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A STUDY OF THE ACCELERATED MOTION OF SMALL DROPS THROUGH A VISCOUS MEDIUM.[1]

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NOT long ago Professor E. P. Lewis[2] developed a new method of determining the amplitude of a sound wave in air. It was necessary to assume that if we have a small particle falling freely through a viscous medium under the action of weight, and at the same time acted upon by a simple harmonic force at right angles to the vertical fall, the equations of the vertical and horizontal components of the motion may be written respectively,

$$mg = k\frac{dy}{dt}, \tag{1}$$

$$m\frac{d^2x}{dt^2} - k\frac{dx}{dt} = A \sin \omega t, \tag{2}$$

where m represents the mass of the particle, x and y the displacement at any instant, A the amplitude of the simple harmonic force, g the acceleration of gravity and k a constant based upon the assumption that the viscous resisting force, in both cases, bears the same proportion to the first power of the velocity. Three general questions are involved. In the first place what are the limitations under which equation (1) holds if the particle moves under the influence of a constant force in one dimension only? Stokes[3] has stated this problem as that of the motion of an incompressible fluid, infinite in extent and of uniform density, moving with a small velocity and without slipping, past the surface of a small sphere, and he arrives at the well-known formula,

$$F = 6\pi\mu a v, \tag{3}$$

where F is the resultant viscous force, μ the coefficient of viscosity of the medium, a the radius of the sphere and v the terminal velocity. The fall of solid and fluid spheres in air and in liquids has been very carefully studied experimentally by Millikan,[4] Nordlund,[5] Jones,[6] Allen,[7]

[1] Accepted in partial satisfaction of the requirements for the degree of doctor of philosophy in the University of California.

[2] Lewis and Farris, PHYS. REV., N.S., Vol. VI., p. 492.

[3] Stokes, Math. and Phys. Papers, Vol. III., p. 57.

[4] Millikan, PHYS. REV., Vol. 4, Apr., 1911, p. 349; Vol. 2, Aug., 1913, p. 109.

[5] Nordlund, Zeitschrift für Physikalische Chemie, 87, 1914, p. 40.

[6] Jones, Phil. Mag., 37, p. 45, 1914.

[7] Allen, Phil. Mag., 1900, p. 323.

Zeleny,[1] Arhold,[2] Ladenburg,[3] Silvey,[4] Ellis,[5] and others, and the theoretical development given by Stokes has been extended and modified to fit certain conditions not contained in the earlier treatment. The work of these men shows that equation (1) represents the motion of a particle with a uniform linear velocity V, so long as the quantity $\frac{Va}{\nu}$ is small, where a is the radius of the particle and ν is the kinematic coefficient of viscosity.[6]

In the second place, no record of experimental evidence has been found by the writer in justification of using equation (2) as an expression of the horizontal vibratory motion, even though the particle be confined to move in one dimension. Although in the early theoretical work of Stokes, he arrives at an equation for the force acting on a pendulum bob, assuming that it oscillates in a straight line through a viscous medium, and later, after investigating the problem of the small sphere moving in a similar medium, he points out that the expression for the resisting force on the spherical bob reduces to that for the force acting on a small sphere, which moves with uniform velocity, when the period of the pendulum approaches infinity. Thus far the two equations (1) and (2) have been discussed independently one of the other, as expressions representing only a one-dimensional motion, and it has been pointed out that we cannot justly assume equation (2) without further experimental evidence. There still remains the third question, *i. e.*, granted that equations (1) and (2) hold separately, when the motion is linear, will they still serve to express the component relations when the motion is two-dimensional? The experiment described below was designed to answer this third question, and to determine to what extent, if any, the terminal vertical velocity component of a small sphere will be changed if a simple harmonic force is applied at right angles to the original motion.

Charged particles are made to fall between two condenser plates connected to the terminals of a Thoradson transformer, and illuminated in such a way that a photographic trace of the path of each small sphere may be obtained. In this case equation (2) becomes

$$m\frac{d^2x}{dt^2} + k\frac{dx}{dt} = neX \sin \omega t, \tag{4}$$

where m and k have the same meaning as previously and X is the potential gradient of the field, e the elementary charge and n the number of

[1] Zeleny, PHYS. REV., Vol. 30, May, 1910, p. 535.
[2] Arnold, Phil. Mag., 22, 6, p. 755.
[3] Ladenburg, Ann. der Physik, 22, p. 287 (1907), and 23, p. 447, 1907.
[4] Silvey, PHYS. REV., N. S., Jan. 1916, p. 106.
[5] Ellis, Phil. Mag., 6th Series, Vol. 29, p. 526.
[6] Rayleigh, Phil. Mag., Vol. 36, 1893, p. 365.

elementary charges on a given particle. The complete solution of this equation is

$$x = \frac{neX}{k\omega} \sin \delta \sin (\omega t - \delta) + Be^{-(kt/m)} + B_1, \tag{5}$$

where δ is the phase difference between the motion of the particle and the varying simple harmonic disturbance and B and B_1 the integration constants.

The second term soon becomes negligible in the actual case, whereas the third term, which is a constant, merely determines the point about which oscillations take place, depending upon the phase of the driving force at the instant the field is turned on. After a short time, under these conditions, the particles would vibrate back and forth in a horizontal line, but for the fact that they are falling at the same time so that the trace becomes an approximate sine curve.

From equation (5) the maximum displacement from the mean position $x = B$ is seen to vary directly as the number of elementary charges on the falling drop and directly as the magnitude of the field gradient X, providing of course, that k remains a constant for such motion. Thus it is possible to test in a very simple way the involved assumption in regard to k, since the displacement, x and y, may be read from the plate and ne and X may be determined experimentally. From equation (5), neglecting the last two terms, the horizontal motion, after a very short interval of time, is practically expressible as

$$V = A \sin \omega t, \quad \text{where} \quad A = \frac{ne\, X \sin \omega t}{k\omega}. \tag{6}$$

The experimental arrangement was as follows: Two parallel brass plates, each 17.78 cm. × 3.81 cm. × .5 cm., were supported on an ebonite base. The distance between them was accurately fixed at 3.8 cm. by means of ebonite bars, and grooves in the base. The plates were then connected to the terminals of a transformer and the air condenser thus formed was set parallel to, and about 7 cm. from the lens of the camera. The particles were illuminated by a carbon arc enclosed in a sheet iron cylinder, and the angle between the direct rays of the arc and the perpendicular to the face of the camera was adjusted to approximately 28°. A large sheet of paper in which a narrow slit 2 mm. × 10 mm. had been cut, was arranged about an inch above the condenser so that when a water or oil spray was made with an atomizer above the paper, the small drops fell into the illuminated field near the center of the condenser.

Needle points were placed at the top of the condenser so that as the drops fell past them they would pick up one or more elementary charges and then when the drops came between the plates they would oscillate back and forth as they fell, thus tracing a sine-like path. After careful focusing on a glass fiber, held for the instant at the center of the condenser but withdrawn after final adjustment, a spray was made above the paper and so soon as the first droplets entered the field the shutter was opened from ½ to 5 seconds. Since the image on the plate was very narrow, each plate could be withdrawn a small distance at a time and thus made to serve for several exposures.

A lens of 1.5 cm. aperture and .9 cm. focal length was used. It was found necessary to employ the very sensitive Seed Graflex plate and the largest stop was used for most of the work, but a very faint narrow trace could be obtained with a stop .5 mm. in diameter.

To exclude all light coming from other sources, it was found helpful to place a screen immediately in front of the condenser. In every case, the best photographic results were obtained when the particles fell so as to be viewed against a background in which there was no reflecting material of any kind. For instance, if the screen was placed too near the condenser, some light would be reflected from its surfaces into the camera and fog the plates, even though all screens and the condenser itself was painted black.

Owing to convection currents, very small particles soon drifted out of focus, so that in the present work only comparatively large drops were used. At first an X-ray tube was employed to furnish a constant supply of ions between the plates in the hope that the particle would pick up charges as they fell, which would, as we have seen from equation (6), result in a change ΔA of the amplitude A, which would bear an integral relation to the previous amplitude. The average potential maintained between the condenser plates, as determined by the sparking distance between two needle points was about 28,000 volts, and calculation shows that for an increase of only one elementary charge the increase in A would be small, so that it would be very difficult to measure the amplitudes with a sufficient degree of accuracy to test the ratio $\Delta A/A$.

In view of this fact it was decided to increase the field gradient in a known manner, keeping the charge on the particle constant. This was done by making the distance between the plates at the top different from that at the bottom. A heavy brass strip 3.81 cm. × 1.27 cm. × 8.9 cm. was screwed to the lower edge of the condenser, so that with this in place the two field gradients were in the ratio 2 to 3. The same ratio should, therefore, exist between the amplitudes of the particle's motion at the top and the bottom of the condenser.

Four representative photographs are reproduced. In each of these, traces of a large number of particles are shown. Many of the drops drift out of focus and in the measurements which follow these were discarded. The data for thirty-six particles is recorded in

TABLE I.

Drop No.	No. of Waves at Top of Plate.	Dist. in Mm. on Plate.	No. of Waves at Bottom of Plate.	Dist. in Mm. on Plate.	Ave. λ_t.	Ave. λ_b.	Amp. at Top in Mm.	Amp. at Bottom.	Ratio of Amplitudes.
1	4	32.5	5	45.3	8.12	9.06	2.4	3.6	.666
2	4	37.5	4	39.5	9.37	9.87	2.6	3.5	.666
3	3	34.2	4	47.2	11.40	11.80	1.6	2.6	.616
4	3	33.5	4	47.7	11.16	11.92	1.6	2.0	.800
5	2	39.0	2	40.0	19.50	20.00	.6	1.0	.600
6	3	33.0	3	33.5	11.00	11.16	1.5	2.3	.652
7	3	46.0	3	48.7	15.33	16.23	1.3	1.7	.764
8	4	26.7	6	43.2	6.67	7.20	.8	1.0	.800
9	4	32.6	5	41.6	8.15	8.32	.5	.8	.633
10	6	41.6	6	43.5	6.93	7.25	1.0	1.5	.666
11	3	45.5	2	30.9	15.16	15.45	2.0	2.7	.740
12	5	47.2	4	38.7	9.44	9.67	1.3	2.0	.650
13	3	42.7	2	28.5	14.23	14.25	1.0	1.5	.666
14	11	50.0	9	38.7	4.54	4.30	1.9	2.7	.705
15	4	47.2	3	37.2	11.80	12.40	1.3	2.0	.650
16	3	38.2	3	41.5	12.73	13.83	2.1	3.2	.656
17	5	31.2	6	41.0	6.24	6.83	1.0	1.5	.666
18	6	50.5	7	59.3	8.42	8.47	1.8	2.4	.750
19	5	39.5	6	45.5	7.90	7.60	1.8	2.5	.720
20	6	48.5	6	52.0	8.08	8.66	2.0	3.0	.666
21	4	40.0	6	61.5	10.00	10.25	.8	1.5	.537
22	5	54.4	5	55.0	10.88	11.00	1.4	2.0	.700
23	5	30.0	4	24.0	6.00	6.00	2.3	3.5	.656
24	4	32.3	5	40.5	8.07	8.10	1.6	2.0	.800
25	6	45.5	2	13.2	7.58	7.60	1.9	2.5	.760
26	3	34.0	4	45.0	11.30	11.25	1.0	1.8	.555
27	4	34.0	6	52.5	8.50	8.75	1.0	1.8	.555
28	4	47.0	4	47.0	11.75	11.75	2.0	2.8	.714
29	5	49.0	6	59.7	9.80	9.95	.6	1.0	.600
30	5	56.5	5	58.3	11.30	11.66	1.3	2.0	.650
31	7	49.5	6	46.5	7.07	7.75	1.0	1.5	.666
32	8	61.5	8	65.5	7.71	8.19	.9	1.5	.600
33	8	60.0	8	62.5	7.50	7.81	1.5	2.1	.715
34	5	54.5	5	58.5	10.90	11.70	1.0	1.5	.666
35	8	54.0	9	65.0	6.77	7.22	.6	1.0	.600
36	9	45.0	12	60.0	5.00	5.00	1.0	1.6	.625

Columns 5 and 6 show the mean wave-lengths for the top and the bottom of the condenser, a wave-length being taken as the distance between successive maximum and minimum in the approximate sine curve reproduced on the plate. The frequency of the 60-cycle alternating current feeding the primary of the transformer was found to be approximately constant at night when most of the work was done, therefore we may compare the wave-lengths of the top and bottom, which should be equal, if the terminal vertical component of velocity is not changed at the lower part of the condenser where the gradient becomes greater.

It will be noted that with three exceptions corresponding to drops Nos. 14, 19 and 26 the wave-lengths are longer in the region where the field gradient is a maximum, and careful observation of the plates showed that in each of these three cases the particles had drifted out of focus so that at the lower end of the fall, the distance measured on the plate was only the projection of the actual motion, which accounts for the apparent discrepancy in the results. We should therefore conclude that a particle falls faster when it is simultaneously vibrating at right angles to the line of fall.

The objection might be raised that the particles under consideration had not reached a constant vertical velocity when they first entered the region of the condenser and so to make sure that this error did not occur photographs were taken of particles falling between parallel condenser plates. The results for six representative particles are shown in the following table and indicate that the terminal vertical velocities had been reached before the particles came under observation.

TABLE II.

Drop No.	No. of Waves at Top of Plate.	Wave-length in Mm.	No. of Waves at Bottom of Plate.	Wave-length in Mm.
1	3	11.29	3	11.29
2	3	12.06	3	12.03
3	4	9.20	4	9.20
4	5	8.02	5	8.01
5	6	8.00	4	8.01
6	3	11.86	3	11.88

Furthermore, the field was somewhat distorted near the middle of the condenser where the half plate terminates. The force lines are curved upward at this point and thus the amplitude of oscillation begins to increase slightly above this region, but this distortion should not give rise to the qualitative increase in fall in the part of the condenser where the field intensity is strongest.

In reference to the amplitudes of the particles, it is interesting to note that a large number of the ratios given in Table I. are approximately two thirds, which would indicate that k is a constant. It is impossible, however, to draw any definite conclusions from these ratios, for it was observed that the particles usually picked up additional charges near the middle of the condenser, owing to the constant discharge taking place from the sharp edge of the half plate.

If the motions in the two perpendicular directions are independent of each other, we may write the component displacements as

$$x = a \sin \omega t \quad \text{and} \quad y = ct,$$

or the resultant path is given by the equation

$$x = a \sin \frac{\omega y}{c},$$

where c is a constant. The resultant velocity at any instant is therefore

$$o = \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} = \sqrt{c^2 + a^2\omega^2 \cos^2 \omega t},$$

and makes an angle with the vertical equal to

$$\frac{dx}{dy} = \frac{a\omega}{c} \cos \frac{\omega y}{c}.$$

If we assume a frictional viscous force $F = kv^n$ tangent to the path and resisting the motion of the particle it will have a normal component F_v equal to

$$F \cos \theta = kc(c^2 + a^2\omega^2 \cos^2 \omega t)^{n-1}.$$

If $n = 1$ or if the resistance is proportional to the first power of the velocity, then $F \cos \theta = kc =$ const. If $n = 1$ then the mean value of $F \cos \theta$ for a half period interval is constant, so that the mean vertical component velocity should be constant within the same interval, and this mean value should diminish as the amplitude increases for the vertical component of the force becomes larger. This would produce a result opposite to the one actually observed.

The maximum horizontal component velocity of the particles was from 10 cm. to 75 cm. per sec., values far in excess of the limit within which Stokes's law holds, so that the resistance in this direction varied as some higher power of the velocity.

If F_x and F_y represent the viscous forces, where each component motion is taken separately, it is evident that they do not represent the component forces when the displacement is a resultant of the separate displacements.

In conclusion it is pointed out that if we assume the resultant viscous force to be exerted tangential to the path at any point, then the terminal vertical component will be decreased rather than increased, so that it is necessary to assume a turbulent motion of the medium in order to explain this discrepancy from observed results.

If F_y is the resistance offered to a small particle moving with a constant velocity c in the y direction and F_x is the viscous resistance offered to a particle oscillating in the x direction, then it does not follow that F_x and F_y are the components of the viscous tangential resistance when both motions are compounded, unless this resistance depends upon the first power of the velocity.

The experimental results of this paper indicate that if a small sphere, moving through a viscous medium with a terminal velocity v under a constant force is suddenly caused to vibrate at right angles to the original line of motion the vertical velocity component v is increased.

Certain problems[1] relating to the two-dimensional flow of viscous liquids have been solved, but the solution under the boundary conditions set in the present problem involves certain mathematical difficulties. The general equations of motion of an incompressible fluid is of the type

$$\frac{\partial u}{\partial t} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} - v\frac{\partial u}{\partial y} - \omega\frac{\partial u}{\partial z} + \surd\Delta^2 u$$

where u, v, ω, x, y, z, are the components of velocity and of body force respectively, p the mean pressure, ρ the density, and $\surd$ the kinematic coefficient of viscosity. Two general types of solution have been obtained, one in which terms involving the second power of velocity, such as $u(\partial u/\partial x)$ are omitted, and the other in which the terms $\surd\Delta^2 u$, etc., are neglected. Thus, for steady one dimensional motion, there is a large range of velocities for which no solution has been found to apply. It should be expected, therefore, that the broader problem of motion of a small sphere in two dimensions should present greater difficulties.

A study of the experimental and theoretical problem suggests three possible explanations of the results observed in this paper.

1. "Cavitation" at the edge of the sphere would give rise to the phenomenon observed, since even a small decrease in pressure on the lower side of the drop would give rise to a greater vertical component of velocity.

2. From the experiments of Mr. Ellis Williams[2] the stream lines for large values of Va/v indicate that for a comparatively long distance in the wake of the particle the liquid has the same velocity as the small sphere. If the particle suddenly stops as it does at the end of the oscillation and starts to return the pressure will evidently be greater on the upper side and will increase the vertical fall.

[1] Jeffery, Phil. Mag., 29, 455, 1915.

[2] Phil. Mag., 29, 526, 1915.

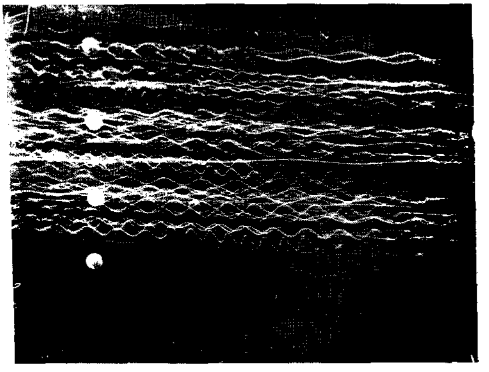

Fig. 1.

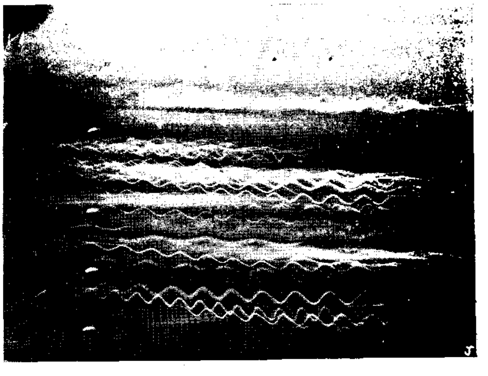

Fig. 2.

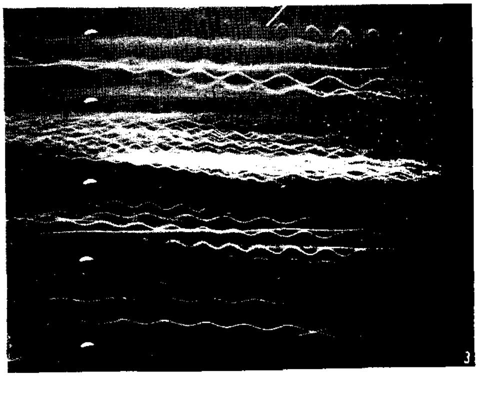

Fig. 3.

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3. If the spherical drop of liquid suffers a distortion when it enters the electric field and becomes flattened, it would tend to set itself with the longer axis at right angles to the line of motion. The effect of the field would be to keep the flat side perpendicular to the sides of the condenser. The resultant effect upon the shape of the drop would be complicated, but it would undoubtedly affect the velocity.

In conclusion, I wish to thank Professor E. P. Lewis for his many helpful suggestions and counsel throughout these experiments and for the facilities placed at my disposal.

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