# ON WEAKLY STABLE TRANSFORMATIONS

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*SUMMARY.* An invertible measure-preserving transformation on a probability space $(\Omega, \mathcal{B}, m)$ is weakly stable if, for every pair of measurable sets $A$ and $B$, the sequence $m(T^{-j}A \cap B)$ is strong Cesaro convergent. In Section 1, certain category theorems are derived and powers and roots of weakly stable transformations are studied. In Section 2, it is shown that $T$ is weakly stable if and only if 1 is the only eigen value of the induced unitary operator. Section 3 deals with the Cartesian square $\bar{T} = T \times T$ of $T$. It is proved here that $T$ is weakly stable if and only if $\bar{T}$ is so and that $T$ is weakly stable if and only if a.e. section of every symmetric $\bar{T}$-invariant set is $T$-invariant. In Section 4, a sufficient condition for a direct integral of ergodic transformations to be weakly stable is obtained. In the last section, necessary and sufficient conditions for a continuous automorphism of a compact topological group $G$ to be weakly stable are derived and examples of weakly stable automorphisms on every $k$-dimensional torus are given.

## 0. Introduction and motivation

If $(\Omega, \mathcal{B}, m)$ is a probability space and $T$ an invertible measure-preserving transformation on $\Omega$, it is a simple consequence of the individual ergodic theorem that, for every two measurable sets $A$ and $B$, the sequence $m(T^{-j}A \cap B), j = 0, 1, 2, \ldots$ is Cesaro convergent. This result is not an incidental corollary to the ergodic theorem but is, in a sense, characteristic of measure-preserving transformations. In fact, one of Dowker's results (Dowker, 1951, Theorem V) can be restated as follows: If $T$ is any invertible, bimeasurable, non-singular and conservative transformation on the probability space $(\Omega, \mathcal{B}, m)$, then $T$ preserves a probability measure equivalent to $m$ if and only if the sequence $m(T^{-j}A \cap B)$ is Cesaro convergent for every pair of measurable sets $A$ and $B$.

Because of this theorem, it is reasonable to expect the structural properties of a measure-preserving transformation $T$ to be related to the convergence properties of the sequences $m(T^{-j}A \cap B)$ and to hope to study $T$ by studying these associated sequences. To support this view one may point out, for instance, the classical results on ergodic and (weakly) mixing transformations —$T$ is ergodic if and only if the Cesaro limit of $m(T^{-j}A \cap B)$ is $m(A)\ m(B)$; by suitably strengthening the convergence to this limit, one obtains the interesting sub-classes of mixing and weakly mixing transformations which are amenable to a more detailed analysis—or the recent study (Maitra, 1966) of transformations $T$ (the stable transformations) for which $m(T^{-j}A \cap B)$ is assumed to be convergent for every $A, B \in \mathcal{B}$.

Motivated by this observation, we study in this paper, the transformations $T$ for which $m(T^{-j}A \cap B)$ is strong Cesaro convergent for every pair of measurable sets $A$ and $B$. We call these the weakly stable transformations. We shall see that, in spite of the apparent artificiality of the defining condition, it is equivalent (as in the case of weakly mixing transformations) to certain very natural analytical conditions.

We may look at the weakly stable transformations from another point of view. Since the weakly mixing transformations are precisely those weakly stable transformations which are also ergodic, it is of interest to know which of the properties of weakly mixing transformations follow from the hypothesis of ergodicity and which from that of weak stability. It will be seen that many of the well-known theorems on weakly mixing transformations have their analogues in terms of weakly stable transformations, which may be seen to simplify, in the presence of ergodicity, to the corresponding theorems on weakly mixing transformations. Some of these analogues are, inevitably, direct generalizations but there are others which are non-trivial and which, we think, help one to understand the structure of weakly mixing transformations better.

## 1. Weak stability

Let $(\Omega, \mathcal{B}, m)$ be a probability space. All transformations which we shall consider on $(\Omega, \mathcal{B}, m)$ will be assumed to be point transformations on $\Omega$ defined at every point of $\Omega$ which are bijective, bimeasurable and measure-preserving. If $T$ is a transformation on $(\Omega, \mathcal{B}, m)$ we shall call a set $A \in \mathcal{B}$ *strictly T-invariant* if $TA = A$ and *T-invariant* if $m(A \Delta TA) = 0$. We denote by $\mathcal{J}_T^0$ the $\sigma$-algebra of strictly $T$-invariant sets and by $\mathcal{J}_T$ the $\sigma$-algebra of $T$-invariant sets. In situations where no confusion is likely, the suffix $T$ will be omitted.

*Definition :* Let $T$ be a transformation on $(\Omega, \mathcal{B}, m)$. $T$ is *weakly stable* if for every $A, B \in \mathcal{B}$, the sequence $m(T^{-j}A \cap B)$, $j = 0, 1, 2, \ldots$ is strong Cesaro convergent, i.e., if there exists a constant $C(A, B)$ such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} |m(T^{-j}A \cap B) - C(A, B)| = 0.$$

$T$ is *stable* (Maitra, 1966) if for every $A, B \in \mathcal{B}$, the sequence $m(T^{-j}A \cap B)$, $j = 0, 1, 2, \ldots$ is convergent, i.e., if there exists a constant $C(A, B)$ such that

$$\lim_{n \to \infty} |m(T^{-n}A \cap B) - C(A, B)| = 0.$$

Every stable transformation is clearly weakly stable. Since strong Cesaro convergence implies Cesaro convergence, it follows from the individual ergodic theorem that, if $T$ is a weakly stable transformation, then for every $A, B \in \mathcal{B}$, $C(A, B) = \int_B P(A \mid \mathcal{J}) dm$, where $P(A \mid \mathcal{J})$ is the conditional probability of $A$ given $\mathcal{J}$.

The simplest example of a weakly stable transformation is the identity. If $\Omega$ is a finite set, $\mathcal{B}$ the class of all subsets of $\Omega$ and $m$ a probability measure which gives positive measure to singletons, then the identity is the only weakly stable transformation on $(\Omega, \mathcal{B}, m)$. On the unit interval, however, we can find non-trivial examples of weakly stable transformations. In fact, we can find non-trivial examples of weakly stable transformations which are not weakly mixing, as the ensuing discussion shows.

Let $(\Omega, \mathcal{B}, m)$ be the probability space associated with the unit interval (here $\mathcal{B}$ is the $\sigma$-algebra of Borel sets) and $T$ a weakly stable transformation on $\Omega$. We shall show that $\Omega$ is essentially the union of two disjoint sets on one of which $T$ is the identity and on the other, antiperiodic. In other words, we shall prove that, for every integer $n \geqslant 2$, the set $A_n$ of all periodic points of period $n$ has measure zero. Clearly $A_n$ is measurable and strictly $T$-invariant. If $m(A_n) > 0$, then by an argument similar to that of Halmos (1956, p. 70), we may find a measurable subset $E$ of $A_n$ such that $E, TE, \ldots, T^{n-1}E$ are disjoint, $T^nE = E$ and $m(E) = \frac{1}{n} m(A_n)$. But then the sequence $m(T^{-j}E \cap E)$ is not strong Cesaro convergent—a contradiction. (This argument shows, incidentally, that no periodic transformation of period greater than one, on the unit interval, is weakly stable. This is true even generally. See Corollary 2.2.)

This result might justifiably make one wonder whether the weak stability of a transformation $T$ which is not weakly mixing is due only to the occurrence of a set of positive measure on which $T$ is the identity. This is not so. In Section 5, we give examples of families of antiperiodic, weakly stable (in fact, stable) but not weakly mixing transformations on every $k$-dimensional torus. Since the normalised measure space of a torus is point isomorphic to that of the unit interval, we may conclude that there exist such transformations even on the latter.

We now give some results regarding the category of sets of stable and weakly stable transformations. Let $(\mathcal{B}^*, m^*)$ be the measure algebra induced by $(\Omega, \mathcal{B}, m)$ and $\mathcal{G}$ the group of all automorphisms of $(\mathcal{B}^*, m^*)$ equipped with the weak topology $\mathcal{T}$ (Halmos, 1956, p. 61). Then with the obvious definitions of weak stability etc., for elements of $\mathcal{G}$, we have the following observations:

(i) The set of all weakly stable automorphisms contains the set of all weakly mixing automorphisms and is hence a dense set of the second category in $(\mathcal{G}, \mathcal{T})$. It is not the whole of $\mathcal{G}$, however. In fact, any ergodic but not weakly mixing automorphism is an example of an automorphism which is not weakly stable.

(ii) The set of all weakly stable but not weakly mixing automorphisms, being a subset of the complement of a dense $G_\delta$, is of the first category. It is dense. To see this, let $T$ be an automorphism induced by an antiperiodic, weakly stable but not weakly mixing transformation on $(\Omega, \mathcal{B}, m)$ (such a one exists, as we noted earlier), observe that any automorphism conjugate to $T$ is also weakly stable but not weakly mixing and apply the Conjugacy Lemma of Halmos (1956, p. 77).

(iii) The set of all stable automorphisms is a dense set of the first category. This is because the set of mixing automorphisms is a dense set of the first category and the set of stable, but not mixing automorphisms, is, as an argument similar to that in (ii) shows, also a dense set of the first category.

(iv) The set of all weakly stable automorphisms which are not stable is a set of the second category. It is dense since it contains the conjugacy class of every weakly mixing non-mixing automorphism.

We now return to the study of stable and weakly stable transformations on an arbitrary probability space. A few properties of these may be deduced from the definition. In proving these and later too, we shall make use of the following two facts about strong Cesaro convergence without explicit mention.

(a) A bounded sequence $a_n$ of complex numbers is strong Cesaro convergent to $a$ if and only if there exists a set $D$ of natural numbers of density one such that $a_n$ converges to $a$ on $D$.

(b) If a bounded sequence $a_n$ of complex numbers is strong Cesaro convergent to $a$ and $D$ is any set of natural numbers of positive density, then $a_n$ is strong Cesaro convergent to $a$ on $D$.

Theorem 1.1 : *If $T$ is weakly stable, then so is $T^k$ for every integer $k$.*

*Proof :* Let $T$ be weakly stable. If $k = 0$, $T^0 = I$ is weakly stable. Since $T^{-1}$ is easily seen to be weakly stable, to complete the proof, it is enough to show that $T^k$ is weakly stable for every positive integer $k$. But this is true because the set of positive multiples of $k$ has density $1/k$. The theorem is proved.

*Remark :* The analogue of Theorem 1.1 is true and trivial for stable transformations.

Theorem 1.2 : *If $T$ is weakly stable, then for every non-zero integer $k$, $\mathcal{J}_T = \mathcal{J}_{T^k}$.*

*Proof :* It is enough to prove that $\mathcal{J}_{T^k} \subseteq \mathcal{J}_T$ for every positive integer $k$. Let $A \in \mathcal{J}_{T^k}$ and let for arbitrary $B \in \mathcal{B}$, $C(A, B)$ denote the strong Cesaro limit of the sequence $m(T^{-j} A \cap B)$. Then $C(A, B)$ is also the strong Cesaro limit of the sequence $m(T^{-jk}A \cap B)$. But, for every $j$, $m(T^{-jk}A \cap B) = m(A \cap B)$ and therefore $C(A, B) = m(A \cap B)$. On the other hand, $C(T^{-1}A, B) = C(A, B)$. It follows that $A \in \mathcal{J}_T$. The theorem is proved.

Corollary 1.1 : *No weakly stable automorphism of a measure algebra can have finite orbits of order $> 1$.*

While it is true that every power of a weakly stable transformation is weakly stable, a similar statement does not hold for roots. E.g., if $T$ is a periodic transformation of period $n > 1$, then $T^n = I$ is weakly stable but $T$ is not weakly stable (see Corollary 2.2). We can however give a necessary and sufficient condition.

Given a transformation $T$, we shall call a measurable function $f$ *$T$-invariant* if $f(T\omega) = f(\omega)$ a.e.

Theorem 1.3 : *If $T$ is weakly stable and $S$ is a root of $T$, then $S$ is weakly stable if and only if $\mathcal{J}_S = \mathcal{J}_T$.*

*Proof :* If $S$ is weakly stable, then Theorem 1.2 implies that $\mathcal{J}_S = \mathcal{J}_T$. Suppose, conversely, that $\mathcal{J}_S = \mathcal{J}_T$ and that $S^k = T$, $k$ a positive integer. First note that for all $A \in \mathcal{B}$, the function $\alpha_A = P(A \mid \mathcal{J}_T) = P(A \mid \mathcal{J}_S)$ is $S$-invariant. Now, for every fixed $r$, $0 \leqslant r < k$ and arbitrary measurable sets $A$ and $B$, $m(S^{-r-kj}A \cap B) = m(T^{-j}A \cap S^{r}B)$ and hence the strong Cesaro limit of the sequence $m(S^{-r-kj}A \cap B)$, $j = 0, 1, 2, \ldots$ exists and is equal to $\int_{S^r B} \alpha_A dm = \int_B \alpha_A dm$. Since this limit is independent of $r$, it follows that the sequence $m(S^{-j}A \cap B)$ is itself strong Cesaro convergent for every $A$, $B \epsilon \mathcal{B}$. $S$ is therefore weakly stable.

Corollary 1.2 (Blum and Friedman, 1966): *If $T$ is weakly mixing and $S$ is a root of $T$, then $S$ is weakly mixing.*

*Proof:* Since $S$ is a root of $T$, $\mathcal{J}_S \subset \mathcal{J}_T$. Since $\mathcal{J}_T$ is trivial, $\mathcal{J}_T \subseteq \mathcal{J}_S$ and therefore $\mathcal{J}_S = \mathcal{J}_T$ and is trivial. By Theorem 1.3, $S$ is weakly stable and hence weakly mixing.

*Remark:* The analogue of Theorem 1.3 is true for stable transformations and the proof is, if anything, easier in this case. We therefore have the following corollary.

Corollary 1.3: *A weakly stable root of a stable transformation is stable.*

## 2. Spectral characterization

Every transformation $T$ on the probability space $(\Omega, \mathcal{B}, m)$ induces in a natural way a unitary operator on $\mathcal{L}_2(\Omega)$, the Hilbert space of equivalence classes of square-integrable complex-valued functions on $(\Omega, \mathcal{B}, m)$. (We shall not be too careful, in what follows, to maintain the distinction between functions and equivalence classes of functions.) This induced operator we shall denote by $U_T$ or by $U$ if no confusion is likely; $Uf = f.T$ for all $f \epsilon \mathcal{L}_2$. We shall let $P_T(P)$ stand for the subspace (in $\mathcal{L}_2$) of $T$-invariant square-integrable functions. ($P$ is precisely the subspace of square-integrable functions measurable with respect to $\mathcal{J}$.) $P \neq 0$ whatever be $T$, since the constant functions always belong to $P$. We shall use the letter $P$ to denote also the projection in $\mathcal{L}_2$ on the subspace $P$.

We now characterize the weak stability of a transformation $T$ in terms of the spectral properties of the induced unitary operator $U$. It is this characterization which will be most helpful to us in our study of weakly stable transformations from now on.

Lemma 2.1: *$T$ is weakly stable if and only if for every $f, g \epsilon \mathcal{L}_2(\Omega)$, there exists a constant $C_{f,g}$ such that*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} |(U^j f, g) - C_{f,g}| = 0.$$

*If $T$ is weakly stable, then $C_{f,g} = (Pf, g)$ for all $f$ and $g \epsilon \mathcal{L}_2(\Omega)$.*

The proof of the first part is straightforward. The second part is a consequence of the mean ergodic theorem.

Lemma 2.2: *$T$ is weakly stable if and only if for every $f \epsilon P^{\perp}$,*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} |(U^j f, f)| = 0.$$

*Proof :* The 'only if' part is a consequence of Lemma 2.1. To prove the 'if' part, notice that the given condition implies that for every fixed $f \epsilon P^{\perp}$, $|(U^j f, U^k f)|$ is Cesaro convergent to zero for every fixed non-negative integer $k$ and hence that $|(U^j f, g)|$ is Cesaro convergent to zero for every $g$ in the subspace, say $S$, spanned by $f, Uf, U^2 f, \ldots$. Since the same holds trivially for $g$ in $S^{\perp}$, it follows that $|(U^j f, g)|$ is Cesaro convergent to zero for every $g$ in $\mathcal{L}_2(\Omega)$ and fixed $f \epsilon P^{\perp}$. On the other hand, for $f \epsilon P$ and $g$ arbitrary, $(U^j f, g) = (Pf, g)$ for all $j$. It is now easy to verify that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} |(U^j f, g) - (Pf, g)| = 0$$

for every $f, g \epsilon \mathcal{L}_2(\Omega)$. $T$ is therefore weakly stable.

Theorem 2.1 : *$T$ is weakly stable if and only if 1 is the only eigen value of $U$.*

*Proof :* If 1 is not the only eigen value of $U$ and $f$ is an eigen function corresponding to an eigen value $\lambda \neq 1$ with $\|f\| = 1$, then $f \epsilon P^{\perp}$ but $|(U^j f, f)| = 1$ for all $j$. By virtue of Lemma 2.2, $T$ cannot be weakly stable.

Conversely, if 1 is the only eigen value of $U$ and $E(\cdot)$ denotes the spectral measure associated with $U$, then for every function $f \epsilon P^{\perp}$, the measure defined on the unit circle by $\mu(\cdot) = (E(\cdot)f, f)$ is non-atomic and hence (Halmos, 1956, p. 40) $|(U^j f, f)|^2 = |\int \lambda^j d\mu|^2$ is Cesaro convergent to zero. Since strong Cesaro convergence is equivalent to quadratic strong Cesaro convergence for bounded sequences, invoking Lemma 2.2 again, we conclude that $T$ is weakly stable.

Corollary 2.1 : *If $T$ is weakly stable, then $U$ has no finite orbits of order $> 1$ on $\mathcal{L}_2(\Omega)$.*.

*Proof :* If $U$ has a finite orbit of order $k > 1$, then there will exist a $k$-dimensional invariant subspace for $U$ on which $U$ is different from the identity. $U$ will then have eigen values different from 1.

Corollary 2.2 : *If there is a separating sequence of sets (Halmos and von Neumann, 1942) in $\Omega$, then no periodic transformation $T$ of period greater than 1 can be weakly stable.*

*Proof :* If $T^n = I$, $n > 1$, then $U^n = I$. $U$ will have finite orbits of order $> 1$ since $U = I$ would mean $T = I$.

The converse of Corollary 2.1 above is not tue in general. Any ergodic rotation on the circle group will serve as a counter-example. But if $(\Omega, \mathcal{B}, m)$ is the probability space associated with a compact abelian group $G$ and $T$ is a continuous automorphism of $G$, then the converse does hold. See Corollary 5.1'.

## 3. Cartesian squares

Let $(\bar{\Omega}, \bar{\mathcal{B}}, \bar{m})$ be the Cartesian square of the probability space $(\Omega, \mathcal{B}, m)$; i.e., let $\bar{\Omega} = \Omega \times \Omega$, $\bar{\mathcal{B}} = \mathcal{B} \times \mathcal{B}$ and $\bar{m} = m \times m$. Every transformation $T$ on $(\Omega, \mathcal{B}, m)$ induces a transformation $\bar{T} = T \times T$ on $(\bar{\Omega}, \bar{\mathcal{B}}, \bar{m})$ defined by $\bar{T}(u, v) = (Tu, Tv)$ for all $u, v \epsilon \Omega$. We shall write $\bar{U}$ for $U_{\bar{T}}$, $\bar{\mathcal{J}}$ for $\mathcal{J}_{\bar{T}}$ etc.

Theorem 3.1 : *$T$ is weakly stable if and only if $\bar{T}$ is weakly stable.*

*Proof:* If $\bar{T}$ is weakly stable, then for every $A, B \in \mathcal{B}$, $m(T^{-j}A \cap B) = \bar{m}(\bar{T}^{-j}(A \times \Omega) \cap (B \times \Omega))$ is strong Cesaro convergent and hence $T$ is weakly stable.

Conversely, let $T$ be weakly stable. If $A$ and $B$ are rectangles in $\bar{\mathcal{B}}$ say $A = C \times D$ and $B = E \times F$, then

$$\bar{m}(\bar{T}^{-j}A \cap B) = m(T^{-j}C \cap E)\, m(T^{-j}D \cap F)$$

is strong Cesaro convergent. It follows that for every two sets $A, B \in \bar{\mathcal{B}}_0$, the algebra of finite disjoint unions of measurable rectangles, $\bar{m}(\bar{T}^{-j}A \cap B)$ is strong Cesaro convergent. If now, $A, B \in \bar{\mathcal{B}}$ be arbitrary, for a given $\varepsilon$, $0 < \varepsilon < 1$, choosing $A_0, B_0$ in $\bar{\mathcal{B}}_0$ such that $\bar{m}(A \,\Delta\, A_0) < \frac{\varepsilon^2}{64}$ and $\bar{m}(B \,\Delta\, B_0) < \frac{\varepsilon^2}{64}$ and using the inequality

$$\begin{aligned} |\bar{m}(\bar{T}^{-j}A \cap B) - (\bar{P}\chi_A, \chi_B)| &\leq |\bar{m}(\bar{T}^{-j}A \cap B) - \bar{m}(\bar{T}^{-j}A_0 \cap B_0)| \\ &+ |\bar{m}(\bar{T}^{-j}A_0 \cap B_0) - (\bar{P}\chi_{A_0}, \chi_{B_0})| \\ &+ |(\bar{P}\chi_A, \chi_B) - (\bar{P}\chi_{A_0}, \chi_{B_0})| \end{aligned}$$

it is not difficult to show that

$$\frac{1}{n}\sum_{j=0}^{n-1} |\bar{m}(\bar{T}^{-j}A \cap B) - (\bar{P}\chi_A, \chi_B)| < \varepsilon$$

for all $n$ sufficiently large and hence to conclude that $\bar{T}$ is weakly stable.

*Remark:* The following generalization of the above theorem is true and may be proved in a similar way: If $\{(\Omega_i, \mathcal{B}_i, m_i)\ i = 1, 2, \ldots, k\}$ is a finite family of probability spaces and $T_i$ is a transformation of $(\Omega_i, \mathcal{B}_i, m_i)$, then $T_1 \times T_2 \times \ldots \times T_k$ is weakly stable if and only if $T_i$ is weakly stable for each $i$.

Theorem 3.1 above is analogous to the well-known result that $T$ is weakly mixing if and only if $\bar{T}$ is weakly mixing. But in the context of weak mixing something more is true—the weak mixing of $T$ follows even from the ergodicity of $\bar{T}$ (Halmos, 1956, p. 39). What might be an analogue of this in the case of weak stability? A plausible conjecture is that $T$ is weakly stable if and only if $\bar{\mathcal{J}} = \mathcal{J} \times \mathcal{J}$. The 'if' part of this conjecture is true and is not difficult to prove. We do not know whether the 'only if' part is true or not.* We observe, however, that if true, it will explain the occurrence of ergodicity in Halmos's mixing theorem.

There is another, equally interesting, mixing theorem due to Hopf (1948) which involves what may be called the symmetrized Cartesian square of $T$. Call a measurable subset $A$ of $\bar{\Omega}$ *symmetric* if $m(A \Delta \tilde{A}) = 0$ where $\tilde{A} = \{(u, v) : (v, u) \in A\}$. Let $\bar{\mathcal{B}}_s$ be the $\sigma$-algebra of symmetric sets and $\bar{m}_s$ the measure $\bar{m}$ restricted to $(\bar{\Omega}, \bar{\mathcal{B}}_s)$.

*We prove that it is true for a particular case in Theorem 5.3. Dr. J. K. Ghosh has proved it in the general case; his proof will appear elsewhere.

Since $\tilde{T}$ takes symmetric sets to symmetric sets, it may be restricted to a transformation $\tilde{T}_s$ on the measure space $(\dot{\Omega}, \tilde{\mathcal{B}}_s, \tilde{m}_s)$. Hopf's mixing theorem (Hopf, 1948, p. 37) says that $T$ is weakly mixing if and only if $\tilde{T}_s$ is ergodic.

We shall now obtain a generalization of this theorem to the case of weakly stable transformations. This generalization, we think, explains, at least partly, the occurrence of ergodicity in Hopf's mixing theorem.

Theorem 3.2 : *$T$ is weakly stable if and only if a.e. section of every symmetric $\tilde{T}$-invariant set is $T$-invariant.*

*Proof:* Let $T$ be weakly stable and let $A$ be any symmetric $\tilde{T}$-invariant set. To prove that a.e. section of $A$ is $T$-invariant, it is enough to prove that a.e. $v$-section of $K(u, v)$, the characteristic function of the set $A$, is a $T$-invariant function. Now $K(u, v)$ is a real, symmetric and square-integrable function on $(\tilde{\Omega}, \tilde{\mathcal{B}}, \tilde{m})$. Consider the compact hermitian operator $L$ on $\mathcal{L}_2(\Omega)$ defined by

$$(Lf)(u) = \int K(u, v)f(v)dm(v)$$

for every $f \in \mathcal{L}_2$. Let $\lambda_0 = 0, \lambda_1, \lambda_2, \ldots$ be the eigen values of $L$ and $S_n$, $n = 0, 1, 2, \ldots$ be the corresponding eigen subspaces. Since $L$ commutes with $U$, the unitary operator induced by $T$, $U$ is invariant on every $S_n$. But $S_n$, for $n \geqslant 1$, is finite dimensional, and the weak stability of $T$ implies that $U = I$ on every $S_n$, $n \geqslant 1$. It follows that $UL = L$ on $\mathcal{L}_2(\Omega)$. Therefore, for every $f \in \mathcal{L}_2(\Omega)$,

$$\int K(Tu, v)f(v)\, dm(v) = \int K(u, v)\, f(v)\, dm(v).$$

Standard measure-theoretic arguments prove that $K(Tu, v) = K(u, v)$ a.e. on $\tilde{\Omega}$ and therefore that a.e. $v$-section of $K(u, v)$ is $T$-invariant.

Conversely, let a.e. section of every symmetric $\tilde{T}$-invariant set be $T$-invariant. This implies that a.e. section of every real symmetric $\tilde{T}$-invariant function is $T$-invariant. To prove that $T$ is weakly stable, we shall prove that if $f$ is a non-zero eigen function corresponding to the eigen value $\lambda$ of $U$, then $f$ is, in fact, $T$-invariant.

Let $f(u) = f_1(u)+if_2(u)$ where $f_1$ and $f_2$ are respectively the real and imaginary parts of $f$. Consider the function $\dot{f}$ on $\dot{\Omega}$, defined by $\dot{f}(u, v) = f(u)\overline{f(v)}$. Then $\text{Re}\,\dot{f}(u, v) = f_1(u)f_1(v)+f_2(u)f_2(v)$ is a real symmetric $\tilde{T}$-invariant function. If there exists a $v_0 \epsilon \Omega$ such that $f_1(v_0) = 0, f_2(v_0) \neq 0$ (respectively $f_1(v_0) \neq 0, f_2(v_0) = 0$) and such that the $v_0$-section of $\text{Re}\,\dot{f}$ is $T$-invariant, then $f_2$ (respectively $f_1$) is $T$-invariant and hence $f$ is itself $T$-invariant. We have therefore to consider only the case when there exist two disjoint $T$-invariant sets $A$ and $B$ of total measure 1 such that $f \equiv 0$ on $A$ and neither $f_1$ nor $f_2$ vanishes at any point of $B$. Since $f$ is non-zero, $m(B) > 0$. We may assume without loss of generality that for every $v \epsilon B$, the corresponding $v$-section of $\text{Re}\,\dot{f}$ is $T$-invariant. If $B$ is a singleton set, then $f$ is trivially $T$-invariant. If there exist two points $v$ and $w$ in $B$ such that $f_1(v)/f_2(v) \neq f_1(w)/f_2(w)$, then since $f_1(u)f_1(v) + f_2(u)f_2(v)$ and $f_1(u)f_1(w)+f_2(u)f_2(w)$ are both $T$-invariant functions, we may conclude that $f$ is itself $T$-invariant. If lastly, $f_1(v)/f_2(v) = f_1(w)/f_2(w) = k$ for every two points

$v, w \in B$, then $f_2(u) = kf_1(u)$ a.e. and therefore $f$ is a complex multiple of the real function $f_1$. Hence $Uf_1 = \lambda f_1$. Considering now the symmetric $\tilde{T}$-invariant function $\tilde{f_1}(u, v) = f_1(u)\overline{f_1(v)} = f_1(u)f_1(v)$ and taking a suitable section, we see that $f_1$ and hence $f$ is $T$-invariant. The proof of the theorem is complete.

Corollary 3.1 (Hopf, 1948): *$T$ is weakly mixing if and only if $\tilde{T}_s$ is ergodic.*

*Proof:* Let $\tilde{T}_s$ be ergodic and let $A$ be any $\tilde{T}_s$-invariant set. Then a.e. section of $A$ has measure 1 if $\tilde{m}(A) = 1$ and measure 0 if $\tilde{m}(A) = 0$. In either case a.e. section of $A$ is $T$-invariant. By Theorem 3.2 $T$ is weakly stable. Since the ergodicity of $T$ is an immediate consequence of that of $\tilde{T}_s$, $T$ is, in fact, weakly mixing.

Let now $T$ be weakly mixing. $T$ is then ergodic and weakly stable. Therefore if $A$ is any $\tilde{T}_s$-invariant set, then a.e. section of $A$ has measure 0 or 1. But then $A$ itself must have measure 0 or 1. Hence $\tilde{T}_s$ is ergodic.

We owe to the referee the interesting observation that as a consequence of our conjectured generalisation of Halmos's mixing theorem and Theorem 3.2, $\tilde{\mathcal{J}}_s \subseteq \mathcal{J} \times \mathcal{J}$ if and only if $\tilde{\mathcal{J}} = \mathcal{J} \times \mathcal{J}$.

## 4. Direct integrals

It is a well-known result in ergodic theory that, under suitable conditions, a measure-preserving transformation on a probability space may be expressed as a 'direct integral' of ergodic transformations in an essentially unique fashion (Halmos, 1941). Since, in a manner of speaking, ergodicity is to measure-preservingness what (weak) mixing is to (weak) stability, it is reasonable to ask whether (weakly) stable transformations are expressible as direct integrals of (weakly) mixing transformations. Maitra (1966) has considered this question and answered it in the negative. He has an example of a stable transformation none of whose (ergodic) components is even weakly mixing.

In this section we show that, while the components of a weakly stable transformation need not be weakly mixing, it is nevertheless true that if $T$ is a transformation almost all of whose components are weakly mixing, then $T$ is necessarily weakly stable. In fact, in Theorem 4.1 below, we prove that the weak stability of $T$ is implied even by a much weaker condition on collections of components of $T$. (It would be interesting to know if this sufficient condition is also necessary.) We note that, as we use the spectral characterization of weak stability in proving this theorem, our methods do not yield a similar theorem for stable transformations.

Let us first give a brief sketch of the decomposition theory.

Let $(\Omega, \mathcal{B}, m)$ be a probability space for which the following two conditions hold:

(a) $\mathcal{B}$ is countably generated.

(b) For every countably generated sub $\sigma$-algebra $\mathcal{A}$ of $\mathcal{B}$, there exists a real-valued function $Q(.,.)$ on $\Omega \times \mathcal{B}$ such that

(i) for each $\omega \in \Omega$, $Q(\omega, .)$ is a probability measure on $(\Omega, \mathcal{B})$ ·

(ii) for each $B \in \mathcal{B}$, $Q(., B)$ is an $\mathcal{A}$-measurable function and

(iii) for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$,

$$\int_A Q(\omega, B) dm = m(A \cap B).$$

Let $T$ be a transformation on $(\Omega, \mathcal{B}, m)$ with the property that $\mathcal{J}^0$, the $\sigma$-algebra of strictly $T$-invariant sets is countably generated. Let $Q(.,.)$ be the function given by condition (b) above for $\mathcal{A} = \mathcal{J}^0$. One can then show that there exists a set $N \in \mathcal{J}^0$ with $m(N) = 0$ such that for every $A \in \mathcal{J}^0$, $Q(\omega, A) = \chi_A(\omega)$ for $\omega \notin N$.

Since $\mathcal{J}^0$ is countably generated, it has atoms and every set in $\mathcal{J}^0$ is a union of atoms. Let the atoms of $\mathcal{J}^0$ which are disjoint with $N$ be indexed by a set $X$ and let for $x \in X$, $Y_x$ denote the corresponding atom. Each $Y_x$ is made a probability space by requiring the measurable subsets $\mathcal{B}_x$ of $Y_x$ to be of the form $Y_x \cap B$ with $B \in \mathcal{B}$ and defining a measure $\nu_x$ on $(Y_x, \mathcal{B}_x)$ by $\nu_x(Y_x \cap B) = Q(\omega, B)$ where $\omega \in Y_x$ is arbitrary. ($\nu_x$ is well-defined because the function $Q(., B)$ is constant on the $Y_x$'s for every $B \in \mathcal{B}$.)

We convert $X$ itself into a probability space $(X, \mathcal{S}, \mu)$ by declaring a subset of $X$ to be measurable if and only if the union of the corresponding atoms of $\mathcal{J}^0$ is in $\mathcal{J}^0$ and defining the measure $\mu$ of this subset to be the $m$-measure of the corresponding set in $\mathcal{J}^0$.

Now for each $x \in X$, $Y_x$ is a strictly $T$-invariant set and hence the transformation $T$ may be restricted to a (bijective) mapping $T_x$ of $Y_x$. The decomposition theorem of Halmos (1941) says that for a.e. $x \in X$, $T_x$ is a transformation on $(Y_x, \mathcal{B}_x, \nu_x)$ and, indeed, an ergodic transformation. The transformations $T_x$ are called the components of $T$ and $T$ is said to be the direct integral of the $T_x$'s over the measure space $(X, \mathcal{S}, \mu)$.

We are now in a position to present our theorem. Let $X_0 \subseteq X$ be the set (of measure 1) of all $x$ with the property that $T_x$ is an ergodic transformation on $(Y_x, \mathcal{B}_x, \nu_x)$. For each $x \in X_0$, let $U_x$ denote the unitary operator on $\mathcal{L}_2(Y_x)$ induced by $T_x$.

Theorem 4.1 : *Let for every complex number $\lambda$ of modulus 1, $A_\lambda$ denote the set of all $x \in X_0$ such that $\lambda$ is an eigen value of $U_x$. If $\mu_*(A_\lambda) = 0$ for every $\lambda \neq 1$, then $T$ is weakly stable.*

*Proof :* Suppose $T$ to be not weakly stable and let $\lambda \neq 1$ be an eigen value of $U$. Choose and fix a bounded measurable function $f$ defined everywhere on $\Omega$ and a strictly $T$-invariant set $M \subseteq \Omega - N - \bigcup_{x \notin X_0} Y_x$ of positive measure with the property that for every $\omega \in M$, $f(T\omega) = \lambda f(\omega) \neq 0$. Then for every $Y_x \subseteq M$, the function $f$ restricted to $Y_x$, say $f_x$, is a non-zero bounded measurable function on $(Y_x, \mathcal{B}_x, \nu_x)$ and $f_x T_x(\omega) = \lambda f_x(\omega)$ for all $\omega \in Y_x$, so that $\lambda$ is an eigen value of $U_x$. If now $A$ is the set of all $x \in X_0$ for which $Y_x \subseteq M$, then $A \subseteq A_\lambda$ and $\mu(A) = m(M) > 0$. Hence $\mu_*(A_\lambda) > 0$.

Corollary 4.1 : *If $T_x$ is weakly mixing for almost all $x \in X$, then $T$ is weakly stable.*

## 5. Automorphisms of compact groups

In this section we shall take $\Omega$ to be a compact group $G$, $\mathcal{B}$ the $\sigma$-algebra of Borel subsets of $G$ and $m$ the normalised Haar measure on $G$. All automorphisms of $G$ which we shall consider are assumed to be continuous. An automorphism of $G$ is a transformation on the probability space $(G, \mathcal{B}, m)$ in our sense (see Section 1).

For convenience of presentation and for motivating the results in the general case we consider the case of abelian groups first.

*Abelian case.* It is well-known that for automorphisms of $G$ the conditions of ergodicity and (weak) mixing are equivalent and that an automorphism $T$ of $G$ is ergodic if and only if the induced unitary operator $U$ on $\mathcal{L}_2(G)$ does not have finite orbits on the set of non-constant characters in $\Gamma$, the character group of $G$ (Halmos, 1956, p. 53). Motivated by this result, Maitra (1966) has shown that $T$ is stable if and only if $U$ has no finite orbits of order $> 1$ on $\Gamma$. The following theorem and its corollary on weakly stable transformations are immediate consequences of Maitra's theorem and Corollary 2.1.

Theorem 5.1′ : *$T$ is weakly stable if and only if $U$ has no finite orbits of order $> 1$ on $\Gamma$.*

Corollary 5.1′ : *$T$ is weakly stable if and only if $U$ has no finite orbits of order $> 1$ on $\mathcal{L}_2(G)$.*

Note that Theorem 5.1′ and Maitra's result quoted above together imply that every weakly stable automorphism of $G$ is stable. An example of an automorphism of $G$ which is not weakly stable is the automorphism which takes every element of $G$ to its inverse.

If $G$ is the circle group, then the only weakly stable automorphism of $G$ is the identity automorphism. On the tori, however, we can find examples of non-trivial weakly stable automorphisms. To see this let, for any integer $k \geqslant 2$, $G^{(k)}$ be the $k$-dimensional torus. We know (see Jacobs, 1963) that there exists a one-to-one correspondence between automorphisms of $G^{(k)}$ and matrices of order $k$ with integral entries and determinant $\pm 1$ such that if $T$ is an automorphism and $M = (m_{ij})$ the corresponding matrix, then for every point $(x_1, x_2, \ldots, x_k) \epsilon G^{(k)}$, $T(x_1, x_2, \ldots, x_k) = (x_1', x_2', \ldots, x_k')$ is given by $x_j' = \prod_{i=1}^{k} x_i^{m_{ij}}$ $1 \leqslant j \leqslant k$. Moreover the character group $\Gamma^{(k)}$ of $G^{(k)}$ may be identified with $Z^{(k)}$, the $k$-fold direct product of the group of integers $Z$ with itself such that the action of $U$ on $\Gamma^{(k)}$ coincides with the action of $M$ on $Z^{(k)}$ defined by $M(n_1, \ldots, n_k) = (n_1', \ldots, n_k')$ where $n_i' = \sum_{j=1}^{k} n_j m_{ij}$. $T$ is therefore weakly stable as soon as $M$ does not have finite orbits of order $> 1$ on $Z^{(k)}$. But for this, a sufficient condition is that no eigen value $\lambda \neq 1$ of $M$ should be a root of unity. (For, if $T$ were not weakly stable, then there would exist a periodic element $n \epsilon Z^{(k)}$ for $M$ of period $p > 1$. It is then not difficult to conclude that $M$ must admit an eigen value different from unity which is, however, a $p$-th root of unity.) This condition is mild enough to enable us to construct many weakly stable automorphisms of $G^{(k)}$. In fact, we can have families of weakly stable automorphisms none of which is weakly mixing. For example, if $M = (m_{ij})$ is an integral-entried matrix of order $k$ such that $m_{ii} = 1$ for $1 \leq i \leq k$, $m_{ij} = 0$ for $i > j$, $1 \leq i, j \leq k$ and $m_{12} \neq 0$, then the corresponding automorphism on $G^{(k)}$ is an antiperiodic weakly stable transformation which (see Jacobs, 1963) is not weakly mixing.

We end our discussion of weakly stable automorphisms of abelian groups with the following

Theorem 5.2′ : *Let $T$ be an automorphism of $G$. The subspace $P$ of $T$-invariant functions in $\mathcal{L}_2(G)$ is spanned by the $T$-invariant characters in $\Gamma$ if and only if $T$ is weakly stable.*

*Proof :* Let $T$ be weakly stable and let $\{f_j\}$ be the family of non-$T$-invariant characters in $\Gamma$. It is enough to prove that if $f = \sum_j c_j f_j$, $\sum_j |c_j|^2 < \infty$, is any $T$-invariant function, then $f = 0$. But this is an immediate consequence of the fact that $U$ has only infinite orbits on the set $\{f_j\}$.

If $T$ is not weakly stable and $g$ is any non-$T$-invariant character in $\Gamma$ such that for some integer $p > 1$, $U^p g = g$, $Ug, \ldots, U^{p-1}g$ are all distinct, then the function $h = g + Ug + \ldots + U^{p-1}g$ is a non-zero $T$-invariant function. $h$ is therefore in $P$ but is not in the span of the $T$-invariant characters.

*General case.* In the remainder of this section we shall assume that the reader is familiar with the elementary theory of representations of an arbitrary compact group $G$.

Every automorphism $T$ of $G$ induces in a natural way a mapping $\pi$ of the set of all irreducible representations of $G$ onto itself ($\pi$ is given by $\pi V = \{\pi V(g)\} = \{V(Tg)\}$ for every irreducible representation $V = \{V(g)\}$). If now the set of all equivalence classes of irreducible representations of $G$ be indexed by $\alpha$, then by the well-known Peter-Weyl theorem, to every $\alpha$ there corresponds a (unique) subspace $S_\alpha$ of $\mathcal{L}_2(G)$ (obtained as the span of the matrix functions of any representation of type $\alpha$) such that $\mathcal{L}_2(G) = \bigoplus_\alpha S_\alpha$. Let $\mathcal{F}$ denote the family of all $S_\alpha$'s. Since $\pi$ takes equivalent representations to equivalent representations, it in turn induces a mapping which we shall continue to denote by $\pi$ of $\mathcal{F}$ onto itself. (In fact, $\pi(S_\alpha)$, for every $\alpha$, is the image of $S_\alpha$ under $U$, the unitary operator induced by $T$.) It is easy to check that the action of $\pi$ on $\mathcal{F}$ may be identified with the action of $U$ on $\Gamma$, the set of all irreducible characters $\chi_\alpha$ of $G$ and hence that $U$ maps $\Gamma$ onto itself.

Kaplansky (1949) has shown that even for a non-abelian group $G$, the conditions of ergodicity and (weak) mixing are equivalent for automorphisms $T$ of $G$ and that $T$ is ergodic if and only if $U$ has no finite orbits on the set of non-constant characters in $\Gamma$. (Incidentally, this implies that if $G$ admits ergodic automorphisms and if for every positive integer $n$, $\rho_n$ is the number (possibly infinite) of inequivalent irreducible representations of degree $n$, then $\rho_1 = 1$ or infinity and for $n \geqslant 2$, $\rho_n = 0$ or infinity.) Now, looking at this result and Theorem 5.1′, one is tempted to conclude that, even in this generality, an automorphism $T$ is weakly stable if and only if $U$ has no finite orbits of order $> 1$ on $\Gamma$. But this would be rash. For, if $T$ is any inner automorphism of $G$, then $U$ is the identity on $\Gamma$ but *no inner automorphism, except the identity automorphism, can be weakly stable. In particular no inner automorphism of $G$ can be ergodic.* (If $T$ is any inner automorphism of $G$, then, $\pi(S_\alpha) = S_\alpha$ for every $\alpha$ and hence every $S_\alpha$ is a (finite-dimensional) invariant subspace for $U$. If $T$ is to be weakly stable, then $U$ must be the identity on every $S_\alpha$ and hence on $\mathcal{L}_2(G)$.) The following theorem, however, is true.

Theorem 5.1 : *An automorphism $T$ of $G$ is weakly stable if and only if for every irreducible representation $V$ of $G$, $\pi(V) \sim V$ implies $\pi(V) = V$ and $U$ has no finite orbits of order $> 1$ on $\Gamma$.*

*Proof :* Let $T$ be weakly stable. Suppose that $V$ is any irreducible representation of $G$ and is of type $\alpha$. If $\pi V \sim V$, then $S_\alpha$ is invariant under $U$ and hence $U$ is the identity on $S_\alpha$. If now, for every $g \epsilon G, (v_{ij}(g))$ is the matrix $V(g)$, then for all $i$ and $j$, the function $v_{ij}(g) \epsilon S_\alpha$ and hence is equal to $v_{ij}(Tg)$ a.e. Since the functions $\{v_{ij}\}$ are all continuous, $v_{ij}(g) = v_{ij}(Tg)$ everywhere. It follows that $\pi(V) = V$. The second part is a consequence of Corollary 2.1 and the fact that $\Gamma$ is invariant under $U$.

Conversely, let $T$ be such that the conditions of the theorem are satisfied. Let $\Gamma_0$ be the set of $T$-invariant characters in $\mathcal{S}$ and $\mathcal{S}_0$ the corresponding subset of $\mathcal{S}$. The first condition implies that $U$ is the identity on every $S_\alpha \epsilon \mathcal{S}_0$. The second condition implies that $U$ has only infinite orbits on $\Gamma - \Gamma_0$ and hence that $\pi$ has only infinite orbits on $\mathcal{S} - \mathcal{S}_0$. It follows that for every function $f \epsilon S_\alpha \epsilon \mathcal{S} - \mathcal{S}_0$, the functions $f, Uf, U^2f, \ldots$ are mutually orthogonal. A proof similar to that in the abelian case (Maitra, 1960) yields the result that $T$ is, in fact, stable.

Corollary 5.1 : *$T$ is weakly stable if and only if for every irreducible representation $V$ of $G$, $\pi(V) \sim V$ implies $\pi(V) = V$ and $U$ has no finite orbits of order $> 1$ on $\mathcal{L}_2(G)$.*

Corollary 5.2 : *Every weakly stable automorphism of $G$ is stable.*

We now have the following generalization of Theorem 5.2'.

Theorem 5.2 : *Let $T$ be an automorphism of $G$. The subspace $P$ of $T$-invariant functions in $\mathcal{L}_2(G)$ is the direct sum of the $S_\alpha$'s corresponding to the $T$-invariant characters of $G$ in $\Gamma$ if and only if $T$ is weakly stable.*

*Proof :* Let $\Gamma_0$ be the set of $T$-invariant characters in $\Gamma$ and $\mathcal{S}_0$ the set of the corresponding $S_\alpha$'s. Let $S = \oplus\{S_\alpha : S_\alpha \epsilon \mathcal{S}_0\}$. We have to prove that $S = P$ if and only if $T$ is weakly stable.

Let $T$ be weakly stable. If $S_\alpha \epsilon \mathcal{S}_0$ then $U$ leaves $S_\alpha$ invariant and hence is the identity on $S_\alpha$. Therefore $S \subseteq P$. Thus, to prove that $S = P$, it is enough to prove that if $f \epsilon S^\perp$ and is $T$-invariant then $f = 0$. Now any $f \epsilon S^\perp$ is (uniquely) of the form $\sum_\alpha f_\alpha$ with $f_\alpha \epsilon S_\alpha \epsilon \mathcal{S} - \mathcal{S}_0$. If $f$ is $T$-invariant, then $\sum_\alpha f_\alpha = \sum_\alpha U^n f_\alpha$, $n = 1, 2, \ldots$. The fact that $\pi$ has only infinite orbits on $\mathcal{S} - \mathcal{S}_0$ now implies that for every $f_\alpha$, there exists a sequence $\alpha_k$ of distinct $\alpha$'s such that $\|f_\alpha\| = \|f_{\alpha_k}\|$ for all $k$. Since $\sum_\alpha \|f_\alpha\|^2 = \|f\|^2 < \infty$, it follows that $f = 0$.

Conversely, let $S = P$. If $V$ is any irreducible representation of $G$, of type $\alpha$ say, such that $\pi(V) \sim V$, then $\pi(S_\alpha) = S_\alpha \subseteq S = P$ and hence $U$ is the identity on $S_\alpha$. This in turn implies that $\pi(V) = V$. Since we can show that $U$ has no finite orbits on $\Gamma - \Gamma_0$ just as in the abelian case, an application of Theorem 5.1 yields the result that $T$ is weakly stable.

We show, finally, that the conjectured generalization of Halmos's mixing theorem (see Section 3) is true for automorphisms of a compact group $G$. The Cartesian square $\bar{G}$ of $G$ is again a compact group and the normalised Haar measure on $\bar{G}$ is only the product measure. For two subspaces $S_1$ and $S_2$ of $\mathcal{L}_2(G)$, we denote by $S_1 \times S_2$ the closed linear span in $\mathcal{L}_2(\bar{G})$ of functions of the type $f_1(g)\, f_2(h)$ with $f_1 \epsilon S_1$ and $f_2 \epsilon S_2$. Recall that if the Peter-Weyl decomposition of $\mathcal{L}_2(G) = \oplus\, S_\alpha$, $\alpha \epsilon A$, then the Peter-Weyl decomposition of $\mathcal{L}_2(\bar{G}) = \oplus\, \bar{S}_{\alpha,\beta}$ where $\bar{S}_{\alpha,\beta} = S_\alpha \times S_\beta$, $\alpha, \beta \epsilon A$. Also the character of the equivalence class of representations corresponding to the pair $(\alpha, \beta)$ is $\bar{\chi}_{\alpha,\beta}(g, h) = \chi_\alpha(g)\chi_\beta(h)$. It is easy to see that the Cartesian square $\bar{T}$ of an automorphism $T$ of $G$ is an automorphism of $\bar{G}$.

Theorem 5.3 : *An automorphism $T$ of $G$ is weakly stable if and only if $\bar{\mathcal{J}} = \mathcal{J} \times \mathcal{J}$.*

*Proof :* Let $\bar{\mathcal{J}} = \mathcal{J} \times \mathcal{J}$ and $f$ an eigen function with eigen value $\lambda$ for $U$. Then $\bar{f}(g, h) = f(g)\overline{f(h)}$ is a $\bar{T}$-invariant function and hence a.e. section of $\bar{f}$ is $\mathcal{J}$-measurable, i.e., $T$-invariant. Taking a suitable $h$-section we see that $f$ is $T$-invariant and so $\lambda = 1$.

If now $T$ is weakly stable, we know that $\bar{T}$ is weakly stable and by Theorem 5.2, $\bar{P}(= P_{\bar{T}})$ is the direct sum of those $\bar{S}_{\alpha,\beta}$'s which correspond to $\bar{T}$-invariant characters. If the character $\chi_\alpha(g)\chi_\beta(h)$ is $\bar{T}$-invariant, then $\chi_\alpha$ as well as $\chi_\beta$ are $T$-invariant and hence the corresponding subspaces $S_\alpha$ and $S_\beta$ are both contained in $P$. Thus $\bar{P} \subseteq P \times P$. Since obviously $P \times P \subseteq \bar{P}$ we have $\bar{P} = P \times P$. From this it follows immediately that $\bar{\mathcal{J}} = \mathcal{J} \times \mathcal{J}$.

*Added in proof :* Some of the results of this paper have been generalized to semi-groups of transformations by one of the authors.

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