Quality Control

By W. A. SHEWHART

Introduction

A MANUFACTURER is interested in producing a controlled product—one in which the deviations about the average level of quality are no larger than can be accounted for as a result of chance. The present paper gives simple detailed methods for determining from inspection data whether or not a product is being controlled in the sense of indicating the presence of assignable causes of variation. Naturally the inspection data constitutes a sample of the effects of the manufacturing causes and hence the interpretation of these data in terms of what may be expected in the future is a statistical problem.

A controlled product is defined as one for which the frequency of deviations from the expected quality can be estimated by probability theory. To make such estimates, however, it is necessary to characterize or specify the distribution of quality which the manufacturer wishes to maintain. These specifications of the desired quality must be arrived at by methods customarily used in setting engineering standards, but when once they have been established the statistical methods amplified in this paper make possible the most economical control of this quality.

The limits within which quality may be controlled with a given amount of inspection depend upon the standards adopted for the quality to be maintained.

This paper interprets quality specifications in terms of five different types of constant systems of manufacturing causes. The five types chosen are sufficient to cover the entire range and it is believed that only five types are necessary because sampling theory indicates that little practical advantage would be derived by endeavoring to subdivide one or more of these. It is shown that quality control can be maintained with the fewest number of measurements and within the closest limits through the adoption of Type V.

Specification of Control

One of the principal objects of inspection is the detection of lack of control of manufactured product, that is, the detection of the presence of assignable causes of variation in the quality. A recent paper in this *Journal* [1] describes a quality control chart designed to attain this ob-

[1] Shewhart, W. A., "Quality Control Charts," October 1926.

ject and some of the results obtained through the application of the chart have also been presented.[2] In general the detection of the existence of assignable causes of variation leads to their elimination at a minimum of cost.

As a basis for this chart we start with the conception of a constant system of causes as being one such that the probability of a unit of product having the quality X within the range X to $X + dX$ is independent of time. For convenience in the present discussion we may represent this probability dP as a function f of the quality X and m parameters. Thus

$$dP = f(X, \lambda_1, \lambda_2, \cdots \lambda_m)dX. \tag{1}$$

The present paper presents different ways of specifying the constant system of causes and of detecting lack of control upon the basis of the different specifications principally by setting sampling limits on the parameters. In this way it is shown that the best control can be secured when all of the parameters together with the function f in Eq. 1 are specified. We shall assume, in what follows, one set of specifications after another for the constant system of causes and then show for each set how sampling limits may be established. Nomograms are presented to make the determination of the limits possible without the use of even a slide rule. We shall start with the simplest specification, usually referred to as Type I, which has found extensive use.

Type I often gives a satisfactory basis of control although it makes use, as we shall see, of only a fraction of the information given by the data used in connection with Specification Type V, which is the ideal set wherever the manufacturer is warranted economically in trying to secure the highest degree of control. The choice of specification to be adopted in a given case depends entirely upon the economic advantage attainable through the detection and elimination of assignable causes of variation. In particular the use of Type V specification in the initial stages of the development of the manufacturing process is almost always warranted, because it materially assists in arriving at a controlled process with a minimum of labor.

Specification Type I: The probability of the production of a defective piece of apparatus shall be p'.[3]

To set limits in this case is very simple indeed, particularly if we choose the probability P associated with the limits to exceed .9. It

[2] Jones, R. L., "Quality of Telephone Materials," *Bell Telephone Quarterly*, Vol. 6, pp. 32–46, January 1927.

[3] The primed notation is used throughout to denote parameters of the universe as contrasted with the estimates of these determined from the sample.

has been found satisfactory in many cases to take $P \doteq .99$ and so, upon this basis, we shall present the method of setting limits upon the expected fraction defective in a sample of size n. It is well known that the probability of an observed value of p lying within the limits $p' \pm 3\sigma_{p'}$ is approximately equal to .997 provided the fraction defective p' is approximately equal to the fraction non-defective q', and n is large. It can be shown, however, that irrespective of the magnitudes of p' and n the value of P so determined lies between .95 and 1.00 and for most cases met in practice P does not differ from .997 by as much as 1 per cent.

It is obvious, therefore, that, if we construct an alignment chart on which we may read directly the standard deviation $\sigma_{p'}$ when p' and n are given, then the average p' plus or minus three times the standard deviation $\sigma_{p'}$ gives the corresponding values of the limits.

Let us consider a practical problem, see how the question of whether or not a product is controlled really arises and see how control limits can be found from the alignment chart of Fig. 1 to answer this question.

Table 1 represents the observed fraction found defective over a period of 12 months for two kinds of product designated here as Type A and Type B. The table gives for each month the sample size n, the number defective m and the fraction defective $p = m/n$. The average fractions defective for the 12-month period are $\bar{p}_A = .0109$ and $\bar{p}_B = .0095$. Subject to later consideration we shall assume $p'_A = \bar{p}_A$ and $p'_B = \bar{p}_B$.

TABLE 1

Apparatus Type *A*				Apparatus Type *B*			
Month	n No. Insp.	m No. Def.	$p = \frac{m}{n}$ Fraction Def.	Month	n No. Insp.	m No. Def.	$p = \frac{m}{n}$ Fraction Def.
Jan......	527	4	.0076	Jan.....	169	1	.0059
Feb......	610	5	.0082	Feb.....	99	3	.0303
Mar.....	428	5	.0117	Mar....	208	1	.0048
Apr......	400	2	.0050	Apr.....	196	1	.0051
May....	498	15	.0301	May....	132	1	.0076
June....	500	3	.0060	June....	89	1	.0112
July.....	395	3	.0076	July....	167	1	.0060
Aug.....	393	2	.0051	Aug. ...	200	1	.0050
Sept.....	625	3	.0048	Sept....	171	2	.0117
Oct......	465	13	.0280	Oct.....	122	1	.0082
Nov.....	446	5	.0112	Nov....	107	3	.0280
Dec.....	510	3	.0059	Dec.....	132	1	.0076
Average.	483.08	5.25	.0109		149.33	1.42	.0095

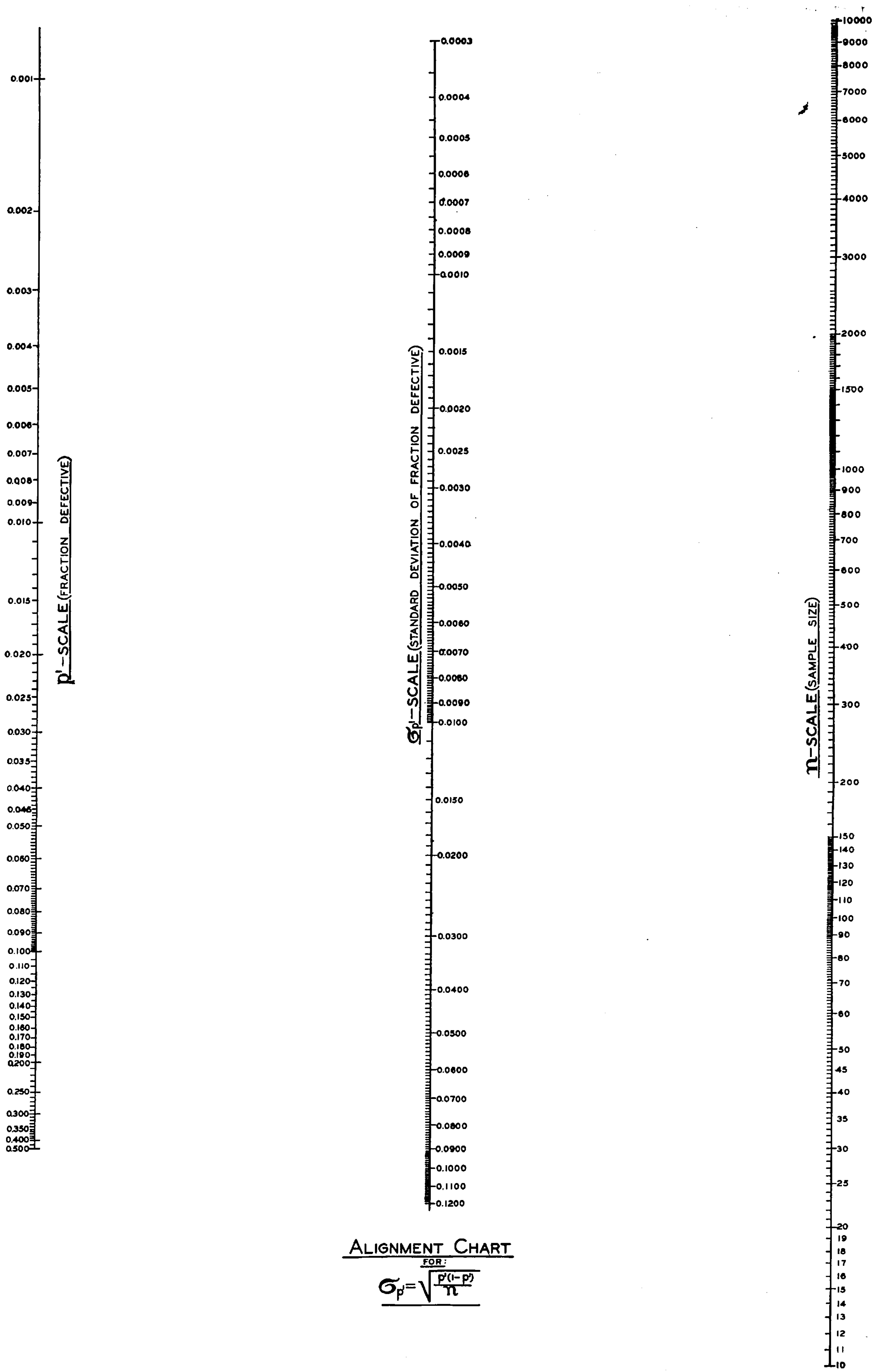
p'-SCALE (FRACTION DEFECTIVE)
0.001
0.002
0.003
0.004
0.005
0.006
0.007
0.008
0.009
0.010
0.015
0.020
0.025
0.030
0.035
0.040
0.045
0.050
0.060
0.070
0.080
0.090
0.100
0.110
0.120
0.130
0.140
0.150
0.160
0.170
0.180
0.190
0.200
0.250
0.300
0.350
0.400
0.500
σp'-SCALE (STANDARD DEVIATION OF FRACTION DEFECTIVE)
0.0003
0.0004
0.0005
0.0006
0.0007
0.0008
0.0009
0.0010
0.0015
0.0020
0.0025
0.0030
0.0040
0.0050
0.0060
0.0070
0.0080
0.0090
0.0100
0.0150
0.0200
0.0300
0.0400
0.0500
0.0600
0.0700
0.0800
0.0900
0.1000
0.1100
0.1200
n-SCALE (SAMPLE SIZE)
10000
9000
8000
7000
6000
5000
4000
3000
2000
1500
1000
900
800
700
600
500
400
300
200
150
140
130
120
110
100
90
80
70
60
50
45
40
35
30
25
20
19
18
17
16
15
14
13
12
11
10
ALIGNMENT CHART
FOR:
$\sigma_{p'} = \sqrt{\frac{p'(1-p')}{n}}$

Fig. 1

Is there any indication that the observed fluctuations in the fraction defective p could have been produced by other than chance causes? In other words, were apparatus Type A and apparatus Type B controlled over the given period? Furthermore, is there any indication that the product could have been improved during this period without changing the process of its manufacture?

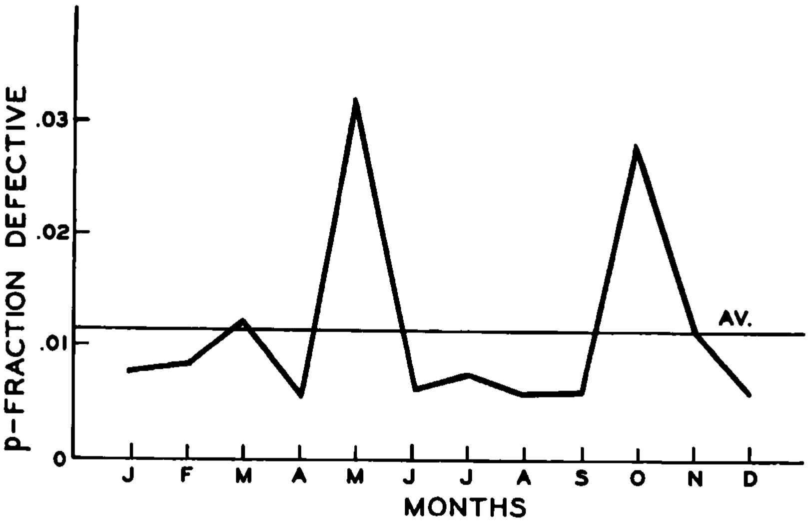

Fig. 2*a*. Apparatus Type A

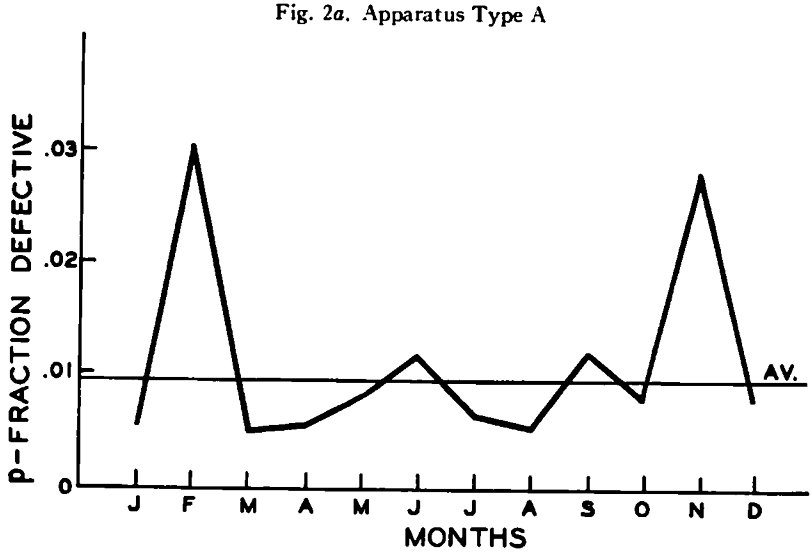

Fig 2*b*. Apparatus Type B.

To better visualize the fluctuations in p, the data of Table 1 are shown graphically in Fig. 2*a* and Fig. 2*b*. It may appear that during the months of May and October there existed some assignable cause of variation in the production process of Type *A* apparatus. The same may appear to be true for Type *B* apparatus during the months of February and November.

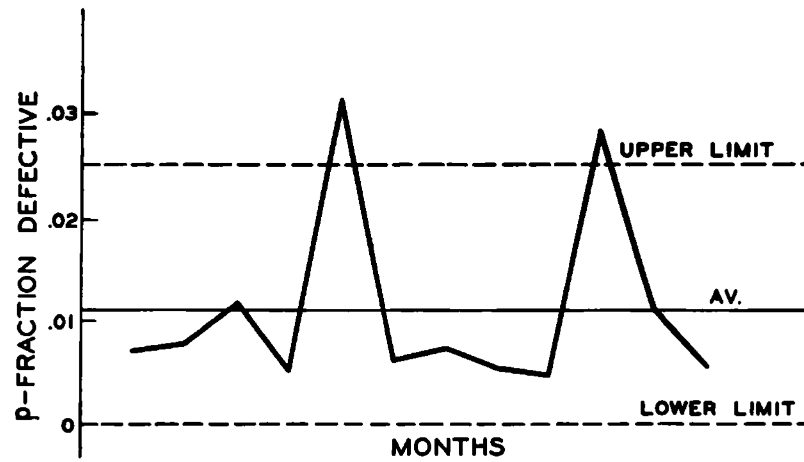

Fig. 3*a*. Apparatus Type A

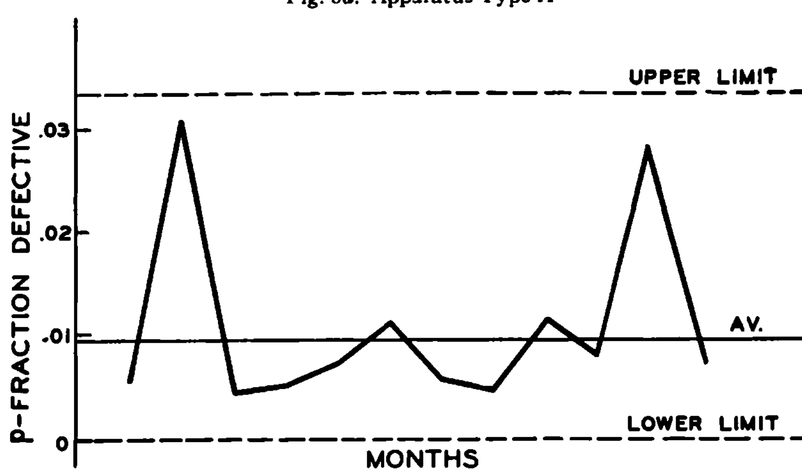

Fig. 3*b*. Apparatus Type B

We shall see upon investigation that there is evidence of lack of control of apparatus Type *A* but not any evidence of lack of control of apparatus Type *B*.

Taking n equal to the average sample size (483 for Type A), we connect by a straight line the point 483 on the n scale of Fig. 1 with the point .0109 on the p' scale. We read the intersection of this straight line with the $\sigma_{p'}$ scale as .0047. Hence the upper limit for p' is $.0109 + 3\sigma_{p'} = .0250$, a value which is exceeded during the months of May and October; and the lower limit is $.0109 - 3\sigma_{p'} = -.0032$. Of course negative values of p have no significance; hence we take the lower limit as zero. Following the same procedure for Type B apparatus, we get limits 0 and .0332.

We see that twice during the year Type A apparatus appears to have been out of control whereas at no time during the year can we say this of Type B.[4]

Now, we shall take up successively the method of finding limits corresponding to specifications involving:

a. Only one parameter (Type II).
b. Only two parameters (Type III).
c. Two parameters and a restriction on the function f over a certain range (Type IV).
d. Four parameters and a specific function f (Type V).

We shall find that the limits become progressively smaller in the above order. In fact for Specification Type II no limits can be set and for Specifications Type III and IV the limits are so large as to be in most instances impractical.

Specification Type II: The expected or average quality shall be $\overline{X}'$.

There is an indefinitely large number of constant systems of causes which would meet this requirement. Associated with each constant system of causes there are specific sampling limits. Sufficient information, however, is not called for in the Specification Type II to fix sampling limits on the quality of a single unit or on the expected or average quality.

In other words, Specification Type II is useless insofar as it does not provide for the detection of lack of control in the sense now under discussion.

Specification Type III: The expected or average quality shall be $\overline{X}'$ and the standard deviation shall be σ'.

Again there is an indefinitely larger number of different cause systems which would satisfy this requirement. However, it is re-

[4] Strictly speaking statistical theory only shows that two of the observed deviations in p_A are highly improbable upon the assumption that the product had been controlled about p'_A. It should be noted, of course, that the sample size is not the same from month to month and hence that the limits for a given month should really have been based upon the sample size for that month. However, in the present instance, this method of procedure leads to the same conclusion as given above and hence was not introduced because of necessary complications.

markable, even though this be true, that the work of Tchebycheff[5] makes it possible for us to give a lower bound to the probability that a unit of product will be produced with a quality X lying within the range $\bar{X}' \pm L_1$ and also therefore to the probability that an observed average quality of a sample of n units will lie within any given range $\bar{X}' \pm L_n$.

Taking $L_1 = c\sigma'$ $(c > 1)$, the probability $P_{c\sigma'}$ that the constant system of causes Type III will produce a unit of product having a quality X within the range $\bar{X}' \pm L_1$ is given by the expression

$$P_{c\sigma'} \geq 1 - \frac{1}{c^2}. \tag{2}$$

Expression 2 also defines the probability that the average quality of n units of product coming from the constant system of causes Type III will lie within the range $\bar{X}' \pm L_n$ where

$$L_n = \frac{c\sigma'}{\sqrt{n}}.$$

Let us illustrate the method of finding the limits under these specifications. Assume that the specified average resistance $\bar{X}'$ of a relay is 150 ohms and the standard deviation σ' is 5 ohms. What is the range within which we may expect 90 per cent of the product (i.e. $P_{c\sigma'} = .90$) to lie, assuming no assignable causes of variation in product? What is the similar range for the average of 1000 relay windings?

Turning to the nomogram of Fig. 4, we connect by a straight line the point $P_{c\sigma'} = .90$ and the point A near the center of the chart. The point on the c scale fixed by the intersection of the straight line so determined with the c scale is 3.15. The required values of L_1 and L_{1000} are therefore $L_1 = 3.15 \times 5 = 15.75$ ohms and $L_{1000} = \frac{3.15 \times 5}{\sqrt{1000}} = .50$ ohm. Hence the limits are 150 ± 15.75 ohms and $150 \pm .50$ ohms.

To avoid the slide rule computations in obtaining $c\sigma'$ and $c\sigma'/\sqrt{n}$ we can use the nomogram of Fig. 5. We enter this nomogram by the value $c = 3.15$ and find a point on the $c/\sqrt{n}$ scale which lies on a straight line with the point $c = 3.15$ on the c scale and $n = 1$ on the n scale. Connecting the point thus fixed on the $c/\sqrt{n}$ scale with the point $\sigma' = 5$, we read on the L scale 15.75 ohms. Carrying through the same procedure, but starting with $n = 1000$ instead of $n = 1$, we read on the L scale .50. These values give the limits found above.

[5] Tchebycheff, *Liouville Journal*, 1867. "Des valeurs moyennes," *Journal de Mathematiques* (2), Vol. 12, pp. 177–84.

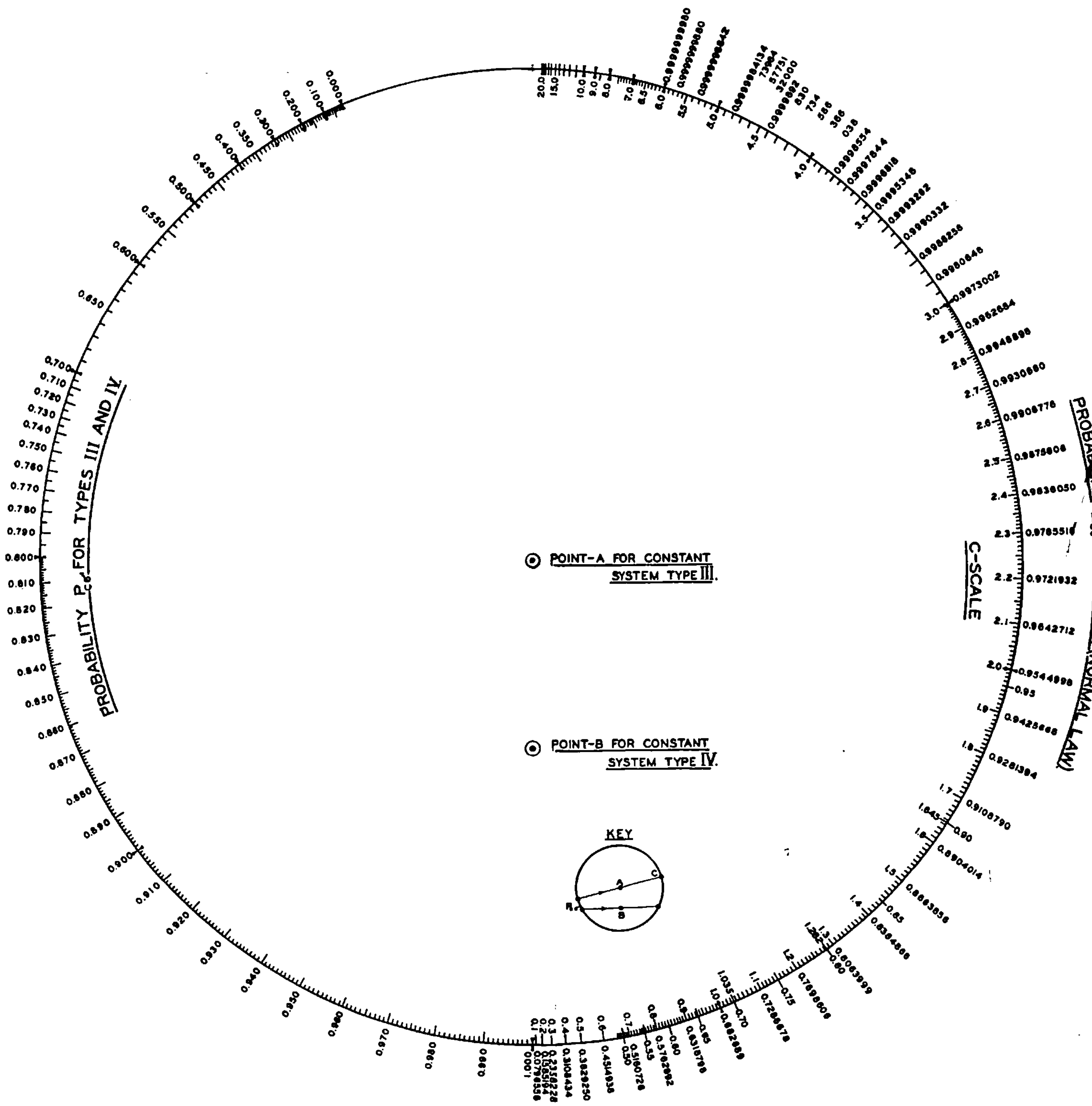
PROBABILITY P_{C_0} FOR TYPES III AND IV
PROBABILITY P_{C_0} FOR TYPE V (NORMAL LAW)
C-SCALE
POINT-A FOR CONSTANT SYSTEM TYPE III.
POINT-B FOR CONSTANT SYSTEM TYPE IV.
KEY

Fig. 4

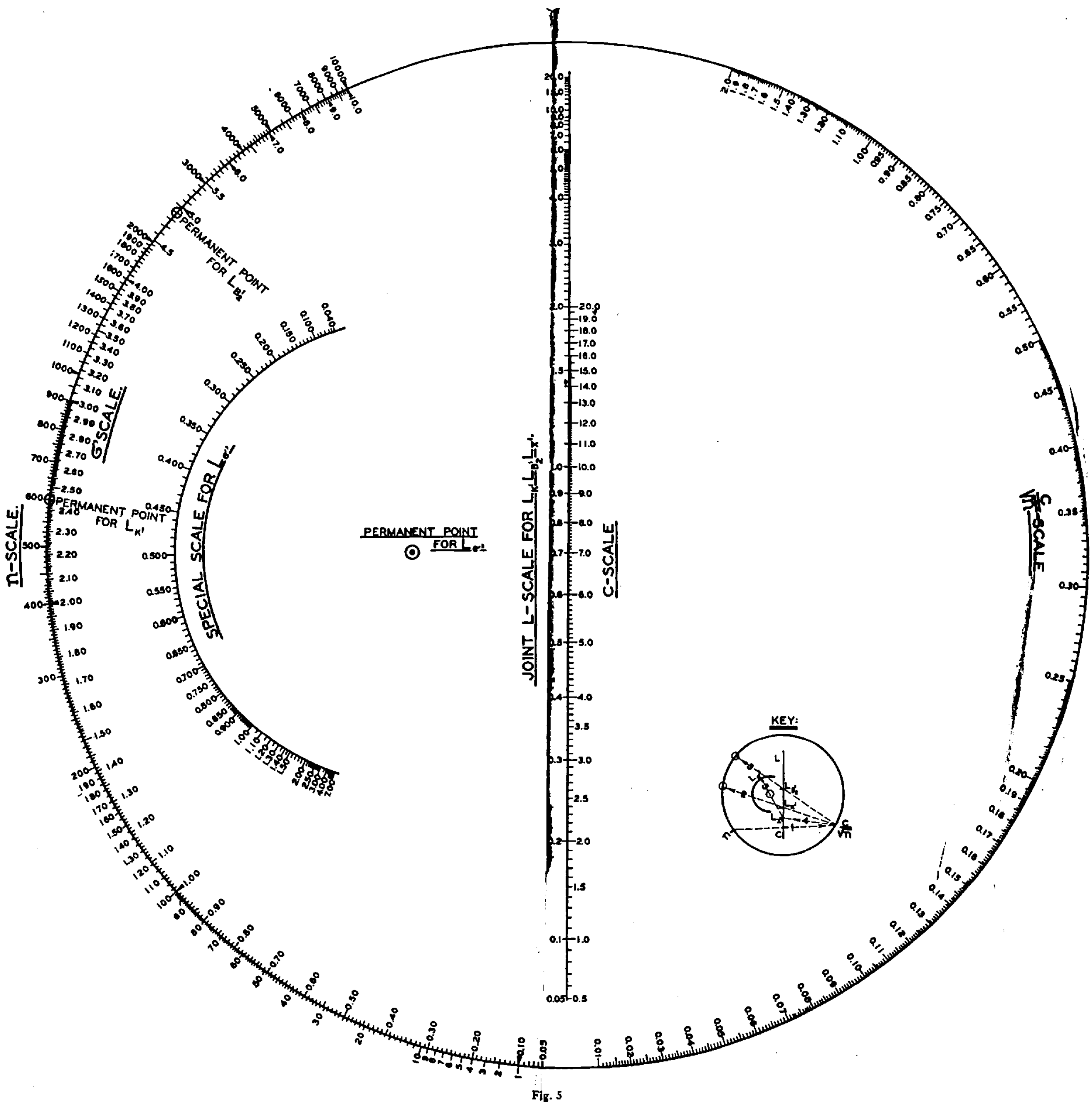
n-SCALE.
G'SCALE.
PERMANENT POINT FOR L
PERMANENT POINT FOR L
SPECIAL SCALE FOR L
PERMANENT POINT FOR L
JOINT L-SCALE FOR
C-SCALE
-SCALE
KEY:

Fig. 5

Specification Type IV: The expected or average values of quality and standard deviation shall be $\bar{X}'$ and σ' respectively. The expected modal and average qualities shall coincide and the probability function for the constant system of causes shall be monotonically decreasing for all values of x where x is measured from the mean.

In this case the lower bound to the probability $P_{c\sigma'}$ is given by the expression [6]

$$P_{c\sigma'} \geq 1 - \frac{1}{2.25c^2}. \tag{3}$$

The limits can be obtained just as in case of Type III except that we use point B in Fig. 4 instead of point A. It may be easily verified by this nomogram that the Type IV values of L_1 and L_n for the special problem considered for Type III are $L_1 = 10.4$ ohms and $L_n = 0.33$ ohm respectively.

This shows that the additional requirements placed upon Type IV over those of Type III make for better control in the sense that the associated sampling limits are thereby decreased. By going further in adding restrictions upon the cause system, we gain even more marked improvements in the condition for control. In fact it is the system now to be described that has been found to be the most useful practical standard where the quality is measured as a variable.

Specification Type V: The system of causes shall yield a product distributed according to the Gram Charlier series [7] with arithmetic mean $\bar{X}'$, standard deviation σ', skewness k' and kurtosis β_2'.

With the use of the four parameters we can detect lack of control of product through the failure of the observed value of any parameter determined from a sample of size n to fall within its sampling limits. It may happen that lack of control will be indicated by deviation beyond the sampling limits for only one of the four parameters. This case has already been illustrated in the article referred to in footnote 1. We shall now present, however, a method of setting these limits which is very easily applied.

As a specific example, let us assume the following expected values:

$$\bar{X}' = 0,$$
$$\sigma' = 1,$$
$$k' = 0,$$

[6] Camp, Burton H., "A New Generalization of Tchebycheff's Statistical Inequality," *Bulletin of the Amer. Math. Soc.*, December 1922, pp. 427–432. Eq. 3 is a special case of the general theorem of Camp. This theorem may be extended to determine lower bound to the probability of an error of the average as is done in this paper.

[7] Of course we might use certain other functions involving the same parameters.

and

$$\beta_2' = 3.$$

Also let us assume that the size of the sample n for which the limits are to be established is 1000 and that we wish to establish limits upon the basis of a probability $P \doteq .997$.

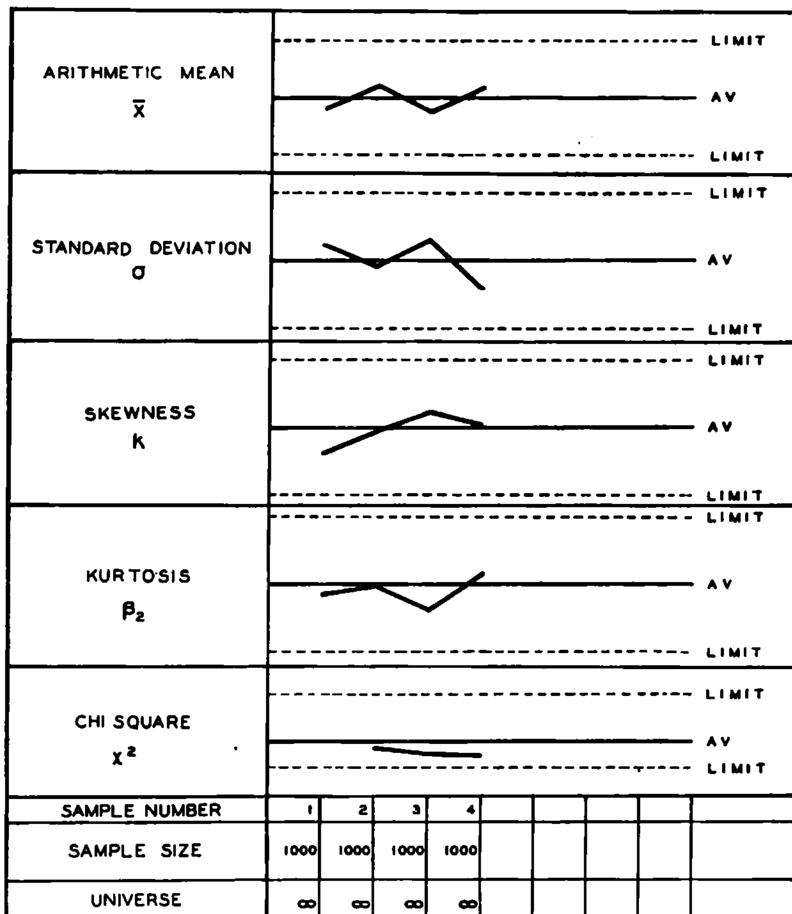

Fig. 6

The nomogram of Fig. 4 gives us immediately that $c = 3.0$ for $P = .997$. Hence we enter the nomogram of Fig. 5 on the c scale $c = 3.0$. The best way in which all four limits can be found by using the nomogram of Fig. 5 is then as follows, where the limits are set in the order $L_{k'}$, $L_{\beta_2'}$, $L_{\bar{X}'}$, and $L_{\sigma'}$. Join the point $n = 1000$ on the n scale and $c = 3.0$ on the c scale by a straight line and thus find a pivot point on the $c/\sqrt{n}$ scale. Holding the ruler on this pivot point, join it successively with the permanent points of $L_{k'}$ and $L_{\beta_2'}$ and with σ' taken all on the σ' scale and read accordingly $L_{k'}$, $L_{\beta_2'}$, and $L_{\bar{X}'}$, on the L scale. After reading $L_{\bar{X}'}$, release the pivot point and turn the ruler around the $L_{\bar{X}'}$ point so as to join it with the permanent point for $L_{\sigma'}$. Then read the intersection of the ruler with the inner circular scale $L_{\sigma'}$, hereby obtaining the limit for σ'. Thus in five movements of the ruler we find all four limits:[8]

$$\begin{aligned} 0 \pm L_{k'} &= 0 \pm .23, \\ 3 \pm L_{\beta_2'} &= 3 \pm .46, \\ 0 \pm L_{\bar{X}'} &= 0 \pm .095, \\ 1 \pm L_{\sigma'} &= 1 \pm .067. \end{aligned}$$

Figure 6 presents the graphical representation of the limits thus determined together with limits on χ^2 assuming that the theoretical frequency distribution was broken up into 13 cells.[9] The irregular lines show the fluctuations in the estimates of these parameters determined from four samples of 1000 each drawn under conditions satisfying the specifications just described for $\bar{X}' = 0$, $\sigma' = 1$, $k' = 0$ and $\beta_2' = 3$. Incidentally it should be noted that in every case the observed fluctuations in the estimates of the parameters are well within the sampling limits. This was to be expected because every effort was made in the sampling process to come as close as practicable to the ideal case where no assignable causes of variation were present. In this respect the data of Fig. 6 form an interesting contrast to the data of Fig. 4 of the article referred to in footnote 1, where evidence of lack of control was found.

Figure 7 makes it possible for us to set limits about the average or expected χ^2 corresponding to a probability of either .98 or .80. Thus for the data of Fig. 6 the limits for χ^2 corresponding to probability .98 are approximately 3 and 26 respectively as read from this chart. If limits corresponding to any other probability are desired, they can be readily obtained from tables for goodness of fit.[10]

We are now in a position to consider more in detail the advantages

[8] In case the given data bring the readings on the extreme points of the scale (where $\sigma' > 10$) it is advisable to take $\sigma'/10$ and multiply the final results obtained by ten. It is also helpful to remember that the L-scale on the nomogram of Fig. 5 can be considered as a regular scale of the product of two factors read on σ' scale and $c/\sqrt{n}$ scale.

[9] For the significance of χ^2 as here used, see paper, footnote 1.

[10] Elderton's Tables for Goodness of Fit reproduced in Pearson's "Tables for Statisticians and Biometricians" and also R. A. Fisher's "Tables for Goodness of Fit" given in his recent book "Statistics for Research Workers" will be found very helpful in the construction of curves similar to those of Fig. 7.

from a control viewpoint of Type V specification over the other suggested specifications. We have seen in Fig. 4 of the previous article on the control chart, footnote 1, that evidence of lack of control may be obtained through deviations in one parameter and not in

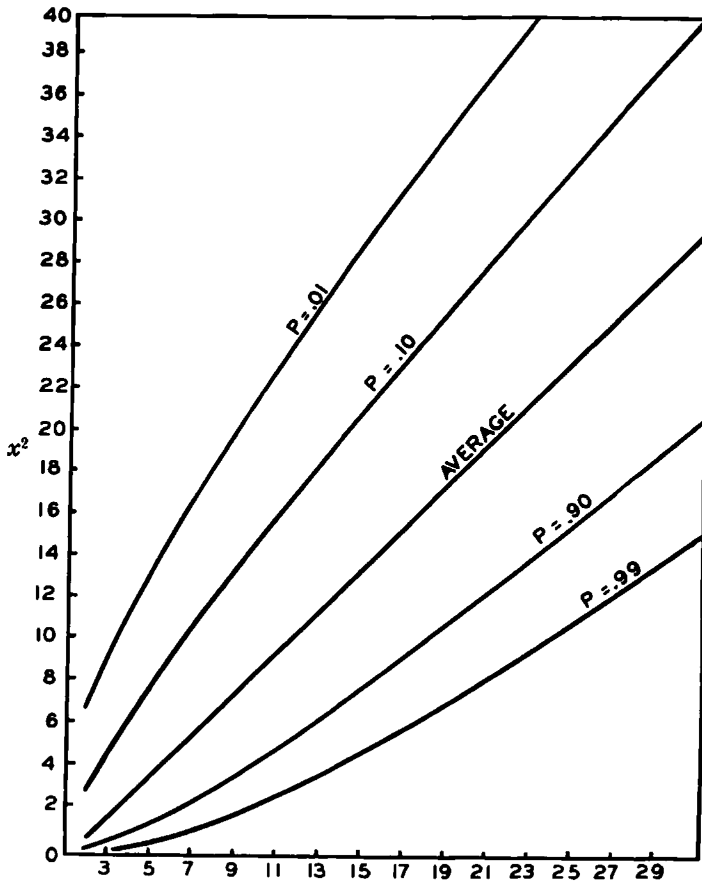

Fig. 7

another. For example, in this figure the per cent defective part of the chart corresponded to Specification Type I. Only 4 of the 12 points on this chart were outside the control limits whereas more than 4 points were outside the control limits for every other parameter and for the χ^2 part of the chart every one of the points was outside the control limits. Of course it is to be expected that the χ^2 test would be much more stringent than the test applied under Type I specification because the control limits established under the Type I specification are merely the limiting case of the limits set on χ^2 for the case of two cells. We see, however, when samples are actually drawn from a constant system of causes, as was done as nearly as possible in obtaining the data for Fig. 6 of the present paper, all of the estimates of the parameters remain well within the sampling limits at least the expected proportion of the time.

To show that the limits set by means of Specification Type V upon the expected or average value of the data in Fig. 4 of the article on control charts just referred to are much smaller than could have been set by means of either Specification Type III or Specification Type IV, Fig. 8 is given. The limits based upon Specifications Type III and IV

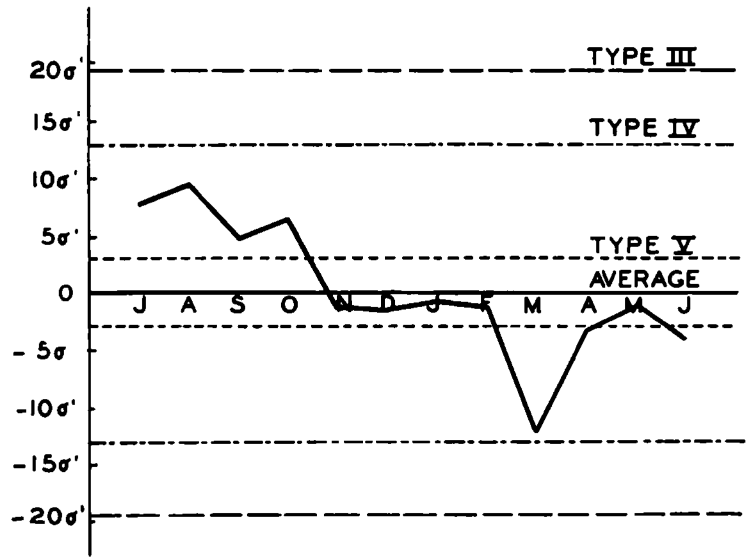

Fig. 8

were obtained directly from the nomogram of Fig. 4. The magnitudes of L_n stand in the order 19.3, 12.8 and 3.0. We see at a glance that lack of control, not indicated at all upon the basis of either Types III or IV, appears probable upon the basis of Type V.

Of course the use of the nomogram of Fig. 5 involves certain assumptions which now should be considered. The sampling limits are based upon the assumption that the sample is drawn from a normal universe. Even under these conditions the distributions of the values of estimates of the four parameters considered above are skew with the exception of that of the average, but approach normality as the size of the sample is increased. Theoretical and practical considerations lead us to believe, however, that satisfactory limits can be established by the method just described making use of the nomogram of Fig. 5 provided the following restrictions as to the size of the sample are made.

(*a*) The expected distribution of the averages of samples of any size n is normal about the expected value $\overline{X}'$.

(*b*) Comparatively small error [11] will be made in fixing the limits on the parameter σ' by means of the nomogram of Fig. 5 provided n is 25 or more.

(*c*) For a sample of size n of 500 or more the nomogram of Fig. 5 may be used in fixing limits on all four parameters.[12]

These limitations do not require necessarily that the distribution of the estimate of a parameter must be normal for n as large or larger than specified; instead they merely require that it may be represented by the first few terms of the Gram Charlier series for which the normal law integral over a range equally divided by the expected value of the parameter is a close approximation to the integral of the Gram Charlier series over the same range.

Fixing the Parameters

There are various ways of arriving at the values of the parameters to be accepted as the basis for quality control. Sometimes they may be fixed by the economics of the problem. Such is the case for the Type I specification when the economic standard fraction defective or p' is known. At other times the parameters are fixed by technical considerations such for example as in the case of an induction coil whose inductance must lie within well-defined limits in order to obtain a proper functioning of the entire circuit, for this would effectively fix $\overline{X}'$ and σ'. In most practical instances the technical considerations tend to fix only the average and standard deviation. At other times we may empirically choose the observed estimates of these parameters determined from the data obtained within the fixed interval of time wherein we have reason to believe the quality has been produced under essentially the same conditions. Irrespective, however, of what period is chosen as a base in fixing p' or any other parameter, the control chart serves to show whether or not the product has been controlled over this period. In any case the parameters are accepted at least as

[11] Pearson, Karl, "On the Distribution of Standard Deviations of Small Samples," *Biometrika*, Vol. X, Part IV, May 1915, pp. 522–529.

[12] Pearson, Karl, and others, "On the Probable Errors of Frequency Constants," *Biometrika*, Vol. XIV, 1903, p. 273 seq., Vol. IX, 1913, p. 22 seq. Isserlis, L., "On the Conditions under Which the Probable Errors of Frequency Distributions Have a Real Significance," *Proceedings of the Royal Society*, Series A, Vol. XCII, 1915, pp. 23–41.

temporary standards. In every case the choice of the fixed values calls for the exercise of engineering judgment. The statistical problem enters after these standards have been fixed. It is to determine whether or not the observed fluctuations in the observed estimates of the parameters are explainable upon the basis of chance. In general, the method of fixing the limits closely corresponds to that whereby a manufacturer sets up specifications for any kind of product.

It should be noted that from a statistical standpoint the control charts are based upon a priori reasoning. The type of cause system specified by the engineer is taken as a standard a priori system which is accepted as an ideal which the manufacturer hopes to maintain. The control chart thus makes it possible to differentiate between deviations in quality which can reasonably be accounted for on the basis of sampling and those deviations which cannot be so accounted for.

It will have been noted that the limits are a function of the size of the sample n. The question is therefore often raised: How large a sample shall be chosen?

So long as we are willing to risk our engineering judgment that the system of causes is controlled, we need take no samples. If, however, we have reason to believe that the quality has not been controlled or at least wish to make sure that it is being controlled to the extent that the deviation introduced by the assignable cause shall not escape detection if greater than some chosen value, it is necessary for us to take a sample of sufficient size to reduce the limits of sampling fluctuations in the particular parameter under study to just less than this same value.

In those cases where customary practice calls for the inspection of a certain number of units of product for reasons other than control, these data may be used in the manner outlined above to indicate the degree of control. In many instances the number of units of product to be inspected is so fixed as to insure with a known degree of probability that the apparatus passing from one stage of the manufacturing process to another meets a given tolerance for defects. This practice serves to fix the number to be inspected in order to maintain a given quality of apparatus as it passes through the stages of the manufacturing process. The use of the data so obtained in the form of a control chart serves to fix attention upon the assignable causes of variation in the quality. The presence of these causes having been detected, it generally becomes a comparatively simple matter to find and eliminate them. In this way we can secure a controlled product usually requiring less inspection and hence involving the lowest cost of manufacture.

I am indebted to Mr. V. A. Nekrassoff for the construction of the nomograms presented in this paper.