SOME RESULTS ABOUT BASIC ELEMENTS, CANCELLATION AND EFFICIENT GENERATION OF MODULES OVER LAURENT POLYNOMIAL RINGS

By

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Thesis submitted to the Indian Statistical Institute (Calcutta)
in partial fulfilment of the requirements for the
award of the degree of

Doctor of Philosophy

1982

ACKNOWLEDGEMENT

While writing an acknowledgement to one's thesis one has roughly two choices: One may like to recall all the people who contributed to one's education in one way or another, or else one can cut it short and acknowledge only his thesis supervisor. This author admits his preference to the first choice. However to keep it within reasonable length I shall have to avoid mentioning individual names and acknowledge the people concerned in groups.

I express my deepest gratitude to my thesis supervisor Professor Amit Roy. Apart from training me in this subject, suggesting problems and giving excellent guidance, he also provided me with some amount of emotional shelter at my desperate moments. I think I have absolutely no hope of finding appropriate words to thank him.

I am also thankful to my other teachers and colleagues at TIFR who have directly helped me in my learning of mathematics and while doing this work. I thank all my friends at TIFR who have made my life smooth and enjoyable by keeping their warm company. I owe a vote of thanks to my other friends in Bombay who have provided me with their company whenever I needed a change.

I often think that I have been much influenced by my friends and acquaintances during various phases of my life. Their reactions while interacting with me and the casual answers they gave to my questions were revealing, exciting and subtly educational. At times it appears as if they all reacted in a planned and motivated way to push me to this "goal" of mine. I cannot explicitly thank all of them. Still, I would like to single out the contribution of those friends of mine with whom I have shared my life during my college days and that of my teachers and friends of my days at the Indian Statistical Institute, Calcutta.

At this moment I recall my late father with deep emotions. To acknowledge my mother's contributions, I really run out of words. How can I ever express my gratitude for the struggles and sacrifices she went through in order to give me a decent upbringing! Also, this acknowledgement would not be complete without mentioning my brothers' assistance and encouragements.

Finally, I thank Mrs. Lalitha Krishnan and Mr D.B. Sawant for their excellent typing.

CONTENTS

INTRODUCTION AND SUMMARY

Unless otherwise stated all rings are assumed to be commutative and Noetherian with finite Krull dimension and modules are finitely generated.

There are three related questions in commutative algebra. They are about the existence of unimodular elements in projective modules, cancellation of projective modules and minimal number of generators for finitely generated modules. In particular, determining the minimal number of generators of ideals is also of great interest in this area.

In this thesis we shall be concerned with the above questions in the case of Laurent polynomial rings.

The thesis consists of three Chapters. The main results are presented in Chapters II and III.

We shall brifely recall some classical results in the above mentioned area. Serre [Sr] proved that : *if* P *is a projective module over a ring* R *with* rank $P \geq \dim R+1$ *then* P *has a unimodular element*. Cancellation theorem of Bass [Ba-2] says that : *if* P *is a projective module over* R *with* rank $P \geq \dim R+1$ *then* P *has* cancellation property. Forster [F] and Swan [Sw] proved that : *for an* R-*module* M, *if* $n = \max\{\mu(p,M) + \dim(R/p) | p$ *is a prime ideal with* $M_p \neq 0\}$ *then* M *is generated by* n *elements*.

A unified treatment to all these problems was given by Eisenbud and Evans [EE-2]. They introduced the idea of *basic* elements for modules, extending the concept of unimodular elements for projective modules, and deduced all the results mentioned above.

It was also known that the above mentioned results are the best possible in the general situation. However one expected to improve these results for special kinds of rings. Eisenbud and Evans [EE-1] suggested the following three conjectures for polynomial rings:

EEC I. If M is a finitely generated $R[T]$- module such that $\mu(p, M) \geq \dim R + 1$ then M has a basic element.

EEC II. If P is a finitely generated projective module over $R[T]$ of rank $\geq \dim R + 1$ and if P' and Q are finitely generated projective modules such that $P \oplus Q \simeq P' \oplus Q$ then $P \simeq P'$.

EEC III. Let M be a finitely generated $R[T]$- module and let $e(M) = \max\{ \mu(p,M) + \dim(R[T]/p) \mid p \in \mathrm{Spec}(R[T]) \text{ with } \dim(R[T]/p) \leq \dim R\}$. Then M is generated by $e(M)$ elements.

EEC III was first proved by Sathaye [Sa] in the case when the base ring is an affine domain over an infinite field and then Mohan Kumar [MK-2] proved it completely. EEC I and II were proved by Plumstead [P] in his thesis. He also gave an alternative proof of EEC III.

In an obvious way one can make statements analogous to EEC I-III for Laurent polynomial rings.

In Chapter II of this thesis we shall discuss these Laurent polynomial anologous of EEC I-III. We shall see that all these analogues of EEC I-III for Laurent polynomial rings are settled affirmatively.

In Chapter III we shall consider a problem about minimal number of generators of ideals. There is a well known result ([MK-1], Lemma)that : for an ideal I of a ring R , $\mu(I/I^2) \leq \mu(I) \leq \mu(I/I^2) + 1$. It would be desirable to have $\mu(I) = \mu(I/I^2)$. But this is not true in general. However it is expected that this equality holds for a large class of rings and ideals.

Mohan Kumar [MK-2] proved that:if R is a polynomial ring over a field in several variables and $\mu(I/I^2) \geq \dim(R/I) + 2$ then I is generated by $\mu(I/I^2)$ elements.

Inspired by this result one expects that : when R is a polynomial ring or a Laurent polynomial ring in several variables over a commutative ring A and I is an ideal of R with height(I) > dim A and $\mu(I/I^2) \geq \dim(R/I) + 2$, then $\mu(I) = \mu(I/I^2)$. Our main result in Chapter III will establish this proposition when R is a Laurent polynomial ring with atleast one Laurent polynomial variable. It is interesting to note that our method does not work in the polynomial case (see Chapter III, Remark 2.6).

About the techniques used, one of our main tools is a technique of patching isomorphisms of modules introduced by Plumstead [P], developed from an idea of Quillen [Q].

Now we shall briefly mention the chapterwise organization of the thesis and the main results.

Chapter I. In this chapter we shall fix up some preliminaries. In § 1 we shall discuss the patching techniques of Plumstead. The rest of the preliminaries and notations are in § 2.

Chapter II. As we have mentioned, Chapter II consists of analogues of EEC I-III for Laurent polynomial rings.

Theorem 1.1 of this chapter is the $R[T,T^{-1}]$- analogue of EEC I. It states that : if M is a finitely generated $R[T,T^{-1}]$- module with $\mu(p,M) \geq \dim(R[T,T^{-1}]/p)$ for all minimal prime p of $R[T,T^{-1}]$ then M has a basic element. This theorem is an easy consequence of the validity of EEC I ([P], p.19, Theorem 2), unlike the analogues of EEC II and III.

The Laurent polynomial analogue of EEC II is established in Theorem 1.2 of this chapter. Namely the theorem is : if P, P' and Q are finitely generated projective $R[T,T^{-1}]$- modules with rank $P \geq \dim R + 1$, then $P \oplus Q \approx P' \oplus Q$ implies $P \approx P'$. The proof of this theorem is accomplished by actually proving a kind of cancellation theorem for torsion-free modules over polynomial rings, which we mention as proposition 1.5 in this chapter. Apart from generalizing Plumstead's cancellation theorem (i.e. validity of EEC II), this proposition seems to be of some further interest. In fact it was later used by Bhatwadekar and Roy ([BR], § 4, Theorem 4.3)to prove a cancellation theorem for projective modules over certain class of overrings of polynomial rings. After that Ravi Rao ([R], §4, Theorem 1.1(B)) has used a modified version of this proposition to prove a cancellation theorem for projective modules over all overrings of polynomial rings.

Theorem 2.1 of Chapter II will establish the Laurent polynomial analogue of EEC III. More precisely, the theorem says: if M is a finitely generated module over $R[T,T^{-1}]$, then M is generated by e(M) elements, where

$$e(M) = \max\{\mu(p,M) + \dim(R[T,T^{-1}]/p) \mid p \in \operatorname{Spec}(R[T,T^{-1}]) \text{ with } \dim(R[T,T^{-1}]/p) \leq \dim R\} .$$

Chapter III. The main theorem (Theorem 2.3) of this chapter states that : if $R = A[X_1,\ldots,X_n, T_1^{\pm 1},\ldots, T_r^{\pm 1}]$ $n \geq 0$ and $r \geq 1$ is a Laurent polynomial ring with atleast one Laurent polynomial variable and I is an ideal of R with height(I) > dim A and $\mu(I/I^2) \geq \dim(R/I) + 2$, then $\mu(I) = \mu(I/I^2)$.

For proving Theorem 2.3, one of our key results in this Chapter is Theorem 1.2. It says : if I is an ideal in R[T] with $\mu(I/I^2) \geq \dim(R[T]/I) + 2$ and if I contains a monic polynomial whose constant term is a unit, then $\mu(I) = \mu(I/I^2)$. We use Theorem 1.2 to deduce its Laurent polynomial analogue (Theorem 2.2), namely : if $R = A[T,T^{-1}]$ and I is an ideal of R which contains a monic polynomial in T and a monic polynomial in T^{-1} and if $\mu(I/I^2) \geq \dim(R/I) + 2$, then I is generated by $\mu(I/I^2)$ elements.

Theorem 2.3 follows immediately from this theorem and another result (Lemma 2.4) of this chapter. The results states that:

if $R = A[X_1,\ldots,X_n, T_1^{\pm 1},\ldots, T_r^{\pm 1}]$ *with* $n \geq 0$ *and* $r \geq 1$ *is a Laurent polynomial ring in several variables over a commutative ring* A *and* I *is an ideal of* R *with* $\text{height}(I) > \dim A$, *then* I *will contain* (*after a change of variables*) *a monic polynomial in* T_1 *and a monic polynomial in* T_1^{-1}

This lemma is a generalization of a lemma of Suslin ([Su-2], § 7, Lemma 7.1) dealing with the case $n = 0$.

CHAPTER I

Preliminaries

In this chapter we shall record some preliminaries and notations. In § 1 we shall recall some Lemmas of Plumstead [P] about patching of isomorphisms of modules. As we feel there is a flaw in Plumstead's proof of these lemmas concerning the use of his version of Quillen's Lemma, we shall also give the proofs of these lemmas along with an appropriate version (§ 1, Lemma 1.2) of Quillen's Lemma ([Q], Lemma 1).

Rest of the prerequisites and notations are in § 2.

§ 1. Plumstead's Patching Lemmas.

The main lemma (Lemma 1.5) in this section is a lemma of Plumstead ([P], page 11, Lemma 1) for patching isomorphisms of modules. We shall start with a relative version of Quillen's Lemma.

Lemma 1.1 (Quillen's Lemma). *Let* R *be a commutative ring and* A *be an algebra over* R *(which is not necessarily commutative). Let* s *be an element of* R *and* F *be an element of* $(1 + TA_s[T])^{\bullet}$*, the units of* $A_s[T]$ *which are congruent to* 1 *module* T*. Then there exists an integer* $k \geq 0$ *such that for any* r_1, r_2 *in* R *with* $r_1 - r_2$ *in* s^kR*, there exists* G *in* $(1 + TA[T])^{\bullet}$ *with* $G_s(T) = F(r_1T)F(r_2T)^{-1}$*. Further, if* $h : A \to B$ *is a ring homomorphism and that the image of* F *in* $B_s[T]$ *is one, then* G *may be taken with the property that the image of* G *in* $B[T]$ *is one.*

Proof. The absolute case is proved in [Q]. In the relative case we see that the following diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A_s & \longrightarrow & B_s \end{array}$$

of rings and homomorphisms is commutative. Given any element d in A_s with its image in B_s zero, we can choose a c in A with its image in B zero and $(c)_s = s^k d$ in A_s. In Quillen's proof ([Q], Lemma 1)if c_{ij}'s are chosen in A so that their images in B is zero, then the relative version of the lemma is established as in [Q].

The following special case of this Lemma will be used to prove the main lemma.

Lemma 1.2. Let R *be a noetherian commutative ring and* s *an element in* R. *Suppose* M *is a finitely generated module over* R *and* $M[T]$ *denotes* $M \otimes_R R[T]$. *Given any isomorphism* $\varphi : M_s[T] \xrightarrow{\sim} M_s[T]$ *of* $R_s[T]$*- modules with* $\varphi \equiv \mathrm{Id} \pmod{T}$, *there is an integer* $k \geq 0$ *such that for* r_1, r_2 *in* R *with* $r_1 - r_2$ *in* $s^k R$ *there is an isomorphism* $\theta : M[T] \xrightarrow{\sim} M[T]$ *of* $R[T]$*- modules such that* $\theta_s(T) = \varphi(r_1 T)\varphi(r_2 T)^{-1}$. *Further if* $h : R \to R'$ *is a homomorphism of commutative rings with* $\varphi \otimes R'_s = \mathrm{Id}$, *then* θ *may be taken such that* $\theta \otimes R' = \mathrm{Id}$.

Proof. Follows from Lemma 1.1 by taking $A = \text{End}_R(M)$ and $B = \text{End}_{R'}(M \otimes R')$.

Definition 1.3. Let R be a commutative ring and M , M' be modules over R. Suppose $f, g : M \tilde{\to} M'$ are isomorphisms of R-modules. We say f is isotopic to g if there is an isomorphism $\varphi : M[T] \tilde{\to} M'[T]$ over $R[T]$ with $\varphi(0) = f$ and $\varphi(1) = g$. If $h : R \to R'$ is a homomorphism of commutative rings then f is isotopic to g relative to h if there is an isotopy φ of f to g with $\varphi \otimes_{R[T]} R'[T] = f \otimes_R R'[T]$, a constant map.

We give an example of isotopic isomorphisms.

Example 1.4. Let $R = A[T]$ be a polynomial ring and let $M[T]$ be an R-module extended from an A-module M. If f is an R-automorphism of $M[T]$ with $f(0) = \text{Id}_M$, then f is isotopic to $\text{Id}_{M[T]}$. Moreover the isotopy can be chosen to be relative to the homomorphism $R \xrightarrow{T=0} A$. In fact the automorphism $\varphi(X) = f(XT)$ of the $R[X]$- module $M[T][X]$ provides the desired isotopy.

The proof of the main lemma in this section is just the reproduction of Plumstead's proof but for the use of the appropriate version (Lemma 1.2) of Quillen's lemma.

Lemma 1.5. ([P], page 11, Lemma 1). Let R be a commutative noetherian ring and let s_1, s_2 be elements in R with $Rs_1 + Rs_2 = R$. Let M, M' be finitely generated R-modules and let $\varphi_i : M_{s_i} \tilde{\to} M'_{s_i}$ be isomorphism for $i = 1,2$. If $(\alpha_1)_{s_2}$ and $(\alpha_2)_{s_1}$ are isotopic, then there exists an isomorphism $\theta : M \tilde{\to} M'$. Further, if $h : R \to R'$ is

<u>a homomorphism of commutative rings such that</u> $(\alpha_1)_{s_2}$ <u>and</u> $(\alpha_2)_{s_2}$ <u>are isotopic relative to</u> $h_{s_1 s_2} : R_{s_1 s_2} \to R'_{s_1 s_2}$ <u>then</u> θ <u>may be chosen with</u> $\alpha_{s_i} \otimes R' = \alpha_i \otimes R$ <u>for</u> $i = 1,2$.

<u>Proof</u>. Let $\pi = (\alpha_2)^{-1}_{s_1} \circ (\alpha_1)_{s_2}$. If $\pi = (\psi_2)_{s_1} \circ (\psi_1)_{s_2}$ where $\psi_i \in \mathrm{Aut}(M_{s_i})$, then by setting $\alpha_1' = \alpha_1 \circ \psi_1^{-1}$ and $\alpha_2' = \alpha_2 \circ \psi_2$ we have $(\alpha_1')_{s_2} = (\alpha_2')_{s_1}$. So there is an isomorphism $\theta : M \tilde{\to} M'$ with $\theta_{s_i} = \alpha_i'$.

Now since $(\alpha_2)_{s_1}$ is isotopic to $(\alpha_1)_{s_2}$ (<u>resp</u>. relative to $h_{s_1 s_2}$), Id is isotopic to π (<u>resp</u>. relative to $h_{s_1 s_2}$). Suppose φ is an isotopy with $\varphi(0) = \mathrm{Id}$ and $\varphi(1) = \pi$. We show that $\pi = \varphi(1)\,\varphi(a)^{-1}\,\varphi(a)$, where $\varphi(1)\,\varphi(a)^{-1}$ is in the image of $\mathrm{Aut}(M_{s_2})$ and $\varphi(a)$ is in the image of $\mathrm{Aut}(M_{s_1})$ for a properly chosen a in R. By applying Lemma 1.2 twice we see that there exists $k \geq 0$ such that $\varphi(T)\,\varphi(aT)^{-1}$ is in image $(\mathrm{Aut}(M_{s_2}[T]))$ if $1-a$ is in $s_1^k R$ and $\varphi(aT) = \varphi(aT)\,\varphi(oT)^{-1}$ is in image $(\mathrm{Aut}(M_{s_1}[T]))$ if a is in $s_2^k R$. As $Rs_1 + Rs_2 = R$ there is a in R such that a belongs to $s_2^k R$ and $1-a$ belongs to $s_1^k R$. So $\varphi(1)\,\varphi(a)^{-1}$ is in image $(\mathrm{Aut}(M_{s_2}))$ and $\varphi(a)$ in image $(\mathrm{Aut}(M_{s_1}))$ for some a in R.

In the relative case we have $\varphi \otimes R_{s_1 s_2}[T] = 1$. By applying relative version of Lemma 1.2 we may assume that the lifting ψ_2 and ψ_1 of $\varphi(T)\,\varphi(aT)^{-1}$ and $\varphi(aT)$ may be taken with $\psi_i \otimes R' = 1$. But now $\alpha_1' \otimes R' = (\alpha_1 \otimes R') \circ (\psi_1^{-1} \otimes R') = \alpha_1 \otimes R'$ and similarly $\alpha_2' \otimes R' = \alpha_2 \otimes R'$. Thus the isomorphism θ will satisfy

$\theta_{s_i} \otimes R' = \alpha_i \otimes R'$ for $i = 1,2$.

Remark 1.6. In Plumstead's proof of Lemma 1.5, we have $\varphi : M_{s_1s_2}[T] \tilde{\to} M_{s_1s_2}[T]$ an isotopy of $\varphi(0) = \mathrm{Id}$ to $\varphi(1) = \pi$ with $\varphi \otimes R'[T] = \mathrm{Id}$. To apply Quillen's lemma we consider φ as an element of $\mathrm{End}_{R_{s_1s_2}}(M_{s_1s_2})[T]$ and since $\varphi \otimes R'[T] = \mathrm{Id}$, the image of φ in $\mathrm{End}_{R_{s_1s_2}}(M_{s_1s_2} \otimes R'_{s_1s_2})[T]$ is one. As for a ring homomorphism $R \to R'$ one does not necessarily have $\mathrm{End}_{R_{s_1s_2}}(M_{s_1s_2}) \otimes R'_{s_1s_2} = \mathrm{End}_{R'_{s_1s_2}}(M_{s_1s_2} \otimes R'_{s_1s_2})$, Plumstead's version of Quillen's lemma ([P], p.10) is not applicable.

We shall conclude this section by quoting the following lemma of Plumstead which is a consequence of Lemma 1.5 and Example 1.4.

Lemma 1.7. ([P], p.12, Lemma 2). *Let* A *be a commutative noetherian ring and* $R = A[T]$ *be a polynomial ring over* A. *Let* M, M' *be finitely generated* R-*modules*. *Let* s_1, s_2 *be elements of* A *with* $As_1 + As_2 = A$. *Assume* $\alpha_i : M_{s_i} \tilde{\to} M'_{s_i}$ *are isomorphisms for* $i = 1,2$ *satisfying*

1. $(\alpha_2)_{s_1}^{-1} \circ (\alpha_1)_{s_2} \equiv 1 \mod T$
2. $M_{s_1s_2}$ *is extended from* $A_{s_1s_2}$.

Then there exists an isomorphism $\theta : M \tilde{\to} M'$ *such that* $\theta_{s_i} \equiv \alpha_i \mod T$ *for* $i = 1,2$.

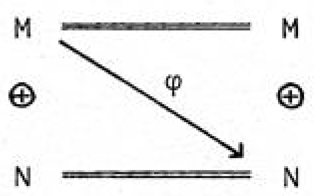
§ 2. Some more Preliminaries and Notations

Before we go into any further preliminaries we shall fix up some notations.

Throughout this section and in the subsequent Chapters A and R will always denote a commutative noetherian ring with finite Krull dimension. $A[T]$ and $A[T,T^{-1}]$ respectively denote the polynomial ring and the Laurent polynomial ring over A with T as the variable. By dim A we shall mean the Krull dimension of A.

Unless otherwise specified all our modules are finitely generated. If M is a module over R, then $\mu(M)$ or $\mu_R(M)$ will denote the minimal number of generators of M as an R-module. If p is a prime ideal of R, then $\mu_{R_p}(M_p)$ will be denoted by $\mu(p, M)$.

For R-modules M and N and R-linear map $\varphi : M \to N$, we can define an automorphism of $M \oplus N$ by setting $(m,n) \to (m, n+\varphi(m))$ for m in M and n in N. We shall describe this automorphism diagramatically as



We also recall that for an element m in an R-module M, $O(m,M)$ denotes the ideal

$$\{\ \varphi(m)\ |\varphi : M \to R \ \text{ an } \ R\text{-linear map}\ \}.$$

Sometimes we write $O(m)$ for $O(m,M)$.

We shall recall some definitions.

Definition 2.1. An element m in an R-module M is said to be unimodular if $O(m,M) = R$.

Definition 2.2. An element m in an R-module M is said to be basic at a prime ideal p of R if m does not belong to pM_p . And m is said to be basic if it is basic at all primes of R.

We remark that a unimodular element is basic and in the case of finitely generated projective modules an element is unimodular if and only if it is basic.

The set of all non-negative integers will be denoted by $\mathbb{N}$.

Definition 2.3. If P is a subset of Spec(R) and $d : P \to \mathbb{N}$ is a function, then for p , q in P define $p \ll q$ if $p \subset q$ and $d(p) > d(q)$. This defines a partial order on P . Such a function is called a generalized dimension function if for any ideal I of R, $V(I) \cap P$ has only finitely many minimal elements with respect to the ordering $\ll$.

Before we give any example of generalized dimension function we shall quote two theorems. The first one is an improved version of Eisenbud-Evans theorem ([EE-2], §3, Theorem A(b)) on the existence of basic elements of modules and the other theorem is an improvement on Eisenbud-Evans theorem ([EE-2], §7, Theorem B) on number of generators of modules using the notion of generalized dimension functions.

Theorem 2.4 ([P], page 6, EISENBUD-EVANS THEOREM). Suppose M is a finitely generated R-module. Let P be a subset of $\mathrm{Spec}(R)$ and $d : P \to \mathbb{N}$ be a generalized dimension function. Assume $\mu(p, M) \geq 1 + d(p)$ for all p in P. Let (r,m) be an element of $R \oplus M$, basic at all primes p of P. Then there is an element m' of M such that $m + rm'$ is basic at all primes p of P.

Theorem 2.5 ([P], Section I, Theorem 0). Let R be a commutative ring, M a finitely generated R-module, P a set of primes of R and $d : P \to \mathbb{N}$ a generalized dimension function. Let $x_1, x_2, \ldots, x_n$ be elements of M and N a submodule of M such that $(Rx_1 + Rx_2 + \cdots + Rx_n + N)_p = M_p$ for each p in P. Assume $n \geq \mathrm{Sup}\{\mu(p, M) + d(p) | p$ in $P\}$. Then there exist elements $y_i = x_i + m_i$ for $i = 1,2,\ldots,n$ where m_i belongs to N such that $M' = Ry_1 + Ry_2 + \cdots + Ry_n$ is fully basic in M at all p in P i.e. $M_p = M'_p$ for all p in P.

Theorem 2.4 and Theorem 2.5 can be used very efficiently to find generators of modules on prime sets where we can define some good generalized dimension function.

Now we give some examples of generalized dimension functions.

Example 2.6 ([P], Section I, Example 1). Let $P \subseteq \mathrm{Spec}(R)$ be a set of primes such that for all ideal I of R, $V(I) \cap P$ has only finitely many minimal elements with respect to inclusion. For a prime ideal p of P . define $d(p) = \mathrm{Sup}\{n | p = p_0 \subset p_1 \subset p_2 \cdots \subset p_n$ a chain with $p_0, \ldots, p_n$ in $P\}$. Then d is a generalized dimension function on P . We call it the <u>standard dimension function with respect to</u> P .

<u>Example 2.7</u>. Let $P_1, \ldots, P_r$ be subsets of Spec(R) and $d_i : P_i \to \mathbb{N}$ generalized dimension function on P_i for i = 1 to r. Let $P = P_1 \cup P_2 \cup \cdots \cup P_r$. For a prime p of P , define $d(p) = \mathrm{Sup}\{d_i(p) | p \text{ in } P_i , i = 1, \ldots, r\}$. Then d is a generalized dimension function on P .

<u>Example 2.8.</u> Let R be a noetherian ring and let there be an element s in the radical of R, such that dim(R/sR) < dim R . Suppose $D(T) = \{p \in \mathrm{Spec}(R[T]) | T \notin p\}$. Then there is a generalized dimension function d on D(T) such that for all p in D(T), $d(p) \leq \dim(R[T, T^{-1}]/p_T)$ and $d(p) \leq \dim R$.

<u>Proof</u>. It is enough to find a generalized dimension function d on $\mathrm{Spec}(R[T,T^{-1}])$ such that for all prime p , $d(p) \leq \dim R$ and $d(p) \leq \dim(R[T,T^{-1}]/p)$. Let P_1 be the set of all prime ideals of $R[T,T^{-1}]$ which contains s and $P_2 = \{p \in \mathrm{Spec}(R[T,T^{-1}]) | ht(p) \leq \dim R\}$. Suppose d_1 and d_2 are the standard generalized dimension functions on P_1 and P_2 respectively. Since $\mathrm{Spec}(R[T,T^{-1}]) = P_1 \cup P_2$, if d is the generalized dimension function defined by d_1 and d_2 as in Example 2.7, then d is a generalized dimension function of desired type.

<u>Example 2.9</u>. ([P], Section 1, Example 4). If R is a commutative noetherian ring and s in radical (R) with dim(R/sR) < dim R , then there is a generalized dimension function $d : \mathrm{Spec}(R[T]) \to \mathbb{N}$ such that $d(p) \leq \dim R$ for all p in Spec(R[T]).

We shall conclude this section with a special case of Theorem 2.4.

Theorem 2.10. Let P be a finitely generated R-module of constant rank t. If (r,p) is unimodular in $R \oplus P$, then there is an element q in P such that height $(O(p + rq) \geq t$.

Proof. Use Theorem 2.4 with the set of prime ideals of R of height $< t$ as $\mathcal{P}$, equipped with the standard generalized dimension function (Example 2.6).

CHAPTER II

Analogues of Eisenbud-Evans Conjectures for Laurent Polynomial rings

The aim of this chapter is to prove three theroems about modules over Laurent polynomial rings, namely Theorem 1.1, Theorem 1.2 and Theorem 2.1. Theorem 1.1. is about the existence of basic elements in finitely generated modules over Laurent polynomial rings. Theorem 1.2 is a cancellation theorem for finitely generated projective modules over Laurent polynomial rings. And Theorem 2.1 gives an estimate for number of generators of modules over Laurent polynomial rings. These are the Laurent polynomial analogues of EEC I-III mentioned in the Introduction and Summary.

§1. Basic Elements and Cancellation over Laurent Polynomial Rings.

We begin with the $R[T,T^{-1}]$- analogue of EEC-I.

Theorem 1.1. Let M be a finitely generated $R[T,T^{-1}]$- module with $\mu(p, M) \geq \dim(R[T,T^{-1}]/p)$ for each minimal prime p of $R[T,T^{-1}]$. Then there is an element m of M, which is basic at all primes.

Proof. Let N be a finitely generated $R[T]$- submodule of M such that $N_T = M$. If p is a minimal prime of $R[T]$, then p_T is a minimal prime of $R[T,T^{-1}]$. Further $\dim(R[T,T^{-1}]/p_T) = \dim(R[T]/p)$ and $N = M_{p_T}$. So we have $\mu(p, N) \geq \dim(R[T]/p)$ for each minimal prime p of $R[T]$. So there is an element m in N which is basic in N at all primes of $R[T]$([P], Section III, Theorem 2). It follows that m is basic in M at all primes of $R[T,T^{-1}]$.

The next theorem is the $R[T,T^{-1}]$- analogue of EEC-II.

Theorem 1.2. Let P, P' and Q be finitely generated projective $R[T,T^{-1}]$- modules with rank $P \geq d+1$, where d is the dimension of R. If $P \oplus Q \approx P' \oplus Q$, then $P \approx P'$.

Before going into the proof of Theorem 1.2 we prove a lemma.

Lemma 1.3. If I is an ideal in $R[T,T^{-1}]$ of height dim $R+1$, then I contains an element $g(T)$ such that $g(T) = 1 + Tg'(T)$ for some $g'(T)$ in $R[T]$.

Proof. We may assume I is a root ideal. Suppose $I = p_1 \cap p_2 \cap \cdots p_r$, where p_i's are prime ideals of height dim $R + 1$ in $R[T,T^{-1}]$ for $i = 1$ to r. If $p_i' = p_i \cap R[T^{-1}]$, then height (p_i') = dim $R+1$ as an ideal of $R[T^{-1}]$. So p_i' contains a monic polynomial in T^{-1} for $i = 1$ to r ([Ba], §4, Lemma 3). Hence $I \cap R[T^{-1}] = p_1' \cap \cdots \cap p_r'$ contains a monic polynomial in T^{-1}. Hence by multiplying it by a suitable power of T we get, I contains an element of the desired type.

Proof of Theorem 1.2. By an obvious argument we may assume $Q = R[T,T^{-1}]$. We can also assume that R is reduced.

Let $\varphi : R[T,T^{-1}] \oplus P' \approx R[T,T^{-1}] \oplus P$ be an isomorphism and let $\varphi(1,0) = (a(T),p)$. Without loss of generality we may assume $a(T)$ belongs to $TR[T]$. By Theorem 2.10 of Chapter I there is a q in P such that if $p' = p + a(T)q$, then height$(O(p')) \geq d+1$. Again, by Lemma 1.3, there exists $g'(T)$ in $R[T]$ such that $1 + Tg'(T)$ belongs to $O(p')$. Let $\beta_1 : P \to R[T,T^{-1}]$ be an $R[T,T^{-1}]$- linear map such that $\beta_1(p') = 1 + Tg'(T)$ and let $\hat{q}$ denote the map $R[T,T^{-1}] \to P$ such that $\hat{q}(1) = q$.

Let ψ be the composite isomorphism given by the following diagram.

$$\begin{array}{ccccccc}
R[T,T^{-1}] & & R[T,T^{-1}] & = & R[T,T^{-1}] & = & R[T,T^{-1}] \\
\oplus & \xrightarrow{\varphi} & \oplus & \searrow^{\hat{q}} & \oplus & \nearrow^{\beta_1} & \oplus \\
P' & & P & = & P & = & P
\end{array}$$

$$(1,\ 0) \longrightarrow (a(T),p) \longrightarrow (a(T),p') \longrightarrow (1+Tg'(T)+a(T),p')$$

Then $\psi(1,0) = (1 + Tg'(T)+ a(T),\ p')$. Replacing p' by p and $1 + Tg'(T) + a(T)$ by $a(T)$, we have an isomorphism $\psi : R[T,\ T^{-1}] \oplus P' \to R[T,\ T^{-1}] \oplus P$ such that $\psi(1,0) = (a(T),p)$ and $a(T) = 1 + Tb(T)$ for some polynomial $b(T)$. Again since $1 + Tg'(T)$ belongs to $O(p)$, $g(T) = T + Tg'(T)$ belongs to $O(p)$. Suppose $\beta_2 : P \to R[T,T^{-1}]$ be an $R[T,T^{-1}]$- linear map with $\beta_2(p) = g(T)$. Let $p_1 = T^{-1}p$. Chooose $p_2, p_3,\dots, p_r$ in P such that $p_1, p_2,\dots p_r$ form a system of generators of P and $\beta_2(p_i)$ belongs to $R[T]$for $i = 1$ to r.

Let $M = \sum_{i=1}^{r} R[T]p_i$ and let $\beta : M \to R[T]$ be the restriction of β_2 to M. Then $g(T) = \beta(p) \in O(p,M)$. So we have

(1) M is a finitely generated $R[T]$- submodule of P and $M_T = P$.

(2) $g(T) = T + T^2g'(T) \in O(p,M)$

(3) $p \in TM$ and $a(T) = 1 + Tb(T)$ for some $b(T)$ in $R[T]$.

(4) $\mu(\mathfrak{p},M) \geq d+1$ for all $\mathfrak{p}$ in $\mathrm{Spec}(R[T])$.

(5) $(a(T),p)$ is unimodular in $R[T] \oplus M$ and hence basic at all primes of $R[T]$.

(1), (2), (3) follow by choice. If p belongs to $\mathrm{Spec}(R[T])$ and p' is a minimal prime contained in p , then p'_T is a minimal prime in $R[T,T^{-1}]$. And $M_{p'} = P_{p'_T}$. So $\mu(p, M) \geq \mu(p',M) \geq \mu(p'_T , P) \geq d+1$. Hence (4) holds.

For (5) first note that $a(T) \in O((a(T),p), R[T] \oplus M)$. Now $(a(T),p)$ is unimodular in $R[T,T^{-1}] \oplus P$ and hence $O((a(T),p), R[T,T^{-1}] \oplus P) = R[T,T^{-1}]$. So it follows T^n belongs to $O((a(T),p), R[T] \oplus M)$ for some $n \geq 1$. Hence $R[T] = R[T]\, a(T) + R[T]T^n \subseteq O((a(T),p), R[T] \oplus M)$. So we see that $(a(T),p)$ is unimodular in $R[T] \oplus M$.

Write N for $\dfrac{R[T] \oplus M}{(a(T),p)R[T]}$. Then the sequence

$$0 \longrightarrow R[T] \xrightarrow{(a(T),p)} R[T] \oplus M \longrightarrow N \longrightarrow 0$$

is split exact. So we have

$$N_T \approx \frac{R[T,T^{-1}] \oplus P}{(a(T),p)R[T,T^{-1}]} \approx P' .$$

Since $M_T \approx P$, it is enough to prove $N \approx M$.

As N is a direct summand of $R[T] \oplus M$, it is torsion-free (i.e. non-zero-divisors of $R[T]$ act as non-zero-divisors of N).

Suppose $h : M \to N$ is the natural map. Let barring denote "modulo T" . Then $\bar{h} : \bar{M} \to \bar{N} = \dfrac{R \oplus \bar{M}}{(1,0)R}$ is the natural isomorphism. We shall prove that there is an isomorphism $\theta : M \tilde{\to} N$ such that $\bar{\theta} = \bar{h}$.

Let S be the set of all non-zero-divisors of R. Then $R_S[T] \approx (k_1 \times k_2 \times \cdots \times k_r)[T]$, where $k_1, k_2, \ldots, k_r$ are fields.

Since M_S and N_S are torsion-free over $R_S[T]$, they are extended from R_S. So there is s_1 in S such that M_{s_1} and N_{s_1} are extended from R_{s_1}. It follows that there is an isomorphism $\alpha_1 : N_{s_1} \xrightarrow{\sim} M_{s_1}$ such that $\overline{\alpha}_1 = \overline{h}_{s_1}^{-1}$.

Let $S' = 1 + s_1R$. Then s_1 belongs to $\mathrm{rad}(R_{S'})$ and $\dim(R_{S'}/s_1R_{S'}) < \dim(R_{S'})$. By example 2.9 of Chapter I, there is a generalized dimension function $\delta : \mathrm{Spec}\,(R_{S'}[T]) \longrightarrow \mathbb{N}$ such that $\delta(p) \leq \dim(R_{S'}) \leq d$ for all p in $\mathrm{Spec}\,(R_{S'}[T])$. If $D(T)$ denotes the set of all p in $\mathrm{Spec}\,(R_{S'}[T])$ which does not contain T and δ_1 the restriction of δ to $D(T)$, then δ_1 is a generalized dimension function on $D(T)$. Since $(a(T),p)$ is unimodular and hence basic in $R_{S'}[T] \oplus M_{S'}$ at all p in $D(T)$, we have $(T^2a(T),p)$ is basic at all p in $D(T)$. Further, $\mu(p,M_{S'}) \geq \delta_1(p) + 1$ for all p in $D(T)$. By an application of Eisenbud-Evans Theorem (Chapter I, Theorem 2.4), there is q in $M_{S'}$ such that $p' = p + T^2a(T)q$ is basic in $M_{S'}$ at all primes p in $D(T)$. As a consequence p' is unimodular in $P_{S'} = (M_{S'})_T$. Therefore $T^n \in O(p',M_{S'})$ for some $n \geq 1$. Moreover, by (2), $g(T) = T + T^2g'(T) \in O(p,M_{S'})$. Since $p' = p + T^2a(T)q$, it follows that $T + T^2g''(T) \in O(p',M_{S'})$ for some polynomial $g''(T)$, which shows that $T \in O(p',M_{S'})$ and hence $1 - a(T) \in O(p',M_{S'})$.

If $\lambda : M_{S'} \longrightarrow R_{S'}[T]$ is a homomorphism such that $\lambda(p') = 1 - a(T)$, let $H : R_{S'}[T] \oplus M_{S'} \xrightarrow{\sim} R_{S'}[T] \oplus M_{S'}$, be the composite isomorphism given by the following diagram

$$\begin{array}{ccccccc}
R_{S'}[T] & = & R_{S'}[T] & = & R_{S'}[T] & = & R_{S'}[T] \\
\oplus & \searrow^{T^2\hat{q}} & \oplus & \nearrow^{\lambda} & \oplus & \searrow^{-\hat{p}'} & \oplus \\
M_{S'} & = & M_{S'} & = & M_{S'} & = & M_{S'}
\end{array}$$

$$(a(T),p) \longrightarrow (a(T),p') \longrightarrow (1,p') \longrightarrow (1,0)$$

Then $H((a(T),p)) = (1,0)$ and since $p' \in TM_{S'}$, $\overline{H}$ is given by the diagram

$$\begin{array}{ccc}
R_{S'} & = & R_{S'} \\
\oplus & \nearrow^{\overline{\lambda}} & \oplus \\
\overline{M}_{S'} & = & \overline{M}_{S'}
\end{array}$$

Define $H_1 : N_{S'} \xrightarrow{\sim} M_{S'}$ to be the isomorphism which makes the following diagram commutative

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R_{S'}[T] & \xrightarrow{(a(T),p)} & (R[T] \oplus M)_{S'} & \longrightarrow & N_{S'} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle H_1} & & \\
0 & \longrightarrow & R_{S'}[T] & \xrightarrow{(1,0)} & (R[T] \oplus M)_{S'} & \longrightarrow & M_{S'} & \longrightarrow & 0
\end{array}$$

In this diagram the rows are split exact. If we reduce modulo T we get the following commutative diagram with split exact rows,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R_{S'} & \xrightarrow{(1,0)} & (R \oplus \overline{M})_{S'} & \longrightarrow & \overline{N}_{S'} & \longrightarrow & 0 \\
 & & \Big\downarrow \mathrm{Id} & & \Big\| \; \overline{\lambda} \swarrow \; \Big\| & & \Big\downarrow \overline{H}_1 & & \\
0 & \longrightarrow & R_{S'} & \xrightarrow{(1,0)} & (R \oplus \overline{M})_{S'} & \longrightarrow & \overline{M}_{S'} & \longrightarrow & 0
\end{array}$$

If we identify $\overline{N}_{S'}$ and $\overline{M}_{S'}$ by $\overline{h}_{S'} : \overline{M}_{S'} \longrightarrow \overline{N}_{S'}$, we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R_{S'} & \xrightarrow{(1,0)} & (R \oplus \overline{M})_{S'} & \longrightarrow & \overline{M}_{S'} & \longrightarrow & 0 \\
 & & \Big\downarrow \mathrm{Id} & & \Big\| \; \overline{\lambda} \swarrow \; \Big\| & & \Big\downarrow \overline{H}_1 \circ \overline{h}_{S'} & & \\
0 & \longrightarrow & R_{S'} & \xrightarrow{(1,0)} & (R \oplus \overline{M})_{S'} & \longrightarrow & \overline{M}_{S'} & \longrightarrow & 0
\end{array}$$

Here both the rows consist of natural maps. So it follows that $\overline{H}_1 \circ \overline{h}_{S'} = \mathrm{Id}$. We can find s_2 in S' and an isomorphism $\alpha_2 : N_{s_2} \xrightarrow{\sim} M_{s_2}$ such that $\overline{\alpha}_2 \circ \overline{h}_{s_2} = \mathrm{Id}$.

Since $Rs_1 + Rs_2 = R$, M_{s_1} is extended, $\overline{\alpha}_1 = \overline{h}_{s_1}^{-1}$ and $\overline{\alpha}_2 = \overline{h}_{s_2}^{-1}$, we can apply Lemma 1.7 of Chapter I to the isomorphisms $\alpha_i : N_{s_i} \xrightarrow{\sim} M_{s_i}$ $i = 1,2$ to conclude that N and M are isomorphic. In fact there is an isomorphism $\Theta : M \xrightarrow{\sim} N$ such that $\overline{\Theta} = \overline{h}$ as claimed. So the proof is complete.

<u>Corollary 1.4</u>. <u>Let</u> P, P' <u>and</u> Q <u>be finitely generated projective</u> $R[T,T^{-1}]$<u>-modules with</u> $\mu(p,P) \geq \dim(R[T,T^{-1}]/p)$ <u>for all minimal primes</u> p <u>of</u> $R[T,T^{-1}]$. <u>If</u> $P \oplus Q \approx P' \oplus Q$ <u>then</u> $P \approx P'$.

In the proof of Theorem 1.2 we have seen that following kind of cancellation holds for finitely generated torsion-free modules over polynomial rings :

Proposition 1.5. Let M be a finitely generated torsion-free module over $R[T]$. Suppose M_T is a projective module of rank greater than or equal to $\dim(R[T])$ and $(a(T),p)$ is a unimodular element in $R[T] \oplus M$ with

(1) $p \in TM$ and $a(T) = 1 + Tb(T)$ for some $b(T)$ in $R[T]$.

(2) $O(p,M)$ contains an element $T + T^2 g(T)$ for some $g(T)$ in $R[T]$.

Then there is an isomorphism $\Theta : M \xrightarrow{\sim} \frac{R[T] \oplus M}{(a(T),p)R[T]}$ such that $\overline{\Theta}$ is the natural map $\overline{M} \xrightarrow{\sim} \frac{R \oplus \overline{M}}{(1,0)R}$.

§ 2. Number of Generators for Modules over Laurent Polynomial Rings.

In this section we shall prove the $R[T,T^{-1}]$-analogue (Theorem 2.1) of EEC-III.

Recall that given an R-module M, one defines

$$e(M) = \text{Max}\{\mu(p,M) + \dim(R/p) \mid p \in \text{Spec}(R) \text{ with } \dim(R/p) < \dim R\}$$

$$e'(M) = \text{Max}\{\mu(p,M) + \dim(R/p) \mid p \in \text{Spec}(R) \text{ and } p \text{ is not minimal}\}$$

Theorem 2.1. Suppose M is a finitely generated module over $R[T,T^{-1}]$, the Laurent polynomial ring over R. Then M is generated by $e(M)$ elements.

In view of the following theorem it is enough to prove Theorem 2.1 for certain ideals.

Theorem 2.2. Validity of Theorem 2.1 for all ideals in reduced **Laurent** polynomial rings, not contained in any minimal prime ideal, implies the validity of Theorem 2.1 for any module over a Laurent polynomial ring.

Theorem 2.3. Let $B = R[T,T^{-1}]$ where R is a reduced noetherian commutative ring of finite Krull dimension. Suppose I is an ideal in B, which is not contained in any minimal prime ideal of B. Then I is generated by $e'(I)$ elements.

As mentioned above Theorem 2.1 follows from Theorem 2.2 and 2.3. Theorem 2.2 has an analogous theorem in the polynomial case, which was first proved by Sathaye ([Sa], Theorem 1) in the case of domain and then Mohan Kumar ([MK-2], §3, Theorem 2) proved the theorem in the general case. Our proof of Theorem 2.2 is word for word same as Mohan Kumar's proof. In view of that we shall omit the proof. So we only prove Theorem 2.3.

The following lemma will be used in the proof of Theorem 2.3. This lemma is a very special case of ([MK-2], §3, Corollary 3) .

Lemma 2.4. Let R be a noetherian ring and I an ideal of R with $\mu(I/I^2) = n$. Then we can choose $a_1, a_2, \ldots, a_n$ in I such that

i) $a_1, a_2, \ldots, a_n$ generate I modulo I^2.

ii) For any prime ideal p containing $\sum_{i=1}^{n} a_i R$ and not containing I, height$(p) \geq n$.

<u>Proof of Theorem 2.3.</u> First we do some reduction of the problem. Since I is not contained in any minimal prime of $B = R[T,T^{-1}]$, $(I/I^2)_p = 0$ for all minimal prime p of B. Hence $\mu(p, I/I^2) + \dim(B/p) \leq \mu(p,I) + \dim(B/p) \leq e'(I)$, whenever $(I/I^2)_p \neq 0$. By ([MK-2], § 1, Corollary 2), we have $\mu(I/I^2) \leq e'(I)$. If $\mu(I/I^2) < e'(I)$, then since $\mu(I) \leq \mu(I/I^2) + 1$ ([MK-1], Lemma) we have $\mu(I) \leq e'(I)$. So, we assume $\mu(I/I^2) = e'(I)$. Again since $I_p \neq 0$ for all p in Spec(B), we have $e'(I) \geq \dim(B) = n$ (say). If $e'(I) > n$, then by Lemma 2.4, there are $a_1, a_2, \ldots, a_{e'(I)}$ in I such that $a_1, a_2, \ldots, a_{e'(I)}$ generate I modulo I^2 and any prime p containing $\sum_{i=1}^{e'(I)} a_iB$ also contains I. As $a_1, a_2, \ldots, a_{e'(I)}$ generate I modulo I^2, $I_p = (\sum_{i=1}^{e'(I)} a_iB)_p$ whenever $\sum_{i=1}^{e'(I)} a_iB$ is contained in p and hence for all p in Spec(B). So $a_1, \ldots, a_{e'(I)}$ generate I. Hence we have $\mu(I) \leq e'(I)$.

So in the rest of the proof we shall assume $\mu(I/I^2) = e'(I) = \dim(B) = n$. We write $J = R[T] \cap I$ and denote by $J(0)$, the ideal $\{f(0) \mid f \in J\}$ of R. If p is a minimal prime of R, since J is not contained in any minimal prime, there is an element f in J not belonging to pB. Let $f = a_0 + a_1T + \cdots a_r T^r$ where $a_0, a_1, \ldots, a_{i-1}$ belong to p and a_i does not belong to p for some $i \leq r$. If a is an element in R, not in p and belonging to all other minimal primes of R, then $af = aa_i T^i + aa_{i+1} T^{i+1} + \cdots + aa_r T^r$ belongs to J and since $J = I \cap R[T]$, $g = aa_i + aa_{i+1}T + \cdots + aa_r T^{r-i}$

is in J. Hence aa_i belongs to J(0), which shows that J(0) is not contained in any minimal prime, i.e. there is a non-zero-divisor s_o belonging to J(0).

Now let S denote the set of all non-zero-divisors of R. We have s_o belonging to $J(0) \cap S$. Since $R_S[T]$ is a product of principal ideal domains and J_S is an ideal not contained in any minimal prime, it is a free module of rank one over $R_S[T]$. Hence there is an element s_1 in S such that J_{s_1} is a free module of rank one over $R_{s_1}[T]$. If necessary, by multiplying s_1 by s_o, we may assume s_1 belongs to $J(0) \cap S$. Taking $L = R_{s_1}[T]^{n-1}$, we have $x_1, x_2, \ldots, x_n$ in J such that $0 \to L \to R_{s_1}[T]^n \xrightarrow{(x_i)} J_{s_1} \to 0$ is an exact sequence, where (x_i) denotes the obvious map (for instance take x_1 to be a free generator of J_{s_1} and $x_2 = \cdots = x_n = 0$).

Write $S' = 1 + s_1 R$ and let $P = D(T) \cup D(s_1)$ where $D(T) = \{p \in \mathrm{Spec}(R_{S'}[T]) \mid T \notin p\}$ and $D(s_1) = \{p \in \mathrm{Spec}(R_{S'}[T]) \mid s_1 \notin p\}$. Note that $(\sum_{i=1}^{n} R_{S'}[T]\, x_i + T\, J_{S'})_p = (J_{S'})_p$ for all p in P. We shall construct a generalized dimension function $d : P \to \mathbb{N}$ such that $\mu(p, J_{S'}) + d(p) \leq n$ for all p in P. Since s_1 belongs to $\mathrm{rad}(R_{S'})$ and $\dim(R_{S'}/s_1 R_{S'}) < \dim(R_{S'})$, by example 2.8 in Chapter I, there is a generalized dimension function $d_1 : D(T) \to \mathbb{N}$ such that $d_1(p) \leq \dim(R_{S'})$ and $d_1(p) \leq \dim(R_{S'}[T, T^{-1}]/p_T)$ for all p of D(T). Define $d_2 : D(s_1) \to \mathbb{N}$ such that $d_2(p) = \dim(R_{S'}[t]/p)_{s_1})$ for all p in $D(s_1)$. By example 2.7 in Chapter I $d : P \to \mathbb{N}$

where $d(p) = \max \{d_i(p) | p \in P, i=1,2\}$ defines a generalized dimension function on P. We claim that d has the desired property. To see this let us assume $\mu(p, J_{S'}) + d(p) > n$ for some p in P. Since $d(p) \leq \dim(R_{S'}) < n$ we have $J_{S'} \subseteq p$. As $J_{S'}$ is not contained in any minimal prime ideal, p is not minimal. If $T \in p$, then as $s_1 \in J(0)$ we have $s_1 \in p$ which is impossible. So $T \notin p$, now if $s_1 \in p$ then $\mu(p,J_{S'}) + d(p) = \mu(p_T, I_{S'}) + d_1(p) \leq \mu(p_T, I_{S'}) + \dim(R_{S'}[T,T^{-1}]/p_T) \leq n$.
Therefore we have $s_1 \notin p$ and $T \notin p$. Again as above $d(p) \leq \dim(R_{S'}[T,T^{-1}]/p_T)$ is not possible, so we have $d(p) > \dim(R_{S'}[T,T^{-1}]/p_T) \geq d_1(p)$. Hence $d(p) = d_2(p) = r$(say). As $d_2(p) = \dim((R_{S'}[T]/p)_{s_1})$, we have a chain $p \subset p_1 \subset \cdots \subset p_r$ in $\mathrm{Spec}(R_{S'}[T])$, with $s_1 \notin p_r$. But $\dim(R_{S'}[T,T^{-1}]/p_T) < r$. So $T \in p_r$, since $J_T \subset p \subset p_r$, we have $s_1 \in p_r$, which is again a contradiction. So we have seen that for p in P, $\mu(p, J_{S'}) + d(p) > n$ is not possible.

Therefore, we have a generalized dimension function d on P such that for all p in P, $\mu(p, J_{S'}) + d(p) \leq n$. As $(\sum_{i=1}^{n} R_{S'}[T]x_i + [T]J_{S'})_p = (J_{S'})_p$ for all p in P, we can apply Theorem 2.5 of Chapter I to get $y_1,\ldots,y_n$ in $J_{S'}$, such that $y_i = x_i + T x_i'$ for some x_i' in $J_{S'}$ and $(\sum_{i=1}^{n} R_{S'}[T]y_i)_p = (J_{S'})_p$ for all p in P.

Let $L'' = \ker(R_{S'}[T]^n \xrightarrow{(y_i)} J_{S'})$. Then the sequence $0 \to L'' \to R_{S'}[T]^n \xrightarrow{(y_i)} J_{S'} \to 0$ is exact at all p in P.

Hence the sequences

$$0 \to L''_T \to R_{S'}[T, T^{-1}]^n \xrightarrow{(y_i)} (J_{S'})_T \to 0$$

and

$$0 \to L''_{s_1} \to R_{S'}[T]^n \xrightarrow{(y_i)} (J_{S'})_{s_1} \to 0$$

are exact. Since $(J_{S'})_{s_1}$ is free of rank one, L''_{s_1} is projective and $\text{rank}(L''_{s_1}) = n-1 \geq \dim(R_{S's_1}[T])$. Hence L''_{s_1} is cancellative ([P], Section III, Theorem 1). Hence as L''_{s_1} is stably free, it is free of rank n-1. We can choose $s_2 \in S'$ such that if $L' = \ker(R_{s_2}[T]^n \xrightarrow{(y_i)} J_{s_2})$, then the sequences

$$0 \to L'_{s_1} \to R_{s_1 s_2}[T]^n \xrightarrow{(y_i)} J_{s_1 s_2} \to 0$$

and

$$0 \to L'_T \to R_{s_2}[T,T^{-1}]^n \to J_{s_2 T} \to 0$$

are exact and L'_{s_1} is free of rank n-1. We also have $x_i \equiv y_i \pmod T$.

Let $\alpha' : (\bar{L})_{s_2} \xrightarrow{\sim} (\bar{L}')_{s_1}$ be the isomorphism defined by the following commutative diagram of exact sequences,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\bar{L})_{s_2} & \longrightarrow & R^n_{s_1 s_2} & \xrightarrow{(\bar{x}_i)} & (\bar{J})_{s_1 s_2} & \longrightarrow & 0 \\ & & \wr\downarrow \alpha' & & \wr\downarrow \text{Id} & & \wr\downarrow \text{Id} & & \\ 0 & \longrightarrow & (\bar{L}')_{s_1} & \longrightarrow & R^n_{s_1 s_2} & \xrightarrow{(\bar{y}_i)} & (\bar{J})_{s_1 s_2} & \longrightarrow & 0 \end{array}$$

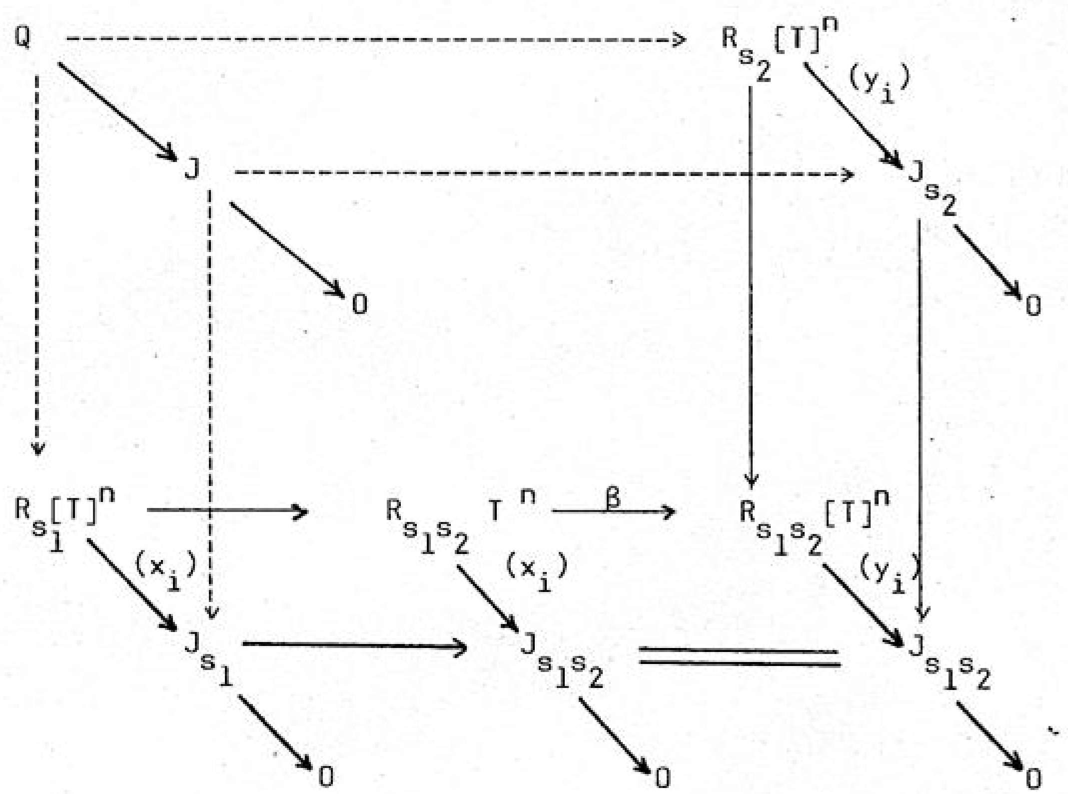
(Barring always denotes "modulo T") . As L'_{s_1} and L_{s_2} are extended there is an isomorphism $\alpha : L_{s_2} \xrightarrow{\sim} L'_{s_2}$ such that

$\alpha(\mathrm{mod}\ T) = \alpha'$. Using α we can define an isomorphism $\beta : R_{s_1 s_2}[T]^n \tilde{\to} R_{s_1 s_2}[T]^n$ with $\beta \equiv \mathrm{Id}\ (\mathrm{mod}\ T)$, to get the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (L)_{s_2} & \longrightarrow & R_{s_1 s_2}[T]^n & \xrightarrow{(x_i)} & J_{s_1 s_2} & \to & 0 \\
 & & \wr\downarrow \alpha & & \wr\downarrow \beta & & \| \ \mathrm{Id} & & \\
0 & \longrightarrow & (L')_{s_1} & \longrightarrow & R_{s_1 s_2}[T]^n & \xrightarrow{(y_i)} & J_{s_1 s_2} & \longrightarrow & 0
\end{array}$$

(This can be done because the rows in the diagram above are split exact).

As $Rs_1 + Rs_2 = R$, we can construct the following fibre product diagram,



In this diagram Q is the fibre product of $R_{s_1}[T]^n$ and $R_{s_2}[T]^n$ given by the maps $R_{s_2}[T]^n \to R_{s_1 s_2}[T]^n \xrightarrow{\beta} R_{s_1 s_2}[T]^n$ and $R_{s_2}[T]^n \to R_{s_1 s_2}[T]^n$. The sequence $Q \to J \to 0$ is got by property of fibre product. If $f_i : Q_{s_i} \xrightarrow{\sim} R_{s_i}[T]^n$ $i = 1,2$ are the obvious isomorphisms, then $(f_2)_{s_1} \circ (f_1)^{-1}_{s_2} = \beta \equiv \mathrm{Id} \pmod{T}$. Hence by Lemma 1.7 of Chapter I, $Q \approx R[T]^n$ i.e. Q is free of rank n.

Again we see that in this diagram all the sequences in the bottom are exact but since

$$R_{s_2}[T]^n \to J_{s_2} \to 0$$

is not necessarily exact,

$$Q \to J \to 0$$

need not be exact. But

$$R_{s_2}[T, T^{-1}]^n \to J_{s_2 T} \to 0$$

is exact. Therefore we see that

$$Q_T \to J_T \to 0$$

is exact . As Q is free of rank n, we see that J_T is generated by n elements. Therefore we conclude that I is generated by n elements, which completes the proof of Theorem 2.3.

Remark 2.4. We shall give an example of a module M over $R[T,T^{-1}]$ for which given any module M' over $R[T]$ with $M'_T = M$ we have $e(M') > e(M)$. This shows that Theorem 2.1 is not an immediate consequence of Mohan Kumar's theorem ([MK-2], § 3, Theorem 3).

Let R be a discrete valuation ring with p a generator of the maximal ideal. Let $f = T + p$ and let M denote the $R[T,T^{-1}]$- module $R[T,T^{-1}]/(f)$. If $\mathfrak{p}$ is a prime ideal of height two of $R[T]$ and contains f, then as height $(\mathfrak{p} \cap R) = 1$, p belongs to $\mathfrak{p}$ and hence T belongs to $\mathfrak{p}$. So we see that f generates a maximal ideal in $R[T,T^{-1}]$. It follows $e(M) = 1$. Let M' be an $R[T]$- module such that $M'_T = M$. If $\mathfrak{p}$ is the ideal generated by f in $R[T]$ then $M'_{\mathfrak{p}} \neq 0$. Hence $\mu(\mathfrak{p}, M') + \dim(R[T]/\mathfrak{p}) = 2$. Hence $e(M) < e(M')$.

<u>Remark 2.6</u>. Mohan Kumar ([MK-3]) has extended Theorem 2.1 for $S^{-1}R[T]$, where S is a multiplicative set of non-zero-divisors in $R[T]$ with $\dim(R[T]) = \dim(S^{-1}R[T])$.

CHAPTER III

Efficient Generation of Ideals in Laurent Polynomial Rings

In this Chapter we discuss some interesting cases of the question that if R is a commutative noetherian ring and I is an ideal of R, then whether I is generated by $\mu(I/I^2)$ elements. In general it is known that $\mu(I/I^2) \leq \mu(I) \leq \mu(I/I^2)+1$ ([MK-1], Lemma).

Mohan Kumar ([MK-2], § 4 , proof of Theorem 5) has proved if $R = A[T]$ is a polynomial ring over a noetherian commutative ring A and I an ideal of R which contains a monic polynomial, with $\mu(I/I^2) \geq \dim(R/I) + 2$, then I is a quotient of a projective R-module of rank $\mu(I/I^2)$. One of our results (Theorem 1.2) in this Chapter is a variant of this result, which says that if we further assume that I contains an element with constant term a unit then I is actually generated by $\mu(I/I^2)$ elements.

Theorem 1.2 will be applied to deduce an analogous result (Theorem 2.2) for Laurent polynomial rings.

The main result (Theorem 2.3) in this Chapter is that if $R = A[X_1,\ldots,X_n , T_1^{\pm 1},\ldots, T_r^{\pm 1}], n \geq 0$ and $r \geq 1$ is a Laurent polynomial ring in several variables over a commutative noetherian ring A and I is an ideal of R with $\mu(I/I^2) \geq \dim(R/I) + 2$ and if $\text{height}(I) > \dim A$, then I is generated by $\mu(I/I^2)$ elements.

We do not know if such a theorem is true for $r = 0$.

Recall $R = A[X_1,\dots,X_n, T_1^{\pm 1},\dots, T_r^{\pm 1}]$ denotes the Laurent polynomial ring over the ring A with n polynomial variables $X_1,\dots,X_n$ and r Laurent polynomial variables $T_1,\dots,T_r$.

§1. A Theorem on Polynomial Rings

In this section we prove Theorem 1.2, the variant of Mohan Kumar's result mentioned above.

Before we state the result we introduce the following definition.

Definition 1.1. A monic polynomial f in $A[T]$ is said to be a special monic polynomial if the constant term of f is equal to one.

Theorem 1.2. Let $R = A[T]$ be a polynomial ring over a commutative noetherian ring A and I an ideal of R. Suppose $\mu(I/I^2) \geq \dim(R/I) + 2$ and I contains a special monic polynomial. Then $\mu(I/I^2) = \mu(I)$.

Before we prove Theorem 1.2 , we shall state a lemma, which is a slight variation of a Lemma of Mohan Kumar ([MK-2], § 3, Lemma 3) and the proof is also similar. Also recall that Lemma 2.4 of Chapter II is again a special case of the same lemma of Mohan Kumar.

Lemma 1.3. Let A be a commutative noetherian ring and I,J be ideals of A, I containing J. Let $n = \mu(I/I^2)$. Assume that $a_1,\dots,a_r$; $r < n$ are elements of I. Further suppose,

(i) $a_1,a_2,\dots,a_r$ form a part of a minimal set of generators of I mod I^2 .

(ii) Whenever p is a prime ideal of A which contains $(\sum_{i=1}^{r} a_i A) + J$ and does not contain I, the image of p in $A/(a_1A + J)$ has height atleast d, for some fixed integer d.

Then we can find a_{r+1} in I such that,

(i) $a_1,\ldots,a_r, a_{r+1}$ form a part of a minimal set of generators of I mod I^2.

(ii) Whenever p is a prime ideal of A, which contains $(\sum_{i=1}^{r+1} a_iA) + J$ and does not contain I, the image of p in $A/(a_1A+J)$ has height atleast d+1.

Proof of Theorem 1.2. Suppose a_1 belongs to a minimal set of generators of I mod I^2. Since I contains a special monic polynomial f, replacing a_1 by $a_1 - a_1(0)f^2 + f^p$ for large enough p, we can assume a_1 is special monic.

Write $J = A \cap I$. Then $A/J \to R/I$ and $A/J \to R/(J,a_1)R$ are integral extensions. So we have $\dim(A[T]/I) = \dim(A/J) = \dim(R/(J,a_1)R)$.

Write $B = R/(J,a_1)R$. By Lemma 1.3 we can choose a_2 in I such that,

(i) a_1,a_2 form a part of minimal set of generators of I mod I^2.

(ii) If a prime ideal p of R contains $a_1R + a_2R + JR$ and does not contain I, then image of p in B has height atleast one.

If necessary by replacing a_2 by $a_2 - a_2(0)a_1^2$ we may assume $a_2(0) = 0$.

If we write $n = \mu(I/I^2)$, then by iterating the above process we can find $a_1,\ldots,a_n$ in I with $a_1(0) = 1$, $a_2(0) = 0,\ldots,a_n(0) = 0$ and such that,

(i) $a_1,a_2,\ldots,a_n$ form a minimal set of generators of I mod I^2.

(ii) Whenever p is a prime ideal of R which contains $(\sum_{i=1}^{n} a_iR) + JR$ and does not contain I, the image of p in B has height atleast $n-1$.

Since $n \geq \dim(R/I) + 2 = \dim B + 2$, by (ii) we have

(iii) For a prime ideal p of R, if p contains $(\sum_{i=1}^{n} a_iR) + JR$ then p also contains I.

Write $B' = R/a_1R$ and consider the multiplicative set $1 + J$ in A. Since J is contained in the radical of A_{1+J} and B'_{1+J} is an integral extension of A_{1+J}, we have J is also contained in radical (B'_{1+J}). In view of (iii) a maximal ideal of B'_{1+J} which contains the images of $a_2,\ldots,a_n$, will also contain I', the image of I in B'_{1+J}. And thus by (i) for a maximal ideal M of B'_{1+J}, which contains the images of $a_2,\ldots,a_n$ in B'_{1+J}, we have I'_M is generated by the images of $a_2,\ldots,a_n$ in B'_{1+J} and hence I' is generated by these elements. So it follows that $I_{1+J} = \sum_{i=1}^{n} a_i R_{1+J}$. Thus we see that $I_{1+s} = \sum_{i=1}^{n} a_i R_{1+s}$, for some s in J.

We shall assume s is not nilpotent (otherwise nothing to prove).

As a consequence the following sequence

$$0 \to K \to R^n_{1+s} \xrightarrow{(a_i)} I_{1+s} \to 0$$

is exact, where the map $R^n_{1+s} \xrightarrow{(a_i)} I_{1+s}$ is the obvious surjection defined by a_i, $i = 1$ to n, and K the kernel of the surjection. As s belongs to I, K_s is projective and since a_1 is a monic polynomial, by Quillen-Suslin Theorem ([Q], Theorem 3/[Su-1], Theorem 1) K_s is free of rank $n-1$.

Since $I_s = R_s$, we have an exact sequence over R_s

$$0 \to K' \to R^n_s \xrightarrow{(1,0,\dots,0)} I_s \to 0$$

where the surjection is the obvious map defined by $(1,0,..,0)$ and K' is the kernel of the surjection which is free.

Let us denote "mod T" by "bar". Now as $a_1(0) = 1$, $a_2(0) = 0,\dots,a_n(0) = 0$, there is an isomorphism $h_1 : \bar{K}_s \xrightarrow{\sim} \bar{K}'_{1+s}$ such that the following diagram

$$\begin{array}{ccccccccc} 0 & \to & \bar{K}_s & \to & A^n_{s(1+s)} & \xrightarrow{(\bar{a}_i)} & \bar{I}_{s(1+s)} & \to & 0 \\ & & \downarrow h_1 & & \| & & \| & & \\ 0 & \to & K'_{1+s} & \to & A^n_{s(1+s)} & \xrightarrow{(1,0,..,0)} & \bar{I}_{s(1+s)} & \to & 0 \end{array}$$

is commutative, where the last and the middle vertical maps are identity.

Since K_s and K'_{1+s} are free, there is an isomorphism $h : K_s \xrightarrow{\sim} K'_{1+s}$ such that $\bar{h} = h_1$.

Using splittings of the surjections

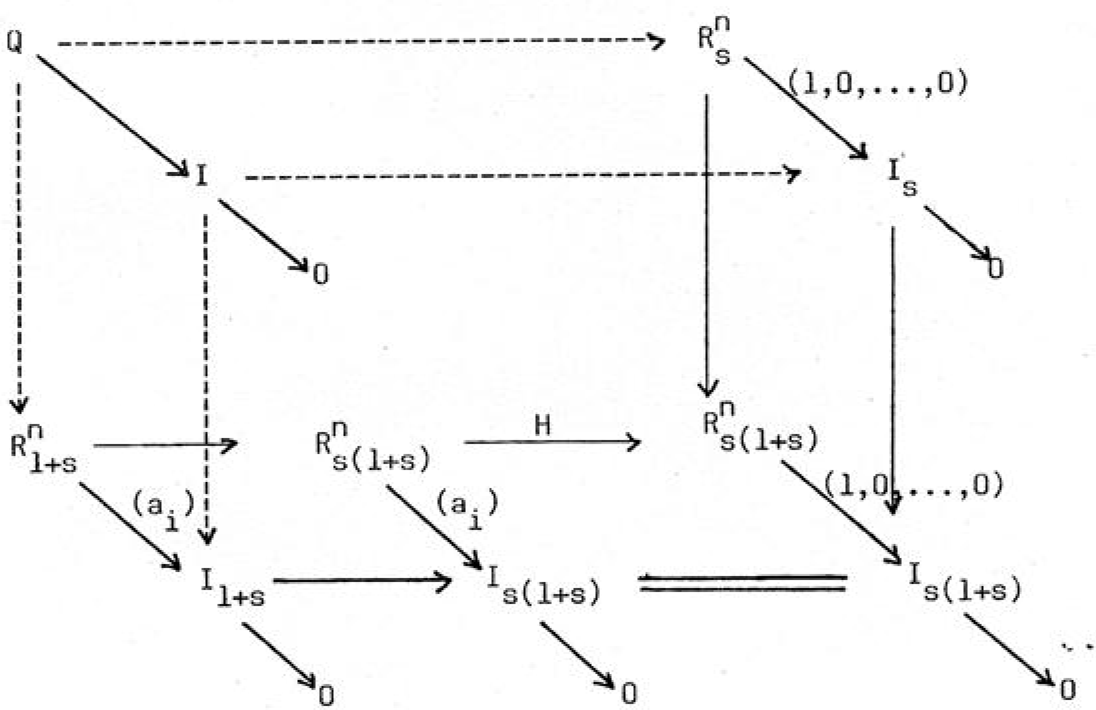
$$R^n_{s(1+s)} \xrightarrow{(a_i)} I_{s(1+s)} \to 0 \quad \text{and} \quad R^n_{s(1+s)} \xrightarrow{(1,0,\ldots,0)} I_{s(1+s)} \to 0$$

which are equal "modulo T", we can define an isomorphism $H : R^n_{s(1+s)} \xrightarrow{\sim} R^n_{s(1+s)}$ such that $H \equiv Id \pmod{T}$ and the following diagram

$$\begin{array}{ccccccccc}
0 \to & K_s & \to & R^n_{s(1+s)} & \xrightarrow{(a_i)} & I_{s(1+s)} & \to 0 \\
 & \wr\downarrow h & & \wr\downarrow H & & \| \; Id & \\
0 \to & K'_{1+s} & \to & R_{s(1+s)} & \xrightarrow{(1,0,..,0)} & I_{s(1+s)} & \to 0
\end{array}$$

is commutative.

As $As + A(1+s) = A$, we can construct the following fibre product diagram,



In this diagram Q is the fibre product of R_s^n and R_{1+s}^n given by the maps $R_s^n \to R_{s(1+s)}^n$ and $R_{1+s}^n \to R_{s(1+s)}^n \xrightarrow{H} R_{s(1+s)}$. The sequence $Q \to I \to 0$ is got by the property of fibre product.

If $g_1 : Q_s \tilde{\to} R_s^n$ and $g_2 : Q_{1+s} \tilde{\to} R_{1+s}^n$ are the obvious isomorphisms, then $(g_1)_{1+s} \circ (g_2^{-1})_s = H \equiv \mathrm{Id} \pmod{T}$. Hence by (Chapter I, Lemma 1.7) $Q \approx R^n$ i.e. Q is free of rank n.

Since upper right hand and lower left hand sequences in the diagram are exact, we see that $Q \to I \to 0$ is exact. Thus I is generated by $n = \mu(I/I^2)$ elements and the proof of Theorem 1.2 is complete.

§2. On Laurent Polynomial Rings.

In this section we shall prove our main result (Theorem 2.3). We also prove the Laurent polynomial analogue (Theorem 2.2) of Theorem 1.2, which will be used in the proof of the main result.

It will be convenient for the subsequent discussion to have the following definition.

Definition 2.1. A Laurent polynomial f in $A[T,T^{-1}]$ is called a doubly monic Laurent polynomial if both the coefficients of the highest degree term and the lowest degree term in f are equal to one.

For example a special monic polynomial is a doubly monic Laurent polynomial.

Theorem 2.2. *Let* $R = A[T,T^{-1}]$ *be a Laurent polynomial ring over a commutative noetherian ring* A *in one variable* T. *Suppose* I *is an ideal of* R, *which contains a doubly monic Laurent polynomial. If* $\mu(I/I^2) \geq \dim(R/I) + 2$, *then* $\mu(I/I^2) = \mu(I)$.

Proof. Write $I_1 = I \cap A[T]$ and $J = A \cap I$. Since I contains a doubly monic Laurent polynomial, I_1 contains a special monic.

Suppose $a_1,\ldots,a_n$ form a minimal set of generators of I mod I^2, where $n = \mu(I/I^2)$. We can assume $a_1,\ldots,a_n$ belong to I_1 and with the help of a special monic in I_1 we can further assume a_1 is a special monic polynomial. We shall see that $a_1,a_2,\ldots,a_n$ generate $I_1 (\text{mod } I_1^2)$. It is enough to see that for every prime ideal p of $A[T]$, $(I_1/I_1^2)_p$ is generated by these elements. If T belongs to p, then a_1 does not belong to p and hence $(I_1/I_1^2)_p = 0$. If T does not belong to p, then $(I_1)_p = (I)_{p_T}$ and hence $(I_1/I_1^2)_p$ is generated by $a_1,a_2,\ldots,a_n$. Hence it follows that $\mu(I/I^2) = \mu(I_1/I_1^2)$.

Now as both R/I and $A[T]/I_1$ are integral extensions of A/J, we have $\dim(R/I) = \dim(A/J) = (\dim A[T]/I_1)$. Thus $\mu(I_1/I_1^2) = \mu(I/I^2) \geq \dim(R/I) + 2 = \dim(A[T]/I_1) + 2$. Therefore by an application of Theorem 1.2 we get $\mu(I_1) = \mu(I_1/I_1^2) = \mu(I/I^2)$. Hence $\mu(I) \leq \mu(I/I^2)$. Thus the proof is complete.

Theorem 2.3. *Suppose* $R = A[X_1,\ldots,X_n, T_1^{\pm 1},\ldots,T_r^{\pm 1}]$ *with* $n \geq 0$ *and* $r \geq 1$ *be a Laurent polynomial ring in several variables over a commutative noetherian ring* A. *Suppose* I *is an ideal of* R

with height(I) > dim A *and* $\mu(I/I^2) \geq \dim(R/I) + 2$. *Then* $\mu(I) = \mu(I/I^2)$.

Proof. Immediate from Theorem 2.2 and the following lemma.

Lemma 2.4. *Let* $R = A[X_1,\ldots,X_n, T_1^{\pm 1},\ldots,T_r^{\pm 1}]$ *with* $n \geq 0$ *and* $r \geq 1$ *be a Laurent polynomial ring in several variables over a commutative noetherian ring* A. *Given any ideal* I *of* R *with* height(I) > dim A, *there is an* A-*automorphism* $\omega : R \tilde{\to} R$ *such that*, $\omega(I)$ *contains a doubly monic Laurent polynomial in* T_1 .

For n = 0 this is a result of Suslin ([Su-2], §7, Lemma 7.1). The proof of this Lemma will be given in the next section (§3). Before we conclude this section we shall record a few remarks.

Remark 2.5. Lemma 2.4 is false in the case of polynomial rings i.e. if I is an ideal of $R = A[X_1,\ldots,X_n]$ with height(I) > dim A, then I need not contain a special monic *via* any A-automorphism. Following is a counter example.

Example. Let R = k[X,Y] be a polynomial ring in two variables over a field k and I = (XY). Then there exists no $\omega : R \tilde{\to} R$ such that $\omega(I)$ contains a special monic in Y. To see this suppose $\omega(X) = f_0(X) + f_1(X)Y + \cdots + f_n(X)Y^n$ and $\omega(Y) = g_0(X) + g_1(X)Y + \cdots + g_m(X)Y^m$ with f_i any g_i belonging to k[X] for i = 0 to n and j = 0 to m, defines a k-automorphism ω of R, such that $\omega(I)$ contains a special monic in Y. It follows that $f_0(X)$ and $g_0(X)$ are units and hence constant and hence the Jacobian of the transformation ω is a multiple of Y.

This contradicts that ω is a k-automorphism.

Remark 2.6. We would like to know if a statement similar to Theorem 2.3 is true for polynomial rings. Namely, if I is an ideal of $R = A[X_1,\ldots,X_n]$ with height(I) > dim A and $\mu(I/I^2) \geq \dim(R/I) + 2$, then whether it is true that $\mu(I) = \mu(I/I^2)$. Obviously a proof like that of Theorem 2.3 does not work in view of Remark 2.5.

§3. The proof of Lemma 2.4.

First we shall set up some notation.

If $R = A[T]$ (resp. $A[T,T^{-1}]$) is a polynomial ring (resp. Laurent polynomial ring) in one variable T over a commutative ring A and I is an ideal of R then $L_T(I)$ denotes the ideal of A, consisting of coefficients of the highest degree term in T of elements in I. Similarly for an ideal I of $R = A[T,T^{-1}]$, $L_{T^{-1}}(I)$ will denote the ideal of A, consisting of coefficients of the lowest degree term in T of the elements in I. In the case of Laurent polynomial rings $R = A[X_1,\ldots,X_n, T_1^{\pm 1},\ldots,T_r^{\pm 1}]$ in several variables, when we write $L_{X_1}(I)$, $L_{T_1}(I)$ or $L_{T_1^{-1}}(I)$, we mean R is considered as a polynomial or a Laurent polynomial ring over the rest of the variables and the notations are used in the above sense.

There is a well known result ([Ba], §4, Lemma 2) which says that <u>if</u> $R = A[T]$ <u>is a polynomial ring and</u> I <u>an ideal of</u> R, <u>then</u> height $(L_T(I)) \geq$ height (I). The following lemma is an easy consequence of this.

Lemma 3.1. Let $R = A[T,T^{-1}]$ be a Laurent polynomial ring over a commutative noetherian ring A and I an ideal of R. Then $\text{height}(L_T(I)) \geq \text{height}(I)$ and $\text{height}(L_{T^{-1}}(I)) \geq \text{height}(I)$.

Proof. It is enough to prove one of these inequalities. We prove $\text{height}(L_T(I)) \geq \text{height}(I)$. Write $J = I \cap A[T]$. Then $\text{height}(J) = \text{height}(I)$ and $L_T(J) = L_T(I)$. Hence $\text{height}(L_T(I)) \geq \text{height}(I)$ by ([Ba], §4, Lemma 2).

Now we are ready to prove Lemma 2.4.

Proof of Lemma 2.4. The proof is by induction in two stages. First we prove the Lemma for $r = 1$ by induction on n and then use induction on r to complete the proof.

Proof of the Lemma when $r = 1$, i.e. $R = A[X_1,\ldots,X_n, T_1, T_1^{-1}]$. If $n = 0$ then $R = A[T_1,T_1^{-1}]$ and in view of Lemma 3.1 we have $L_{T_1}(I) = L_{T_1^{-1}}(I) = A$. So we see that I contains an element of f which is monic in T_1 and an element g which is monic in T_1^{-1}. We can combine f and g suitably to get a doubly monic Laurent polynomial in I.

Assume now $r = 1$ and $n > 0$. We are going to use induction on n to complete the proof in this case. We have $R = A[X_1,\ldots,X_n,T_1,T_1^{-1}]$. Consider the ideal $L_{X_n}(I)$. We see $\text{height}(L_{X_n}(I)) \geq \text{height}(I) > \dim A$. Hence by induction hypothesis we may assume (via an A-automorphism of $A[X_1,\ldots,X_{n-1},T_1,T_1^{-1}]$) that $L_{X_n}(I)$ contains a doubly monic Laurent polynomial f in T_1. In fact we may assume $f = T_1^p + g_1T_1^{p-1} + \cdots + g_{p-1}T_1 + 1$ for some $p \geq 1$ and g_i in

$A[X_1,\ldots,X_{n-1}]$, $i = 1$ to $p-1$. Let $F(X_n)$ be an element in I with f as the coefficient of its highest degree term. Therefore $F(X_n) = fX_n^q + f_1X_n^{q-1} + \cdots + f_q$ for some $q \geq 1$ and f_j in $A[X_1,\ldots,X_{n-1},T_1,T_1^{-1}]$, $j = 1$ to q. Let $s > \max\{\deg_{T_1} f_j , \deg_{T_1^{-1}} f_j$ for $j = 1$ to $q\}$ where $\deg_{T_1} f_j$ and $\deg_{T_1^{-1}} f_j$ denote respectively the T_1-degree and T_1^{-1}- degree of f_j for $j = 1$ to q. Define $\omega : R \tilde{\to} R$ to be the A-automorphism given $\omega(X_i) = X_i$ for $1 \leq i \leq n-1$, $\omega(X_n) = X_n + T_1^s + T_1^{-s}$ and $\omega(T_1) = T_1$. Then $\omega(F(X_n))$ is a doubly monic Laurent polynomial. This completes the proof of the Lemma for $r = 1$ and arbitrary $n \geq 0$.

<u>Proof of the Lemma in the general case</u>. Since we have proved the Lemma when $r = 1$ and $n \geq 0$ arbitrary, here we shall apply induction on r to complete the proof.

Assume $r > 1$ and $n \geq 0$. So, we have $R = A[X_1,\ldots,X_n,T_1^{\pm 1},\ldots T_r^{\pm 1}]$. Look at the ideals $L_{T_r}(I)$ and $L_{T_r^{-1}}(I)$ of $A[X_1,\ldots,X_n,T_1^{\pm 1},\ldots,T_{r-1}^{\pm 1}]$. Since height(I) > dim A, by Lemma 3.1 both $L_{T_r}(I)$ and $L_{T_r^{-1}}(I)$ have heights strictly greater than dim A and hence height($L_{T_r}(I) \cap L_{T_r^{-1}}(I)$)> dim A. By induction hypothesis (<u>via</u>. an A-automorphism of $A[X_1,\ldots,X_n, T_1^{\pm 1},\ldots, T_{r-1}^{\pm 1}]$), $L_{T_r}(I) \cap L_{T_r^{-1}}(I)$ contains a Laurent polynomial f which is doubly monic in T_1. We may write $f = T_1^p + g_1T_1^{p-1} + \cdots + g_{p-1} T_1 + 1$ for some $p \geq 1$ and g_i in $A[X_1,\ldots,X_n, T_2^{\pm 1} ,\ldots,T_{r-1}^{\pm 1}]$. So we can find F and G in I such that, $F(T_r) = f\, T_r^q + f_1\, T_r^{q-1} + \cdots + f_{q-1}\, T_r + f_q$ and

$G(T_r) = f + h_1 T_r + \cdots + h_u\ T_r^u$ for some integers $q \geq 0$ and $u \geq 0$ and f_i, h_j in $A[X_1,\ldots,X_n, T_1^{\pm 1},\ldots, T_{r-1}^{\pm 1}]$ for $i = 1$ to q and $j = 1$ to u.

Let $s > \max\{\deg_{T_1}(f_i), \deg_{T_1^{-1}}(h_j)$ for $i = 1$ to q and $j = 1$ to $u\}$. Define an A automorphism $\omega : R \xrightarrow{\sim} R$ as follows,

$$\omega(X_i) = X_i \quad \text{for } 1 \leq i \leq n$$

$$\omega(T_i) = T_i \quad \text{for } 1 \leq i \leq r-1$$

$$\omega(T_r) = T_r T_1^s \qquad i = r.$$

Then $T_r^{-q}\,\omega(F)$ is monic in T_1 and $\omega(G)$ is monic in T_1^{-1} over the coefficient ring $A[X_1,\ldots,X_n, T_2^{\pm 1},\ldots, T_r^{\pm 1}]$. Hence a suitable linear combination H of $T_r^{-q}\,\omega(F)$ and $\omega(G)$ can be found which is doubly monic in T_1 . As H is an element of $\omega(I)$, the proof of Lemma 2.4 is complete.

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