Teleportation of entanglement

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by

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This is to certify that the dissertation entitled "Teleportation of entanglement" has been carried out by Ajay Nair under my guidance and supervision and is accepted in partial fulfillment of the requirement for the degree of Master of Computer Science.

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Chapter 1
Overview

1.1 Introduction

Quantum computation and quantum information is the study of information processing task that can be accomplished using quantum mechanical laws.Like many simple but profound ideas it was a long time before anybody thought of doing information processing using quantum mechanical system.

The story began at the turn of the twentieth century when a revolution was going in science. Several problem arisen in physics. To explain those problems the modern theory of Quantum mechanics was introduced . Since then quantum mechanics has been an indispensable part of Science, and has been applied with enormous success to everything under and inside Sun, including the structure of the atom, superconductors, the structure of DNA, and the elementary particles of Nature.

What is Quantum mechanics? In a word quantum mechanics is a mathematical framework for the construction of physical theory. For example Quantum electrodynamics can describe the interaction of atoms and light with accuracy. Quantum electrodynamics is built under the framework of quantum mechanics. The relation of a particular physical theory like Quantum electrodynamics with quantum mechanics is just as computer operating system is related to a specific application software.

The rules of quantum mechanics are simple but even the experts find them counterintuitive. One of the major goals of quantum information and quantum computation is to develop tools which sharpen our intuition about quantum mechanics. In the early 1980 the interest arose whether it is possible to signal faster than light using quantum mechanics which is a big no-no according to Einstein's theory of relativity. This problem has a nice implication towards another famous problem of quantum mechanics - can we clone an unknown quantum state? If the answer is 'yes' then it is possible to signal faster than light! Fortunately it was proved that unknown quantum state can not be cloned in general - a landmark result of quantum mechanics which effectively supports Einstein's theory of relativity. Another related historical strand contributing to the development of quantum computation and quantum information is the interest dating to the 1970s, of obtaining complete control over single quantum system. Since the 1970s many technique for controlling single quantum state has been developed. Quantum computation and Quantum information naturally fits into this problem. Despite this intense interest, efforts to build quantum information processing systems have resulted in modest success to date. Small quantum computers, capable of doing dozen of operations on a few qubits (state of a two level quantum mechanical system) represent the current art of practical quantum computing. Experimental prototype of quantum cryptography has been demonstrated and has reached in the level of real world application.

We may think of classical information as being embodied in a physical system which has been prepared in a state unknown to us. By performing a measurement to identify the state we acquire the information. We know that the fundamental unit of classical information, a bit, can have value either 0 or 1. We often allows the receiver to have some prior knowledge about the values, say 0 will be received with probability p_0(respectively p_1). Shannon's theory in this scenario gives a precise mathematical quantification of information and leads to great practical interest.

One of the most striking difference between classical and quantum information is the role of measurement. Given a classical bit we can always with certainty tell that if it's value is 0 or 1. No problem. But the scenario is not so simple in case of qubit. Suppose we are given a qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$. The orthonormal basis is given by $\{|0\rangle, |1\rangle\}$.

What does it mean? It means that one can find the qubit in the state $|0\rangle$ with probability $|\alpha|^2$ and can find it in the state $|1\rangle$ with probability $|\beta|^2$ respectively. But to get the information we have to measure the qubit and it will project the state in one of the two orthonormal basis vector and the information about the original qubit will be lost for ever. This type of feature never arise in classical information processing.

Another most striking difference with classical and quantum information is the "No cloning theorem". Given a classical bit we can make as many copies as required of this bit. But in case of qubit quantum mechanics does not allow us to copy an unknown quantum state perfectly!

Despite of all the interesting feature, the most remarkable feature of quantum mechanics is the nonlocal correlation. Nonlocal correlation comes from the feature of "Quantum Entanglement", the heart of quantum information.

Now consider the situation. Assume Alice and Bob are supplied state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Now they are asked about there individual state. What will be the answer? Note one can not factorized as before. So conclusion is that Alice and Bob exist together, they can not be separated. They are entangled.

Now let us see how entanglement can interact with information retrieval. Assume Alice and Bob are now separated far far away. Alice measures his qubit and gets $|0\rangle$. Then Alice immediately know that Bob's state is projected into state $|0\rangle$. So though Alice and Bob are separated by any distance nonlocal correlation always exists between them! Entangled state or quantum correlation is a peculiar property of a multi particle system that cannot be explained by any classical correlation. This phenomena is manifested through the violation of Bell's inequality. Now the question arises whether this entanglement can have any other manifestation which can be used for some physical process like information processing or distant communication. Surprisingly such kind of manifestation exists nature .. by way of teleportation of an unknown quantum state, super dense coding, quantum key distribution. Teleportation of a quantum state is a very powerful application to information processing as quantum states are indecipherable by any physical means. Dense coding is a process which seems to be the opposite of teleportation where by sending a qubit one send 2 bits of information.

Using this type of nonlocal correlation and entanglement Bennett and Brassard developed quantum teleportation. It is the quantum teleportation which dramatically changed quantum information theory and information theorists were stunned by the power of quantum mechanics. Informally speaking quantum teleportation is just analogous to classical fax machine. By quantum teleportation an unknown quantum state can be transferred to a distance party using only local operation and classical communication. Beside teleportation another remarkable protocol namely Quantum dense coding have been developed. Since then rate of progress of quantum information is enormous. And today scientists are trying mostly to quantify quantum entanglement more accurately, effort have been going to make quantum information more quantified just as classical information (thanks to Shannon). Lot of progress have been made in this field.

Perhaps the most spectacular application of quantum correlation is the quantum computer, which could allow, once realized, an exponential increase of computational speed for certain problem, for example the factorisation of large numbers into primes. Again the superposition principle along with entanglement play the principle role. This offers the possibility of massive parallelism in quantum systems as in quantum theory n qubit(two level) systems can represent 2^n numbers simultaneously due to superposition principle and tensor product Hilbert space structure for multi partite system.

The power of quantum computation comes from the above said quantum parallelism. Classically the time taken to do certain computations can be decreased by using parallel processors. To achieve an exponential decrease in time requires an exponential increase in the number of processors, and hence an exponential increase in the amount of physical space needed. As in quantum systems the amount of parallelism increases exponentially with the size of the system, an exponential increase in parallelism requires a linear increase in space needed.

This incredible features of quantum mechanics has started to interact with the world of secured communication, cryptology. As Shor's algorithm can factorize in polynomial time the widely used crypto system like RSA are seems to be already in trouble. Quantum mechanics has shown the way of designing some incredible crypto protocol which can not be broken even by the use of quantum computer. Some of these protocol have been implemented already. We will come back to all those issues more formally later in this work.

Chapter 2
Basic Quantum Mechanics

2.1 Introduction

In this chapter I am going to describe some basic laws and result of quantum mechanics that is required for the study of Quantum information and Quantum computation. In most of the cases I will describe only the required result or some brief proofs that will be useful for our purpose. Intense mathematical treatment will be avoided in some cases. Rather we will rely more on physical and computing technique to describe the results of quantum mechanics.

2.2 Hilbert Space Formulation of Quantum Mechanics

In a word Quantum Mechanics can be described as a mathematical model of physical world. To understand the model properly Hilbert Space formalism was introduced.

Hilbert Space:

1. It is a vector space H defined over $\mathbb{C}$ (space of complex numbers). Vectors will be indicated generally by Dirac notation e.g $|\psi\rangle$.

2. Inner product is defined as $\langle.|.\rangle : H \otimes H \rightarrow \mathbb{C}$ has the following property

(i) Positivity: $\langle\psi|\phi\rangle > 0$;

(ii) Linearity: $\langle\phi|(a|\psi\rangle + b|\varphi\rangle) = a\langle\phi|\psi\rangle + b\langle\phi|\varphi\rangle$.

(iii) Skew symmetry: $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$

3. It is complete in norms:$||\phi|| = \langle\phi|\phi\rangle^{\frac{1}{2}}$

Meaning of quantum state:

Quantum states are encoded version of a physical reality. In Hilbert space they are treated as a vector.

Observables:

Observable is a property of a physical system that in principle can be measured. In quantum mechanics the observables are self adjoint operators.

Let A be an operator. It's adjoint $A^\dagger$ is defined as:

$$\langle\phi|A\psi\rangle = \langle A^\dagger\phi|\psi\rangle.$$

A is said to be self adjoint iff $A = A^\dagger$. Any self adjoint operator has a spectral decomposition in a Hilbert Space. For example A can be written as:

$$A = \Sigma a_n P_n$$

where a_n is the eigen value and P_n is the corresponding orthogonal projector on the space of eigenvectors with the eigen value a_n.

Measurement

In Quantum mechanics measurement of an observable means getting a eigenvalue of the operator as a outcome with a certain probability. The original quantum state is projected onto a eigen state of the corresponding eigen value. e.g

If $A = \Sigma a_n P_n$ then the probability that the measured outcome will be a_n is given by $P = \langle\phi|P_n|\phi\rangle$ where $|\phi\rangle$ is the original quantum state before the measurement. The final state of the system is projected into $\frac{P_n|\phi\rangle}{\langle\phi|P_n\phi\rangle^{1/2}}$.

Dynamics:

The evolution of a quantum state is unitary i.e. it's dynamics is given by a self adjoint unitary operator called it's Hamiltonian (H). The dynamics of a state is given by

$$\frac{d}{dt}|\psi\rangle = -iH|\psi(t)\rangle.$$

2.3 Qubit

The fundamental unit of information in classical information theory is a bit. Which can take value either 0 or 1 at a time. The fundamental unit of information in Quantum information theory is qubit. If we consider two dimensional Hilbert space with two orthogonal vectors $|0\rangle, |1\rangle$ then the most general form of a qubit defined over that hilbert space is given by;

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \text{ where } |\alpha|^2 + |\beta|^2 = 1.$$

The meaning of the qubit is that if we measure $|\psi\rangle$ in the basis $|0\rangle, |1\rangle$ then the probability that $|0\rangle$ is obtained is $|\alpha|^2$ and the probability that $|1\rangle$ is obtained is $|\beta|^2$.

2.4 Density Operator

We starts with an example. Let we are given a two particle state $|\psi\rangle = a|00\rangle + b|11\rangle$. Now if first particle is measured in the basis $\{|0\rangle, |1\rangle\}$ then if we get result $|0\rangle$ (with probability $|a|^2$) we know that the state of the second particle is immediately projected to $|0\rangle$. Similar cases arise for the other case. So we see that the first and second particle are highly correlated. If we know the state of one then we can deterministically say about the state of the second particle.

Now let us consider an observable acting M acting on the first particle. This is expressed as $M \otimes I$. The expected value of the observable in the state $|\psi\rangle$ is given by:

$$\langle\psi|M \otimes I|\psi\rangle = (a^*\langle 0| \otimes \langle 0| + b^*\langle 1| \otimes \langle 1|)(M \otimes I)(a|0\rangle \otimes |0\rangle + b|1\rangle \otimes |1\rangle) = |a|^2\langle 0|M|0\rangle + |b|^2\langle 1|M|1\rangle$$

This expression can be written in the form:

$$\langle M\rangle = Tr(M\rho_1)$$
$$\rho_1 = |a|^2|0\rangle\langle 0| + |b|^2|1\rangle\langle 1|.$$

The operator ρ_1 is called the density operator of the first particle. It is self adjoint, positive, and has unit trace.

2.5 Pure and Mixed state

If the state of a system is a state in the Hilbert space then it is called a pure state otherwise it will be called a mixed state. If a state is pure then the corresponding density operator ρ satisfies $\rho^2 = \rho$. Because this is simple to check for $\rho = |\psi\rangle\langle\psi|$.

In general the density operator can be expressed as:

$\rho = \Sigma_i p_i|\psi_i\rangle\langle\psi_i|$, where $0 \leq p_i \leq 1$. For mixed state there will be two or more terms. So we see that for a mixed state $\rho^2 \neq \rho$.

Chapter 3
Understanding Entanglement

As we have discussed earlier quantum correlation or quantum entanglement is something which is peculiar in quantum mechanics and has no classical analog. Einstein, Podolsky and Rosen tried to show that quantum mechanics is incomplete using some two particle entangled state. But with a great surprise to many, Bell showed not all the results of quantum mechanics can be reproduced by any local realistic theory however hypothetical it may be. Again entangled state was used to show this result.

Let us consider a system of two particles where Alice holds one and Bob holds the other and they be far apart. Let A, A' denote measurements that Alice can make where outcome will be ± 1 only, B, B' denote measurements that Bob can make where outcome will be ± 1 only.

All Local-Realistic theories will produce the following results of measurements of Alice and Bob which will be bounded above and below.

$$-2 \leq [\langle A, B\rangle + \langle A', B\rangle + \langle A, B'\rangle - \langle A', B'\rangle] \leq 2$$

or, $-2 \leq B_{CHSH} \leq 2$ (called Bell-CHSH inequality)
But if we consider Alice and Bob are sharing singlet states
$|\Psi_s\rangle = \frac{1}{\sqrt{2}}[|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B]$

and they choose the observables A, A', B, B' in such a way that $\hat{a}, \hat{a'}, \hat{b}, \hat{b'}$ lie in the same plane with $\frac{\pi}{4}$ radians seperating $\hat{a'}, \hat{b}$ and $\hat{b}, \hat{a}$ and $\hat{a}, \hat{b'}$.

Then the above inequality will become $-2\sqrt{2}$, a direct contradiction. But for any product state or for any mixture of product states the above inequality is satisfied. Now the question arises: Do all entangled states violate Bell's inequality and in this sense non-local?

Surprisingly there exist states namely which although entangled satisfy Bell's inequality. Let us give an example of such a state which is called the Werner state. $W_F = F|\Psi_s\rangle\langle\Psi_s| + \frac{1-F}{4} I \otimes I$

For $0 \leq F \leq \frac{1}{\sqrt{2}}, W_F$ satisfies Bells inequality. Now it will be shown that for $\frac{1}{3} \leq F \leq \frac{1}{\sqrt{2}}$ the Werner state is entangled but satisfies Bell's inequality. We now

construct a unitary operator called flip operator in the following way:
V on a $2 \otimes 2$ system acts in the following way
$V[|\psi\rangle \otimes |\phi\rangle] = |\phi\rangle \otimes |\psi\rangle$

Hence for all product states
$\langle\psi| \otimes \langle\phi|V|\phi\rangle \otimes |\psi\rangle \geq 0$

Hence for all product states ρ_s, $\mathrm{Tr}(\rho_s V) \geq 0$.

If for any density matrix of a $2 \otimes 2$ system ρ, shows $\mathrm{Tr}(\rho_s V) < 0$ then ρ is definitely entangled.
In case of Werner states W_F, $Tr(W_F V) < 0$ for $\frac{1}{3} < F \leq 1$.
But there is violation of Bell's inequality for $F > \frac{1}{\sqrt{2}}$.

Hence, if for a given density matrix, Bell's inequality is satisfied, the state might have quantum correlation (i.e. entanglement). But just now we constructed the flip operator which can sense entanglement. It is not true that if $\mathrm{Tr}(\rho_s V) \geq 0$ means that ρ is separable. Consider the following state
$\rho_p = p|\Psi_s\rangle\langle\Psi_s| + (1-p)|0\rangle|0\rangle\langle 0|\langle 0|$, for $0 \leq p \leq \frac{1}{2}$, $\mathrm{Tr}(\rho_p V) \geq 0$. But, we will show later that for $p > 0$ this state ρ_p is entangled.
We will now proceed to understand entanglement by using some necessary conditions for separability of Mixed states in general. A breakthrough result is, a separable state remains a positive operator if subjected to a partial transposition, (say on particle 2). A separable state is $\rho_s = \Sigma_i w_i P_i^{(1)} \otimes P_i'^{(2)}$
where 1 and 2 denote the particle or system label. Taking the partial transpose on ρ_s with respect to particle 2, we get
$\rho_s' = \Sigma_i w_i P_i^{(1)} \otimes (P_i'^{(2)})^T$
T denotes the usual transpose operation. After the transpose operation a positive operator remains a positive operator. Hence, $\rho_s = \Sigma_i w_i P_i^{(1)} \otimes Q_i^{(2)} = \rho_s^{T_2}$ where Q_i is also a positive operator.
So for all separable states partial transposition keeps it positive.

We now show an example of a state which is not separable and does not remain positive after partial transposition is done over it. Any density matrix in some basis $|ij\rangle\langle kl|$ can be written as $\rho_{12} = \rho_{ij,kl}|ij\rangle\langle kl|$
$\rho^{T_2} = \rho_{il,kj}|ij\rangle\langle kl|$
Now take $\rho_p^{12} = p|\Psi_s\rangle\langle\Psi_s| + (1-p)|00\rangle\langle 00|$.

$\rho_p^{(12)} =$

$$\begin{pmatrix} 1-p & 0 & 0 & 0 \\ 0 & p/2 & -p/2 & 0 \\ 0 & -p/2 & p/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\rho_p^{(T_2)} =$

$$\begin{pmatrix} 1-p & 0 & 0 & -p/2 \\ 0 & p/2 & 0 & 0 \\ 0 & 0 & p/2 & 0 \\ -p/2 & 0 & 0 & 0 \end{pmatrix}$$

which is definitely negative for $p > 0$. So, ρ_p is entangled for $p > 0$. Interestingly, NPT(non negative under partial transposition) density matrices are entangled and this criterion is necessary and sufficient for $2 \otimes 2$ and $2 \otimes 3$ systems.

Chapter 4

Quantum teleportation and dense coding

4.1 Introduction

Before starting discussion on Quantum Information theory let me quickly review classical information theory very briefly. The basic need of information theory is to encode some amount of news (information) by some means and to decode the encoded version to retrieve the news when required.

Whatever paradigm we choose this is the basic objective of studying Information theory. In classical information theory the fundamental unit of information is bit. We encode an amount of information by the classical bits which can be either 0 or 1 at a time but essentially not both.

To decode the information essential strategy of the receiver is to measure the bit which we can always do easily classically.

The receiver can also have some prior knowledge of the outcome. Say the receiver knows that the probability that 0 will occur is p_0 and the priori probability of 1 is p_1. In this scenario Shannon's remarkable theory gives precise mathematical quantification.

These features of classical information differs dramatically in case of quantum information. The fundamental unit of quantum information is a qubit. In a two level system with orthonormal basis $\{|0\rangle, |1\rangle\}$ the most general form of a qubit is given by $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where $|\alpha|^2 + |\beta|^2 = 1$. The information is encoded in α and β. In contrast to the classical physics, quantum measurement theory places several limitation over the amount of information that we can extract. Most remarkable fact is that although most of the information is inaccessible but still it is useful. And this feature of quantum information makes a huge impact on cryptography and quantum computing.

Another great difference of quantum information with classical information is that like classical information quantum information can not be copied. The fact is guaranteed by one celebrated result of quantum mechanics namely "**No Cloning Theorem**".

The most remarkable difference between classical information and quantum information is due to **Quantum entanglement**. In the chapter 1, I have already described what is meant by quantum entanglement. If two classical system interacts once and then they are kept few light years apart. Can measurement over one of them effect the state of the other? certainly the answer is 'No'. But if we allow two quantum system to interact in such a way that quantum en-

tanglement is established between them, and then we separate them as many **light years** as we wish, the measurement on any one of them changes the state **of the other**. The magical "**Quantum Entanglement**" has no classical counterpart. Quantum entanglement is the heart of quantum information. I will describe two of the major breakthrough results namely "**Quantum Teleportation**" and "**Quantum Dense Coding**", which was made possible because of entanglement, in the upcoming section.

4.2 Quantum no cloning theorem:

Theorem: Unknown quantum state can not be copied

Proof: Let if possible there exists an unitary operator U which is a cloning operator.

$$U(|0\rangle, |\phi\rangle) = |0\rangle|0\rangle$$
$$U(|1\rangle, |\phi\rangle) = |1\rangle|1\rangle$$

In the above expressions $|\phi\rangle$ is the state on which the wanted state will is copied. Now let us apply the linear superposition principle on U. If U exists then the following must be true:

$$U(a|0\rangle + b|1\rangle, |\phi\rangle) = aU(|0\rangle, |\phi\rangle) + bU(|1\rangle, |\phi\rangle) = a|0\rangle|0\rangle + b|1\rangle|1\rangle$$

Which is in general not equal to $(a|0\rangle + b|1\rangle)(a|0\rangle + b|1\rangle)$.

Hence universal quantum cloning machine does not exists.

4.3 Unitary Transformation

Before describing further result of quantum information let me first state some useful single qubit & two qubit quantum state transformations.

Single qubit quantum state transformations: {I , X , Y , Z , H }

$$\mathbf{I} : |0\rangle \rightarrow |0\rangle$$
$$\mathbf{I} : |1\rangle \rightarrow |1\rangle$$

$$\mathbf{X} : |0\rangle \rightarrow |1\rangle$$
$$\mathbf{X} : |1\rangle \rightarrow |0\rangle$$

$$\mathbf{Y} : |0\rangle \rightarrow |1\rangle$$
$$\mathbf{Y} : |1\rangle \rightarrow -|0\rangle$$

$$\mathbf{Z} : |0\rangle \rightarrow |0\rangle$$
$$\mathbf{Z} : |1\rangle \rightarrow -|1\rangle$$

$$\mathbf{H} : |0\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$\mathbf{H} : |1\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Double qubit quantum state transformations: Controlled Not (CNOT)

CNOT is a two qubit quantum gate where the first bit is taken as control bit. If the control bit is 1 then the gates flips the second bit and if the control bit is 0 the gate keeps the second bit as it is. In both the cases it keeps control bit unchanged.

Mathematically this transformation is given by:

$$C_{not} : |00\rangle \rightarrow |00\rangle$$
$$C_{not} : |01\rangle \rightarrow |01\rangle$$
$$C_{not} : |10\rangle \rightarrow |11\rangle$$
$$C_{not} : |11\rangle \rightarrow |10\rangle$$

4.4 Quantum Dense Coding

Say Alice receives two classical bits, encoding the number 0 to 3. If Alice wants to send this information to a distant separated Bob then he has to send 2 classical bit of information. Surprisingly enough Alice can do the same job by sending only one quantum bit (qubit) through quantum channel if he shares one maximally entangled pair with Bob.

Depending on the information Alice has to send Alice performs one of the transformation $\{I, X, Y, Z\}$ on his qubit of the entangled pair $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The resulting state is shown in the following table.

Value	Transformation	New state
0	**I**	$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$
1	**X**	$\frac{1}{\sqrt{2}}(\|10\rangle + \|01\rangle)$
2	**Y**	$\frac{1}{\sqrt{2}}(-\|10\rangle + \|01\rangle)$
3	**Z**	$\frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$

Alice then sends his qubit to the Bob.

Now Bob applies a controlled not gate to two qubits of entangled pair.

Initial state	Controlled not	First bit	Second bit
$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	$\frac{1}{\sqrt{2}}(\|00\rangle + \|01\rangle)$	$\frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$	$\|0\rangle$
$\frac{1}{\sqrt{2}}(\|10\rangle + \|01\rangle)$	$\frac{1}{\sqrt{2}}(\|11\rangle + \|01\rangle)$	$\frac{1}{\sqrt{2}}(\|1\rangle + \|0\rangle)$	$\|1\rangle$
$\frac{1}{\sqrt{2}}(-\|10\rangle + \|01\rangle)$	$\frac{1}{\sqrt{2}}(-\|11\rangle + \|01\rangle)$	$\frac{1}{\sqrt{2}}(-\|1\rangle + \|0\rangle)$	$\|1\rangle$
$\frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$	$\frac{1}{\sqrt{2}}(\|00\rangle - \|10\rangle)$	$\frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$	$-\|0\rangle$

Bob can now measure his qubit without disturbing the quantum state. If the measurement returns $|0\rangle$ then the encoded version was either 0 or 3. And if the measurement returns $|1\rangle$ then it was either 1 or 2.

Then Bob apply Hadamard transform on the first qubit and measures it to distinguish between 0,3 and 1,2.

4.5 Quantum Teleportation

The BBCJPW[9] protocol illustrates the following method. Alice and Bob are two distant separated parties sharing some a priori maximally entangled state say $|\varphi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Alice has a qubit $|\chi\rangle = \alpha|0\rangle + \beta|1\rangle$ which she wants to send to Bob.

What Alice does is that she operates $|\chi\rangle$ with her half of the entangled pair i.e., the state is given by:

$|\chi\rangle \otimes |\varphi\rangle = \frac{1}{\sqrt{2}}\{\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle\} = \frac{1}{2}\{\phi^+(\alpha|0\rangle + \beta|1\rangle) + \phi^-(\alpha|0\rangle - \beta|1\rangle) + \psi^+(\alpha|1\rangle + \beta|0\rangle) + \psi^-(\alpha|1\rangle - \beta|0\rangle)\}$.

Now Alice does bell state measurement on her two qubits and gets outcome either of $\phi^+, \phi^-, \psi^+, \psi^-$ with equal probability. Then Alice makes a phone call to Bob to indicate her outcome i.e., Alice uses two classical bits of information to tell Bob either of the four outcomes.

Bob can now easily retrieve $|\chi\rangle$ accurately by the following decoding process:

Alice's outcome	Bob's state	Decoding
ϕ^+	$\alpha\|0\rangle + \beta\|1\rangle$	I
ϕ^-	$\alpha\|0\rangle - \beta\|1\rangle$	Z
ψ^+	$\alpha\|1\rangle + \beta\|0\rangle$	X
ψ^+	$\alpha\|1\rangle - \beta\|0\rangle$	Y

After the decoding process Bob can successfully retrieve the original state $|\chi\rangle$ to him. Hence by utilizing prior shared entanglement Alice is able to communicate the full quantum information of $|\chi\rangle$ by transmitting merely two bits of classical information to him.

For teleporting an arbitrary unknown qubit the BBCJPW protocol requires one maximally entangled channel and two bits of classical communication. But what if some a priori information is available about the qubit to be teleported? Ghosh et al[8] have shown that even if the state to be teleported is known to be one of the two non commuting qubits, the channel must be maximally entangled i.e., the sender and receiver must share one ebit.

Chapter 5
Teleportation of entanglement

5.1 Our Goal

Suppose instead of a single qubit an arbitrary state of two qubits is to be teleported. The arbitrary state may or may not be entangled. Our motivation here is to sort out the protocols, channels for teleporting arbitrary entangled states in particular of two qubits. The resources required in terms of channel entanglement(ebits) and classical communication (cbits) shall also be investigated.

5.2 Situations regarding requirement of resources for teleporting entanglement using LOCC on the subsystems

Consider now a different situation. A source delivers an arbitrary two qubit entangled state to Alice which must be finally shared between Bob_1 and Bob_2. Instead of state teleportation Alice has the task of entanglement teleportation. It would suffice if Alice shares a maximally entangled state with Bob_1 and another with Bob_2. Alice would then just teleport the two qubits using the BBCJPW protocol. But what if Alice shares with Bob_1-Bob_2, less than two ebits of entanglement? Suppose Alice shares with Bob_1 and Bob_2 the GHZ state. $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. Would the same feat be possible now? Gorbachev and Trubilko[10] have considered this case and shown that if the state has been prepared in the Schmidt basis $\{|0'0''\rangle, |1'1''\rangle\}$ i.e., if Alice knows that the state is of the form $|\chi\rangle = \alpha|0'0''\rangle + \beta|1'1''\rangle$ with known $|0'0''\rangle, |1'1''\rangle$ but unknown Schmidt coefficients α, β then this state can be made to share between Bob_1 and Bob_2.

Ghosh et al[12] have shown that even if Alice and the two Bobs share the state $|ghz\rangle = \frac{1}{\sqrt{2}}(|0\phi 0\rangle + |1\phi' 1\rangle)$ where $|\phi\rangle$ and $|\phi'\rangle$ are not necessarily orthogonal, it is possible for Alice and Claire to make the two Bobs share the state $|\chi'\rangle = \alpha|\phi 0'\rangle + \beta|\phi' 1'\rangle$.

5.3 Description of exact teleportation of an arbitrary entangled state of two qubits via two maximally entangled channels by using the BBCJPW protocol

Suppose $Alice_1$ and $Alice_2$ share with Bob_1 and Bob_2 two maximally entangled channels in state $|\psi^-\rangle$. In addition $Alice_1$ has qubit C_1 and $Alice_2$ has qubit C_2 where C_1 and C_2 are arbitrary entangled state. Thus systems A_1 and C_1 are in

Alice$_1$'s possession, A_2 and C_2 with Alice$_2$, B_1 with Bob$_1$ and B_2 with Bob$_2$. Now if Alice$_1$ and Alice$_2$ perform a Bell measurement on their systems C_1A_1 and C_2A_2 and communicate classically the results to Bob$_1$ and Bob$_2$ respectively who then apply the respective unitary transformations resulting in the original entangled state of C_1 and C_2 to be shared between Bob$_1$ and Bob$_2$. The number of ebits required is two and the cbits required are four.

5.4 Teleportation of entanglement via GHZ states

Suppose Alice shares with Bob$_1$-Bob$_2$ system less than two ebits of entanglement, as in the case of the GHZ state, $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. Gorbachev and Trubilko[10] have considered this case and shown that if the state has been prepared in the Schmidt basis $\{|0'0''\rangle, |1'1''\rangle\}$ i.e., if Alice knows that the state is of the form $|\chi\rangle = \alpha|0'0''\rangle + \beta|1'1''\rangle$ with known $|0'0''\rangle$, $|1'1''\rangle$ but unknown Schmidt coefficients α, β then this state can be made to share between Bob$_1$ and Bob$_2$.

5.5 Teleportation of entanglement via two same Bells mixture channels

Consider the following situation[13] $\rho_{BM} = \omega P[|\phi^+\rangle] + (1-\omega)P[|\phi^-\rangle]$,$|\phi_{12}\rangle = |\phi\rangle = \alpha|00\rangle + \beta|11\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$. Now both pairs Alice$_1$ and Bob$_1$ and Alice$_2$ and Bob$_2$ share the channel in the state ρ_{BM}. Then Alice$_1$, Alice$_2$, Bob$_1$ and Bob$_2$ together follow the teleportation protocol P_B. The output state $\sigma_{B_1B_2}$ is $|\alpha|^2|00\rangle\langle 00| + (2\omega-1)^2\alpha\beta^*|00\rangle\langle 11| + (2\omega-1)^2\beta\alpha^*|11\rangle\langle 00| + |\beta|^2|11\rangle\langle 11|$, which remain inseparable for each $\omega \neq \frac{1}{2}$. Interestingly only for ω= 1/2 does the channel itself become separable.

Chapter 6

Searching for channel states as well as protocols with lower entanglement as resource

6.1 Smolin's state

Smolin[11] discovered a state shared between four parties which has some interesting properties. If four parties are kept apart, no entanglement can be distilled between any two parties by LOCC. In any 2 parties vs 2 parties cut the state is separable. We shall study the teleportation of entanglement through Smolin's state. It will be shown that any two parties sharing the Smolin's state can teleport some pure entanglement (though not exactly)in a proper basis. Then we study and compare their initial and final entanglement. After that we study the teleportation of entanglement through the channel state $\omega P[|\phi^+\rangle \otimes |\phi^+\rangle] + (1-\omega) P[|\phi^+\rangle \otimes |\phi^+\rangle]$ and will show that it teleports all pure entangled states in a given basis exactly. It also teleports entanglement in all maximally entangled states (in any basis).

6.2 Teleportation of entanglement via Smolin's state

Consider the Smolin's state [11] shared between four parties Alice$_1$, Alice$_2$, Bob$_1$ and Bob$_2$

$\rho^{(s)}_{A_1B_1C_1D_1} = \frac{1}{4}[P[|\phi^+\rangle \otimes |\phi^+\rangle] + P[|\phi^-\rangle \otimes |\phi^-\rangle] + P[|\psi^+\rangle \otimes |\psi^+\rangle] + P[|\psi^-\rangle \otimes |\psi^-\rangle]]$

and consider the state that to be teleported as

$$|\psi\rangle_{AB} = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

where Alice$_1$ has qubits A$_1$ and A in her possession and Alice$_2$ has qubits B$_1$ and B in her possession. The joint state state is given by $P[|\psi\rangle_{AB}] \otimes \rho^{(s)}_{A_1B_1C_1D_1}$.

Now Alice$_1$ and Alice$_2$ perform Bell measurement on A,A$_1$ and B,B$_1$ and communicate the result via classical communication to Bob$_1$ and Bob$_2$ respectively using the BBCJPW protocol who then perform the corresponding unitary transformations on their qubits, namely C$_1$ and D$_1$.

Protocol using the result of the Bell Measurement.

Measurement	Operator
$\|\phi^+\rangle$	I

$\|\phi^-\rangle$	σ_z
$\|\psi^+\rangle$	σ_x
$\|\psi^+\rangle$	$i\sigma_y$

The final state shared by Bob_1 and Bob_2 is a Bell mixture:
$\sigma_{C_1D_1} = |\frac{\alpha+\delta}{\sqrt{2}}|^2 P[|\phi^+\rangle] + |\frac{\alpha-\delta}{\sqrt{2}}|^2 P[|\phi^-\rangle] + |\frac{\beta+\gamma}{\sqrt{2}}|^2 P[|\psi^+\rangle] + |\frac{\beta-\gamma}{\sqrt{2}}|^2 P[|\psi^-\rangle]$

We find that if the initial state is any of the four Bell states in either $(|00\rangle, |11\rangle)$ basis or $(|01\rangle, |10\rangle)$ basis then it is teleported exactly. The entanglement of initial state is

$$E(|\psi_{AB}\rangle) = -\lambda_1 log\lambda_1 - \lambda_2 log\lambda_2$$

where $\lambda_1, \lambda_2 = \frac{1\pm\sqrt{1-4|\beta\gamma-\alpha\delta|^2}}{2}$

The output state has entanglement of formation
$E(\sigma_{C_1D_1}) = \varepsilon(C(\rho))$
where $\varepsilon(x) = -\frac{1+\sqrt{1-x^2}}{2} log_2 \frac{1+\sqrt{1-x^2}}{2} - \frac{1-\sqrt{1-x^2}}{2} log_2 \frac{1-\sqrt{1-x^2}}{2}$
and $x = C(\rho) = max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$
where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$

$$\sigma_{C_1D_1} = \begin{pmatrix} |\frac{\alpha+\delta}{\sqrt{2}}|^2 & & & \\ & |\frac{\alpha-\delta}{\sqrt{2}}|^2 & & \\ & & |\frac{\beta+\gamma}{\sqrt{2}}|^2 & \\ & & & |\frac{\beta-\gamma}{\sqrt{2}}|^2 \end{pmatrix}$$

Hence the class of input states for which the output state is entangled, satisfy $\lambda_1 \geq \lambda_2 + \lambda_3 + \lambda_4$ where λ_i's are the eigenvalues of the above matrix.

Case for input states with known Schmidt basis.

Take the input state $|\Psi_\lambda\rangle_{AB} = \sqrt{\lambda}|\phi\rangle_A \otimes |\chi\rangle_B + \sqrt{1-\lambda}|\phi^\perp\rangle_A \otimes |\chi^\perp\rangle_B$

Now $\rho^s{}_{A_1B_1C_1D_1} = \frac{1}{4}\Sigma^4{}_{i=1} P[|B_i\rangle_{A_1B_1} \otimes |B_i\rangle_{C_1D_1}]$ where
$|B_1\rangle = |\phi^+\rangle, |B_2\rangle = |\phi^-\rangle, |B_3\rangle = |\psi^+\rangle, |B_4\rangle = |\psi^-\rangle$

Also,
$|\phi\chi\rangle + |\phi^\perp\chi^\perp\rangle = |B_1'\rangle$

$|\phi\chi\rangle - |\phi^{\perp}\chi^{\perp}\rangle = |B_2'\rangle$
$|\phi\chi^{\perp}\rangle + |\phi^{\perp}\chi\rangle = |B_3'\rangle$
$|\phi\chi^{\perp}\rangle - |\phi^{\perp}\chi\rangle = |B_1'\rangle$

The transformation of basis $|B_i\rangle \mapsto |B_i'\rangle$ is by the following unitary transformations.

$U_{A_1}|0\rangle_{A_1} = |\phi\rangle_{A_1}$
$U_{A_1}|1\rangle_{A_1} = |\phi^{\perp}\rangle_{A_1}$
$U_{B_1}|0\rangle_{B_1} = |\chi\rangle_{B_1}$
$U_{B_1}|1\rangle_{B_1} = |\chi^{\perp}\rangle_{B_1}$
$U_{C_1}|0\rangle_{C_1} = |\phi\rangle_{C_1}$
$U_{C_1}|1\rangle_{C_1} = |\phi^{\perp}\rangle_{C_1}$
$U_{D_1}|0\rangle_{D_1} = |\chi\rangle_{A_1}$
$U_{D_1}|1\rangle_{D_1} = |\chi^{\perp}\rangle_{A_1}$

$$\rho^{s'} = U_{A_1} \otimes U_{B_1} \otimes U_{C1} \otimes U_{D_1} \quad \rho^{s} \quad U^{\perp}{}_{A_1} \otimes U^{\perp}{}_{B_1} \otimes U^{\perp}{}_{C_1} \otimes U^{\perp}{}_{D_1},$$
$$\rho^{s'} = \tfrac{1}{4}\Sigma^4{}_{i=1} P[|B_i'\rangle_{A_1B_1} \otimes |B_i'\rangle_{C_1D_1}].$$

$$|\Psi\rangle_{AB} = \tfrac{\sqrt{\lambda}+\sqrt{1-\lambda}}{\sqrt{2}}|B_1'\rangle_{AB} + \tfrac{\sqrt{\lambda}-\sqrt{1-\lambda}}{\sqrt{2}}|B_2'\rangle_{AB}.$$

The output

$$\rho_{C_1D_1} = |\tfrac{\sqrt{\lambda}+\sqrt{1-\lambda}}{\sqrt{2}}|^2 P[|B_1'\rangle] + |\tfrac{\sqrt{\lambda}-\sqrt{1-\lambda}}{\sqrt{2}}|^2 P[|B_2'\rangle]$$

Now $|\Psi\rangle_{AB} = \sqrt{\lambda}|\phi_A\rangle \otimes |\chi_B\rangle + \sqrt{1-\lambda}|\phi_A^{\perp}\rangle \otimes |\chi_B^{\perp}\rangle$

$$\rho_A = \lambda|\phi_A\rangle\langle\phi_A| + (1-\lambda)|\phi_A^{\perp}\rangle\langle\phi_A^{\perp}|$$

The corresponding eigenvalues are $\lambda, 1-\lambda$.

$\therefore \quad E(|\Psi_{AB}\rangle) = -\lambda log\lambda - (1-\lambda)log(1-\lambda)$

The final state between Bob_1 and Bob_2 is

$$\rho_{C_1D_1} = |\tfrac{\sqrt{\lambda}+\sqrt{1-\lambda}}{\sqrt{2}}|^2 P[|B_1'\rangle] + |\tfrac{\sqrt{\lambda}-\sqrt{1-\lambda}}{\sqrt{2}}|^2 P[|B_2'\rangle]$$

Now $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$

in our case $\rho^* = \rho$.

Here $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

and $R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}} \;=\; \rho \quad$ as $\tilde{\rho} = \rho$

R gives the eigenvalues $\omega_1, \omega_2, \omega_3, \omega_4 = |\frac{\sqrt{\lambda}+\sqrt{1-\lambda}}{\sqrt{2}}|^2, |\frac{\sqrt{\lambda}-\sqrt{1-\lambda}}{\sqrt{2}}|^2, 0, 0$

Now $E(\rho) = \varepsilon(C(\rho))$ where

$$\varepsilon(x) = -\frac{1+\sqrt{1-x^2}}{2}log_2\frac{1+\sqrt{1-x^2}}{2} - \frac{1-\sqrt{1-x^2}}{2}log_2\frac{1-\sqrt{1-x^2}}{2}$$
$$where \;\; x = C(\rho) = max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$$

Also $\omega_1 > \omega_1 + \omega_2 + \omega_3$

$$\therefore x \;=\; 2\sqrt{\lambda - \lambda^2}$$

$$E(\rho) = -\lambda log_2\lambda - (1-\lambda)log_2(1-\lambda)$$

or that the entanglement is preserved in case of teleporting a state in known Schmidt basis. Though the teleportation of the pure entangled state in a given basis has been changed to a mixed state the entanglement of formation(which is a measure of entanglement) remains unchanged.

6.3 Teleportation of entangled states via the state:

Consider $\rho_{A_1B_1C_1D_1} = \omega P[|\phi^+\rangle_{A_1C_1} \otimes |\phi^+\rangle_{B_1D_1}] + (1-\omega)P[|\phi^-\rangle_{A_1C_1} \otimes |\phi^-\rangle_{B_1D_1}]$

This state has the interesting property that in a 2 party vs 2 party cut it has 1 ebit of entanglement. Consider the input state

$|\Psi\rangle_{AB} = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$ which is to be teleported.
Using Bell measurement and LOCC the output state is

$$\sigma_{C_1D_1} = \omega P[\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle] + (1-\omega)P[\alpha|00\rangle - \beta|01\rangle - \gamma|10\rangle + \delta|11\rangle]$$

Now $\sigma_{C_1D_1}$ is entangled iff the self adjoint matrix $\sigma^{T_{B_1}}{}_{C_1D_1}$ has atleast one negative eigenvalue.

$\sigma^{T_{B_1}}{}_{C_1D_1} =$

$$\begin{pmatrix} |\alpha|^2 & (2\omega-1)\alpha\beta^* & (2\omega-1)\gamma\alpha^* & \gamma\beta^* \\ (2\omega-1)\beta\alpha^* & |\beta|^2 & \delta\alpha^* & (2\omega-1)\delta\beta^* \\ (2\omega-1)\alpha\gamma^* & \alpha\delta^* & |\gamma|^2 & (2\omega-1)\gamma\delta^* \\ \beta\gamma^* & (2\omega-1)\beta\delta^* & (2\omega-1)\delta\gamma^* & |\delta|^2 \end{pmatrix}$$

Now for the case when $\omega = 1/2$ the state $\rho_{A_1B_1C_1D_1}$ is not entangled in general and not all input entangled states give the corresponding states entangled. An example of this would be the maximally entangled state

$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$ where

$\alpha = \frac{1}{\sqrt{2}}e^{i\theta}$

$\beta = \frac{1}{\sqrt{2}}e^{i\omega}$

$\gamma = \frac{1}{\sqrt{2}}e^{i\theta+\phi-\omega}$

$\delta = \frac{1}{\sqrt{2}}e^{i\phi}$

where the output is not entangled.

Now interestingly one can teleport pure entanglement either in $(|00\rangle, |11\rangle)$ basis or $(|01\rangle, |10\rangle)$ basis exactly for any ω. For $\omega \neq 1/2$, let us choose,

$\alpha = r_1 e^{i\theta_1}$

$\beta = r_2 e^{i\theta_2}$

$\gamma = r_3 e^{i\theta_3}$

$\delta = r_4 e^{i\theta_4}$

and $\theta = \theta_1 + \theta_4 - \theta_2 - \theta_3$

The eigenvalue expression gives us:

$$\lambda^4 - \lambda^3 + 4\omega(1-\omega)(r_1^2 + r_4^2)(r_2^2 + r_3^2)\lambda^2 - \lambda[(r_2^2 r_3^2 - r_1^2 r_4^2)(r_1^2 + r_4^2 - r_2^2 - r_3^2) + 2(2\omega-1)^2 r_1 r_2 r_3 r_4 Cos\theta - r_1^2 r_4^2(r_2^2 + r_3^2) - r_2^2 r_3^2(r_1^2 + r_4^2)] - [(r_1 r_4 - r_2 r_3)^2 + 4(2\omega-1)^2 r_1 r_2 r_3 r_4 Sin^2\theta/2][(r_1 r_4 + r_2 r_3)^2 - 4(2\omega-1)^2 r_1 r_2 r_3 r_4 Cos^2\theta/2] = 0$$

Clearly by analysis of sign the expression always has a negative eigenvalue unless the original state is itself not entangled. This indicates that the output is entangled if the input is entangled.

6.4 Entanglement in the channel state $\rho_{A_1B_1C_1D_1}$ in the A_1B_1 vs C_1D_1 cut

Now what about the entanglement in the channel state $\rho_{A_1B_1C_1D_1}$ in the A_1B_1 vs C_1D_1 cut? There is a standard result that distillable entanglement is bounded above by log neg calculated for the state $\rho_{A_1B_1C_1D_1}$ where log neg is defined as the sum of the negative eigenvalues of the partially transposed matrix w.r.t either A_1B_1 or C_1D_1.

log neg (E_n) has been calculated for $\rho_{A_1B_1C_1D_1}$ and it was found to be $E_n = 2 - H(p)$ or distillable entanglement E_d of $\rho_{A_1B_1C_1D_1}$ should satisfy
$E_d \leq 2 - H(p)$

Again there is a well known conjecture by Eisert et.al [14] which tells that if Alice and Bob share some pair of qubits, with some classical information about the state, then if they lose that classical information quantified as DI, then the amount of distillable entanglement they lose is related to DI as
$\Delta E_D \leq DI$

Using this result one can show that E_D for $\rho_{A_1B_1C_1D_1}$satisfies (here, $p = \omega$
$E_d \geq 2 - H(p)$
So
$E_d = 2 - H(p)$
and $E_d > 1$ for $p \neq 1/2$
SO the channel state with $p \neq 1/2$ has distillable entanglement more than 1 ebit and interestingly in these cases only $\sigma_{C_1D_1}$ remains entangled for all input pure entangled states.

6.5 Discussion:

If A_1, B_1, C_1, D_1 all are far apart and when A_1B_1 share some pure entanglement, then, let A_1B_1 try to teleport the state to C_1D_1 so that the teleported state remains entangled for all input pure entangled states. The relevant question is what is the required entanglement of the channel state $\rho_{A_1B_1C_1D_1}$ in the cut A_1B_1 vs C_1D_1
One obvious solution is
$\rho_{A_1B_1C_1D_1} = |\phi^+\rangle_{A_1C_1}\langle\phi^+| \otimes [x|\phi^+\rangle\langle\phi^+| + \frac{1-x}{4} I \otimes I]_{B_1D_1}$

For this channel and standard Bennett protocol all pure entangled states between A_1B_1 will be teleported to C_1D_1 keeping some entanglement. Obviously the entanglement of the above channel state E_D is greater than 1.
But it is an asymmetric channel state. Our previous example is a symmetric channel state but one thing is common. Required entanglement should be greater than 1 ebit. We conjecture that for some distant parties, if two parties want to teleport some pure entanglement(inexactly with the aim that the teleported pair remains entangled) then more than 1 ebit of entanglement is necessary.

Chapter 7

Bibliography

For Further reading:

[1] *Quantum computation and Quantum information* by Michael A. Nielsen and Isacc L. chuang.

[2] *Introduction to quantum computation and information* edited by T Spliller, H.-K. Lo and S.Popescu.

[3] *Quantum computing* by J.Gruska.

[4] John Preskill's lecture notes in quantum information and quantum computation available at http://www.theory.caltech.edu/people/preskill/ph229.

[5] Umesh Vazirani's lecture notes in quantum computation available at http://www.cs.berkeley.edu/ vazirani.

[6] Many useful papers are available regularly at http://xxx.lanl.gov/archive/quant-ph.

[7] R.P. Feynman, Int. J. Theor. Phys. 21,467 (1982).

For our problem

[8] Ghosh.et.al., quant-ph/0012118 Teleporting non commuting qubits requires maximal entanglement

[9] C.H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres and W.K Wotters, Phys. Rev. Lett. 70,1895 (1993).

[10] Gorbachev V.N and Trubilko A I 2000 Quantum teleportation of EPR pair by three party entanglement

[11] J.A. Smolin, quant-ph/0001001

[12] Ghosh.et.al.,New Jr of physics;4(2002) p 48.1-48.9

[13] Ghosh.et.al., Physical Review A 66,024301 (2002)

[14] J.Eisert,T.Felbinger,P.Papadopoulos,M.B.Plenio,M.Wilkens, Phys. Rev. Lett. 84,1611 (2000)