(\mathbf{e}^t\beta_{mk}\alpha_{ij}^*\mathbf{g})^t & 0 \\ 0 & I_m & 0 \\ \mathbf{e}^t\beta_{mk}\alpha_{ij}^*\mathbf{g} & -\frac{1}{2}\mathbf{e}^t\beta_{mk}\alpha_{ij}^*\mathbf{g}\phi^{-1}(\mathbf{e}^t\beta_{mk}\alpha_{ij}^*\mathbf{g})^t & I_m \end{pmatrix}$$

$$= \left[E^*_{(\mathbf{e}^t\beta_{mk}\phi^{-1}\alpha_{ij}^t\mathbf{j}\mathbf{h}^t\alpha_{ij})}, E^*_{(\frac{\mathbf{e}^t\beta_{mk}}{2})}\right] \left[E^*_{(\mathbf{h}^t\alpha_{ij})}, E^*_{(\mathbf{e}^t\beta_{mk})}\right]$$

$$\left[E_{(\mathbf{j}^t\alpha_{ij})}, E^*_{(\mathbf{e}^t\beta_{mk})}\right] E^*_{(\mathbf{e}^t\beta_{mk}\phi^{-1}\alpha_{ij}^t\mathbf{g})}.$$

Corresponding to the elementary generator $[E_{\alpha_{mk}}, E_{\delta_{jl}}]$, we have

$$T^{-1}[E_{\alpha_{mk}}, E_{\delta_{jl}}]T = \begin{pmatrix} I_n & 0 & \phi^{-1}\mathbf{g}^t\delta_{jl}\alpha_{mk}^*\mathbf{j} \\ -\mathbf{j}^t\alpha_{mk}\delta_{jl}^*\mathbf{g} & I_m - \mathbf{j}^t\alpha_{mk}\delta_{jl}^*\mathbf{h} & \mathbf{j}^t(\delta_{jl}\alpha_{mk}^* - \alpha_{mk}\delta_{jl}^*)\mathbf{j} \\ 0 & 0 & I_m + \mathbf{h}^t\delta_{jl}\alpha_{mk}^*\mathbf{j} \end{pmatrix}$$

$$= \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_m & \frac{1}{2}\mathbf{j}^t\alpha_{mk}\delta_{jl}^*\mathbf{h}\mathbf{j}^t\delta_{jl}\alpha_{mk}^*\mathbf{j} - \frac{1}{2}\mathbf{j}^t\alpha_{mk}\delta_{jl}^*\mathbf{j}\mathbf{h}^t\delta_{jl}\alpha_{mk}^*\mathbf{j} \\ 0 & 0 & I_m \end{pmatrix}$$

$$\begin{pmatrix} I_n & 0 & 0 \\ 0 & I_m - \mathbf{j}^t\alpha_{mk}\delta_{jl}^*\mathbf{h} & 0 \\ 0 & 0 & I_m + \mathbf{h}^t\delta_{jl}\alpha_{mk}^*\mathbf{j} \end{pmatrix} \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_m & \mathbf{j}^t(\delta_{jl}\alpha_{mk}^* - \alpha_{mk}\delta_{jl}^*)\mathbf{j} \\ 0 & 0 & I_m \end{pmatrix}$$

$$\begin{pmatrix} I_n & 0 & \phi^{-1}\mathbf{g}^t\delta_{jl}\alpha_{mk}^*\mathbf{j} \\ -\mathbf{j}^t\alpha_{mk}\delta_{jl}^*\mathbf{g} & I_m & -\frac{1}{2}\mathbf{j}^t\alpha_{mk}\delta_{jl}^*\mathbf{g}\phi^{-1}\mathbf{g}^t\delta_{jl}\alpha_{mk}^*\mathbf{j} \\ 0 & 0 & I_m \end{pmatrix}$$

$$= \left[E_{(\frac{1}{2}\mathbf{j}^t\alpha_{mk})}, E_{(\mathbf{j}^t\alpha_{mk}\delta_{jl}^*\mathbf{h}\mathbf{j}^t\delta_{jl})}\right]\left[E_{(\mathbf{j}^t\alpha_{mk})}, E^*_{(\mathbf{h}^t\delta_{jl})}\right]\left[E_{(\alpha_{mk})}, E_{(\mathbf{j}^t\delta_{jl})}\right] E_{(-\mathbf{j}^t\alpha_{mk}\delta_{jl}^*\mathbf{g})}.$$

Corresponding to the elementary generator $[E^*_{\beta_{mk}}, E^*_{\gamma_{jl}}]$, we have

$$T^{-1}[E^*_{\beta_{mk}}, E^*_{\gamma_{jl}}]T = \begin{pmatrix} I_n & \phi^{-1}\mathbf{d}^t\gamma_{jl}\phi^{-1}\beta_{mk}^t\mathbf{e} & 0 \\ 0 & I_m + \mathbf{f}^t\gamma_{jl}\phi^{-1}{\beta_{mk}}^t\mathbf{e} & 0 \\ -\mathbf{e}^t\beta_{mk}\phi^{-1}\gamma_{jl}^t\mathbf{d} & \mathbf{e}^t(\gamma_{jl}\beta_{mk}^* - \beta_{mk}\gamma_{jl}^*)\mathbf{e} & I_m - \mathbf{e}^t\beta_{mj}\phi^{-1}\gamma_{jl}^t\mathbf{f} \end{pmatrix}$$

$$= \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_m & 0 \\ 0 & \mathbf{e}^t\beta_{mk}{\gamma_{jl}}^*(\frac{\mathbf{e}\mathbf{f}^t - \mathbf{f}\mathbf{e}^t}{2})\gamma_{jl}{\beta_{mk}}^*\mathbf{e} & I_m \end{pmatrix}$$

$$\begin{pmatrix} I_n & 0 & 0 \\ 0 & I_m & 0 \\ 0 & \mathbf{e}^t(\gamma_{jl}\beta_{mk}^* - \beta_{mk}\gamma_{jl}^*)\mathbf{e} & I_m - \mathbf{e}^t\beta_{mj}\phi^{-1}\gamma_{jl}^t\mathbf{f} \end{pmatrix}$$

$$\begin{pmatrix} I_n & 0 & 0 \\ 0 & I_m + \mathbf{f}^t\gamma_{jl}\phi^{-1}{\beta_{mk}}^t\mathbf{e} & 0 \\ 0 & 0 & I_m - \mathbf{e}^t\beta_{mj}\phi^{-1}\gamma_{jl}^t\mathbf{f} \end{pmatrix}$$

$$\begin{pmatrix} I_n & \phi^{-1}\mathbf{d}^t\gamma_{jl}\phi^{-1}\beta_{mk}^t\mathbf{e} & 0 \\ 0 & I_m & 0 \\ -\mathbf{e}^t\beta_{mk}\phi^{-1}\gamma_{jl}^t\mathbf{d} & \frac{1}{2}\mathbf{e}^t\beta_{mk}\phi^{-1}\gamma_{jl}^t\mathbf{d}\phi^{-1}\mathbf{d}^t\gamma_{jl}\phi^{-1}\beta_{mk}^t\mathbf{e} & I_m \end{pmatrix}$$

$$= [E^*_{(\frac{\mathbf{e}^t\beta_{mk}}{2})}, E^*_{(\mathbf{e}^t\beta_{mk}\gamma_{jl}^*\mathbf{e}\mathbf{f}^t\gamma_{jl})}][E^*_{(\mathbf{e}^t\beta_{mk})}, E^*_{(\mathbf{e}^t\gamma_{jl})}]$$

$$[E^*_{(\mathbf{e}^t\beta_{mk})}, E_{(\mathbf{f}^t\gamma_{jl})}]E_{(-\mathbf{e}^t\beta_{mk}\gamma_{jl}^*\mathbf{d})}.$$

These equations prove that C_m and U_m^- are normalized by $\mathrm{O}_A(Q \perp H(A)^{m-1})$. Hence the theorem follows. □

We can immediately deduce the following stability result.

Corollary 5.2.5. EO_A *is a normal subgroup of* O_A.

5.3 Normality of Roy's Elementary Group under a Condition on Hyperbolic Rank

In this section, we prove that the elementary orthogonal group $\mathrm{EO}_A(Q \perp H(A)^m)$ is normal in the orthogonal group $\mathrm{O}_A(Q \perp H(A)^m)$ under a condition on the hyperbolic rank. First, we prove the normality when the hyperbolic rank at least $d+2$, where $d = \dim \mathrm{Max}\,(A)$.

In the following theorem, let q denote the quadratic form on Q.

Theorem 5.3.1 ([48, Corollary 6.4])**.** *Let A be a Noetherian ring with* $\dim \mathrm{Max}\,(A) = d < \infty$. *Let P be a finitely generated projective A-module of* rank $\geq d+1$ *and Q be a quadratic A-space. If Q contains a non-singular element w, then the orthogonal transformations of $Q \perp H(P)$ act transitively on the elements of norm $q(w)$.*

Remark 5.3.2 ([48, Remark 5.6])**.** Let w be an element of $Q \perp H(P)$ with its P-component unimodular. Then there exists an orthogonal transformation E_β^* which maps w into $H(P)$. For, let w be written as (z, x, f) with $z \in Q, x \in P$, and $f \in P^*$. Since x is unimodular, there exists an A-linear map $\beta' : P \to Q$ satisfying $\beta'(x) = z$. Let $\beta : Q \to P^*$ be an A-linear map such that $\beta^* = \beta'$. Then

$$\begin{aligned} E_\beta^*(z, x, f) &= \left(z - \beta^*(x), x, f + \beta(z) - \frac{1}{2}\beta\beta^*(x)\right) \\ &= \left(0, x, f + \beta(z) - \frac{1}{2}\beta\beta^*(x)\right). \end{aligned}$$

Theorem 5.3.3 ([48, Theorem 7.1])**.** *Let Q be a quadratic A-space of hyperbolic* rank *larger than $d+2$. Then the orthogonal transformations of Q act transitively on*

(i) *the non-singular elements of Q of a given norm and*

(ii) *the set of hyperbolic planes in* Q.

Theorem 5.3.4 ([48, Theorem 8.1′]). *Let* A *be a semilocal ring and let* Q *be a quadratic space over* A *of rank at least* 1*. Then the orthogonal transformations of* Q *act transitively on the non-singular elements of* Q *of a given norm.*

We now prove the following normality result.

Theorem 5.3.5. *The elementary orthogonal group* $\mathrm{EO}_A(Q \perp H(A)^m)$ *is normal in the orthogonal group* $\mathrm{O}_A(Q \perp H(A)^m)$ *when* m *is at least* $d+2$*, where* $d = \dim \mathrm{Max}\,(A)$.

Proof. By Theorem 5.3.3, it follows that the group $\mathrm{EO}_A(Q \perp H(A)^m)$ acts transitively on hyperbolic pairs. In the case of semilocal rings, by Theorem 5.3.4, the same holds for $m \geq 1$.

For, if $\alpha \in \mathrm{O}_A(Q \perp H(A)^m)$ and (e_1, f_1) is a hyperbolic pair, then, by Theorem 5.3.1, $(\alpha e_1, \alpha f_1)$ and (e_1, f_1) are in the same orbit of $\mathrm{EO}_A(Q \perp H(A)^m)$. Let e be a map which takes one orbit to the other. Therefore $e\alpha$ fixes (e_1, f_1) and hence $e\alpha \in \mathrm{O}_A(Q \perp H(A)^{m-1})$, whence so does $(e\alpha)^{-1}$. Now, by Lemma 5.2.4, it follows that $(e\alpha)^{-1}$ normalizes the group $\mathrm{EO}_A(Q \perp H(A)^m)$. But then α^{-1} normalizes the group $\mathrm{EO}_A(Q \perp H(A)^m)$. □

5.4 A Decomposition Theorem

In this section, we prove a decomposition of Roy's elementary group under the stable range condition. Assume that A satisfies the stable range condition SA_l.

We start with the following lemma.

Lemma 5.4.1. *The elementary orthogonal group* $\mathrm{EO}_A(Q \perp H(A)^m)$ *is generated by* G_m *and* Y^-.

Proof. It follows from the commutator relations

$$[E^*_{\beta_{ir}}, E^*_{\gamma_{mj}}] = \Big[[E^*_{\eta_{ml}}, E^*_{\gamma_{m-1,k}}], [E_{\alpha_{m-1,s}}, E^*_{\theta_{iq}}]\Big] \text{ for } 1 \leq i \leq m-1, 1 \leq r, j, k, l, s, q \leq n,$$

$$[E^*_{\beta_{ir}}, E^*_{\gamma_{js}}] = \Big[[E^*_{\eta_{jl}}, E^*_{\gamma_{mt}}], [E_{\alpha_{mk}}, E^*_{\theta_{iq}}]\Big] \text{ for } 1 \leq i, j \leq m-1, i \neq j, 1 \leq r, s, l, t, k, q \leq n,$$

$$E^*_{\eta_{kj}} = \left[E_{\alpha_{ij}}, [E^*_{\beta_{kl}}, E^*_{\gamma_{ir}}]\right] [E_{\alpha_{ij}}, E^*_{\frac{\eta_{kj}}{2}}]^{-1} \text{ for } 1 \leq i, k \leq m, i \neq k, 1 \leq j, l, r \leq n,$$

that the subgroup generated by G_m and Y^- contains all the generators of the elementary orthogonal group $\mathrm{EO}_A(Q \perp H(A)^m)$. □

Lemma 5.4.2. *Let the subgroups U^+, U^-, Y^+ and Y^- are as defined in Section 5.1. Then we have the following inclusions involving these subgroups:*

(i) $Y^-U^+ \subseteq U^+Y^-Y^+$,

(ii) $Y^+U^- \subseteq U^-Y^+Y^-$,

(iii) $Y^-U^+U^- \subseteq U^+U^-Y^+Y^-$.

Proof. (i) Let $\sigma \in U^+$. Then $\sigma = \eta\mu$, where η lies in the subgroup generated by $[E_{\alpha_{ik}}, E^*_{\beta_{jl}}]$, where $1 \leq i \leq m-2, 1 \leq j \leq m$ and $1 \leq k, l \leq n$ and $\mu \in Y^+$. Then from the commutator relations, it follows that for any $\xi \in Y^-$, the element $\xi\eta\xi^{-1}$ lies in U^+. Thus $\xi\sigma = \xi\eta\xi^{-1} \cdot \xi \cdot \mu \in U^+Y^-Y^+$.

(ii) Similar proof as (i).

(iii) Follows from (i) and (ii). □

We denote by S the set consisting of elements $\sigma \in L_m$ such that the matrix corresponding to σ has the $(n+m-1, n+m)^{th}$ and $(n+m, n+m)^{th}$ entries zero.

The following lemma is a crucial one since it depends on the stability conditions. The rest of the proof of the decomposition theorem is independent of the stable range condition.

Lemma 5.4.3. *Let $m \geq l+2$. Then, for every $\sigma \in L_m$, there exist elements $\varphi_\sigma \in V^+, \psi_\sigma \in V^-$ such that $\psi_\sigma\varphi_\sigma\sigma \in S$.*

Proof. Let $\sigma \in L_m$ and let v be the $(n+m)^{th}$ column of the matrix corresponding to σ. From the definition of stable rank, it follows that there exists a matrix $\gamma \in M(m-2, 2, A)$ such that $\begin{pmatrix} 0 & I_{m-2} & \gamma & 0 \end{pmatrix} v \in A^{m-2}$ is unimodular. Hence we get an element $\varphi_\sigma \in V^+$

such that the first $n+m-2$ coordinates of $v' = \varphi_\sigma v$ form a unimodular column, where

$$\varphi_\sigma = \begin{pmatrix} I_n & 0 & 0 \\ 0 & \begin{pmatrix} I_{m-2} & \gamma \\ 0 & I_2 \end{pmatrix} & 0 \\ 0 & 0 & \begin{pmatrix} I_{m-2} & 0 \\ -\gamma^t & I_2 \end{pmatrix} \end{pmatrix}.$$

Now, there exists another matrix $\kappa \in M(2, m-2, A)$ and $\psi_\sigma \in V^-$ such that $v'' = \psi_\sigma v'$ has the coordinates $v''_{n+m-1} = v''_{n+m} = 0$, where

$$\psi_\sigma = \begin{pmatrix} I_n & 0 & 0 \\ 0 & \begin{pmatrix} I_{m-2} & 0 \\ \kappa & I_2 \end{pmatrix} & 0 \\ 0 & 0 & \begin{pmatrix} I_{m-2} & -\kappa^t \\ 0 & I_2 \end{pmatrix} \end{pmatrix}.$$

Hence $\psi_\sigma \varphi_\sigma \sigma \in S$. □

Corollary 5.4.4. *Let $m \geq l+2$. Then we have the following inclusion*

$$U_m U_m^- L_m \subseteq U^+ U^- S.$$

Proof. Let $\sigma \in L_m$. Then, by Lemma 5.4.3, there exists $\varphi_\sigma \in V^+$. Since φ_σ normalizes U_m^-, we have

$$U_m U_m^- \sigma = (U_m \varphi_\sigma^{-1})(\varphi_\sigma U_m^- \varphi_\sigma^{-1} \cdot \psi_\sigma^{-1})(\psi_\sigma \cdot \varphi_\sigma \sigma) \subseteq U^+ U^- S. \quad \square$$

Lemma 5.4.5. *Let $m \geq 2$. Then we have the following inclusion*

$$Y^- U^+ U^- S \subseteq U^+ U^- L_m F_m.$$

Proof. Let $\theta \in S$ and $\tau \in (Y^-)^\theta$. Then τ is of the form $\tau = \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_m & 0 \\ 0 & \gamma & I_m \end{pmatrix}$ for some skew-symmetric matrix γ. Now it follows from the definition of S that the $(n+m)^{th}$ column of

θ remains unchanged if we multiply θ on the left by an element of Y^-. Hence the $(n+m)^{th}$ column of θ coincides with that of the identity matrix and the m^{th} column of γ is zero. Since γ is a skew-symmetric matrix, we get that the m^{th} row of γ is also zero. We now get

$$\tau \in U_m^- \cap F_m \subseteq F_m.$$

Now, by Corollary 5.4.4 and Lemma 5.4.5, we have the following inclusions.

$$Y^-U^+U^-\theta \subseteq U^+U^-Y^+Y^-\theta \subseteq U^+U^-\theta(Y^+)^\theta(Y^-)^\theta \subseteq U^+U^-L_mF_m. \qquad \square$$

We now have enough machinery to prove the following decomposition theorem.

Theorem 5.4.6 (Decomposition Theorem). *Let $m \geq l+2$. Then every element of $\mathrm{EO}_A(Q \perp H(A)^m)$ has a $G_mU_m^-F_m$-decomposition.*

Proof. Since L_m normalizes both U_m and U_m^-, we have

$$G_mU_m^-F_m = U_mL_mU_m^-F_m = U_mU_m^-L_mF_m.$$

To prove $G_mU_m^-F_m = \mathrm{EO}_A(Q \perp H(A)^m)$, it is enough to prove that $G_mU_m^-F_m$ is stable under left multiplication by the generators of $\mathrm{EO}_A(Q \perp H(A)^m)$. Now, by Lemma 5.4.1, it is enough to show that $Y^-G_mU_m^-F_m \subseteq G_mU_m^-F_m$. We now get

$$Y^-G_mU_m^-F_m = Y^-U_mU_m^-L_mF_m \subseteq Y^-U^+U^-SF_m \subseteq U^+U^-L_mF_m \subseteq G_mU_m^-F_m. \qquad \square$$

5.5 Normality under Λ-Stable Range

In this section, we prove the normality under the assumption that A satisfies the 0-stable range condition 0-SA_l. i.e., A satisfies the stable range condition SA_l and for every unimodular vector $(a_1, \dots a_{l+1}, b_1, \dots b_{l+1})^t \in A^{2l+2}$, there exists an $(l+1)\times(l+1)$ skew-symmetric matrix β such that $(a_1, \dots a_{l+1})^t + \beta(b_1, \dots b_{l+1})^t \in A^{l+1}$ is unimodular.

Lemma 5.5.1. *Let $m \geq l+1$. Then, for any $\sigma \in \mathrm{O}_A(Q \perp H(A)^m)$, there is an element $\varrho \in G_m$ such that $\sigma\varrho$ has 1 in its $(n+m, n+m)^{th}$ position.*

We shall use the following theorem of L.N. Vaserstein in the proof of Lemma 5.1. For completeness, we include its proof.

Theorem 5.5.2 (L.N. Vaserstein, [61, Theorem 1] **).** *Let R be an associative ring of finite stable rank l. Then, for any natural number $n > l$ and any unimodular row $(b_i)_{1\leq i\leq n}$, there exist $c_i \in R$ such that $(b_i + c_i b_n)_{1\leq i\leq n-1}$ is R- unimodular and $c_i = 0$ when $i > l$.*

Proof. Let $n > l$. Since the stable range condition SA_l holds, we have $\sum_{i=1}^{n} a_i b_i = 1$ for some $a_i \in R$. Now let $b'_i = b_i$ $(1 \leq i \leq l)$ and $b'_{l+1} = \sum_{i=l+1}^{n} a_i b_i \in R$. Then the vector $b' = (b'_i)_{1\leq i\leq l+1}$ is R-unimodular and by the stable range condition, there exist $c'_i \in R$ $(1 \leq i \leq l)$ such that $\sum_{i=1}^{l} Rb''_i = R$, where $b''_i = b'_i + c'_i b'_{l+1} = b_i + c'_i \sum_{j=l+1}^{n} a_j b_j$. We set $B_{i,j} = c'_i a_j$ $(1 \leq i \leq l < j \leq n-1)$, $c_i = c'_i a_n$ $(1 \leq i \leq l)$ and $c_i = 0$ when $i > l$. Then $b''_i = b_i + c_i b_n + \sum_{j=l+1}^{n-1} B_{i,j} b_j$ $(1 \leq i \leq l)$. We also set $b''_i = b_i$ when $l < i < n$ and $B = I_{n-1} + \sum_{i=1}^{l}\sum_{j=l+1}^{n-1} B_{i,j} e_{i,j} \in \mathrm{GL}_{n-1}(R)$, where $B_{i,j}e_{i,j}$ is the matrix with $B_{i,j}$ in position i, j and with zeros elsewhere. Since the vector $b'' = (b''_i)_{1\leq i\leq n-1}$ is R-unimodular, the vector $B^{-1}b'' = (b_i + c_i b_n)_{1\leq i\leq n-1}$ is also unimodular. □

Proof of Lemma 5.5.1. Let σ be the 3×3 block matrix corresponding to the orthogonal transformation $\sigma \in \mathrm{O}_A(Q \perp H(A)^m)$ given by

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix},$$

where σ_{11} is an $n \times n$ matrix, σ_{12}, σ_{13} are $n \times m$ matrices, σ_{21}, σ_{31} are $m \times n$ matrices and $\sigma_{22}, \sigma_{23}, \sigma_{32}, \sigma_{33}$ are $m \times m$ matrices. Since $\sigma^{-1} \in \mathrm{O}_A(Q \perp H(A)^m)$, it also has a similar matrix description. Now $(\sigma_{21}, \sigma_{22}, \sigma_{23})$ is a unimodular vector in $M_n(A) \times (M_m(A))^2$. Let $v = (u, v_2, v_3)$ be the bottom row of $(\sigma_{21}, \sigma_{22}, \sigma_{23})$. It is unimodular in A^{n+2m}. Then there exists a vector $v' \in A^{n+2m}$ such that $\langle v, v'\rangle = 1$.

The unimodular vector (u, v_2, v_3) can be written as $\big(u, \{\langle f_i, v\rangle\}_{1\leq i\leq m}, \{\langle e_i, v\rangle\}_{1\leq i\leq m}\big)$. Then, by unimodularity condition, we have

$$\sum_{i=1}^{m}\langle f_i, v\rangle\langle v', e_i\rangle + \sum_{i=1}^{m}\langle e_i, v\rangle\langle v', f_i\rangle + \langle v', u\rangle = 1$$

which implies that

$$\sum_{i=1}^{m} A\langle f_i, v\rangle + \sum_{i=1}^{m} A\langle e_i, v\rangle + A\langle v', u\rangle = A.$$

i.e., $\big(\{\langle e_i, v\rangle\}_{1\le i\le m}, \{\langle f_i, v\rangle\}_{1\le i\le m}, \langle v', u\rangle\big)$ is unimodular in A^{2m+1}.

Since $m \ge l+1$ and A has stable rank l, by Theorem 5.5.2, there exist $c_i \in A$ $(1 \le i \le m)$ such that

$$\sum_{i=1}^{m} A\langle f_i, v\rangle + A\left(\sum_{i=1}^{m}\langle e_i, v\rangle + c_i\langle v', u\rangle\right) = A. \tag{5.5.1}$$

Now set $v'' = v' - \sum_{i=1}^{m}(\langle f_i, v'\rangle e_i + \langle e_i, v'\rangle f_i)$. Then $\langle f_i, v''\rangle = 0$, $\langle e_i, v''\rangle = 0$, $\langle v'', u\rangle = \langle v', u\rangle$. Now take $\mu_1 = \prod_{i=1}^{m} E^*_{\beta_i} = \prod_{i=1}^{m} T_{f_i, c_i v''} \in G_m$ and denote $\mu_1(v)$ by (u', v_2', v_3').

$$\begin{aligned} E^*_{\beta_i}(v) &= v + \beta_i(v) - \beta_i^*(v) - \frac{1}{2}\beta_i\beta_i^*(v) \\ &= v - \langle v, f_i\rangle c_i v'' + \langle c_i v'', u\rangle f_i - q(c_i v'')\langle v_2, f_i\rangle f_i. \end{aligned}$$

Set $a_i = \langle f_i, v\rangle$, and $b_i = \langle e_i, v\rangle$. Then $a_i' = \langle f_i, E^*_{\beta_i}(v)\rangle = \langle f_i, v\rangle = a_i$ and $b_i' = \langle e_i, E^*_{\beta_i}(v)\rangle = \langle e_i, v\rangle + c_i\langle u, v''\rangle - q(c_i v'')\langle v, f_i\rangle = \langle e_i, v\rangle + c_i\langle v', u\rangle - c_i^2 q(v'')\langle v, f_i\rangle = b_i + c_i\langle v', u\rangle + r_i a_i$ for $r_i \in A$. Hence, by equation 5.5.1, we get

$$\sum_{i=1}^{m} Aa_i' + Ab_i' = \sum_{i=1}^{m}\big(A\langle f_i, E^*_{\beta_i}(v)\rangle + A\langle e_i, E^*_{\beta_i}(v)\rangle\big) = A.$$

Thus, by multiplying σ with $\mu_1 = \prod_{i=1}^{m} E^*_{\beta_i} = \prod_{i=1}^{m} T_{f_i, c_i v''}$, we can assume that (v_2', v_3') is unimodular in A^{2m}.

Since A satisfies the 0-stable range condition 0-SA_l and $m \ge l+1$, there exists a skew-symmetric matrix $\gamma \in M_m(A)$ such that $v_2' + v_3'\gamma$ is unimodular in A^m. Now set

$$\mu_2 = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & \gamma & I \end{pmatrix} = \prod_{1\le i,k\le m}\ \prod_{1\le j,l\le n}\left[E^*_{\beta_{ij}}, E^*_{\eta_{kl}}\right] \in G_m,$$

where I denotes the identity matrix and 0 denotes the zero matrix of the corresponding block size.

Since A satisfies stable range condition SA_l and $m \ge l+1$, there is a product ϵ of elementary matrices such that $(v_2' + v_3'\gamma)\epsilon = (0, \ldots, 0, 1)$.

Set

$$\mu_3 = \begin{pmatrix} I & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon^{t^{-1}} \end{pmatrix} = \prod_{1\le i,k\le m} \prod_{1\le j,l\le n} [E_{\alpha_{ij}}, E^*_{\beta_{kl}}] \in G_m.$$

Then $\sigma\mu_1\mu_2\mu_3$ has $(n+m)^{th}$ row $(u', 0, \ldots, 0, 1, v_3'\varepsilon^{t^{-1}})$. This completes the proof of the lemma. □

Theorem 5.5.3. *Let A be a commutative ring in which 2 is invertible. Suppose A satisfies the stable range condition 0-SA_l. Then, for all $m > l$, the elementary group $\mathrm{EO}_A(Q \perp H(A)^m)$ is normal in $\mathrm{O}_A(Q \perp H(A)^m)$.*

Proof. Let $\eta \in \mathrm{EO}_A(Q \perp H(A)^m)$, where $\operatorname{rank}(Q) = n$. By Lemma 5.5.1, there is an element ϱ_1 in $G_m \subseteq \mathrm{EO}_A(Q \perp H(A)^m)$ such that the $(n+m, n+m)^{th}$ coefficient of $\eta\varrho_1$ is 1. Then there is a matrix $\varrho_2 = \prod_{i=1}^{m-1}\prod_{1\le j,k\le n}[E_{\alpha_{mj}}, E^*_{\beta_{ik}}]$ such that $\eta\varrho_1\varrho_2$ has 0 in the first $n+m-1$ entries of its $(n+m)^{th}$ row and 1 in the $(n+m)^{th}$ entry of this row. It follows that there is a matrix $\varrho_3 = \left(\prod_{i=1}^{m-1}\prod_{1\le r,k\le n}[E^*_{\beta_{ir}}, E^*_{\gamma_{mk}}]\right)\left(\prod_{i=1}^{m-1}\prod_{1\le r,k\le n}[E_{\alpha_{ir}}, E^*_{\beta_{mk}}]\right)\left(\prod_{j=1}^{n} E^*_{\gamma_{mj}}\right)$ such that $\varrho_3\eta\varrho_1\varrho_2$ has the same m^{th} row as $\eta\varrho_1\varrho_2$ and the same m^{th} column as the $(n+2m)\times(n+2m)$ identity matrix. For any matrix

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \in \mathrm{O}_A(Q \perp H(A)^m),$$

it follows from equation (5.2.1) that the $(n+2m, n+2m)^{th}$ coefficient of $\varrho_3\eta\varrho_1\varrho_2$ is 1. Then there is a matrix

$$\varrho_4 = \left(\prod_{i=1}^{m-1}\prod_{1\le r,k\le n}[E_{\alpha_{mk}}, E^*_{\beta_{ir}}]\right)\left(\prod_{i=1}^{m-1}\prod_{1\le r,k\le n}[E^*_{\beta_{ir}}, E^*_{\gamma_{mk}}]\right)\left(\prod_{i=1}^{m-1}\prod_{1\le r,k\le n}[E_{\alpha_{ir}}, E_{\delta_{mk}}]\right)\left(\prod_{j=1}^{n} E_{\zeta_{mj}}\right)$$

such that $\varrho_4\varrho_3\eta\varrho_1\varrho_2$ has the same $(n+m)^{th}$ row and $(n+m)^{th}$ column as $\varrho_3\eta\varrho_1\varrho_2$ and the same $(n+2m)^{th}$ column as the $(n+2m, n+2m)$ identity matrix. Now, it follows that $\varrho_4\varrho_3\eta\varrho_1\varrho_2$ has the same $(n+2m)^{th}$ row as the $(n+2m, n+2m)$ identity matrix. Thus, by the stabilization homomorphism, we have $\varrho_4\varrho_3\eta\varrho_1\varrho_2 \in O_A(Q \perp H(A)^{m-1})$, where rank $(Q) = n$. Let $\rho = \varrho_4\varrho_3\eta\varrho_1\varrho_2$. By Proposition 5.2.4, it follows that ρ normalizes $\mathrm{EO}_A(Q \perp H(A)^m)$, where rank $(Q) = n$. Since $\eta = \varrho_3^{-1}\varrho_4^{-1}\rho\varrho_2^{-1}\varrho_1^{-1}$, it follows that η normalizes $\mathrm{EO}_A(Q\perp H(A)^m)$. Thus $\mathrm{EO}_A(Q\perp H(A)^m)$ is normal in $\mathrm{O}_A(Q\perp H(A)^m)$. $\square$

5.6 Stability of K_1

In this section, we prove the following stability theorem using the normality theorem of the previous section and the decomposition theorem under the 0-stable range condition and injective stability of K_1 of $\mathrm{O}_A(Q\perp H(A)^m)$ under the usual stable range condition.

Theorem 5.6.1. *Let A be a commutative ring of 0-stable rank l in which 2 is invertible. Then, for all $m \geq l+1$, the coset space $KO_{1,m}(Q \perp H(A)^m)$ is a group. Further, the canonical map*

$$KO_{1,r}(Q\perp H(A)^r) \longrightarrow KO_{1,m}(Q\perp H(A)^m)$$

is surjective for $l \leq r < m$, and when $m \geq l+2$, the canonical homomorphism

$$KO_{1,m-1}(Q\perp H(A)^{m-1}) \longrightarrow KO_{1,m}(Q\perp H(A)^m)$$

is an isomorphism.

Proof. By Theorem 5.5.3, we get that $KO_{1,m}(Q\perp H(A)^m)$ is a group and the map

$$KO_{1,m-1}(Q\perp H(A)^{m-1}) \longrightarrow KO_{1,m}(Q\perp H(A)^m)$$

is surjective. By induction on $m-l$, we obtain that the map

$$KO_{1,r}(Q\perp H(A)^r) \longrightarrow KO_{1,m}(Q\perp H(A)^m)$$

is surjective for $l \leq r < m$.

To prove the final assertion, let $\sigma \in \mathrm{O}_A(Q\perp H(A)^{m-1}) \cap \mathrm{EO}_A(Q\perp H(A)^m)$. Let $\eta\xi\mu$ be an $F_{(m)}U^-_{(m)}G_{(m)}$-decomposition of σ. Since the $(n+m)^{th}$ row of η coincides with that of the $(n+2m)\times(n+2m)$ identity matrix, it follows that the $(n+m)^{th}$ row of $\eta\xi\mu$ coincides with the $(n+m)^{th}$ row of $\xi\mu$. Thus the $(n+m)^{th}$ row of $\xi\mu$ coincides with that of the $(n+2m)\times(n+2m)$ identity matrix. We can write the matrix μ as

$$\mu = \begin{pmatrix} I & \gamma & 0 \\ 0 & \varepsilon & 0 \\ \vartheta & \psi & \varepsilon^{t^{-1}} \end{pmatrix},$$

where I is an $n \times n$ identity matrix, γ is an $n \times m$ matrix, ε is an $m \times m$ invertible matrix, ϑ and ψ are matrices of size $m \times n$ and $m \times m$ respectively.

If (u, v, w) denotes the $(n+m)^{th}$ row of ξ, then the $(n+m)^{th}$ row of $\xi\mu$ is

$$\begin{pmatrix} u, & v, & w \end{pmatrix} \begin{pmatrix} I & \gamma & 0 \\ 0 & \varepsilon & 0 \\ \vartheta & \psi & \varepsilon^{t^{-1}} \end{pmatrix} = \begin{pmatrix} u + w\vartheta, & u\gamma + v\varepsilon + w\psi, & w(\varepsilon^t)^{-1} \end{pmatrix}.$$

Since the $(n+m)^{th}$ row of $\xi\mu$ is same as that of the $(n+2m)\times(n+2m)$ identity matrix, we get $w(\varepsilon^t)^{-1} = 0$. Now, by the invertibility of $(\varepsilon^t)^{-1}$, we get $w = 0$. This implies that $u = 0$. Thus $\xi \in G_m$.

Now write $\eta = \eta_1\mu_1$, where $\eta_1 \in \mathrm{EO}_A(Q \perp H(A)^{m-1})$ and $\mu_1 \in C_m \subseteq G_m$.

Then $\sigma = \eta_1\mu_1\xi\mu$ and $\mu_1\xi\mu \in G_m \cap \mathrm{O}_A(Q \perp H(A)^{m-1})$. Now it suffices to show that $\mu_1\xi\mu$ lies in $\mathrm{EO}_A(Q \perp H(A)^{m-1})$. In fact, we show that $\mu_1\xi\mu \in G_{m-1}$.

Write

$$\mu_1\xi\mu = \begin{pmatrix} I & \gamma & 0 \\ 0 & \varepsilon & 0 \\ \vartheta & \delta & \varepsilon^{t^{-1}} \end{pmatrix}.$$

Since $\mu_1\xi\mu \in \mathrm{O}_A(Q \perp H(A)^{m-1})$, it follows that γ and δ have their last column 0 and ϑ, δ have their last row 0. Also, it follows that $\varepsilon \in \mathrm{GL}_{m-1}(A)$. From the definition of G_m, we see that ε is an $m \times m$ matrix of the form

$$\varepsilon = \begin{pmatrix} \varepsilon' & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{E}_m(A).$$

Thus $\varepsilon' \in \mathrm{E}_m(A) \cap \mathrm{GL}_{m-1}(A)$. Since A satisfies the stable range condition, by the stability for K_1 of the general linear group [15, Chapter V, Theorem 4.2], we get $\varepsilon' \in \mathrm{E}_{m-1}(A)$.

Thus $\mu_1\xi\mu$ lies in G_m. Hence the canonical homomorphism

$$KO_{1,m-1}(Q\perp H(A)^{m-1}) \longrightarrow KO_{1,m}(Q\perp H(A)^m)$$

is an isomorphism. □

Now, we prove the injective stability for K_1 of the orthogonal group $\mathrm{O}_A(Q\perp H(A)^m)$ under the usual stable range condition.

Theorem 5.6.2. *Let A be a commutative ring of stable rank l in which 2 is invertible and let $m \geq l+2$. Then the canonical map*

$$KO_{1,m-1}(Q\perp H(A)^{m-1}) \longrightarrow KO_{1,m}(Q\perp H(A)^m)$$

is injective.

Proof. Let σ be an element of $\mathrm{O}_A(Q\perp H(A)^{m-1}) \cap \mathrm{EO}_A(Q\perp H(A)^m)$. Then, by Theorem 5.4.6, σ can be written as a product $\tau\nu\mu$, where

$$\tau = \begin{pmatrix} I_n & 0 & t_{13} \\ t_{21} & t_{22} & t_{23} \\ 0 & 0 & t_{33} \end{pmatrix} \in G_m, \qquad \nu = \begin{pmatrix} I_n & u_{12} & 0 \\ 0 & I_m & 0 \\ u_{31} & u_{32} & I_m \end{pmatrix} \in U_m^-, \qquad \mu \in F_m.$$

Since the $(n+m)^{th}$ column of μ coincides with that of identity matrix, we get

$$t_{33}(u_{32})_{i1} = 0 \qquad \text{for } i = 1, \ldots, m.$$

Since t_{33} is invertible, we get

$$(u_{32})_{i1} = 0 \qquad \text{for } i = 1, \ldots, m.$$

Hence $\mu \in F_m$. Thus we can assume that $\sigma = \tau\mu$ and $\tau, \mu \in \mathrm{O}_A(Q\perp H(A)^{m-1})$.

Now proceeding as in Theorem 5.6.1, we get that $t_{22} = \begin{pmatrix} t'_{22} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{E}_m(A)$. Thus $t'_{22} \in \mathrm{E}_m(A) \cap \mathrm{GL}_{m-1}(A)$. Since $m \geq$ s-rank $A+2$, the injective stability theorem for

K_1 of the general linear group [15, Chapter V, Theorem 4.2], we have $t'_{22} \in \mathrm{E}_{m-1}(A)$ and hence $\sigma \in \mathrm{EO}_A(Q \perp H(A)^{m-1})$. Thus the canonical map

$$KO_{1,m-1}(Q \perp H(A)^{m-1}) \longrightarrow KO_{1,m}(Q \perp H(A)^m)$$

is injective. □

Publications

1. S.D. Adhikari, A.A. Ambily and B. Sury, *Zero-sum problems with subgroup weights*, Proc. Indian Acad. Sci. Math. Sci.120 (2010), no.3, 259–266.
2. A.A. Ambily and Ravi A. Rao, *Extendability of quadratic modules over a polynomial extension of an equicharacteristic regular local ring*, J. Pure Appl. Algebra, 218 (2014), no.10, 109–121.
3. A.A. Ambily, *Yoga of commutators in Roy's elementary orthogonal group*, Preprint, `arXiv:1305.2826[math.AC]`.
4. A.A. Ambily, *Normality and K_1-stability of Roy's elementary orthogonal group*, Communicated, `arXiv:1401.0822[math.AC]`.
5. A.A. Ambily, *Equality of Roy's elementary orthogonal group and odd hyperbolic unitary group*, in preparation.
6. A.A. Ambily and B. Sury, *Comparison of Roy's elementary orthogonal group with the group of ESD transvections*, in preparation.

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