M. Tech. (Computer Science) Dissertation Series

Decomposition And Packing Of Cycles Of A Given Length In A Complete Graph

a dissertation submitted in partial fulfilment of the requirements for the M. Tech. (Computer Science) degree of the Indian Statistical Institute

By

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CERTIFICATE

This is to certify that the work described in the dissertation entitled "Decomposition and Packing of Cycles of a Given Length in a Complete Graph" has been undertaken by Sharmistha Chakraborty under my guidance and supervision. The dissertation is found worthy of acceptance for the award of the Degree of Master of Technology in Computer Science.

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Chapter 1

INTRODUCTION

Existence of an edge-disjoint collection of cycles of given length m, which partition a complete undirected graph K_n of order n, depends on both m & n. In 1847, T. P. Kirkman[4] determined the *spectrum* of 3-cycle systems (the set of all n such that a 3-cycle system of order n exists). In 1892, *spectrum problem* for n-cycle systems of K_n was settled[7]. Kotzig[5] in 1965 and Rosa[10] in 1966 determined the *spectrum* of m-cycle systems, for all even m. But the general problem of *packing* is still unsolved.

In this work, we have handled *odd-cycle systems* and *even-cycle systems* separately. It has been known that the *spectrum* of 3-cycle systems is precisely the set of all $n \equiv 1$ or $3 \pmod 6$. We have considered those n where $n \equiv 0$ or 2 or $5 \pmod 6$. Our objective is to pack largest possible no. of 3-cycles in a complete graph where no. of nodes is of the form $6t$, $6t+2$ or $6t+5$. The *packing problem* in case of 4-cycle systems is handled in a different way. The *spectrum* of 4-cycle systems is precisely the set of all $n \equiv 1 \pmod 8$. In this work, we have considered the problem of *packing* 4-cycles in a complete graph of even order. A construction, that produces an edge-disjoint collection of largest possible no. of 4-cycles from a complete graph of even order, is given. It has also been shown that this construction is equivalent to *Spouse-avoiding* variant of *Oberwolfach Problem* proposed by Huang, Kotzig and Rosa[3] in 1979.

Chapter 2

m-CYCLE SYSTEMS : EXISTENCE & CONSTRUCTION

This chapter contains all the important definitions and results regarding m-cycle systems and decomposition of complete graph.

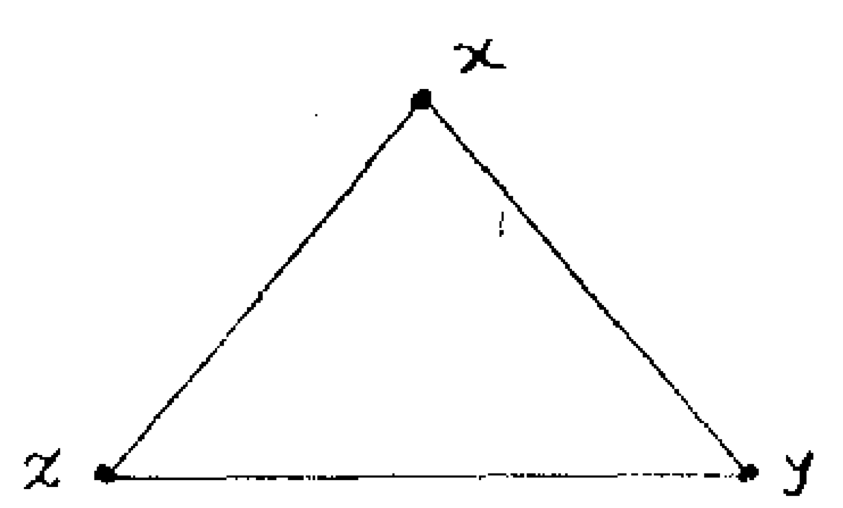
An *m-cycle system* is a pair (S, C), where C is an edge-disjoint collection of m-cycles which partition K_n with vertex set S. The number $n = |S|$ is called the *order* of the m-cycle system (S, C). Of course $|C| = n(n-1)/2m$.

In this work, we have considered two special cases, viz. *3-cycle systems* or *triple systems* and *4-cycle systems*.

2.1 STEINER TRIPLE SYSTEMS

A *Steiner triple system* (more simply, triple system) is a pair (S, T), where T is a collection of edge-disjoint triangles (or triples) which partition the complete undirected graph K_n with vertex set S.For a triple system (S, T) of order n, $|T| = n(n-1)/6$.

In this work, we will denote the 3-cycle



by $\{x, y, z\}$ or simply xyz in any order.

Example 2.1 *(1) The unique triple system of order 3.*

1 2 3

(2) The unique (to within isomorphism) triple system of order 7.

1 2 4	3 4 6	5 6 1	7 1 3
2 3 5	4 5 7	6 7 2	

(3) The unique (to within isomorphism) triple system of order 9.

1 2 3	1 4 7	1 5 9	1 6 8
4 5 6	2 5 8	2 6 7	2 4 9
7 8 9	3 6 9	3 4 8	3 5 7

(4) The non-cyclic triple system of order 13*[8].*

1 2 3	2 4 6	3 5 12	4 11 12	7 8 13
1 4 5	2 5 7	3 6 13	5 6 10	7 10 12
1 6 7	2 8 10	3 7 11	5 8 11	
1 8 9	2 9 12	3 9 10	5 9 13	
1 10 11	2 11 13	4 7 9	6 8 12	
1 12 13	3 4 8	4 10 13	6 9 11	

5) Triple system of order 15*[8].*

1 4 5	2 5 7	3 6 11	5 6 13	8 11 15
1 6 7	2 8 10	3 7 9	5 9 11	9 13 14
1 8 9	2 9 12	3 10 13	5 10 15	7 10 14
1 10 11	2 11 14	3 12 15	6 8 14	6 9 15
1 12 13	2 13 15	4 7 15	6 10 12	5 8 12
1 14 15	3 4 8	4 9 10	7 8 13	4 11 13
2 4 6	3 5 14	4 12 14	7 11 12	1 2 3

2.2 CONSTRUCTION OF TRIPLE SYSTEMS

Let $Q = \{1, 2, 3, \ldots, 2k\}$ and $H = \{\{1,2\}, \{3,4\}, \{5,6\}, \ldots, \{2k-1, 2k\}\}$. The 2-element subsets in H are called *holes*. Let $(Q, \circ)$ be a commutative quasigroup[1] with the property that, for each hole $h \in H$, $(h, \circ)$ is a subquasigroup. Such a quasigroup is called *commutative quasigroup with holes* H, and exists for every $2k \geq 6$[12].

[1]A groupoid $(G, \circ)$ is said to be a *quasigroup* if for any two elements $a, b \in G$, each of the equations $a \circ x = b$ and $y \circ a = b$ has a unique solution in G.

An example of commutative quasigroup of order 8 with holes $H = \{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}$ is shown below:

$\circ$	1	2	3	4	5	6	7	8
1	1	2	8	5	4	7	6	3
2	2	1	6	7	8	3	4	5
3	8	6	3	4	7	2	5	1
4	5	7	4	3	1	8	2	6
5	4	8	7	1	5	6	3	2
6	7	3	2	8	6	5	1	4
7	6	4	5	2	3	1	7	8
8	3	5	1	6	2	4	8	7

The $6t+1$ Construction. Let $(Q,\circ)$ be a commutative quasigroup of order $2t$ with holes H, set $S = \{\infty\} \cup (Q \times \{1,2,3\})$ and define a collection of triples T as follows:

(1) For each hole $h \in H$, construct a copy of the triple system of order 7 on $\{\infty\} \cup (h \times \{1,2,3\})$ and place these 7 triples in T, and

(2) if x and y belong to *different* holes of H, the three triples $\{(x,1),(y,1),(x \circ y,2)\}$, $\{(x,2),(y,2),(x \circ y,3)\}$, and $\{(x,3),(y,3),(x \circ y,1)\} \in T$.

Then (S,T) is a triple system of order $6t+1$.

The $6t+3$ Construction. Let $(Q,\circ)$ be a commutative quasigroup of order $2t$ with holes $H = \{h_1,h_2,h_3,\ldots,h_t\}$, set $S = \{\infty_1,\infty_2,\infty_3\} \cup (Q \times \{1,2,3\})$ and define a collection of triples T as follows:

(1) Construct a copy of the triple system of order 9 on $\{\infty_1,\infty_2,\infty_3\} \cup (h_1 \times \{1,2,3\})$ and place these 12 triples in T,

(2) for each of the remaining holes $h_2,h_3,\ldots,h_t$ construct a copy of the triple system of order 9 on $\{\infty_1,\infty_2,\infty_3\} \cup (h_i \times \{1,2,3\})$ such that $\{\infty_1,\infty_2,\infty_3\}$ is one of the triples, and place the 11 triples in T, and

(3) the same as (2) in the $6t+1$ Construction.

Then (S,T) is a triple system of order $6t+3$.

These two constructions produce triple systems of every admissible order $n \equiv 1$ or $3 \pmod 6$, $n \geq 19$. Also we already have examples of triple systems of order 3, 7, 9, 13 and 15. Thus we can produce any triple system of order n, where $n \equiv 1$ or $3 \pmod 6$, $n \geq 3$.

Theorem 2.1 *The spectrum for Steiner triple systems (that is, 3-cycle systems) is precisely the set of all $n \equiv 1$ or $3 \pmod 6$.*

2.3 ODD-CYCLE SYSTEMS

The obvious necessary conditions for the existence of an m-cycle system (S, C) of order $|S| = n$ are

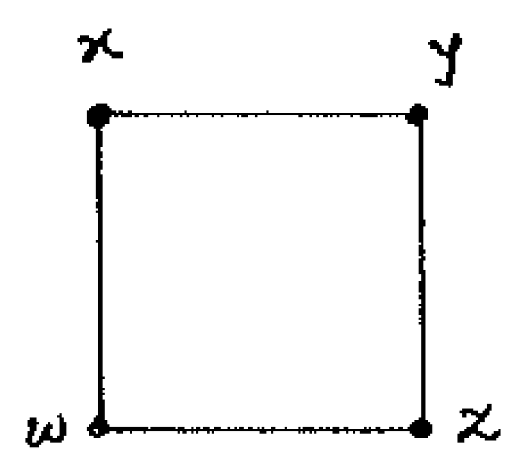
$$\begin{cases} (1) & n \geq m, \text{ if } n > 1, \\ (2) & n \text{ is odd,and} \\ (3) & n(n-1)/2m \text{ is an integer.} \end{cases}$$

Let $[n] = \{x \mid x \equiv n \pmod{2m}\}, 1 \leq n \leq 2m$.
Let $S[n]$ be the order of m-cycle systems that belong to $[n]$.

Theorem 2.2 *[2] Let $m \geq 3$ be odd. If $1 \leq n \leq 2m$ satisfies the necessary conditions (2) and (3) for the existence of an m-cycle system and if the smallest number in $S[n]$ is n or $n + 2m$, then $S[n] = [n]$ or $[n] \setminus n$ respectively.*

2.4 4-CYCLE SYSTEMS

In this work, we will denote the 4-cycle



by any cyclic shift of (x, y, z, w) or (y, x, w, z).

Example 2.2 *4-cycle system of order 9.*
Let $C = \{(i, 1 + i, 8 + i, 5 + i) \mid i \in Z_9\}$.
Then (Z_9, C) is a 4-cycle system of order 9.

The $n + 8$ Construction. Let $(\{\infty\} \cup X, C_1)$ be a 4-cycle system of order n and $(\{\infty\} \cup Y, C_2)$ a 4-cycle system of order 9. Let $H = \{h_1, h_2, \ldots, h_{(n-1)/2}\}$ and $G = \{g_1, g_2, g_3, g_4\}$ be partitions of X and Y, respectively, into "holes" of size 2 and set $C_3 = \{(a, c, b, d) \mid$ all $h_i = \{a, b\} \in H$ and $g_i = \{c, d\} \in G\}$. Then $(\{\infty\} \cup X \cup Y, C_1 \cup C_2 \cup C_3)$ is a 4-cycle system of order $n + 8$.

Starting with the 4-cycle system of order 9, the $n+8$ Construction produces a 4-cycle system of every order $\equiv 1 \pmod 8$.

Theorem 2.3 *The spectrum for 4-cycle system is precisely the set of all $n \equiv 1 \pmod 8$.*

2.5 EVEN-CYCLE SYSTEMS

The necessary conditions for the existence of an even-cycle system are the same as for odd-cycle systems. The idea here is to extrapolate the $n+8$ Construction for 4-cycle systems to even-cycle systems in general.

Theorem 2.4 *[11] The complete bipartite graph $K_{X,Y}$ can be decomposed into edge-disjoint cycles of length $2k$ if and only if*
(1) $|X| = x$ and $|Y| = y$ are even,
(2) $x \geq k$ and $y \geq k$, and
(3) $2k$ divides $x \cdot y$.

The $n+2m$ Construction. Let $m = 2k$ and let $(\{\infty\} \cup X, C_1)$ be an m-cycle system of order n and $(\{\infty\} \cup Y, C_2)$ an m-cycle system of order $2m+1$. (Such an m-cycle system is known to exist for all $m = 2k \geq 4$ [5, 10].) Let C_3 be a decomposition of $K_{X,Y}$ into m-cycles ($K_{X,Y}$ satisfies the necessary conditions for such a decomposition.) Then $(\{\infty\} \cup X \cup Y, C_1 \cup C_2 \cup C_3)$ is an m-cycle system of order $n+2m$.

Theorem 2.5 *[9] If m is even and there exists an m-cycle system of order n, then there exists an m-cycle system of every order $n+2mx$, x is any positive integer.*

Chapter 3

PACKING OF 3-CYCLES IN A COMPLETE GRAPH

The necessary conditions for the existence of a triple system (S, C) of order $|S| = n$ are

$$\begin{cases} (1) & n \geq 3, \\ (2) & n \text{ is odd, and} \\ (3) & n(n-1)/6 \text{ is an integer.} \end{cases}$$

In this chapter, we will consider those cases where n is even and/or $n(n-1)/6$ is not an integer.

3.1 PACKING OF CYCLES

By *packing* of cycles of length m in a complete graph of order n we mean the generation of largest possible no. of edge-disjoint m-cycles from that complete graph.

In general, the *upper bound* of number of 3-cycles, that can be packed in a complete graph K_n of order n, is given by $\left\lfloor \frac{n(n-1)}{6} \right\rfloor$. But, for even n, number of 3-cycles can never exceed $\left\lfloor \frac{n(n-2)}{6} \right\rfloor$. This is shown in the following lemma.

Lemma 3.1 *For even n, the upper bound of number of 3-cycles that can be packed in a complete graph K_n of order n is $\left\lfloor \frac{n(n-2)}{6} \right\rfloor$.*

Proof : Let r_i be the number of occurrences of ith vertex(element) in the system of 3-cycles packed in the complete graph, assumed to be undirected. Each vertex can appear with two distinct vertices at a time. As n is *even*, no vertex can appear in more than $(n-2)/2$ places. Also there are n such vertices. Thus

$$\sum_{i=1}^{n} r_i \leq \frac{n(n-2)}{2}$$

. Hence number of cycles produced is atmost

$$\left\lfloor \frac{n(n-2)/2}{3} \right\rfloor = \left\lfloor \frac{n(n-2)}{6} \right\rfloor$$

□

Illustration : Let us consider a complete graph of order 8 i.e. here $n = 8$. There are

$$\binom{8}{2} = 28$$

edges in the complete graph. So no. of edge-disjoint triangles should be less than or equal to $\lfloor 28/3 \rfloor = 9$.
But each vertex can appear with two distinct vertices at a time, thus any vertex can appear in atmost 3 triples.
Total no. of occurrences of all the vertices is $8 \times 3 = 24$. Thus, we can have atmost $\frac{24}{3} = 8$ triples.

The unique (to within isomorphism) *packing* of order 8.

1	2	4	3	4	6	5	6	8	7	8	2
2	3	5	4	5	7	6	7	1	8	1	3

Similarly, for $n = 10$, upper bound for no. of 3-cycles is

$$\left\lfloor \frac{n(n-2)}{6} \right\rfloor = \left\lfloor \frac{10 \times 8}{6} \right\rfloor = 13$$

The unique (to within isomorphism) *packing* of order 10.

1	2	3	2	4	6	3	4	7	4	8	10
1	4	5	2	5	8	3	6	8	5	7	10
1	6	7	2	7	9	3	5	9	6	9	10
1	8	9									

Lemma 3.1 can be generalised for all $m \geq 3$.

Lemma 3.2 *For* even n, *the* upper bound *of number of m-cycles that can be packed in a complete graph K_n of order n is* $\left\lfloor \frac{n(n-2)}{2m} \right\rfloor$.

3.2 GENERAL CONSTRUCTIONS FOR $n = 6t, 6t+2$

The following construction produces a maximal packing of 3-cycles in a complete graph of every admissible order $n \equiv 0$ or $2 \pmod 6$, for every $n \geq 2$.

The $6t+k$ Construction ($k = 0$ or 2). Let S be a set of $6t+k$ elements and $S' = S\cup\{\infty\}$, where ∞ is a dummy element. Define a collection of triples T as follows :

(1) Construct a copy of the triple system of order $6t+1$ or $6t+3$ on S', depending on value of k, (for $k = 0$, construct triple system of order $6t + 1$; otherwise, triple system of order $6t+3$ should be constructed) and place all these triples in T, and

(2) delete those triples from T where one of the three elements is ∞.

Then (S,T) is a packing of 3-cycles in a complete graph of order $n = 6t + k, k = 0$ or 2.

We will prove that this packing is *maximal*.

Theorem 3.1 *The $6t+k$ Construction produces maximal packing of 3-cycles in a complete graph of order $6t+k, k = 0, 2$.*

Proof : By lemma 3.1, upper bound of number of 3-cycles that can be packed in a complete graph of order n is $\left\lfloor \frac{n(n-2)}{6} \right\rfloor$, when n is even.

Case 1. $k = 0, n = 6t$. Then

$$\left\lfloor \frac{n(n-2)}{6} \right\rfloor = \left\lfloor \frac{6t(6t-2)}{6} \right\rfloor = t(6t-2) = 6t^2 - 2t.$$

By definition, $|S'| = 6t + 1$.
Now, for a triple system of order $6t+1$, number of triples generated —

$$\frac{(6t+1)6t}{6} = t(6t+1) = 6t^2 + t$$

The dummy element ∞ will appear in $\frac{6t}{2} = 3t$ triples.
Thus $|T| = 6t^2 + t - 3t = 6t^2 - 2t$, which is equal to the upper bound of number of 3-cycles, that can be packed in a complete graph of order $6t$.

Case 2. $k = 2, n = 6t + 2$. Then

$$\left\lfloor \frac{n(n-2)}{6} \right\rfloor = \left\lfloor \frac{(6t+2)6t}{6} \right\rfloor = t(6t+2) = 6t^2 + 2t.$$

As $|S'| = 6t + 3$, number of triples generated following $6t+3$ Construction is

$$\frac{(6t+3)(6t+2)}{6} = (2t+1)(3t+1) = 6t^2 + 5t + 1$$

The dummy element ∞ will appear in $\frac{6t+2}{2} = 3t + 1$ triples.
Thus $|T| = (6t^2 + 5t + 1) - (3t + 1) = 6t^2 + 2t$, which is equal to the upper bound of number of 3-cycles, that can be packed in a complete graph of order $6t + 2$.

The above results show that *the $6t+k$ Construction* generates the maximal packing of 3-cycles in a complete graph of order $6t+k$, for $k = 0, 2$. □

3.3 GENERAL CONSTRUCTIONS FOR $n = 6t + 5$

Next, we consider complete graphs of order n where n is of the form $6t + 5, t = 0, 1, \ldots$. Then n can also be expressed as $18t + k$ where $k = 5, 11$ or $17, t = 0, 1 \ldots$. We will handle these three cases separately. But, first the definition of Latin Square.

Definition 3.1 *An array*

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rn} \end{matrix}$$

is an $r \times n$ Latin rectangle *if each row contains the numbers* $1, 2, \ldots, n$ *in some order and if each column does not contain any digit repeated. In general,* $r \leq n$, *and if* $r = n$, *the array is a* Latin Square *and each number* $1, 2, \ldots, n$ *occurs exactly once in each row and column [1].*

The following construction produces a packing of 3-cycles in a complete graph of every admissible order $n \equiv 5 \pmod{18}$.

The $18t + 5$ Construction. Let S_1, S_2, S_3 be three mutually disjoint sets, containing $6t + 1$ elements each (we assume that all the elements in a particular set are distinct.) Set $S = S_1 \cup S_2 \cup S_3 \cup \{\infty_1, \infty_2\}$. Then $|S| = 18t + 5$. Define a collection of triples as follows :

(1) For $t = 0$, construct a packing of 3-cycles on S as follows :

```
1 2 3    1 4 5
```

and place these 2 triples in T.

(2) For $t \geq 1$, do the following :

(a) construct a copy of the triple system of order $6t + 3$ on $S_1 \cup \{\infty_1, \infty_2\}$ and place these triples in T,

(b) for each of S_2 and S_3, construct a copy of the triple system of order $6t + 3$ on $S_i \cup \{\infty_1, \infty_2\}, i = 2, 3$ and place all but the triple $\{\infty_1, \infty_2, a\}$, (where $a \in S_i, i = 2, 3$) in T, and

(c) construct a latin square of order $(6t + 1) \times (6t + 1)$ based on elements of S_3 and place the triples $\{a_i, b_j, c_{ij}\}$ in T, where a_i is the ith element of S_1, b_j is the jth element of S_2 and c_{ij} is the element in the ith row and jth column of the latin square, $i, j = 1, 2, \ldots, 6t + 1$.

Then (S, T) is a packing of 3-cycles in a complete graph with $18t + 5$ vertices, $t = 0, 1, \ldots$.

Definition 3.2 *Let* $a(n)$ *denote the upper bound of no. of edge-disjoint 3-cycles that can be packed in a complete graph of order* n, *and* $b(n)$ *denote the maximum no. of edge-disjoint cycles that can be generated from a complete graph of order* n, *using some method of construction.*

Theorem 3.2 *The $18t+5$ Construction produces a packing of 3-cycles in any complete graph of order $n \equiv 5 \pmod{18}$ with $a(n) - b(n) = 1$.*

Proof : When the $18t+5$ Construction is followed, no pair of elements can appear in more than one triples. So, it produces a packing of 3-cycles in complete graph of order $\equiv 5 \pmod{18}$.

Now, for $t = 0$, $a(n) = a(5) = \left\lfloor \frac{5 \times 4}{6} \right\rfloor = 3$.
But $b(n) = 2$,

(follows from rule-(1), described in the construction.)

Thus, $a(n) - b(n) = 1$.

For $t \geq 1$,

$$\begin{aligned} a(n) &= \left\lfloor \frac{n(n-1)}{6} \right\rfloor \\ &= \left\lfloor \frac{(18t+5)(18t+4)}{6} \right\rfloor \\ &= \left\lfloor \frac{324t^2 + 162t + 20}{6} \right\rfloor \\ &= 54t^2 + 27t + 3 \end{aligned} \tag{3.1}$$

We calculate $b(n)$ as follows :
By rule-(2a),

$$\frac{(6t+3)(6t+2)}{6} = (2t+1)(3t+1) = 6t^2 + 5t + 1$$

triples are produced.
Rule-(2b) will produce $6t^2 + 5t$ triples from each of S_2 & S_3.
Also, rule-(2c) can produce $(6t+1)^2$ triples.
Thus,

$$\begin{aligned} b(n) &= (6t^2 + 5t + 1) + 2(6t^2 + 5t) + (6t+1)^2 \\ &= 54t^2 + 27t + 2 \end{aligned} \tag{3.2}$$

From equations 3.1 & 3.2 we get,

$$a(n) - b(n) = 1.$$

Hence the theorem. □

Next, we will discuss about a construction that can produce a packing of 3-cycles in a complete graph of every admissible order $n \equiv 17 \pmod{18}$.

The $18t+17$ Construction. Let S_1, S_2, S_3 be three mutually disjoint sets, containing $6t+5$ elements each (we assume that all the elements in a particular set are distinct.) Set $S = S_1 \cup S_2 \cup S_3 \cup \{\infty_1, \infty_2\}$. Then $|S| = 18t + 17$. Define a collection of triples as follows :

(1) construct a copy of the triple system of order $6(t+1)+1$ on $S_1 \cup \{\infty_1, \infty_2\}$ and place these triples in T,

(2) for each of S_2 and S_3, construct a copy of the triple system of order $6(t+1)+1$ on $S_i \cup \{\infty_1, \infty_2\}, i = 2,3$ and place all but the triple $\{\infty_1, \infty_2, a\}$ (where $a \in S_i, i = 2,3$) in T, and

(3) construct a latin square of order $(6t+5) \times (6t+5)$ based on elements of S_3 and place the triples $\{a_i, b_j, c_{ij}\}$ in T, where a_i is the ith element of S_1, b_j is the jth element of S_2 and c_{ij} is the element in the ith row and jth column of the latin square, $i, j = 1, 2, \ldots, 6t+5$.

Then (S, T) is a packing of 3-cycles in a complete graph with $18t+17$ vertices, $t = 0, 1, \ldots$.

Let $a(n)$ and $b(n)$ be as defined in definition 3.2.

Theorem 3.3 *The $18t+17$ Construction produces a packing of 3-cycles in any complete graph of order $n \equiv 17 \pmod{18}$ with $a(n) - b(n) = 1$.*

Proof : From the construction, it is clear that any pair of elements can appear in atmost one triple. So, the above construction produces a packing of 3-cycles in complete graph of every order $\equiv 17 \pmod{18}$.

Now,

$$\begin{aligned} a(n) &= \left\lfloor \frac{n(n-1)}{6} \right\rfloor \\ &= \left\lfloor \frac{(18t+17)(18t+16)}{6} \right\rfloor \\ &= \left\lfloor \frac{324t^2 + 594t + 272}{6} \right\rfloor \\ &= 54t^2 + 99t + 45 \end{aligned} \quad (3.3)$$

We calculate $b(n)$ as follows :
Rule-(1) will produce,

$$\frac{[6(t+1)+1][6(t+1)]}{6} = 6(t+1)^2 + (t+1)$$

triples from $S_1 \cup \{\infty_1, \infty_2\}$.
Rule-(2) can produce one less triples from each of $S_2 \cup \{\infty_1, \infty_2\}$ and $S_3 \cup \{\infty_1, \infty_2\}$, i.e. a total of

$$2[6(t+1)^2 + (t+1) - 1]$$

triples will be produced. Using rule-(3) we can produce $(6t+5)^2$ triples. Thus,

$$\begin{aligned} b(n) &= 6(t+1)^2 + (t+1) + 2[6(t+1)^2 + (t+1) - 1] + (6t+5)^2 \\ &= 54t^2 + 99t + 44 \end{aligned} \quad (3.4)$$

From equations 3.3 & 3.4, it follows that,

$$a(n) - b(n) = 1.$$

Hence the theorem. □

We are left with those n where n is of the form $18t + 11, t = 0, 1, \ldots$. We suggest one construction that can produce a packing of 3-cycles in a complete graph of every admissible order $n \equiv 11 \pmod{18}$ with $a(18t + 11) - b(18t + 11) \leq 2t + 1$.

The $18t + 11$ Construction. We construct the packing inductively.

(1) For $t = 0$, construct a packing of 3-cycles on S with 11 vertices as follows :

1 2 3	2 4 8	3 4 6	4 7 10
1 4 5	2 6 10	3 7 9	4 9 11
1 6 7	2 5 9	3 8 10	5 7 8
1 8 9	2 7 11	3 5 11	6 8 11
1 10 11			

and place these 17 triples in T.

(2) Let us assume that a packing of 3-cycles can be constructed from any complete graph of order $n = 18t + 11$, where $t \leq k - 1$. Also, using $18t + 5$ & $18t + 17$ Constructions, we can produce a packing of 3-cycles from any complete graph of order $n \equiv 5$ or $17 \pmod{18}$. For $t = k$, the packing is constructed as follows :

Let S_1, S_2, S_3 be three mutually disjoint sets, containing $6t + 3$ elements each (we assume that all the elements in a particular set are distinct.) Set $S = S_1 \cup S_2 \cup S_3 \cup \{\infty_1, \infty_2\}$. Then $|S| = 18t + 11$. Define a collection of triples as described below :

(a) For each $S_i, i = 1, 2, 3$, construct a packing of 3-cycles on $S_i \cup \{\infty_1, \infty_2\}$ such that the pair (∞_1, ∞_2) does not appear in any triple. As $|S_i \cup \{\infty_1, \infty_2\}| = 6k + 5$ and $6k + 5 \leq 18(k - 1) + 11$ for all $k \geq 1$, such a packing exists by our assumption. Place all the triples thus produced in T,

(b) construct a latin square of order $(6t + 3) \times (6t + 3)$ based on elements of S_3 and place the triples $\{a_i, b_j, c_{ij}\}$ in T, where a_i is the ith element of S_1, b_j is the jth element of S_2 and c_{ij} is the element in the ith row and jth column of the latin square, $i, j = 1, 2, \ldots, 6t + 3$.

Then (S, T) is a packing of 3-cycles in a complete graph with $18t + 11$ vertices, $t = 0, 1, \ldots$.

Before proving that for the above construction $a(18t + 11) - b(18t + 11) \leq 2t + 1$, we should prove the following lemma :

Lemma 3.3 *Given a complete graph of order n, if for some construction $a(n) - b(n) \geq 1$, then there are at least 3 pairs of vertices that will never appear in any triples of the packing.*

Proof : No. of edges in a complete graph of order n —

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

Let $a(n) - b(n) = 1$.
Now, $a(n) = \left\lfloor \frac{n(n-1)}{6} \right\rfloor$.
So, $b(n) = \left\lfloor \frac{n(n-1)}{6} \right\rfloor - 1$.
Thus, no. of triples obtained by the construction is $\left\lfloor \frac{n(n-1)}{6} \right\rfloor - 1$.
Each triple contains three distinct edges. So, atmost

$$\frac{n(n-1)}{6} \times 3 - 3 = \frac{n(n-1)}{2} - 3$$

edges can appear in the packing, i.e. 3 edges will never appear in any triples of the packing.
Similarly, for $a(n) - b(n) = k \geq 1$, no. of triples obtained by the construction is

$$\left\lfloor \frac{n(n-1)}{6} \right\rfloor - k$$

Thus, $3k \geq 3$ edges will never appear in any triples of the packing.

Hence, we can always get at least three pairs of vertices (terminals of edges) that appear in no triples. □

Theorem 3.4 *(1) The* $18t+11$ *Construction produces a 3-cycle packing in a complete graph of every order* $n \equiv 11 \pmod{18}$*, and*
(2) $a(18t+11) - b(18t+11) \leq 2t+1$.

Proof : (1) **Case 1.** $t = 0$.
All the 17 triples shown in the construction are mutually edge-disjoint.
So, it is a packing.

Case 2. $t \geq 1$.
For all $i = 1, 2, 3$, we can always have such a packing on $S_i \cup \{\infty_1, \infty_2\}$ that a particular pair (∞_1, ∞_2) never appears in any triple.

(follows from lemma 3.3.)

Thus, all the triples generated from $S_i \cup \{\infty_1, \infty_2\}, i = 1, 2, 3$ are edge-disjoint.
Again, triples generated from latin square are also edge-disjoint.

(follows from defn of latin square.)

Hence, the $18t+11$ Construction generates a packing of 3-cycles in a complete graph of every admissible order $n \equiv 11 \pmod{18}$.

(2) We prove the second part by induction.

Case 1. $t = 0, n = 18t + 11 = 11$.

Now, $a(11) = \left\lfloor \frac{11 \times 10}{6} \right\rfloor = 18$.

But, $b(11) = 17$,

(from construction.)

Thus, $a(11) - b(11) = 1 = 2 \times 0 + 1$,

i.e. $a(18t + 11) - b(18t + 11) \leq 2t + 1$ for $t = 0$.

Case 2. Let us assume that

$$a(18t + 11) - b(18t + 11) \leq 2t + 1$$

for all $t \leq k - 1$, $k \geq 1$.

Now, for $t = k$,

$$\begin{aligned} a(18k + 11) &= \left\lfloor \frac{(18k + 11)(18k + 10)}{6} \right\rfloor \\ &= \left\lfloor \frac{324k^2 + 378k + 110}{6} \right\rfloor \\ &= 54k^2 + 63k + 18 \end{aligned} \tag{3.5}$$

The packing is constructed from three $(6k + 5)$-packings.

Subcase 2.1. Let $6k + 5 \equiv 18m + 5$ for some m. Then $k = 3m$.

According to the $18t+5$ Construction, $54m^2 + 27m + 2$ triples are produced from a complete graph of order $18m + 5$.

(See equation 3.2.)

Thus,

$$\begin{aligned} b(18k + 11) &= (6k + 3)^2 + 3(54m^2 + 27m + 2) \\ &= (6k + 3)^2 + 3(6k^2 + 9k + 2) \\ &= 54k^2 + 63k + 15 \end{aligned} \tag{3.6}$$

From equations 3.5 & 3.6 we get,

$$\begin{aligned} a(18k + 11) - b(18k + 11) &= 3 \\ &\leq 2k + 1, \text{as } k \geq 1 \end{aligned} \tag{3.7}$$

Subcase 2.2. Let $6k + 5 \equiv 18m + 17$ for some m. Then $k = 3m + 2$.

By the $18t + 17$ Construction, $54m^2 + 99m + 44$ triples can be produced from a complete graph of order $18m + 17$.

(See equation 3.4.)

Thus,

$$\begin{aligned} b(18k+11) &= (6k+3)^2 + 3(54m^2 + 99m + 44) \\ &= (6k+3)^2 + 3(6k^2 + 9k + 2) \\ &= 54k^2 + 63k + 15 \qquad (3.8) \end{aligned}$$

So, here also

$$\begin{aligned} a(18k+11) - b(18k+11) &= 3 \\ &\leq 2k+1, \text{as } k \geq 1 \qquad (3.9) \end{aligned}$$

(follows from equations 3.5 & 3.8.)

Subcase 2.3. Let $6k + 5 \equiv 18m + 11$ for some m.
Then $k = 3m + 1$, i.e. $3m = k - 1$.
Thus, $m \leq k - 1$, for all $k \geq 1$, and

$$a(18m + 11) - b(18m + 11) \leq 2m + 1$$

(by our assumption.)

Now, $a(18m + 11) = 54m^2 + 63m + 18$. (from equation 3.5.)

So, at least

$$(54m^2 + 63m + 18) - (2m + 1) = 54m^2 + 61m + 17$$

triples can be produced from each of $S_i \cup \{\infty_1, \infty_2\}, i = 1, 2, 3$.
Thus,

$$\begin{aligned} b(18k+11) &\geq (6k+3)^2 + 3(54m^2 + 61m + 17) \\ \text{i.e. } b(18k+11) &\geq 54k^2 + 61k + 17 \qquad (3.10) \end{aligned}$$

From 3.5 & 3.10, it follows that,

$$\begin{aligned} a(18k+11) - b(18k+11) &\leq (54k^2 + 63k + 18) - (54k^2 + 61k + 17) \\ &= 2k + 1 \qquad (3.11) \end{aligned}$$

Thus, $a(18t + 11) - b(18t + 11) \leq 2t + 1$, for $t = k$.

(follows from results 3.7, 3.9 & 3.11.)

So, the statement is true for all $t \geq 1$. Hence the theorem. □

In most of the cases, the difference is much less than 2t+1.
The following table shows the difference between $a(18t+11)$ & $b(18t+11)$ for different values of t.

t	$18t+11$	$6t+5$	$a(18t+11)-b(18t+11)$
0	11	5	1
1	29	11	3
2	47	17	3
3	65	23	3
4	83	29	9
5	101	35	3
6	119	41	3
7	137	47	9
8	155	53	3
9	173	59	3
10	191	65	9
11	209	71	3
12	227	77	3
13	245	83	27
14	263	89	3
15	281	95	3
22	407	137	27
31	569	191	27
40	731	245	81
49	893	299	27
58	1055	353	27
67	1217	407	81
76	1379	461	27
85	1541	515	27
94	1703	569	81
103	1865	623	27
112	2027	677	27
121	2189	731	243
130	2351	785	27
139	2513	839	27
148	2675	893	81
157	2837	947	27
166	2999	1001	27
175	3161	1055	81
184	3323	1109	27
193	3485	1163	27

t	$18t+11$	$6t+5$	$a(18t+11)-$ $b(18t+11)$
202	3647	1217	243
211	3809	1271	27
220	3971	1325	27
229	4133	1379	81
238	4295	1433	27
247	4457	1487	27
256	4619	1541	81
265	4781	1595	27
274	4943	1649	27
283	5105	1703	243
292	5267	1757	27
301	5429	1811	27
310	5591	1865	81
319	5753	1919	27
328	5915	1973	27
337	6077	2027	81
346	6239	2081	27
355	6401	2135	27
364	6563	2189	729
373	6725	2243	27
382	6887	2297	27
391	7049	2351	81
400	7211	2405	27
409	7373	2459	27
418	7535	2513	81
427	7697	2567	27
436	7859	2621	27
445	8021	2675	243
454	8183	2729	27
463	8345	2783	27
472	8507	2837	81
481	8669	2891	27
490	8831	2945	27
499	8993	2999	81

For all other $t \leq 500, a(18t+11) - b(18t+11) = 3$ or 9.

Chapter 4

PACKING OF 4-CYCLES IN A COMPLETE GRAPH

A necessary condition for the existence of a 4-cycle system of order n is $n \equiv 1 \pmod 8$. In 1979, Huang, Kotzig & Rosa[3] formulated *spouse-avoiding* variant of the *Oberwolfach Problem*. The problem is equivalent to decomposing the cocktail-party graph $K_n - F$ (the complete graph K_n minus one 1-factor) into isomorphic edge-disjoint cycles. Decomposition of $K_n - F$ into a collection of edge-disjoint 4-cycles is possible only when $n \equiv 0 \pmod 4$.

In this chapter, we will consider the problem of packing 4-cycles in complete graphs of order $n \equiv 0 \pmod 2$.

4.1 GENERAL CONSTRUCTIONS FOR $n = 2t$

The following construction generates a maximal packing of 4-cycles in a complete graph of order $n \equiv 0 \pmod 2$, for every $n \geq 2$.

The $2t$ Construction. Let $S' = \{0, 1, \ldots, (t-1)\}$. Consider a complete graph with vertex set $S = S' \times \{1, 2\}$. Then $|S| = 2t$. An element (a, j) of $S' \times \{1, 2\}$ will be denoted by a_j. Define set T_k as follows :

$T_k = \{(i_1, (i+k)_2, i_2, (i+k)_1) \mid i = 0, 1, \ldots, (t-k-1)\}$ $\qquad k = 1, 2, \ldots, (t-1)$.

Let $T = \bigcup_{k=1}^{t-1} T_k$.

Then (S, T) is a packing of 4-cycles in a complete graph of order $2t$, $t = 1, 2, \ldots$.

Theorem 4.1 *The $2t$ Construction produces maximal packing of 4-cycles in a complete graph of order $2t$,$t = 1, 2, \ldots$.*

Proof : According to lemma 3.2, the upper bound of number of 4-cycles that can be packed in a complete graph K_{2t} of order $2t$ is

$$\left\lfloor \frac{2t(2t-2)}{8} \right\rfloor = \frac{t(t-1)}{2}$$

Now $|T_k| = t - k$

Thus,

$$\begin{aligned} |T| &= \sum_{k=1}^{t-1}(t-k) \\ &= (t-1)+(t-2)+\cdots+3+2+1 \\ &= \frac{t(t-1)}{2} \end{aligned}$$

which is equal to the upper bound of maximum number of edge-disjoint 4-cycles that can be generated from a complete graph of order $2t$.

Hence, the $2t$ Construction generates the maximal packing of 4-cycles in a complete graph of order $n = 2t, t = 1, 2, \ldots$. □

Equivalent Construction for $n = 2t$, $t =$ **even.** Let $t = 2m$ and $S' = \{0, 1, \ldots, (2m-2)\}$. Define $S = (\{\infty\} \cup S') \times \{1, 2\}$. Thus $|S| = 4m = 2t$.

Any element (a, j) of S will be denoted by a_j.

Let $T_0 = \{(\infty_1, 0_2, \infty_2, 0_1), (1_1, (2m-2)_2, 1_2, (2m-2)_1), \ldots, ((m-1)_1, m_2, (m-1)_2, m_1)\}$ and,

$$\begin{aligned} T_k = &\{(\infty_1, (0+k)_2, \infty_2, (0+k)_1), ((1+k)_1, (2m-2+k)_2, (1+k)_2, (2m-2+k)_1), \ldots \\ &\ldots, ((m-1+k)_1, (m+k)_2, (m-1+k)_2, (m+k)_1)\} \end{aligned}$$

where addition is taken modulo $2m-1$, $k = 1, 2, \ldots, (2m-2)$.

Define $T = \bigcup_{k=0}^{2m-2} T_k$

Then (S, T) is a packing of 4-cycles in a complete graph of order $2t, t = 1, 2, \ldots$.

As $|T| = \sum_{k=0}^{2m-2} |T_k| = \sum_{k=0}^{2m-2} m = m(2m-1) = \frac{t(t-1)}{2}$, this also produces a maximal packing.

We will call the above construction as **The $4m$ Construction.**

4.2 OBERWOLFACH PROBLEM AND ITS VARIATIONS

The well-known *Oberwolfach problem*(OP) was formulated by Ringel at a graph theory meeting in 1967: "Is it possible to seat an *odd* number $2n+1$ of people at s round tables $T_1, T_2, \ldots, T_s$ (where T_i can accommodate exactly $k_i \geq 3$ people and $\sum k_i = 2n+1$) for m different meals so that each person has every other for a neighbour exactly once?" The problem is equivalent to decomposing the complete graph K_{2n+1} into isomorphic edge-disjoint

2-factors. Several authors gave solutions in many cases, but the problem remains open in general.

The *spouse-avoiding* variant of the *Oberwolfach problem*(NOP) is as follows: "At a gathering there are n couples. Is it possible to arrange a seating for the *even* number $2n$ of people present at s round tables $T_1, T_2, \ldots, T_s$ (where T_i can accommodate exactly $k_i \geq 3$ people and $\sum k_i = 2n$) for m different meals so that each person has every other person except his spouse for a neighbour exactly once?" This problem is equivalent to decomposing the graph $K_{2n} - F$ (where F is one 1-factor) into isomorphic edge-disjoint 2-factors.

Formulation of Oberwolfach problem(OP) and Spouse-avoiding Oberwolfach problem(NOP) : OP (NOP, respectively) consists of decomposing the complete graph K_v (the cocktail-party graph $K_v - F$, respectively) into isomorphic edge-disjoint 2-factors, each consisting of s circuits having length $k_1, k_2, \ldots, k_s$; here $k_i \geq 3$ for each $i = 1, 2, \ldots, s$, and $\sum_{i=1}^{s} k_i = v$. The problem is denoted by OP$(v; k_1, k_2, \ldots, k_s)$, and by NOP$(v; k_1, k_2, \ldots, k_s)$, respectively. If $k_1 = k_2 = \cdots = k_s = k$, we write simply NOP$(v; k)$ instead of NOP$(v; k_1, k_2, \ldots, k_s)$.

If a solution to either of the two problems exists, we say that OP$(v; k_1, k_2, \ldots, k_s)$ (or NOP$(v; k_1, k_2, \ldots, k_s)$) exists.

Result 4.1 *An* NOP$(k; k)$ *exists if and only if k is even.*

Result 4.2 *Let k be even. If there exists an* NOP$(v; k)$ *and a resolvable decomposition of $K_{v,v}$ into k-circuits then there exists an* NOP$(2v; k)$.

Result 4.3 *The necessary condition for the existence of an* NOP$(v; k)$ *is*

$$v \equiv 0 \pmod{k} \text{ if } k \text{ is even.}$$
$$v \equiv 0 \pmod{2k} \text{ if } k \text{ is odd.}$$

Result 4.4 *An* NOP$(v; 4)$ *exists if and only if* $v \equiv 0 \pmod 4$.

We will show that *the* $4m$ *Construction* is equivalent to *Spouse-avoiding* variant of *Oberwolfach problem*.

Theorem 4.2 *The* $4m$ *Construction is equivalent to* NOP$(4m; 4)$.

Proof :We will give the proof in two parts —
(I) a_1 and a_2 ($a \in \{\infty\} \cup S'$) will never appear together (as neighbours) in any 4-cycle.
(II) Given any a_1 and $b_2 (a \neq b)$, we can always find one 4-cycle where a_1 and b_2 appear as neighbours.

Proof of (I) : For $a = \infty$, it is trivially true. Let us assume that $a \neq \infty$.
Any 4-cycle in T_k, not containing the vertices ∞_1, ∞_2, is of the form

$$((i+k)_1, (2m-(i+1)+k)_2, (i+k)_2, (2m-(i+1)+k)_1)$$

where $i = 1, 2, \ldots, (m-1)$ and $k = 0, 1, \ldots, (2m-2)$,

and addition is taken modulo $2m-1$.
Now,

$$(i+k) - (2m-(i+1)+k) = 2i \pmod{2m-1} \tag{4.1}$$
$$(2m-(i+1)+k) - (i+k) = (2m-1) - 2i \pmod{2m-1} \tag{4.2}$$

As $1 \leq i \leq m-1$; none of $2i$ & $(2m-1)-2i$ can ever become zero.
Thus, a_1 and a_2 can never occur as neighbours in any 4-cycle.

Proof of (II) :
Case 1. : Let us assume that $a = \infty, b \neq \infty$.
Then a_1 and b_2 will appear as neighbours in the 4-cycle $(\infty_1, b_2, \infty_2, b_1)$.

Case 2. : Let $a \neq \infty, b = \infty$.
Then, in the 4-cycle $(\infty_1, a_2, \infty_2, a_1)$, a_1 and b_2 will appear as neighbours .

Case 3. : Let $a \neq \infty, b \neq \infty$.
Let $a - b = r \pmod{2m-1}$, where $0 < r < 2m-1$.
First, we assume that r is even. Then $r = 2p$ for some p, $1 \leq p \leq m-1$. Let $a - p = q \pmod{2m-1}, 0 \leq q < 2m-1$.
Then a_1 and b_2 will be neighbours in the 4-cycle

$$((p+q)_1, (2m-(p+1)+q)_2, (p+q)_2, (2m-(p+1)+q)_1)$$

But, if r is odd, then r can be expressed as $(2m-1)-2p'$ for some p', $1 \leq p' \leq m-1$. Let $a - (2m-(p'+1)) = q' \pmod{2m-1}, 0 \leq q' < 2m-1$.
In that case, a_1 and b_2 will appear as neighbours in the 4-cycle

$$((p'+q')_1, (2m-(p'+1)+q')_2, (p'+q')_2, (2m-(p'+1)+q')_1)$$

In all the three cases, a pair (a_1, b_2) appears exactly once.

Thus, if we define $2m$ couples as $(a_1, a_2), a \in \{\infty\} \cup S'$, and assume that there are m round tables, each of which can accommodate exactly 4 people, then each of $4m$ persons will have every other person except his/her spouse for a neighbour exactly once, that is, *the 4m Construction* is equivalent to NOP$(4m; 4)$. Hence the theorem. □

Nearly Spouse-avoiding Oberwolfach Problem. We assume that, at a gathering there are t couples, t is odd. We modify *the 4m Construction* as follows :

Let $t = 2m-1$ and $S' = \{0, 1, \ldots, (2m-2)\}$. Define $S = S' \times \{1, 2\}$. Then, $|S| = 2t = 4m-2$. Any element (a, j) of S is denoted by a_j.

Define

$$T_k = \{((1+k)_1, (2m-2+k)_2, (1+k)_2, (2m-2+k)_1), \ldots \\ \ldots, ((m-1-k)_1, (m+k)_2, (m-1+k)_2, (m+k)_2)\}$$

where addition is taken modulo $2m-1, k = 0, 1, \ldots, 2m-2$.

Let $T = \bigcup_{k=0}^{2m-2} T_k$.

Then

$$|T| = \sum_{k=0}^{2m-2} |T_k| = (m-1)(2m-1) \tag{4.3}$$

By lemma 3.2, the upper bound of number of 4-cycles that can be packed in a complete graph of order $4m-2$ is

$$\left\lfloor \frac{(4m-2)(4m-4)}{8} \right\rfloor == (2m-1)(m-1) \tag{4.4}$$

From 4.3 & 4.4, it follows that (S,T) is a maximal packing of 4-cycles in a complete graph of order $4m-2$. So, the above construction gives a nearmost solution to *Spouse-avoiding Oberwolfach problem.*

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