Correction of Data for Errors of Measurement

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INTRODUCTION

EVERY measurement is subject to error. This universally accepted truth is the result of every-day experience. From the simplest type of measurement, such as determining the length of a board with an ordinary tape measure, to the most refined type of measurement, such as determining the charge on an electron, errors are bound to creep in.

Now, a manufacturer must constantly make measurements of one kind or another in an effort to control his production processes and to measure the quality of his finished product in terms of certain of its characteristics, but, before he can safely determine the significance of observed differences in his production processes or in the quality of his product as given by these measurements, he must make allowance for his errors of measurement; i.e., for the fact that the observed differences may be larger or smaller than the true differences. To make such allowances for the errors of measurement of any characteristic, to find out what the true magnitude of the characteristic most probably is, to find out, as it were, what a thing most probably is from what it appears to be, presents an endless chain of interesting problems to be solved.

Three important types of problems arising in engineering practice are discussed in this paper. They are:

1. Error correction of data taken to show the quality of a particular lot.

2. Error correction of data taken periodically to detect significant changes in quality of product.

3. Error correction of data taken to relate observed deviations in quality of product to some particular cause.

The solution of the first one is presented here for the first time. The solution of the second has been generalized to include cases not previously solvable. All three types of problems are illustrated.

PART I

Type 1—Error Correction of Data Taken to Show the Quality of a Particular Lot

Let us take a specific problem first. Assume that we have a lot consisting of 15,000 transmitters [1] and a machine with which to measure the efficiency of each instrument. Suppose we make one observation on each transmitter—a total of 15,000 measurements. Suppose we find, as in the distribution illustrated in Fig. 1, that one measurement is in the efficiency range −1.75 to −1.50, 17 within the range

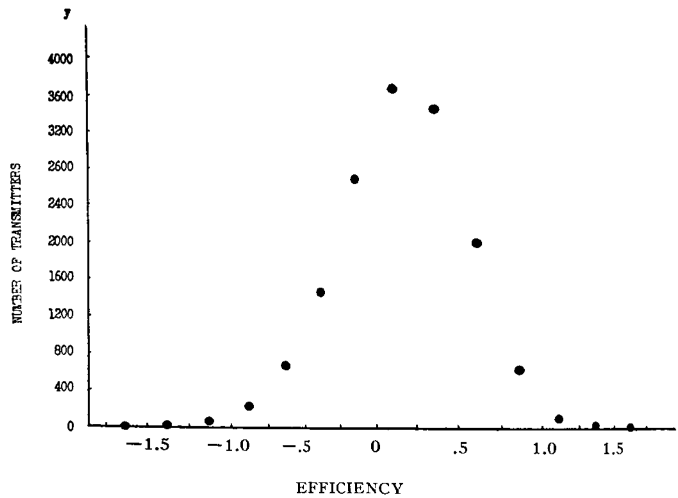

Fig. 1—Typical frequency distribution. Chart showing observed number of transmitters versus efficiency

−1.50 to −1.25 units, and so on. The vertical height of a point represents the number or frequency of occurrence of observations falling within the corresponding interval laid off on the horizontal axis of the chart.

So far so good, but suppose a customer wants to buy these transmitters. We know that some transmitter which appeared to have an efficiency within the range of 1.25 to 1.50 units say, *may* actually have had an efficiency within some other interval. We know too that, because of the errors of measurement, the transmitters appear to differ more among themselves than they really do. We therefore

[1] Of course, the efficiency of a transmitter does not remain constant during a series of tests but these inherent variations in the transmitter may be considered, for our purpose, as forming a component part of the resultant error of measurement.

desire to find the most probable numbers of transmitters within the different intervals indicated in Fig. 1.

Analytical Statement of Problem

Let us assume that the most probable number of transmitters within the interval of efficiency from X to $X+dX$ is $f_T(X)dX$. It is this function $f_T(X)$ that we want to find. Similarly let us assume that there is some function $f_o(X)$ such that $f_o(X)dX$ gives the observed number of transmitters appearing to have efficiencies within the interval X to $X+dX$ where the measurements are made by a method wherein the probability of making an error within the interval x to $x+dx$ is $f_E(x)dx$. It is reasonable to expect that, if two of these functions are known, the third can be easily determined. We shall proceed to show that this is the case. Let us first find the law of error experimentally.

Finding the Law of Error

The problem is to determine the chance of making an error of a given magnitude in measuring the efficiency of any transmitter. Naturally, the only way of doing this is to make a series of measurements on a single transmitter from which we can determine the observed frequency of occurrence of measurements which differ from the average by some fixed amount, and thus find what percentage of the total number of measurements may be expected to fall within any given range on either side of the average. Common sense and intuition may tell us that we may expect to find a large percentage of the measurements within a narrow range on either side of the average, that there will be just as many measurements greater than the average by a certain amount as there are less than the average by the same amount, and that large deviations from the average may be expected to occur with less frequency than small deviations. Suppose we make 500 observations of the efficiency of a single transmitter and find the distribution given in Fig. 2. Just as we might have expected, the observed values of the efficiency of the transmitter are grouped symmetrically about the average of all the observed values. We see that the maximum deviation between observations on a single transmitter is quite large (33%) compared with the actual maximum differences observed between the efficiencies of the transmitters.

The results reproduced in Fig. 2 suggest that the deviations for the case in hand are distributed in a manner closely approximating the bell-shaped distribution so familiar in the theory of errors. We often find, as we do in this case, that the observed distribution can be closely approximated by a function $f_E(x)$ of the form

$$f_E(x)dx = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(X-\overline{X})^2}{2\sigma^2}} dx, \qquad (1)$$

where $f_E(x)dx$ is the probability that an error x will lie within the interval x to $x+dx$, σ is the root mean square or standard deviation, $\overline{X}$ is the arithmetic mean value and $(X-\overline{X})$ is the deviation x. The

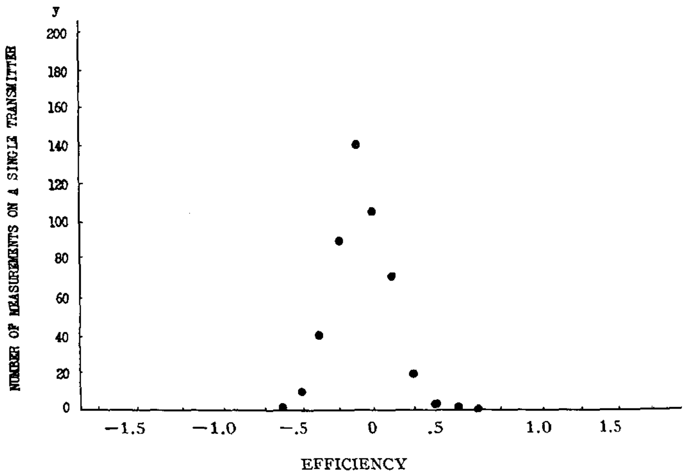

Fig. 2—Typical form of distribution of errors of measurement. Chart showing number of measurements on a single transmitter versus efficiency

function $f_E(x)$ is referred to in the literature as the normal law of error. If we try to fit such a curve to the deviations[2] given in Fig. 2, we obtain the results shown in Fig. 3. This figure is the same as Fig. 2 except for the addition of the smooth normal curve of error calculated for the observed data. Without further consideration, we shall assume the law of error to be normal and hence of the form indicated by Equation (1).

Finding the True Distribution $f_T(X)$

We have next to consider the choice of the function to represent the true distribution $f_T(X)$. Often we have reason to believe that this

[2] If the average of the observed values of the 500 observations of efficiency given in Fig. 3 is assumed to be the true value of the efficiency of the transmitter, then the deviation of an observed value from this mean is also the error of this observed value. We shall use the terms "error" and "deviation" interchangeably in this sense.

is also approximately normal, and hence we shall consider first the method for finding the observed distribution $f_o(X)$ for the special case when both the true distribution $f_T(X)$ and the law of error $f_E(X)$ are normal; i.e., when they are both of the form given by Equation (1).

We shall first obtain an experimental answer to this problem. Suppose we take, say, 1,000 instruments of some kind which are

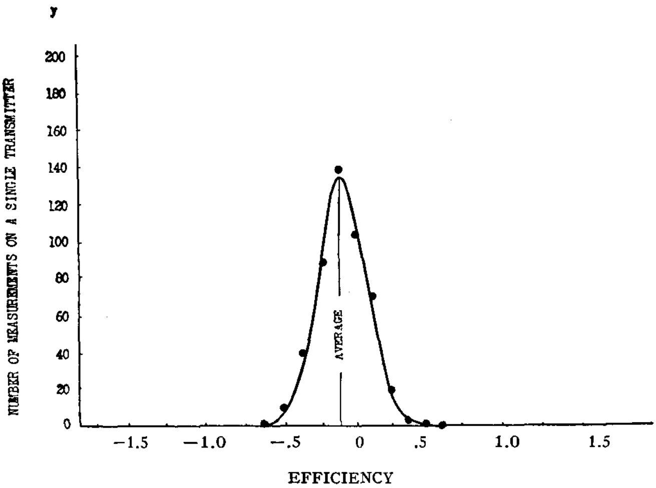

Fig. 3—Chart showing the observed distribution of errors fitted by a typical smooth curve. Data of Fig. 2 fitted by normal law of error, Eq. 1

known to be distributed in normal fashion, in respect to some characteristic, with a standard deviation σ_T. Let us measure each of these instruments by a method subject to the normal law of error whose standard deviation σ_E is $\frac{1}{2}$ σ_T. The results of one such experiment are given in Fig. 4. The observed frequencies of occurrence are represented by the circles. It was found that this observed distribution could be closely approximated by a normal law $f_o(X)$ for which the standard deviation σ_o was $\sqrt{\sigma_T^2+\sigma_E^2}$. This experiment suggests a general theorem which will be demonstrated analytically in a succeeding paragraph. The theorem is: When the true distribution $f_T(X)$ and the law of error $f_E(x)$ are both normal (hence expressible in form indicated by Equation (1)) with root mean square or standard deviations σ_T and σ_E respectively, the most probable observed distribution will be normal in form with a standard deviation $\sigma_o=\sqrt{\sigma_T^2+\sigma_E^2}$.

The observed distribution in Fig. 1 is asymmetrical and hence not normal as it should be if $f_T(X)$ and $f_E(x)$ were both normal. We must therefore, try some other function for $f_T(X)$.

Of course, experiments might be performed for other types of true and error distributions, but in all such cases the results, as in the illustration just considered, would be subject to errors of sampling.

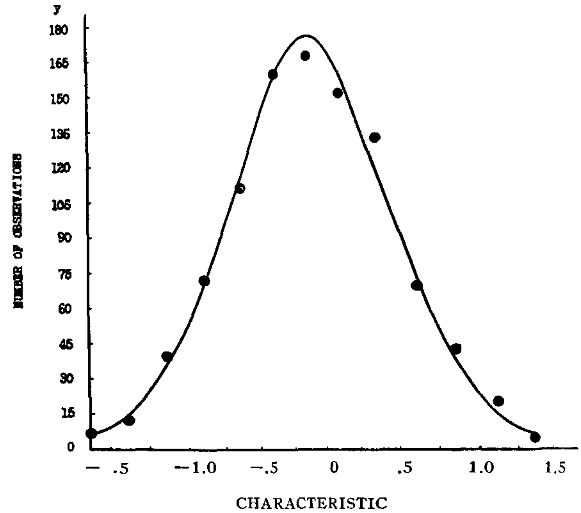

Fig. 4—Experimental results shpwing effects of errors of measurement. Normal curve fitted to observed points, when the true distribution and the law of error are both normal

Hence we shall proceed at once to the analytical treatment of the problem.

Assuming the law of error to be normal, we see that the fraction $f_E(x)dx$ of the number of objects having magnitudes between $X+x$ and $X+x+dx$ will be measured with an error between $-x$ and $-x-dx$ and hence will be observed as of magnitude X (Fig. 5). Thus

$$f_o(X) = \int_{-\infty}^{\infty} f_T(X+x) \frac{1}{\sigma_E\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_E^2}} dx. \tag{2}$$

For the particular case treated in a previous paragraph where both the true distribution $f_T(X)$ and the law of error $f_E(x)$ are normal, we may write Equation (2) in the form

$$f_o(X)dX = \frac{1}{\sigma_T\sigma_E 2\pi} \int_{-\infty}^{\infty} e^{-\frac{(X+x)^2}{2\sigma_T^2}} e^{-\frac{x^2}{2\sigma_E^2}} dX\, dx \tag{3}$$

where σ_T and σ_E are the root mean square or standard deviations of the true and error distributions respectively. Integration of Equation (3) gives [3]

$$f_o(X) = \frac{1}{\sigma_o\sqrt{2\pi}} e^{-\frac{X^2}{2\sigma_o^2}}, \tag{4}$$

where

$$\sigma_o = \sqrt{\sigma_T^2 + \sigma_E^2}. \tag{5}$$

Equations (4) and (5) are the analytical expression for the rule stated previously, for finding the observed distribution $f_o(X)$ when both the true and error distributions are normal, because Equation (4)

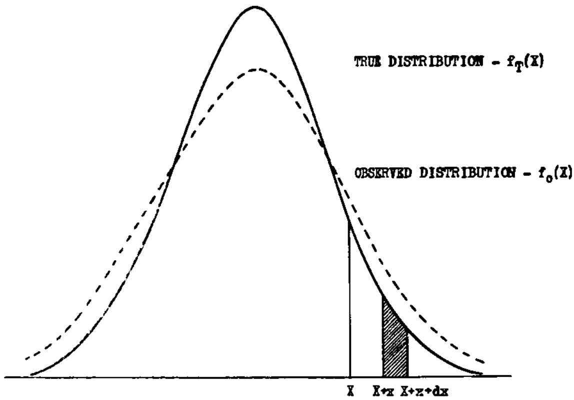

Fig. 5—Chart used in explaining the derivation of $f_o(X)$ in terms of $f_T(X)$

shows it to be normal and Equation (5) expresses the standard deviation σ_o of the observed values in terms of those of the true values and of the errors.

In practice, however, we often find that the true distribution is non-symmetrical or skew and can be more nearly approximated by the function [4]

$$f_T(X) = \frac{1}{\sigma_T\sqrt{2\pi}} e^{-\frac{X}{2\sigma_T^2}}\left[1 - \frac{k_T}{2}\left(\frac{X}{\sigma_T} - \frac{X^3}{3\sigma_T^3}\right)\right] \tag{6}$$

where k_T is a measure of the asymmetry or skewness, the modal or most probable value of X being at a distance $-\frac{k\sigma_T}{2}$ from the average

[3] See Appendix 1 where another method of solution is given.

[4] This is often referred to in the literature of statistics as the second approximation. It is in fact the first two terms of the Gram-Charlier series.

value of X. Substitution of this expression and a normal error function in Equation (2), yields upon integration [5] the following distribution $f_o(X)$ of the observed values

$$f_o(X) = \frac{1}{\sigma_o\sqrt{2\pi}} e^{-\frac{X}{2\sigma_o^2}} \left[1 - \frac{k_o}{2}\left(\frac{X}{\sigma_o} - \frac{X^3}{3\sigma_o^3}\right)\right] \tag{7}$$

where

$$\sigma_o = \sqrt{\sigma_T^2 + \sigma_E^2}, \tag{5}$$

and

$$k_o = k_T \frac{\sigma_T^3}{\sigma_o^3}. \tag{8}$$

We see that the distribution $f_o(X)$, Equation (7), of the observed values is of the same form as that $f_T(X)$, Equation (6), of the true values. The standard deviation of the errors of measurement σ_E, as in the previous case, has equal weight with the standard deviation σ_T in influencing the standard deviation σ_o of the observed values. The degree of asymmetry of the observed distribution as measured by the skewness k_o is, however, less (Equation (8)) than that of the true distribution as measured by the skewness k_T of the true distribution.

Now we can correct the observed distribution, Fig. 1, for the errors of measurement, because we find that the observed frequencies, Fig. 1, can be closely approximated by a function of the type defined by Equation (7). Knowing that the law of error, Fig. 3, is normal we conclude that the true distribution $f_T(X)$ must be a function of the same type as $f_o(X)$ was found to be except that the true standard deviation σ_T will be, from Equation (5), $\sqrt{\sigma_o^2 + \sigma_E^2}$ and the true skewness k_T will be, from Equation (8), $\frac{\sigma_o^3}{\sigma_T^3} k_o$. Now, σ_o and k_o can be calculated from the observed distribution, Fig 1, and σ_E can be determined by the data given in Fig. 3.

Thus finding the values of σ_T and k_T and substituting them in Equation (6), we have the function $f_T(X)$ representing the true distribution which we started out to find. From this knowledge of $f_T(X)$ we can now get the most probable frequencies of occurrence of the different efficiencies. Subtracting these frequencies from those observed and shown in Fig. 1, we get the corrections plotted in Fig. 6, expressed as percentages of the observed frequencies.

[5] This solution is also obtained by another method in Appendix 1.

Summary

We are now in a position to summarize the practical routine to be followed in finding the most probable distribution $f_T(X)$ of quality when the observed distribution is given.

To find $f_T(X)$, we must first know the law of error $f_E(x)$. We must show this to be normal and find the standard deviation σ_E

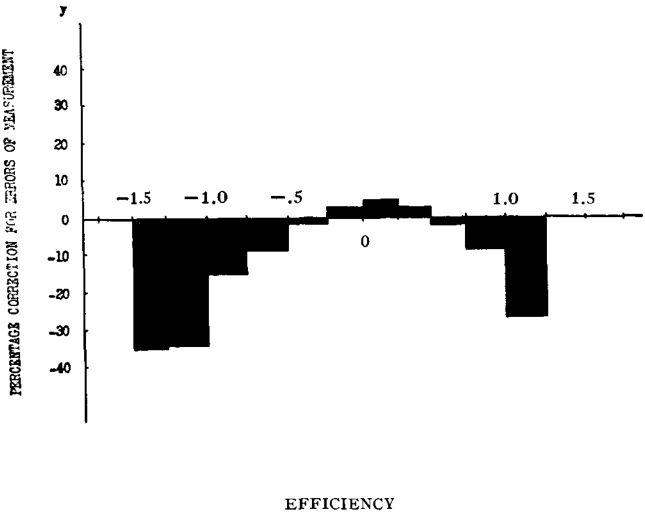

Fig. 6—Correction which must be applied to the observed distribution of transmitters Fig. 1, because of the existence of errors of measurement

by multiple tests on a single unit. The error made in determining the standard deviation σ_E from n observations is $\frac{\sigma_E}{\sqrt{2n}}$. Hence the precision we attain in finding $f_T(X)$ depends upon the number of observations n made in finding σ_E.

Having found σ_E to the required degree of precision, we must next discover whether or not the observed distribution $f_T(X)$ is either normal or the second approximation. Standard statistical methods can be used for this purpose.

If the $f_o(X)$ is normal, we then know that

$$f_T(X) = \frac{1}{\sqrt{2\pi(\sigma_o^2 - \sigma_E^2)}} e^{-\frac{X^2}{2(\sigma_o^2 - \sigma_E^2)}}.$$

and, if $f\,(X)$ is second approximation, we know that $f_T(X)$ is given by Equation (6), where σ_T and k_T can be found with the aid of Equations (5) and (8) in terms of the observed values of σ_E, σ_o and k_o. In other words we have

$$f_T(X) = \frac{1}{\sqrt{2\pi(\sigma_o^2 - \sigma_E^2)}} e^{-\frac{X}{2(\sigma_o^2-\sigma_E^2)}}$$

$$\left[1 - \frac{k_o\sigma_o^3}{(\sigma_o^2-\sigma_E^2)^{\frac{3}{2}}}\left(\frac{X}{(\sigma_o^2-\sigma_E^2)^{\frac{1}{2}}} - \frac{X^3}{3(\sigma_o^2-\sigma_E^2)^{\frac{3}{2}}}\right)\right].$$

PART II

Correction of Data Taken Periodically to Detect Significant Changes in Quality of Product

Irrespective of the care taken in defining and controlling the manufacturing processes, the units of a product will differ among themselves in respect to any measurable characteristic. Random fluctuations in such factors as humidity, temperature, grade of raw material, and wear and tear on machinery may produce such differences between units of a product. Such random variations in the factors underlying the manufacturing process usually yield a product in which the units differ in random fashion according to some law of probability.

Customarily, product is inspected periodically, and the data are analyzed to determine if the observed difference in two samples is greater than can be accounted for as a random variation. If it is, we may assume that the manufacturing processes have changed significantly for some reason which further investigation should disclose. Now, the presence of errors of measurement effectively increases the magnitude of the random differences to be expected from one sample to another and hence makes it harder for us to detect trends or fluctuations in product. Let us investigate this effect of errors of measurement.

Symbolic Statement of Problem

Symbolically we may assume that the probability of production of a unit of product having a characteristic X within any range X to $X+dX$ is $f_T(X)dX$, where the characteristic X is measured by a method subject to a law of error $f_E(x)$, so that $f_E(x)dx$ represents the probability of occurrence of an error x within the range x to $x+dx$. The problem is to find the corresponding distribution $f_o(X)$ for the observed magnitudes.

General Solution of Problem

Obviously the observed magnitude X_o is the algebraic sum of the true value X and the error x. Assuming that there is no correlation between these two quantities, the probability of a unit having a value of X within the range X to $X + dX$ being measured with an error x within the range x to $x + dx$ is $f_T(X)dX\ f_E(x)dx$. Assuming that $X_o = X + x$ we may write the probability

$$y_o = f_o(X_o)dX_o = \int_{-\infty}^{\infty} f_T(X_o - x)dX_o f_E(x)dx,$$

because $f_o(X_o)$ is obtained by taking into account that all possible values of x between $+\infty$ and $-\infty$ may be combined with a given X. This integral is of the same form as that given in Equation (2). Integration for the case where both $f_T(X)$ and $f_E(x)$ are normal gives

$$f_o(X_o) = \frac{1}{\sigma_o\sqrt{2\pi}} e^{-\frac{X_o^2}{2\sigma_o^2}}$$

where as before $\sigma_o = \sqrt{\sigma_T^2 + \sigma_E^2}$. This result is well known as the law of propagation of error.

When $f_E(x)$ is normal and $f_T(X)$ is given by the first two terms of the Gram-Charlier series, Equation (6), with skewness k_T and standard deviation σ_T, the observed distribution $f_o(X_o)$ is of the same functional form as the true distribution $f_T(X)$ and has values of standard deviation σ_o and skewness k_o given by Equations (5) and (8) in Part I. This result appears to be new.

Now for the case where the true distribution $f_T(X)$ and the law of error $f_E(x)$ are both second approximation type, the integration is somewhat tedious, but we can approach a special case of this problem easily from a slightly different angle as indicated in Appendix 2. Under certain special conditions therein set forth, the resultant distribution is also second approximation form with a skewness which is less than that of either $f_T(X)$ or $f_E(x)$ and is equal to $\frac{1}{\sqrt{2}}k_T$ when $k_T = k_E$, the standard deviation σ_o being again equal to $\sqrt{\sigma_T^2 + \sigma_E^2}$.

Example of Applications to Determine Most Economical Way of Measuring Quality

Let us next consider a very simple method of using the above results to indicate the most economical method for determining the quality of product with a given degree of precision.

What is the most economical way of determining the quality of product within some predetermined range $\overline{X} \pm \Delta\overline{X}$ with a known probability P, where $\overline{X}$ is the average quality? Let us assume that:

a_1 = cost of selecting each unit and making it available for measurement,

a_2 = cost of making each measurement,

n_1 = number of units selected,

n_2 = number of measurements made on each unit,

σ_1 = standard deviation of the errors of observation,

$\sigma_2 = \sigma_T$ = standard deviation of the true distribution $f_T(X)$.

Let us take $P = .9973$. Then the range $\overline{X} \pm 3\sigma_{\overline{X}}$ includes 99.73 per cent. of the observations, and hence $\Delta\overline{X} = 3\sigma_{\overline{X}}$.

The average of n_2 measurements made on one unit is the observed value of the magnitude X for that unit, and this average has the standard deviation $\sigma_E = \frac{\sigma_1}{\sqrt{n_2}}$. Hence, from the theory of the preceding section, the standard deviation of the observation is

$$\sigma_o = \sqrt{\sigma_T^2 + \sigma_E^2} = \sqrt{\sigma_2^2 + \frac{\sigma_1^2}{n_2}}.$$

The standard deviation of the average of n_1 observations is $\sigma_{\overline{X}} = \frac{\sigma_o}{\sqrt{n_1}}$ and we find upon solving for n_1,

$$n_1 = \frac{\sigma_2^2 + \frac{\sigma_1^2}{n_2}}{\sigma_{\overline{X}}^2}.$$

The cost of inspection is

$$y = a_1 n_1 + a_2 n_1 n_2,$$

and by customary methods this can be shown to be a minimum when

$$n_2 = \frac{\sigma_1}{\sigma_2}\sqrt{\frac{a_1}{a_2}}.$$

The following values correspond to one practical case:

$\Delta\overline{X} = .3$ unit	$a_1 = \$0.50$
$\sigma_1 = .3$ unit	$a_2 = \$0.02$
$\sigma_2 = .9$ unit	$P = .9973$

Thus with the aid of the above theory we find the most economical method of inspection requires 2 observations on each of 86 units.

Application in Setting Limit Lines

Over 99 per cent. of the averages of samples of size N drawn from a product whose law of distribution is $f_T(X)$ where $f_T(X)$ is either normal or second approximation may be expected to lie within the limits defined by the true average $\overline{X}$ plus or minus $3\frac{\sigma_T}{\sqrt{N}}$. If an average falls outside these limits, this fact is taken as probably indicating the existence of a trend or cyclic fluctuation in product, the cause of which should be sought. The presence of errors of measurement increases the separation of these limits to $6\sigma_o$ from $6\sigma_T$. Our precision of detecting trend or cyclic fluctuation is thereby decreased.

Cases often happen in practice where σ_o is from 15 per cent. to 25 per cent. greater than σ_T. In some instances σ_o has been found to be nearly 50 per cent. greater than σ_T.

PART III

Error Correction of Data Taken to Relate Observed Deviations in Quality of Product to Some Particular Cause

In many practical cases it is not possible to write down an equation to show how the quality of a finished product depends upon the factors controlled by different manufacturing steps. To cite one such case, we may know that the quality of the finished article depends upon the control of the temperature to which some of the piece parts are heated in the process of manufacture. Thus the microphonic properties of carbon depend upon the temperature to which the carbon is heated. In cases where the relationship between quality and some factor (such as temperature in the above illustration) can only be determined through a study of the correlation existing between the quality and the particular factor, use must be made of the correlation coefficient r which is defined as

$$r=\frac{\Sigma yx}{\sigma_x\sigma_y N}$$

where x and y represent respectively deviations from the average quality $\overline{X}$ and the average magnitude $\overline{Y}$ of some factor which is to be controlled by the manufacturing process, and N is the number of observations. Now, if errors of observation are made in determining x and y, the observed correlation coefficient $r_{x_oy_o}$ is known to be given by the expression

$$r_{x_oy_o}=\frac{\sigma_x\sigma_y}{\sigma_{x_o}\sigma_{y_o}}r_{xy} \tag{10}$$

where $\sigma_{x_o}=\sqrt{\sigma_x^2+\sigma_{x_E}^2}$ and $\sigma_{y_o}=\sqrt{\sigma_y^2+\sigma_{y_E}^2}$,

σ_{x_E} and σ_{y_E} being the root mean square errors of observation of x and y respectively.

Attention is directed to Equation (10) which shows that the observed correlation coefficient $r_{x_oy_o}$ is always less than the true correlation coefficient r_{xy} irrespective of the number of observations made. Obviously, this point is of considerable commercial importance as we shall now see.

If the observed correlation is small, we customarily assume that there is little need of trying to control the quality X by controlling the manufacturing factor Y, whereas this conclusion cannot be justified unless it can be shown that the true correlation has not been masked by the errors of measurement.

This point has had to be taken into account in the development of machine methods for testing transmitters and receivers, because the calibration curves of the machines in terms of ear-voice tests depend upon the correlation coefficient.

Appendix I

It may be of some interest to certain readers to note that the results given in Equations (4) and (7) can also be obtained in the following way by the method of moments so often used in statistical investigations.

Assuming that $f_T(X+x)$ is expansible in terms of a Taylor's series, we get

$$f_o(X)=f_T(X)+\frac{\sigma_E^2}{2}f_T''(X)+\frac{1}{\lfloor 2}\left(\frac{\sigma_E^2}{2}\right)^2 f_T''''(X)+\frac{1}{\lfloor 3}\left(\frac{\sigma_E^2}{2}\right)^3 f_T^{vi}(X)+\ldots, \tag{11}$$

If we substitute a normal form for $f_T(X)$ in Equation (11) and solve for the moments of $f_o(X)$, we find that the odd moments are zero and the ratio of the 4th moment to the square of the 2nd is numerically 3 which indicates that $f_o(X)$ is normal in form.

A similar substitution of the 2nd approximation form for $f_T(X)$ in Equation (11) yields a distribution $f_o(X)$ from whose moments we deduce Equation (7). Use is made in this proof of the easily demonstrated theorem that

$$\int_{-\infty}^{\infty} x^i f_E^j(x) = 0$$

if $i < j$, where f_E^j is the jth derivative of the normal law function.

APPENDIX II

It is well known that the normal law of distribution may result from a system of n (n being large) causes each of which produces an increment ΔX measured from some fixed origin with a probability $p=\frac{1}{2}$ and no increment with a probability $q=\frac{1}{2}$. Furthermore the second approximation may result from a similar system in which $p+q$ and n is large. Under such systems of causes, the probabilities of the occurrences of $n, n-1, \ldots 3, 2, 1, 0$ increments are given by the successive terms of the point binomial $(p+q)^n$.

Let us assume that the symbols p_T, q_T, n_T, ΔX and p_E, q_E, n_E, Δx refer to the systems of causes controlling the product and errors respectively. The probabilities of observed combinations $n_T\Delta X + n_E\Delta x$, $(n_T-1)\Delta X + (n_E-1)\Delta x, \ldots$ are given by the successive terms of the expansion $(p_T+q_T)^{n_T}\ (p_E+q_E)^{n_E}$. Now for the special case $p_T = p_E = p$ and $\Delta X = \Delta x$ we have the resultant probability distribution $(p+q)^{n_T+n_E}$ with skewness

$$k_o = \frac{q-p}{\sqrt{pq(n_T+n_E)}}$$

and standard deviation

$$\sigma_o = \sqrt{pq(n_T+n_E)}.$$

Now if $p=q$, the skewness k_o is zero and the observed distribution is more nearly normal than either component, and its standard deviation σ is the square root of the sum of the squares of σ_T and σ_E. This result is similar to that given by Equation (4) of this paper.

We may also consider by this method a case not treated in this paper. When the skewness k_T of the true values is equal to that k_E of the law of error, or, more particularly, when $n_T = n_E = n$, $p_T = p_E = p$, $q_T = q_E = q$, $p = q$, we see that the observed distribution is given by the successive terms of $(p+q)^n$ and the skewness of the observed distribution k_o is $\frac{1}{\sqrt{2}}\,k$, and the standard deviation σ_o is $\sqrt{2}\,\sigma$; i.e. the observed skewness is only $\frac{1}{\sqrt{2}}$ times that of either the true distribution or the law of error, and the observed standard deviation σ_o is $\sqrt{2}$ times the standard deviation of either of the true or error distributions.