Dedicated to

my parents

Acknowledgment

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I feel a deep sense of gratitude for my family who raised me with their unconditional love and taught me to always believe in myself.

Finally, I would also like to thank all the readers of this work, since any piece of academia is useful if it is read and understood by others so that it can become a bridge for further research.

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Chapter 1

Introduction

Proximity graph [JT92; Lio; Tou91] is a graph where the edges between the vertices of the graph depends on the neighborliness of vertices. *Proximity graph* can be intuitively defined as follows: given a point set P in the plane, the vertices of the graphs, there is an edge between a pair of vertices $p, q \in P$ if they satisfy some particular notion of neighborliness.

Proximity graphs can be used in shape analysis and in data mining [JT92; Tou]. In graph drawing, a problem related to proximity graphs is to find the classes of graphs that admit a proximity drawing for some notion of proximity, and whenever possible to efficiently decide, for a given graph, whether such a drawing exists [BETT99; Lio].

In the case of *Gabriel graphs,* GG(P), the notion of neighborliness of a pair of vertices a, b is the closed disk D_{ab} with diameter $\overline{ab}$. An edge ab is in the Gabriel graph of a point set P if and only if $P \cap D_{ab} = \{a, b\}$ (see Figure 1.1(Left)) [ADH10]. Gabriel graphs were introduced by Gabriel and Sokal [GS69] in the context of geographic variation analysis.

In the case of *Delaunay graphs,* DG(P) [ADH11], the region of influence of a pair of vertices a, b is the set of closed disks D_{ab} with chord $\overline{ab}$. An edge ab is in the Delaunay graph of a point set P if and only if there exists a disk $d_{ab} \in D_{ab}$ such that $P \cap d_{ab} = \{a, b\}$

In this thesis, we consider the problems related to the *Witness* graphs (gen-

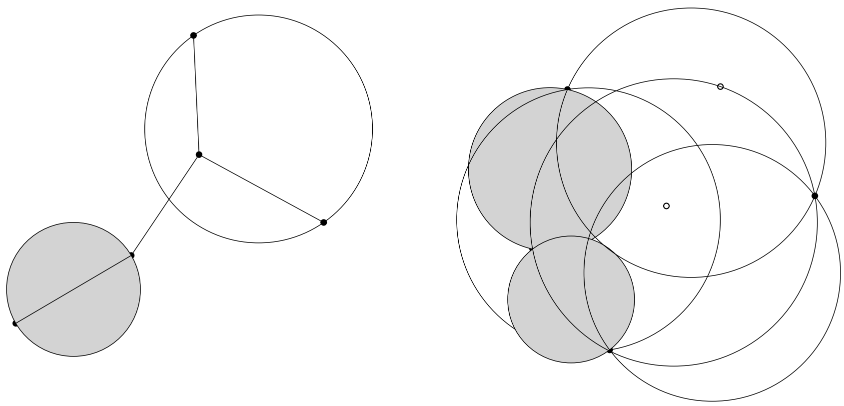

Figure 1.1: Gabriel graph. Left: The vertices defining the shaded disk are adjacent because their disk doesn't contain any other vertex, in contrast to the other vertices defining the unshaded disk. Right: Witness Gabriel graph. Black points are the vertices of the graph, white points are the witnesses. Each pair of vertices defining a shaded disk are adjacent and the pairs defining the unshaded disks are not.

eralization of proximity graphs).

1.1 The Witness Gabriel Graphs

The *witness Gabriel graph* [ADH10] $GG^{-}(P, W)$ is defined by two sets of points P and W; P is the set of vertices of the graph and W is the set of *witnesses.* There is an edge ab in $GG^{-}(P, W)$ if and only if there is no point of W in $D_{ab} \backslash \{a, b\}$ (see Figure 1.1(Right)). The witness Gabriel graphs were introduced by Aronov et al. [ADH10] in 2010.

1.2 The Witness Delaunay Graphs

The *witness Delaunay graph* [ADH11] of a point set P of vertices in the plane, with respect to a point set W of witnesses, denoted $DG^{-}(P, W)$, is the graph with vertex set P in which two points $x, y \in P$ are adjacent if and only if there is an open disk that does not contain any witness $w \in W$ whose bounding circle passes through x and y.

In graph drawing, a problem that is attracting substantial research is to find the number of witness points to remove all the edges of a witness graph. This problem can also be defined independently, as to find the size of the stabbing set for a point set P under some proximity notion. Stabbing set for a point set P is defined as follows: Let S be a family of geometric objects with nonempty interiors, each one associated to a finite subset of P. We say that a point w *stabs* an object $Q \in S$ if w lies in the interior of Q [ADH11]. The problem is to find the minimum number of points required to stab all the elements of S, which we denote by $st_S(P)$. We also alternately call it as stabbing set. We derive bounds on the cardinality of this set for some objects and derive approximation algorithms for some cases.

1.3 Problems considered in this thesis

Problem 1. *For a point set P in general position, derive an upper bound on the size of the stabbing set of $\binom{n}{2}$ disks induced by each pair of points $a, b \in P$ as the diameter of the disks.*

Problem 2. *For a point set P in convex position, derive an upper bound on the size of the stabbing set of $\binom{n}{2}$ disks induced by each pair of points $a, b \in P$ as the diameter of the disks.*

Problem 3. *For a point set P in general position, derive an upper bound on the size of the stabbing set of disks induced by each pair of points $a, b \in P$ as the chord of the disks. Note that here we are not considering finite number of disks.*

Problem 4. *For a point set P, where no two points have the same x or y coordinate, derive an upper bound on the size of the stabbing set of $\binom{n}{2}$ axis-parallel rectangles induced by each pair of points $a, b \in P$ as the diagonal of the rectangles.*

Problem 5. *For a point set P in convex position, derive a lower bound on the size of the stabbing set of $\binom{n}{2}$ axis-parallel rectangles induced by each pair of points $a, b \in P$ as the diagonal of the rectangles.*

Problem 6. *For a point set P in convex position, derive an approximation algorithm for the size of stabbing set of $\binom{n}{2}$ axis-parallel rectangles induced by each pair of points $a, b \in P$ as the diagonal of the rectangles.*

Problem 7. *For a point set P, where no two points have the same x or y coordinate, derive a lower bound on the size of the stabbing set of $\binom{n}{2}$ axis-parallel rectangles induced by each pair of points $a, b \in P$ as the diagonal of the rectangles.*

Problem 8. *For a point set P, where no two points have the same x or y coordinate, derive an approximation algorithm for the size of stabbing set of $\binom{n}{2}$ axis-parallel rectangles induced by each pair of points $a, b \in P$ as the diagonal of the rectangles.*

1.4 Organization of the thesis

We will discuss Problems 1 and 2 in Chapter 2.2, Problem 3 in Chapter 2.3. We discuss Problems 4, 5, 6, 7 and 8 in Chapter 3.2.

Chapter 2

Stabbing Disks Induced by Points on the Plane

2.1 Introduction

Consider a point set P in the plane. Let S be a family of geometric objects with nonempty interiors, each one associated to a finite subset of P. We say that a point w *stabs* an object $Q \in S$ if w lies in the interior of Q [ADH11]. In this chapter, we consider the problem that how many points are required to stab all the elements of S, which we denote by $st_S(P)$, and how large this number can be when all the point sets with $|p| = n$ are considered. We consider this extremal value by $st_S(n) = \max_{|p|=n} st_S(P)$ [ADH11].

2.2 Disks defined by pairs of points as diameter

Let P be a set of n points, and let S be the set of $\binom{n}{2}$ disks induced by each pair of points $a, b \in P$. The diameter of the disk is defined by the line segment $\overline{ab}$.

2.2.1 Sufficiency of Points for Stabbing

In this section, we will prove that for n points in general position, the size of the stabbing set is bounded above by $2n - 2 - k$, where k is the number of points on the boundary of the convex hull of the points of P.

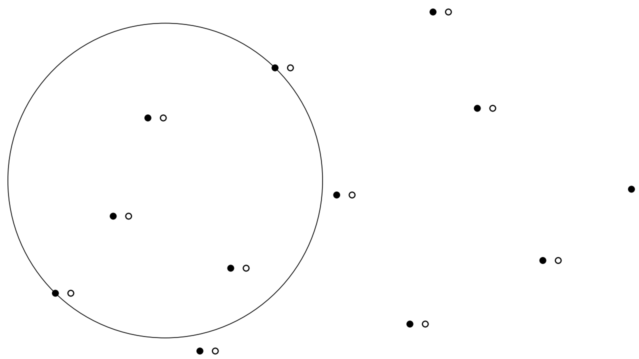

Figure 2.1: Points in P are shown with filled dots and the stabbing points are shown using dots whose interiors are empty.

The earlier bound on stabbing set was due to Aronov et al. [ADH10] which we state here.

Theorem 2.1. *[ADH10]. $n-1$ witnesses are always sufficient to stab all the disks in S.*

Proof. We argue the upper bound here. Without loss of generality, assume no two points of P lie on the same vertical lines, this can be achieved by an appropriate rotation of the coordinate system. Put a witness slightly to the right of each point of P, except for the rightmost one (see Figure 2.1). Every disk with diameter determined by two points of P will contain a witness. □

We now state a few lemmata that would be useful in proving our bound.

Lemma 2.2. *A point sees the diameter of a circle with an angle greater than $\pi/2$ if and only if the point lies inside the circle.*

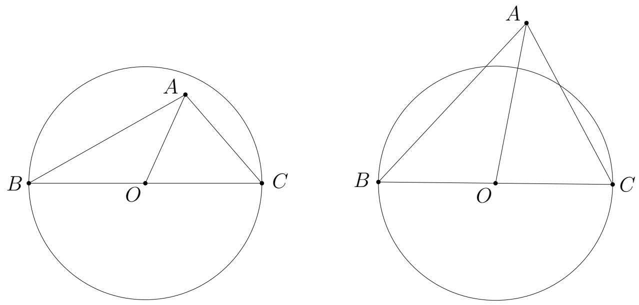

Figure 2.2: Left: Point sees diameter with an angle greater than $\pi/2$. Right: Point sees diameter with an angle less than $\pi/2$.

Proof. [$\Rightarrow$] Let the point A lies inside the circle. We will prove that $\angle BAC > \pi/2$. See Figure 2.2 (left).

Let $\angle OAC = \alpha$, $\angle OAB = \beta$, $\angle ACO = \gamma$, $\angle ABO = \delta$.

As the point A lies inside the circle, $OC > OA$ and $OB > OA$, $\Rightarrow \alpha > \gamma$ and $\beta > \delta$.

$\Rightarrow \alpha + \beta > \gamma + \delta$, $\angle AOB = \alpha + \gamma$.

In $\triangle AOB, \angle AOB + \angle ABO + \angle OAB = \pi$

$\Rightarrow \alpha + \beta + \gamma + \delta = \pi$

$\Rightarrow 2(\alpha + \beta) > \pi$, $\alpha + \beta > \pi/2$.

$\Rightarrow \angle BAC = \alpha + \beta > \pi/2$.

[$\Leftarrow$] Now, to prove the only if part we will prove that if a point lies outside the circle than it will see the diameter of the circle with an angle $< \pi/2$ See Figure 2.2(right).

As the point A lies outside the circle, $OC < OA$ and $OB < OA$, $\Rightarrow \alpha < \gamma$ and $\beta < \delta$

$\Rightarrow \alpha + \beta < \gamma + \delta$, $\angle AOB = \alpha + \gamma$.

In $\triangle AOB, \angle AOB + \angle ABO + \angle OAB = \pi$

$\Rightarrow \alpha + \beta + \gamma + \delta = \pi$

$\Rightarrow 2(\alpha + \beta) < \pi$, $\alpha + \beta < \pi/2$

$\Rightarrow \angle BAC = \alpha + \beta < \pi/2$

So we have proved that a point will see the diameter of a circle with an angle greater than $\pi/2$ if and only if the point lies inside the circle.

□

So by Lemma 2.2, a point will stab a disk if and only if it sees the diameter of the disk with an angle greater than $\pi/2$.

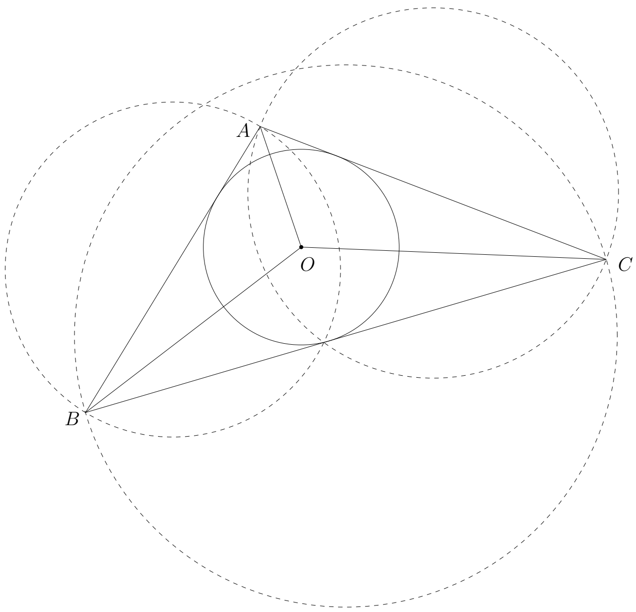

Figure 2.3: Incenter sees the sides of the triangle with an angle greater than $\pi/2$.

Lemma 2.3. *The intersection of three disks having the sides of a triangle as their diameters is always non empty.*

Proof. To prove that the intersection of three disks is non empty, we need to prove that there is at least one point which sees all the three sides of the triangle with angles greater than $\pi/2$.

Let O be the incenter of the triangle ABC (see Figure 2.3).

$\angle OAC + \angle OCA < \pi/2$ (as $\angle BAC + \angle BCA < \pi$)

$\Rightarrow \angle AOC > \pi/2$

This implies that O sees the side AC with an angle greater than $\pi/2$, or in other words O lies inside the disk having AC as its diameter. Similarily, we can prove that O lies inside the disks corresponding to sides AB and BC.

□

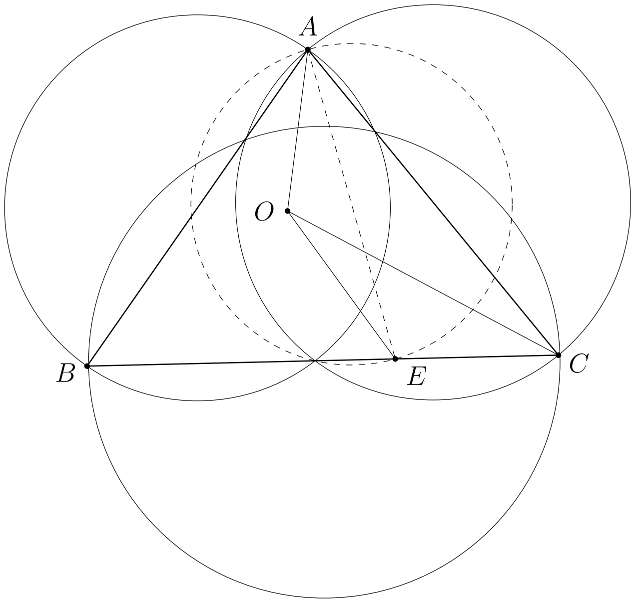

Figure 2.4: Disk corresponding to AE contains the common region of disks corresponding to AB, BC and CA.

Lemma 2.4. *For any $\triangle ABC$, let E be a point on any side of $\triangle ABC$(say BC) (see Figure 2.4). The disk corresponding to diameter AE (vertex A is opposite to side BC) contains the common intersection region of disks, having the sides of $\triangle ABC$ as their diameters.*

Proof. For any point O that lies in the common intersection of three disks corresponding to the sides of the $\triangle ABC$ (see Figure 2.4), there will be the following three cases.

(1) The point O will lie on the line segment AE.

(2) The point O will lie to the left of the line segment AE.

(3) The point O will lie to the right of the line segment AE.

In the first case, when the point O lies on AE, it is trivial to show that it will lie in the disk with AE as diameter. In the second case, when the point O lies to the left of AE, $\angle AOE$ will be equal to the sum of $\angle AOC$ and $\angle EOC$ (see Figure 2.4), which will be greater than $\pi/2$, as $\angle AOC > \pi/2$ and $\angle EOC > 0$. This implies that point O lies inside the disk corresponding to AE. The third case is similar to the second case. $\square$

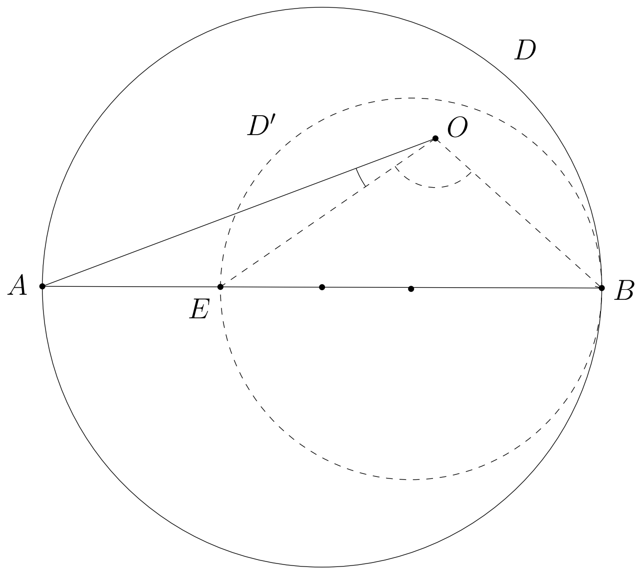

Figure 2.5: Showing that D' lies completely inside D.

Lemma 2.5. *Let D be a disk with AB as its diameter (see Figure 2.5). The disk D' with diameter EB, where E is a point that lies on AB, lies completely inside D.*

Proof. To prove that D' will lie completely inside D, we will show that any point O that lies inside D' sees AB with an angle greater than $\pi/2$. $\angle AOB = \angle AOE + \angle EOB$ (see Figure 2.5). $\angle AOB$ will be greater than $\pi/2$, as $\angle EOB > \pi/2$ and $\angle AOE > 0$. □

Triangulation of a Planar Point Set

Let P be a set of n points in the plane. To be able to formally define a triangulation of P, we first define a *maximal planar subdivision* as a subdivision S such that no edge connecting two vertices can be added to S without destroying its planarity. In other words, any edge that is not in S intersects one of the existing edges.

Definition 1. *A triangulation of P is defined as a maximal planar subdivision whose vertex set is P.*

Lemma 2.6. *[BCKO08]. Let P be a set of n points in the plane, not all collinear, and let k denote the number of points in P that lie on the boundary of the convex hull of P. Then any triangulation of P has $2n - 2 - k$ triangles and $3n - 3 - k$ edges.*

Proof. Let τ be triangulation of P, and let m denote the number of triangles of τ. Note that the number of faces of the triangulation, which we denote by n_f, is $m + 1$. Every triangle has three edges, and the unbounded face has k edges. Furthermore, every edge is incident to exactly two faces. Hence, the total number of edges of τ is $n_e = (3m + k)/2$. Euler's formula tells us that

$n - n_e - n_f = 2$. Plugging the values of n_e and n_f into the formula, we get $m = 2n - 2 - k$, which in turn implies $n_e = 3n - 3 - k$.

□

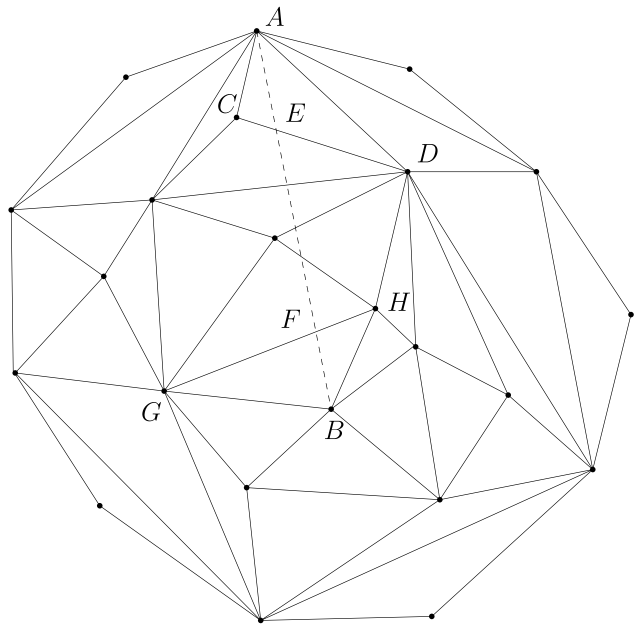

Figure 2.6: $2n - 2 - k$ points is sufficient to stab all the disks.

Theorem 2.7. *Let P be a set of n points in the plane in general position, and let k denote the number of points in P that lie on the boundary of the convex hull of P. Then the size of stabbing set for P is bounded above by $2n - 2 - k$.*

Proof. First we find a triangulation τ of the point set P. According to Lemma 2.6, there will be $2n - 2 - k$ triangles in τ. Then for each triangle we put a stabbing point at the incenter of the triangle. So by Lemma 2.3, these stabbing points will stab all the disks having any edge of the triangulation as its diameter. Now, for the points A and B (see Figure 2.6), which are not adjacent in the triangulation of P, if we draw an edge AB, that edge will intersect one of the opposite side CD of A at point E and the opposite side

GH of B at F in τ (see Figure 2.6). Now according to Lemma 2.4, the disk with diameter AE will be stabbed by the stabbing point of triangle ACD and by Lemma 2.5, the disk with diameter AB is stabbed as the point E lies on AB. So we have stabbed all the disk with $2n - 2 - k$ stabbing points, this completes the proof. □

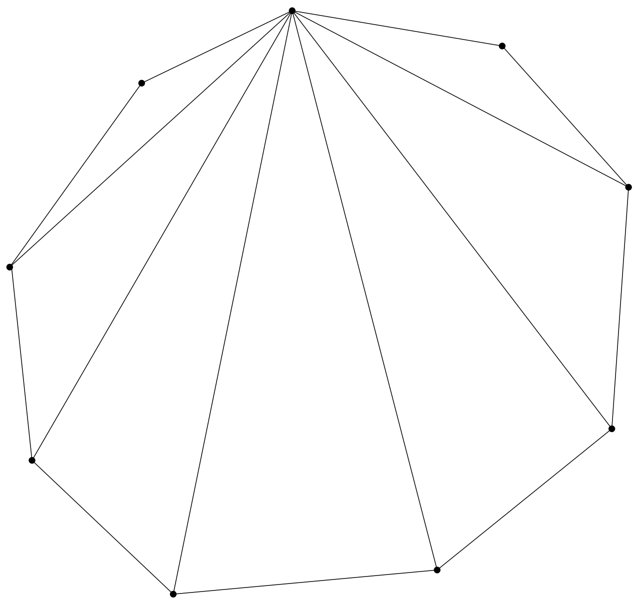

Figure 2.7: For the points in convex position there will be $n - 2$ triangles in any triangulation.

Corollary 2.8. *Let P be a set of n points in the plane in convex position. Then the size of stabbing set for P is bounded above by $n - 2$.*

Proof. If the points of P are in convex position, then we can triangulate P with $n-2$ triangles(see Figure 2.7), and then follow the proof of Theorem 2.7, to prove the sufficiency of $n - 2$ stabbing points. □

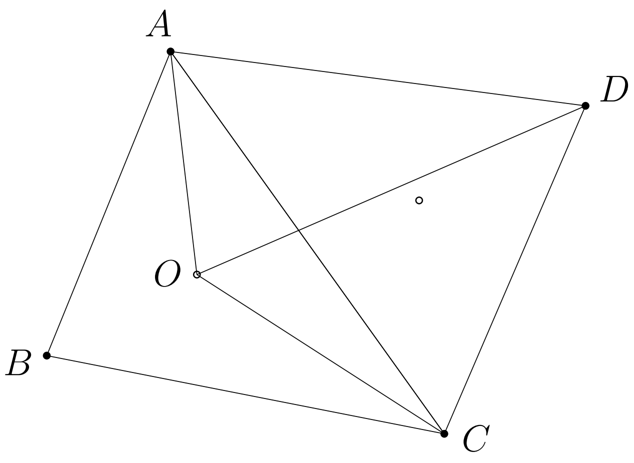

Figure 2.8: Two points are necessary to stab all the disks corresponding to the edges of a quadrilateral.

2.2.2 Necessity of Points for Stabbing

Theorem 2.9. *Two stabbing points are necessary to stab all the disk for a point set P having four points.*

Proof. Let the four points A, B, C and D form a quadrilateral, and there are two stabbing points corresponding to $\triangle ABC$ and $\triangle ACD$ (see Figure 2.8). To prove the necessity of two stabbing points, it is sufficient to prove that any one of these two points can not stab all the disks corresponding to the sides of the quadrilateral. Let O be the stabbing point which stabs the disks with diameters AB, BC, and AC (see Figure 2.8). We prove that O can not stab both the disk with diameters AD and CD as follows: $\angle AOC$ will be strictly less than π, this implies that either $\angle AOD$ or $\angle COD$ will be strictly less than $\pi/2$. So O can not stab both the disk. Similarily, we can prove that the stabbing point lies inside the $\triangle ACD$ can not stab both the disk with diameter as AB and BC. $\square$

2.3 Disks defined by pairs of points as chords

Let P be a set of n points, and let S be the set of disks whose boundary contains at least two points from P. We can easily observe that $|S|$ will be infinite. Now the problem is to find the number of stabbing points to stab all the disks of S.

In this section, we will give an alternate proof to show that $2n-2$ points are sufficient to stab all the disks. The earlier was due to Aronov et al. [ADH11].

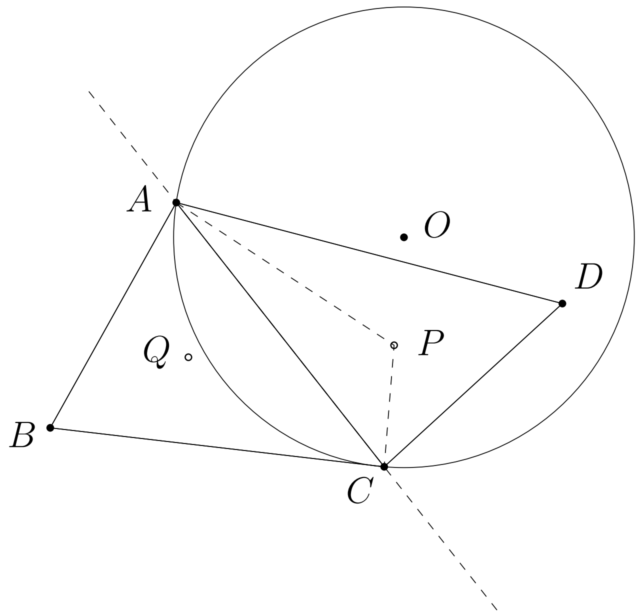

Figure 2.9: Making sure that disk containing A and C on its boundary will be stabbed by one of the stabbing point.

Lemma 2.10. *Let A, B, C and D be four points forming a quadrilateral then any disk having A and C on its boundary will be stabbed by the stabbing point corresponding to $\triangle ABC$ or $\triangle ACD$(see Figure 2.9).*

Proof. The center of the disk passing through point A and C will lie either to the right or to the left of the line passing through A and C. Suppose the center O lies to the right of the line passing through A and C. We can observe that any point will see AC with an angle greater than $\pi/2$ only when the point lies inside the disk. Let P be the stabbing point corresponding to $\triangle ACD$. So $\angle APC > \pi/2$(see Figure 2.9). This implies that P lies inside the disk. Similarily, we can prove the other case when the center of the disk lies to the left of the line passing through A and C. $\square$

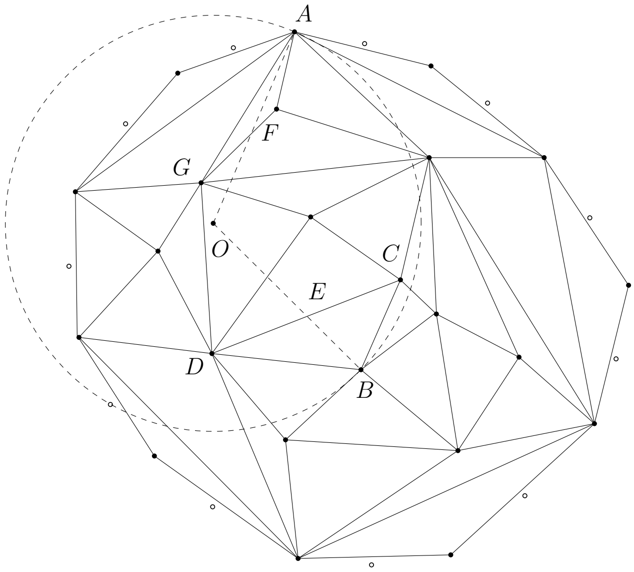

Figure 2.10: Showing that $2n-2$ points will be sufficient to stab all the disks for the case where one of the end points of the chord is an internal point.

Theorem 2.11. *Let P be a set of n points in general position and S be the set of disks contains at least two points of P on its boundary. Then $2n-2$ points will be sufficient to stab all the disks of S.*

Proof. Let τ be any triangulation of points of P. We place a stabbing point at the incenter of each triangle of τ. We also place a stabbing point for every edge of the convex hull, external and very close to its mid point. If the size of the convex hull is k, there will be $2n-2-k$ triangles in τ. So the total number

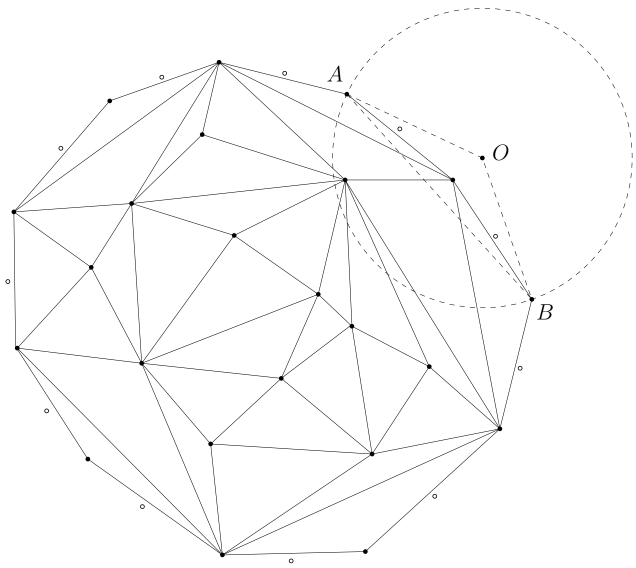

Figure 2.11: Showing that $2n-2$ points will be sufficient to stab all the disks for the case where both the end points of the chord are boundary points.

of stabbing points we have placed is $2n - 2$. Now we will show that, we have stabbed all the disks. According to Lemma 2.10, the disks having any internal edge of τ as a chord have been stabbed by the stabbing points of the triangles sharing that edge. We have also stabbed the disks having a boundary edge as a chord, as we have placed a stabbing point external and close to the middle of each boundary edge. Now we are left with disks containing two non adjacent points A and B of τ (see Figure 2.10) on their boundary. For any such disk there can be following two possibilities:

(1) Both A and B will be the boundary points of τ.

(2) A or B will be an internal point of τ.

In the second case when one of the point A or B is an internal point. Let O be the center of the circle passing through A and B. Then either OA or OB will cut an opposite side of A or B(see Figure 2.10, in this case both

OA and OB intersecting with FG and CD, the opposite sides of A and B respectively). According to Lemma 2.4 and Lemma 2.5, all such disks have been stabbed. In the first case when both the points A and B are boundary points, OA and OB might not cut any opposite side of A and B, but in this case the $\triangle OAB$ will contain at least two stabbing points (see Figure 2.11). So we have stabbed all the disks with $2n-2$ stabbing points as claimed.

$\square$

Chapter 3

Stabbing Rectangles Induced by Points on the Plane

3.1 Introduction

In this chapter, we will introduce another problem for stabbing sets, where the objects are axis-parallel rectangles. We formally define the problem as follows. For a given point set P of n points with no two x or y-coordinates same, for any pair of points $p, q \in P$, there is an axes parallel rectangle in S with one of its diagonal as $\overline{pq}$. So the set of object S for point set P contains $\binom{n}{2}$ axes-parallel rectangles, each corresponding to each pair of points.

3.2 Axis-parallel Rectangles induced by pairs of points as diameter

The problem is again the same as to find the optimal size of stabbing set to stab all such rectangles for a point set P. In this section, we will argue a 4-approximation algorithm for the size of stabbing set of axis-parallel rectangles for a set of points P in convex position.

We start with the upper bound for the size of the stabbing set.

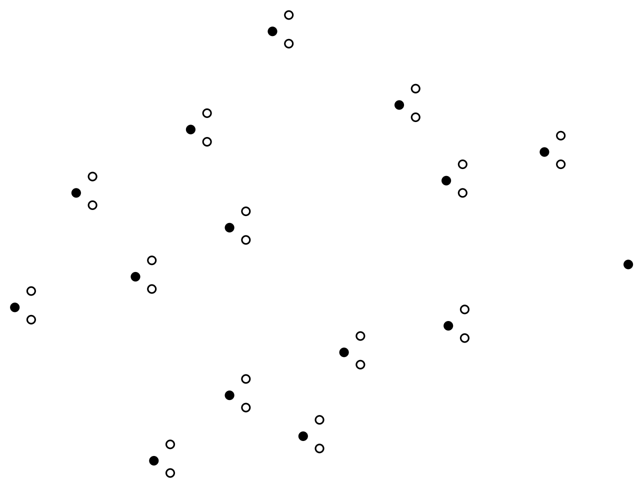

Figure 3.1: Making sure that all the rectangles have been stabbed.

Theorem 3.1. *$2n-2$ stabbing points are always sufficient to stab all the rectangles of S for a point set P having n points in general position.*

Proof. We assume that no two points of P have same x or y coordinate. We place two stabbing points one slightly above and another one slightly below to the right of each point of P except the point having maximum x coordinate (see Figure 3.1). If we consider p as origin, the stabbing point above p will stab all the rectangles corresponding to diagonal $\overline{pq}$ where q belongs to the first quadrant and the stabbing point below p will stab all the rectangles corresponding to diagonal $\overline{pq}$ where q belongs to the fourth quadrant. Every rectangle with diagonal determined by two points of P will contain a stabbing point. So we have proved that $2n-2$ stabbing points will be sufficient to stab all the rectangles. □

Now we will give a lower bound for the stabbing set of S when the points of P are in convex position. We construct an intersection graph $G(V, E)$ for the rectangles corresponding to the edges of the convex hull of the points of P. We call all such rectangles as boundary rectangles. The set V contains one vertex for each boundary rectangle. There is an edge between between two vertices of G if and only if the rectangles corresponding to vertices intersect.

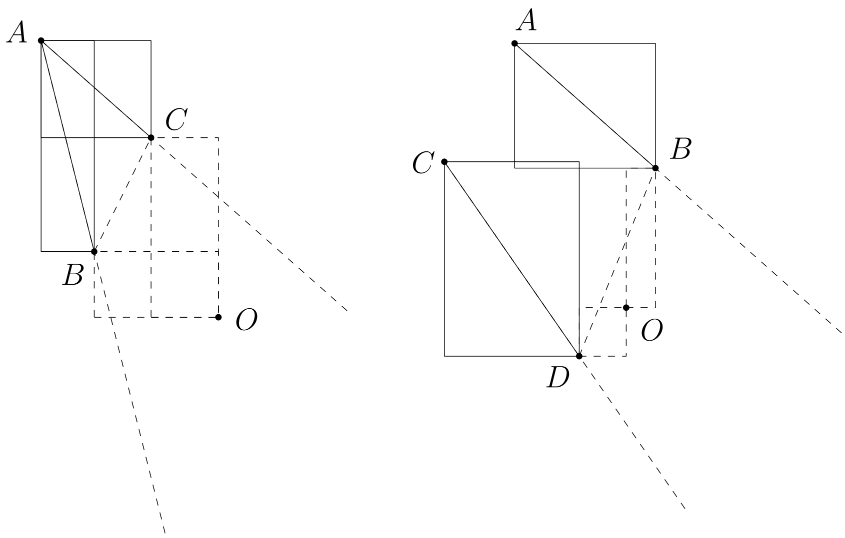

Figure 3.2: Left: No three rectangles corresponding to adjacent edges intersect. Right: No three rectangles corresponding to non adjacent edges intersect.

Theorem 3.2. *Let P be a point set of n points and the points are in convex position. The intersection graph G of the rectangles corresponding to the edges of the convex hull(boundary rectangles) of P does not contain any clique of size three or more.*

Proof. To prove that there is no clique in G of size three or more, we will prove that no three rectangle intersect together. First we need to find out when the two boundary rectangles intersect. There will be two cases. Either the rectangles will be corresponding to adjacent edges or they will be corresponding to non adjacent edges. In the first case when the rectangles are corresponding to adjacent edges, let the rectangles corresponding to edges AB(say R_1) and AC(say R_2) intersect. In this case the point B and C both will lie in the same quadrant if we consider A as origin. Due to the convexity condition all other points must lie in the region R bounded by lines passing through AB, AC and BC (see Figure 3.2). Let point O lies in the region R. Then we can claim that the rectangle corresponding to edge BO can not intersect with the rectangle R_1, as point A and O lies in the different quadrant if we consider point B as origin. Similarly, we can prove that the rectangle corresponding to edges AC and CO can not intersect. The rectangle corresponding to an edge between two points from R can not intersect with rectangles R_1 and R_2, as both the points will lie to the right to B and below to C. So we have seen that in the case when two rectangles corresponding to adjacent edges intersect there does not exist third rectangle which can intersect with both of them. Now we consider the second case when R_1 and R_2 are corresponding to non adjacent edges AB and CD (see Figure 3.2). There will be two regions where the other points can be, the region bounded by the lines AB, CD and AC and the region bounded by the lines AB, CD and BD. Let us consider the region bounded by AB, CD and BD(say R). Let O be a point in R. We can claim that rectangle corresponding to edge OB can not intersect with R_1, as A and O will lie in the different quadrant if we consider B as origin. Similarly, it can be proved that the rectangle corresponding to edge OD can not intersect with R_2. The rectangle corresponding to two points from R will not intersect with R_1 and R_2, as both the points will lie to the right of D and below B. So we have proved that there is no point or a pair of points in R which can make three rectangles intersect. Similarly, it can be proved for the other region too. So as claimed we have proved that there will be no three boundary rectangles intersecting in a common region. This, in turn, implies that there will be no clique of size three or more in graph G. $\square$

Theorem 3.3. *For a point set P in convex position, we derive a 4-factor* *approximation algorithm for the minimum size of the stabbing set of* $\binom{n}{2}$ *axis-parallel rectangles induced by each pair of points* $a, b \in P$ *as the diagonal of the rectangles.*

Proof. By Theorem 3.2, we can say that there will be atleast $n/2$ disjoint rectangles in S. This gives a lower bound on the optimal solution. The above lower bound coupled with the constructive proof of upper bound of $2n - 2$ in Theorem 3.1 gives us a 4-factor approximation algorithm. □

Now we will improve the lower bound of stabbing set, for the points in general position.

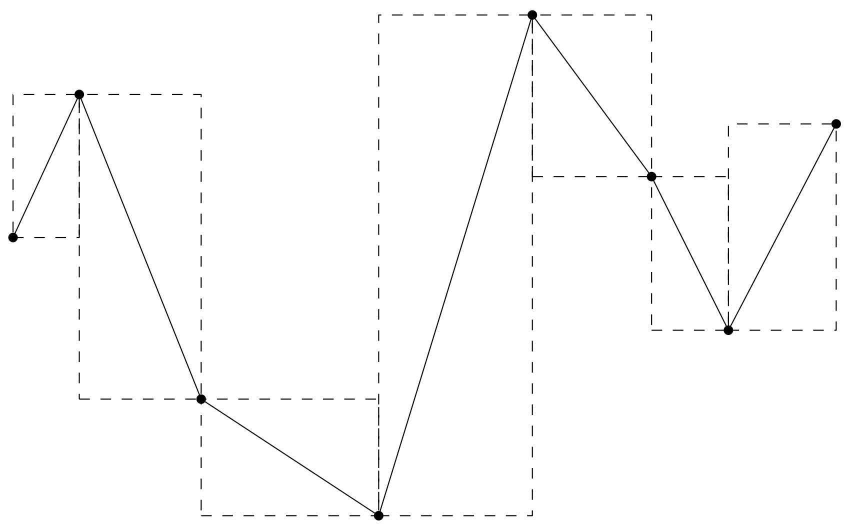

Figure 3.3: Showing that there will be $n - 1$ disjoint rectangles

Theorem 3.4. *Let* P *be a set of* n *points in general position. There will be at least* $n - 1$ *disjoint axis-parallel rectangles in* S.

Proof. We assume that no two points in P have the same x or y coordinate. Sort the points of P with respect to x coordinate. Then the rectangles corresponding to consecutive pairs (see Figure 3.3) in sorted list shall be disjoint. So there will be atleast $n - 1$ disjoint rectangles. □

Theorem 3.5. *For a point set P, where no two points have the same x or y coordinate, we derive a 2-factor approximation algorithm for the size of stabbing set of $\binom{n}{2}$ axis-parallel rectangles induced by each pair of points $a, b \in P$ as the diagonal of the rectangles.*

Proof. Replace Theorem 3.2 with Theorem 3.4 in the proof of Theorem 3.3 to get the result. □

We end the chpater with the following comment. Finding the minimum stabbing set of arbitrary rectangles is NP-Complete [FPT81]. We do not have any hardness result for the special case of rectangles considered here, neither do we have any polynomial time algorithm. Nielsen [Nie00] presented an $O(\log n)$-approximation algorithm for finding the stabbing set of axis-parallel rectangles.

Chapter 4

Conclusion and future directions

In this thesis, we discussed the problem of finding the size of stabbing sets for disks and rectangles. We claimed the upper bound for size of stabbing set for disks induced by points on the plane as diameter, as $2n - 2 - k$. This bound is better than the previous bound given by Aronov et al. [ADH10], when the points are in convex position. We also described an alternate proof for the upper bound of $2n - 2$ for the size of the stabbing set when the disks are induced by the pair of points as chord.

We proved that the upper and lower bounds of the size of stabbing set are $2n - 2$ and $n - 1$ respectively, for the axis parallel rectangles induced by the pair of points as diagonal. We have given a 4-approximation algorithm for stabbing sets for the axis parallel rectangles, when the points are in convex position. We have also given a 2-approximation algorithm for stabbing sets for the axis parallel rectangles, when the points are in general position.

Further work which can be done in future is to improve the bounds given in this thesis and the development of algorithms to find optimal size of stabbing sets for disks and rectangles.

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