QUANTUM SYMMETRIES OF CLASSICAL MANIFOLDS AND THEIR COCYCLE TWISTS

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In the memory of Mohanlal Sinha Roy

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Contents

Notations

$\mathbb{N}$	The set of natural numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathcal{M}_n(\mathcal{C})$	The set of $\mathcal{C}$-valued $n \times n$ matrices for a C^* algebra $\mathcal{C}$
S^1	The circle group
$\mathbb{T}^n$	The n-torus
ev	Evaluation map
id	The identity map
$C^\infty(M)$	The space of smooth functions on a smooth manifold M
$C_c^\infty(M)$	The space of compactly supported smooth functions on M
$\Lambda^k(C^\infty(M))$	Hilbert bimodule of smooth k-forms of a smooth manifold M
$\otimes$	Algebraic tensor product between two vector spaces or algebras
$\hat{\otimes}$	Injective tensor product of two C^* algebras or nice algebras
$\bar{\otimes}$	Exterior tensor product of Hilbert bimodules
$\otimes_{in}$	Interior tensor pruduct of Hilbert bimodules
$\otimes_w$	von Neumann algebraic tensor product
$\oplus$	Direct sum of two vector spaces
$\mathcal{N}(M)$	Total space of normal bundle of an embedded smooth manifold M
$\hat{\mathcal{Q}}$	Dual of a compact quantum group $\mathcal{Q}$
σ_{ij}	Flip map between ith and jth copy of an algebra
$\mathcal{M}(\mathcal{C})$	The multiplier algebra of a C^* algebra $\mathcal{C}$
$\mathcal{L}(E,F)$	The space of adjointable maps from Hilbert modules E to F
$\mathcal{L}(E)$	The space of adjointable maps from a Hilbert module E to itself
$\mathcal{K}(E,F)$	The space of compact operators from Hilbert modules E to F
$\mathcal{K}(E)$	The space of compact operators from a Hilbert module E to itself
$\mathcal{B}(\mathcal{H})$	The set of all bounded linear operators on a Hilbert space $\mathcal{H}$
$\mathcal{A} * \mathcal{B}$	Free product of two C^* algebras $\mathcal{A}$ and $\mathcal{B}$
$<<,>>$	Frechet algebra valued inner product
$<,>$	complex valued inner product

Chapter 0

Introduction

This thesis explores quantum symmetries of spectral triples coming from classical compact, connected, Riemannian manifolds and their cocycle twists. This is a part of a bigger story under the name of "Noncommutative geometry". The root of Non commutative geometry can be traced back to the Gelfand-Naimark theorem which says that there is an anti-equivalence between the category of (locally) compact Hausdorff spaces and (proper, vanishing at infinity) continuous maps and the category of (not necessarily) unital C^* algebras and $*$-homomorphisms. This means that the entire topological information of a locally compact Hausdorff space is encoded in the commutative C^* algebra of continuous functions vanishing at infinity. This motivates one to view a possibly noncommutative C^* algebra as the algebra of "functions on some noncommutative space". If the underlying space has some extra structures, then certain dense subalgebra of the C^* algebras can be specified to capture those structures. It was the remarkable idea of Alain Connes who constructed a spectral triple out of a Riemannian spin manifold so that geometry of the manifold is encoded by the triple. To be precise, the triple consists of an algebra of smooth functions on the manifold, the Hilbert space of square integrable spinors and the Dirac operator. As before, even when there is no space, one can consider a similar spectral triple consisting of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ where $\mathcal{H}$ is a separable Hilbert space, $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is a $*$-subalgebra (not necessarily closed in the C^* norm) and $\mathcal{D}$ is an unbounded operator on $\mathcal{H}$. Such a spectral triple can be viewed as a geometrical object or 'noncommutative manifold'.

Symmetries play a very crucial role in physics. Symmetries of physical systems (classical or quantum) were conventionally modeled by group actions, and after the advent of quantum groups, group symmetries were naturally generalized to symmetries given by quantum group action. In this context, it is natural to think of quantum automorphism or the full quantum symmetry groups of various mathematical and physical structures. The underlying basic principle of defining a quantum automorphism group of a given mathematical structure consists of two steps : first, to identify (if possible) the group of automorphisms of the structure as a universal object in a suitable category, and then, try to look for the universal object in a similar but bigger category by replacing groups by quantum groups of appropriate type.

The formulation and study of such quantum symmetries in terms of universal Hopf algebras date back to Manin [38]. In the analytic set-up of compact quantum groups, it was considered by S. Wang who defined and studied quantum permutation groups of finite sets and quantum automorphism groups of finite dimensional algebras. Subsequently, such questions were taken up by a number of mathematicians including Banica, Bichon, Collins and others (see, e.g. [2], [3], [13], [55]), and more recently in the framework of Connes' noncommutative geometry ([16]) by Goswami, Bhowmick, Skalski, Banica, Soltan, De-Commer, Thibault and many others who have extensively studied the quantum group of isometries (or quantum isometry group) defined in [22] (see also [10], [12], [6] etc.). In this context, it is important to compute quantum isometry groups of classical Riemannian manifolds. One may hope that there are many more quantum symmetries of a given classical space than classical group isometries which will help one understand the space better. However, in all the previous computations of quantum isometry groups of classical compact connected Riemannian manifold M, they turned out to be the same as $C(ISO(M))$, where $ISO(M)$ is the classical isometry group of the manifold M. This led D. Goswami to conjecture that the quantum isometry group of a compact, connected, Riemannian manifold M is same as $C(ISO(M))$, i.e. there are no genuine quantum isometry of such manifolds. The main result of this thesis is a proof of the above conjecture for stably parallelizable compact connected Riemannian manifolds. This also allows us to explicitly describe quantum isometry groups of a very large class of spectral triples obtained by deformation of classical spectral triples. It is worth mentioning that after writing this thesis we have been able to relax the condition of stably parallelizablity on the manifold in [18] and prove the original conjecture.

We begin the thesis by recollecting some necessary preliminaries in Chapter 1 and Chapter 2, giving an overview of the concerned area of the thesis. Chapter 1 mainly recalls basics operator algebras and gives a glimpse of the formulation of noncommutative geometry. Also it collects some well known facts (mostly without proofs) about manifold theory and some topological algebras. Chapter 2 deals with the formulation of quantum isometry group as in [22] and [10]. Although mostly it is a review of the results of these papers a few new results have also been obtained (for example Theorem 2.3.12).

In chapter 3 of this thesis we extend the notion of a C^* action of a compact quantum group on a C^* algebra and adopt an appropriate definition of a smooth action of a compact quantum group on a smooth manifold. Observe that for a smooth manifold M, the algebra of smooth functions (denoted by $C^\infty(M)$) over the manifold is no longer a C^* algebra. Instead, it is a Fréchet algebra. So the right topology in the context of a smooth action turns out to be the Fréchet topology. After defining and discussing some technical issues with smooth action we deduce a necessary and sufficient condition for the 'differential' of the smooth action to make sense as a well defined bimodule morphism on the $C^\infty(M)$ bimodule of one forms. Then we define an inner product preserving action of a compact quantum group and show that an inner product preserving action automatically lifts to a bimodule morphism to the $C^\infty(M)$ bimodule of k-forms.

In Chapter 4 we give two sufficient conditions for a smooth action to be isometric. One of the conditions turns out to be necessary also. We use these results in Chapter 5 to calculate the quantum isometry group of a classical, compact, connected, Riemannian manifold. We restrict our attention to a certain class of manifolds which are stably parallelizable. We show that the quantum isometry group of such a Riemannian manifold coincides with the classical isometry group of the manifold.

Most of the examples of non commutative spectral triples come from suitable deformation of classical spectral triples. In Chapter 6, we consider the problem of describing the quantum isometry group of a particular class of such non commutative spectral triples i.e. the spectral triples obtained from classical spectral triples by twisting using the actions of a CQGs admitting a unitary 2-cocycle. In this chapter we formulate the notion of deformation of a von Neumann algebra on which a CQG with a unitary 2-cocycle has a von Neumann algebraic action. In fact, Nesheveyev *et al* had already considered ([40]) such deformation in the setting of C^* or von Neumann algebras. In a subsequent subsection we compare their formulation with ours and establish the equivalence of the two approaches for the von Neumann algebraic case. We also partially answer one of the questions raised in [40] in the affirmative (in case of a von Neumann algebra). Next we define cocycle twist of a spectral triple and prove that the quantum isometry group of a cocycle twisted spectral triple is same as the cocycle twist of the quantum isometry group of the untwisted spectral triple. Using this result and the main result of [18], we determine all the quantum isometries of the cocycle twisted spectral triples coming from compact, connected, stably parallelizable Riemannian manifolds.

In the 7th Chapter we prove a generalization of the standard technique of averaging a Riemannian metric with respect to a compact group action using the Haar state of the group so that the action becomes inner product preserving. We extend this to the set-up of compact quantum group actions satisfying certain conditions. We believe that this as well as other techniques developed in this thesis will be useful in the study of quantum isometry groups of arbitrary spectral triples.

Chapter 1

Preliminaries

1.1 Classical differential geometry

1.1.1 Smooth manifolds

Definition 1.1.1. *A manifold (without boundary) M of dimension n is a second countable Hausdorff topological space with a set of pairs $\{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ such that $M = \cup_{\alpha\in I} U_\alpha$ and each $\phi_\alpha(U_\alpha)$ is homeomorphic to an open subset of $\mathbb{R}^n$. It is said to admit a smooth structure if on each $\{(U_\alpha \cap U_\beta) : \alpha, \beta \in I\}$, $\phi_\alpha \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ is a smooth map between two open subsets of $\mathbb{R}^n$. A manifold with such a smooth structure is called a smooth manifold. The open sets U_α's are called trivializing neighborhoods and (U_α, ϕ_α) is called a chart.*

Examples

1. Any open subset U of $\mathbb{R}^n$ is an n-dimensional smooth manifold with $\{(U, id)\}$ as the obvious choice of trivializing neighborhood.
2. $S^1 \subset \mathbb{R}^2$ is a smooth manifold of dimension 1. It has two trivializing neighborhoods covering it. One is deleting its north pole and other deleting its south pole. In both cases the corresponding homeomorphism is $\theta(\in (0, 2\pi)) \to (cos(\theta), sin(\theta)) \subset S^1$. It is an example of a compact manifold.
3. If M and N are two smooth manifolds with $\{(U_\alpha) : \alpha \in I\}$ and $\{(V_\beta) : \beta \in J\}$ being corresponding trivializing neighborhoods, then $M \times N$ is a smooth manifold of dimension $2n$ with $\{(U_\alpha \times V_\beta) : (\alpha, \beta) \in I \times J\}$ being the corresponding trivializing neighborhoods. As a result $S^n = S^1 \underbrace{\times ... \times}_{n-times} S^1$ is an n-dimensional manifold.

Let $\mathbb{H}^n = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_n \geq 0\}$. Then we have the following

Definition 1.1.2. *Let M be a second countable topological space with every point M admitting a neighborhood homeomorphic to an open subset of $\mathbb{H}^n$. Then M is said to* *be a manifold with boundary. The boundary defined by* $\partial M = \{m \in M : \phi(m) \in \mathbb{R}^{n-1} \times 0$ *for some chart* $(U, \phi)\}$.

Definition 1.1.3. *A subset* A *of an* n*-dimensional manifold* M *with boundary* ∂M *is called a neat submanifold of* M*, if boundary of* A *denoted by* $\partial A = A \cap \partial M$ *and* A *is covered by charts* (ϕ, U) *of* M *such that* $A \cap U = \phi^{-1}(\mathbb{R}^m)$*, where* $m < n$ *for some* m.

Let M be a smooth manifold of dimension n without boundary with $\{(U_\alpha) : \alpha \in I\}$ being the trivializing neighborhoods and ϕ_α's the homeomorphisms. A real valued function f on M is said to be smooth at a point $m \in M$ if $m \in U_\alpha$ for some α, $f \circ \phi_\alpha^{-1}$ is smooth at $m \in M$. Using the smoothness of M, it can be shown this is indeed well defined. The set of real valued smooth functions is denoted by $C^\infty(M)_\mathbb{R}$. It is an algebra with pointwise multiplication. Its complexification is a $*$-algebra with the involution coming from usual complex conjugation. We denote this $*$-algebra by $C^\infty(M)$. Similarly for the manifolds with boundary the notion of smooth functions can also be defined taking care of the boundary points in an obvious way.

Orientation on manifolds

Definition 1.1.4. *Let* M *be a smooth manifold of dimension* n*. By a pre-orientation* α *on* M *we mean a choice* α_m *of orientation on* $T_m(M)$ *for each* $m \in M$*. A pre-orientation* α *is said to be smooth if it satisfies the following smoothness condition:*

To each $m \in M$*, there is a chart* (U, ϕ) *for* M *at* m *such that* $d\phi_m^{-1} : \mathbb{R}^n \to T_m(M)$ *carries the standard orientation class* E_n *of* $\mathbb{R}^n$ *to* α_m*. A pre-orientation which is smooth will be called an orientation and then we say* M *is orientable. A manifold with a choice of orientation is called an oriented manifold.*

Definition 1.1.5. *A diffeomorphism* $\phi : M \to N$ *between two smooth oriented manifolds* M *and* N *is said to be orientation preserving if at each point* $m \in M$*, we have* $D\phi : T_m(M) \to T_{\phi(m)}N$ *mapping the orientation of* $T_m(M)$ *to that of* $T_{\phi(m)}(N)$.

For the sake of completeness below we recall two standard facts about orientation of smooth manifolds. (see [46])

Proposition 1.1.6. *1. On a connected manifold there can be at most two orientations.*
2. An n*-dimensional manifold is orientable if and only if it has a cover* $\{U_\alpha\}_{\alpha \in I}$ *with* $\{\phi_\alpha\}_\alpha$ *being the charts such that each transition functions* $\phi_\beta \phi_\alpha^{-1}$ *is an orientation preserving diffeomorphisms of open subsets of* $\mathbb{R}^n$.

Smooth vector bundle

Definition 1.1.7. *A locally trivial smooth vector bundle of rank* n *over a smooth manifold* M *is a pair* (E, π) *such that*

1. $\pi : E \to M$ *is a smooth map.*
2. Each $\pi^{-1}(m)$ *is isomorphic as a vector space to* $\mathbb{R}^n$*.*
3. There is a trivializing cover $\{(U_\alpha) : \alpha \in I\}$ *such that* $\pi^{-1}(U_\alpha)$ *is diffeomorphic to* $U_\alpha \times \mathbb{R}^n$ *for all* $\alpha \in I$*. Also if* $\{(\Phi_\alpha) : \alpha \in I\}$ *are the corresponding diffeomorphisms, then if* $(U_\alpha \cap U_\beta)$ *is non empty,* $\Phi_\alpha^{-1} \circ \Phi_\beta : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$ *is* $(id, T_{\alpha\beta})$ *where* $T_{\alpha\beta} : \mathbb{R}^n \to \mathbb{R}^n$ *is a linear map for all* $\alpha, \beta \in I$*.*

Remark 1.1.8. *The manifold* E *is said to be the total space of the vector bundle and* M *is said to be the base space with* π *being called the projection map. A vector bundle is said to be trivial if we can choose only one such trivializing neighborhood. We shall often drop the word "locally trivializing".*

Examples

1. **Tangent and cotangent bundle**

Let M be a smooth manifold of dimension n with (U_α, ϕ_α) being the charts. On each trivializing neighborhood U_α, we can define smooth coordinate functions $\{x_i : 1 \leq i \leq n\}$ which are by definition $x_i := u_i \circ \phi_\alpha$ where $\{u_i : 1 \leq i \leq n\}$'s are coordinate functions for $\mathbb{R}^n$. Note that these are only locally defined functions. Recall the tangent space of a manifold at a point m It can be proved that this is a n-dimensional real vector space with basis given by the canonical derivations (with $m \in U_\alpha$) $\{\frac{\partial}{\partial x_i} : 1 \leq i \leq n\}$ defined by $\frac{\partial}{\partial x_i}|_m(f) := \frac{\partial}{\partial u_i}(f \circ \phi_\alpha^{-1})(0)$ where $\phi_\alpha(0) = m$. The tangent bundle $(T(M), \pi)$ is by definition $T(M) := \{(m, v) : m \in M, v \in T_m(M)\}$ with $\pi(m, v) = m$. The unique smooth structure on $T(M)$ is given by requiring π to be smooth. It can be proved that $T(M)$ is again a smooth manifold of dimension $2n$. $(T(M), \pi)$ is a smooth vector bundle of rank n.

Also we can consider the dual vector space $T_m^*(M)$ of $T_m(M)$ at each $m \in M$. This is again an n-dimensional vector space. We shall denote the dual basis corresponding to the basis $\{\frac{\partial}{\partial x_i} : 1 \leq i \leq n\}$ by $\{dx_i : 1 \leq i \leq n\}$. Then similarly $(T^*(M), \pi)$ is again a smooth locally trivial vector bundle of rank n, where $T^*(M) := \{(m, v) : m \in M, v \in T_m^*(M)\}$ and $\pi(m, v) = m$. This is called the cotangent bundle of M.

If we consider k-fold tensor product of $T_m^*(M)$ and denote it by $T_m^*(M)^{\otimes^k}$, then defining $\Omega^k(M) := \{(m, v) : m \in M, v \in T_m^*(M)^{\otimes^k}\}$ and $\pi : \Omega^k(M) \to M$ by $\pi(m, v) = m$, $(\Omega^k(M), \pi)$ is again a smooth vector bundle over M. If we consider k-fold wedge product of $T_m^*(M)$ and following similar construction we get another smooth vector bundle $(\Lambda^k(M), \pi)$ over M. Note that the cotangent bundle is nothing but $(\Lambda^1(M), \pi)$ which is the same as $(\Omega^1(M), \pi)$.

$C^\infty(M)$ bimodule of sections of a vector bundle

Definition 1.1.9. *A smooth section of a smooth vector bundle* (E, π) *over a smooth manifold* M *is a smooth map* $s : M \to E$ *such that* $\pi \circ s = id$ *on* M*.*

By definition its clear that $s(m) \in \pi^{-1}(m)$ for all $m \in M$. The space of smooth sections of a vector bundle (E, π) is denoted by $\Gamma(E)$. It can be given a $C^\infty(M)$-bimodule structure in the following way:

Let $f, g \in C^\infty(M)$ and $s \in \Gamma(E)$. Then the right and left actions are given by $(f.s.g)(m) := f(m)s(m)g(m)$. Then it is easy to check that $(f.s.g)$ again belongs to $\Gamma(E)$.

Examples

1. **Smooth vector fields**

The space of smooth sections of the tangent bundle $(T(M), \pi)$ over a manifold M is called the vector fields of the manifold M. It is generally denoted by $\chi(M)$. Although in general a smooth vector field does not have a global expression, it has local expression in terms of local coordinates. Let $X \in \chi(M)$ and $m \in M$ such that $m \in U_\alpha$ for some trivializing neighborhood U_α with local coordinates $(x_1, ..., x_n)$. Then on U_α, $X(m) = \sum_{i=1}^n c_i \frac{\partial}{\partial x_i}|_m$. Using the smoothness condition in fact we can show that for all α, $X|_{U_\alpha} = \sum_{i=1}^n f_i^\alpha \frac{\partial}{\partial x_i}$ for some $f_i^\alpha \in C^\infty(U_\alpha)$. Any smooth vector field X maps a $C^\infty(M)$ function to a smooth function . For that take $f \in C^\infty(M)$, then $(Xf) \in C^\infty(M)$ is defined by $(Xf)(m) := X(m)(f)$. It can be shown that a vector field $X \in \chi(M)$ is smooth if and only if $(Xf) \in C^\infty(M)$ for all $f \in C^\infty(M)$. With this Lie Bracket of two smooth vector fields can be defined as follows:

Take $X, Y \in \chi(M)$ and $f \in C^\infty(M)$ and $([X, Y]f)(m) = X(m)(Yf) - Y(m)(Xf)$. It is straightforward to verify that $[X, Y] \in \chi(M)$ for $X, Y \in \chi(M)$. The Lie bracket satisfies $[X, Y] = -[Y, X]$ and $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in \chi(M)$. In particular $[X, X] = 0$.

2. **Space of k-forms: The de-Rham differential**

The $C^\infty(M)$ bimodule of smooth sections of the vector bundles $(\Omega^k(M), \pi)$ and $(\Lambda^k(M), \pi)$ are denoted by $\Omega^k(C^\infty(M))$ and $\Lambda^k(C^\infty(M))$ respectively. $\Lambda^k(C^\infty(M))$ is called the space of k-forms for all k. From the definition it follows that $\Lambda^k(C^\infty(M))$ vanishes for $k \geq (n+1)$ when the manifold M is n-dimensional. Like in the vector fields, any smooth k-form s has a local expression $s|_{U_\alpha} = \sum_{1 \leq i_1 < i_2 < .... < i_k \leq n} f^\alpha_{i_1 i_2 ... i_k} dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k}$, where $\{dx_i : 1 \leq i \leq n\}$ is a basis for $T_m^*(M)$ on U_α as discussed earlier. Alternatively a k-form s at each point $m \in M$, $s(m)$ is a multilinear alternating functional on $T_m^*(M) \underbrace{\times ... \times}_{k-times} T_m^*(M)$. With this we can define the exterior derivative

$d : \Lambda^k(C^\infty(M)) \to \Lambda^{k+1}(C^\infty(M))$ in the following way:

Let θ be a k-form. Let $X_1, ..., X_{k+1} \in \chi(M)$. Then we can define a smooth function

$$\begin{aligned}\tilde{\theta}(X_1, ..., X_{k+1}) &= \sum_i (-1)^{i-1} X_i \theta(X_1, ..., X_{i-1}, X_{i+1}, ..., X_{k+1}) \\ &+ \sum_{i<j} \theta([X_i, X_j], X_1, ...\hat{X}_i, ..., \hat{X}_j, ..., X_{k+1})\end{aligned}$$

Then for $v_1, ..., v_{k+1} \in T^*_m(M)$ choose vector fields $X_1, ..., X_{k+1} \in \chi(M)$ such that $X_i(m) = v_i$ and define $d\theta(v_1, ..., v_{k+1}) := \tilde{\theta}(X_1, ..., X_{k+1})(m)$. It can be shown that this definition is independent of choice of chosen vector fields and $d\theta$ is multilinear linear functional and $d^2 = 0$. Also for $f \in C^\infty(M)$, df is the usual differential of f i.e. in local coordinates $df = \frac{\partial f}{\partial x_i} dx_i$ on each U_α with local coordinates $\{x_i : 1 \le i \le n\}$. This is called the de-Rham differential. Below we state a well known fact connecting the orientability of a smooth manifold and n-form.

Proposition 1.1.10. *A smooth manifold of dimension n is orientable if and only if it has a globally defined smooth n-form called the volume form. We shall denote the globally defined volume form of an oriented manifold by* dvol. *(see [46])*

Riemannian manifolds

Definition 1.1.11. *A Riemannian manifold M is a manifold for which is given at each $m \in M$, a positive definite symmetric bilinear form $<,>$ on each $T_m(M)$ such that for each coordinate $(x_1, ..., x_n)$ the functions $g_{ij} :=< \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i} >$ are smooth functions.*

Remark 1.1.12. *Observe that it is enough to specify the functions g_{ij} to completely determine the metric. Alternatively one can specify the same on the vector spaces $T^*_m(M)$, more precisely one can specify the functions $g^{ij} =< dx_i, dx_j >$ to determine the metric. We shall generally work with the cotangent bundle.*

Classical Hilbert space of forms

Let M be an n-dimensional oriented Riemannian manifold and $\Lambda^k(C^\infty(M))(k = 1, ..., n)$ be the space of smooth k-forms over the manifold. The de-Rham differential d maps $\Lambda^k(C^\infty(M))$ to $\Lambda^{k+1}(C^\infty(M))$. Let $\Omega \cong \Omega(M) = \oplus_k \Lambda^k(C^\infty(M))$. We will denote the Riemannian volume element by dvol. The Hilbert space $L^2(M)$ is obtained by completing $C^\infty_c(M)$ (compactly supported smooth functions) with respect to the pre inner product defined by $< f_1, f_2 >:= \int_M \bar{f}_1 f_2$ dvol. In an analogous way, one can construct a canonical Hilbert space of forms. The Riemannian metric induces an inner product on the vector space $T^*_m(M)$ for all $m \in M$ and hence also on $\Lambda^k(C^\infty(M))$. This gives a pre inner product on the space of compactly supported k-forms by integrating the compactly supported smooth function $m \to < \omega(m), \eta(m) >_m$ over M for $\omega, \eta \in \Lambda^k(C^\infty(M))$. We denote the completion of this space by $\mathcal{H}^k(M)$. Let $\mathcal{H} = \oplus_k \mathcal{H}^k(M)$. Then one can view $d : \Omega(M) \to \Omega(M)$ as an unbounded, densely defined operator (again denoted by d) on $\mathcal{H}$ with domain $\Omega(M)$. We denote its adjoint by d^*.

Isometry groups of classical manifolds

Let M be a Riemannaian manifold of dimension n. Then the collection of all isometries of M has a natural group structure and is denoted by $ISO(M)$. Let C and U be respectively compact and open subsets of M and let $W(C,U) = \{h \in ISO(M) : h.C \subset U\}$. The compact open topology is the smallest topology on $ISO(M)$ for which the sets $W(C,U)$ are open. It follows (see [28]) that under this topology $ISO(M)$ is a closed locally compact group. Moreover, if M is compact, $ISO(M)$ is also compact.

We recall the Laplacian $\mathcal{L}$ on M is an unbounded densely defined self adjoint operator $-d^*d$ on the space of zero forms $\mathcal{H}^0(D) = L^2(M, dvol)$ which has the local expression

$$\mathcal{L}(f) = -\frac{1}{\sqrt{det(g)}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(g^{ij}\sqrt{det(g)}\frac{\partial}{\partial x_i} f)$$

for $f \in C^\infty(M)$ and where $g = ((g_{ij}))$ is the Riemannian metric and $g^{-1} = ((g^{ij}))$. It is well known that on a compact manifold, the Laplacian has compact resolvents. Thus, the set of eigen values of $\mathcal{L}$ is countable, each having finite multiplicities, and accumulating only at infinity. Moreover there exists an orthonormal basis of $L^2(M)$ consisting of eigen vectors of $\mathcal{L}$ which belongs to $C^\infty(M)$. It can be shown (Lemma 2.3 of [22]) that for a compact manifold, the complex linear span of eigen vectors of $\mathcal{L}$ is dense in $C^\infty(M)$ in the sup norm.

The following result is in the form in which it has been stated and proved in [22] (Proposition 2.1).

Proposition 1.1.13. *Let M be a compact Riemannian manifold. Let $\mathcal{L}$ be the Laplacian of M. A smooth map $\gamma : M \to M$ is a Riemannian isometry if and only if γ commutes with $\mathcal{L}$ in the sense that $\mathcal{L}(f \circ \gamma) = (\mathcal{L}(f)) \circ \gamma$ for all $f \in C^\infty(M)$.*

Using this fact, we give an operator theoretic proof of the fact that for a compact manifold, $ISO(M)$ is compact. As the action of $ISO(M)$ commutes with the Laplacian, it has a unitary representation on $L^2(M)$. As the action preserves the finite dimensional eigen spaces of the Laplacian, $ISO(M)$ is a subgroup of $U(d_1) \times U(d_2) \times ....$ (where $\{d_i : i \geq 0\}$ denote the dimensions of the eigen spaces of the Laplacian and $U(d)$ denotes the group of unitary operators on a Hilbert space of dimension d) which is a compact group. As $ISO(M)$ is closed, it is a closed subgroup of a compact group, hence compact.

Examples:
1. The isometry group of the n-sphere S^n is $O(n+1)$ where the action is given by the usual action of $O(n+1)$ on $\mathbb{R}^{n+1}$.
2. The isometry group of the circle S^1 is $S^1 \rtimes Z_2$. Here $Z_2(=\{0,1\})$ action on S^1 is given by $1.z = \bar{z}$ where z is in S^1 while the action of S^1 is its action on itself.

Lie groups

Definition 1.1.14. *A Lie group is a set G which is both a group and a smooth manifold such that the group multiplication and the inverse operation are smooth with respect to the manifold structures of G and $G \times G$.*

Examples

1. The general linear group of invertible linear transformations of $\mathbb{R}^n$ denoted by $GL(n,\mathbb{R})$ is an example of a Lie group considered as an open subset of $\mathbb{R}^{n^2}$. This is a disconnected Lie group. However if we consider $GL(n,\mathbb{C})$, this becomes a connected Lie group.
2. The group of orthogonal linear transformations of $\mathbb{R}^n$ denoted by $O(n,\mathbb{R})$ is an example of a compact Lie group. This is a Lie subgroup(which is both a subgroup and a submanifold) of $GL(n,\mathbb{R})$. Similarly the group of special orthogonal linear transformations $SO(n,\mathbb{R})$ (component of $O(n,\mathbb{R})$ with determinant 1) is also a Lie group.
3. Spin group

Definition 1.1.15. *Let Q be a quadratic form on an n-dimensional vector space V. Then the Clifford algebra $Cl(V,Q)$ is the universal associative algebra $\mathcal{C}$ equipped with a linear map $i : V \to \mathcal{C}$, such that $i(V)$ generates $\mathcal{C}$ as a unital algebra satisfying $i(v)^2 = Q(v).1$.*

Let $\beta : V \to Cl(V,Q)$ be defined by $\beta(x) = -i(x)$. Then, $Cl(V,Q) = Cl^0(V,Q) \oplus Cl^1(V,Q)$ where $Cl^0(V,Q) = \{x \in Cl(V,Q) : \beta(x) = x\}$, $Cl^1(V,Q) = \{x \in Cl(V,Q) : \beta(x) = -x\}$. We will denote by $\mathcal{C}_n$ and $\mathcal{C}_n^{\mathbb{C}}$ the Clifford algebras $Cl(\mathbb{R}^n, -x_1^2 - ... - x_n^2)$ and $Cl(\mathbb{C}^n, -z_1^2 - ... - z_n^2)$.

We will denote the vector space $\mathbb{C}^{2[\frac{n}{2}]}$ by the symbol Δ_n. It follows that $\mathcal{C}_n^{\mathbb{C}} = End(\Delta_n)$ if n is even and equals $End(\Delta_n) \oplus End(\Delta_n)$ if n is odd. There is a representation $\mathcal{C}_n^{\mathbb{C}} \to End(\Delta_n)$ which is the isomorphism when n is even and in the odd case, it is the isomorphism with $End(\Delta_n) \oplus End(\Delta_n)$ followed by the projection onto the first coordinate. This representation restricts to $\mathcal{C}_n$ to be denoted by κ_n and called the spin representation. This representation is irreducible when n is even and it decomposes into two irreducible representations decomposing Δ_n into a direct sum of two vector spaces Δ_n^+ and Δ_n^-.

$Pin(n)$ is defined to be the subgroup of $\mathcal{C}_n$ generated by elements of the form $\{x : ||x|| = 1, x \in \mathbb{R}^n\}$. $Spin(n)$ is the group given by $Pin(n) \cap \mathcal{C}_n^0$. There exists a continuous group homomorphism from $Pin(n)$ to $O(n)$ which restricts to a two covering map $\lambda : Spin(n) \to SO(n)$.
4. The isometry groups of classical compact Riemannian manifolds are Lie groups.

1.1.2 Principal fibre bundle

Definition 1.1.16. *Let G be a Lie group and M be a smooth manifold. Then G is said to act smoothly to the right of M if there is a smooth map $\phi : M \times G \to M$ such that for each $g \in G$, the map $g : M \to M$ given by $g(m) = \phi(m, g)$ is a diffeomorphism and $\phi(m, gh) = \phi(\phi(m, g), h)$. It is said to act transitively if for any two $m, n \in M$ there is a $g \in G$ such that $\phi(g, m) = n$ and it is said to act freely if the only element of the group fixing an element of the manifold is the identity.*

It can be shown that if a Lie group G acts smoothly, transitively and freely then the quotient space of M (denoted by $M//G$) under the action of G can be given a unique manifold structure such that the map $\pi : M \to M//G$ is a smooth map.

Definition 1.1.17. *A principal fibre bundle is a set (P, G, M) such that*
(i) P and M are smooth manifolds and G is a Lie group.
(ii) G acts smoothly and freely to the right of P such that the corresponding quotient manifold is the manifold M so that the projection map $\pi : P \to M$ is smooth and G acts transitively on each fibre $\pi^{-1}(m)$.
(iii) P is locally trivial meaning that for each $m \in M$, there is a neighborhood U of m such that there is a smooth map $F_U : \pi^{-1}(U) \to G$ commuting with the right action of G on P and the map $\pi^{-1}(U) \to U \times G$, $p \to (\pi(p), F_U(p))$ is a diffeomorphism. G is called the structure group.

Example

Bundle of bases: The orthonormal frame bundle
Let M be a smooth Riemannian manifold of dimension n. Then we define $B_M := \{(m, e_1, ..., e_n)\}$ where $(e_1, ..., e_n)$ is a basis of $T_m(M)$. Define $\pi : B_M \to M$ in the obvious way. Then $GL(n, \mathbb{R})$ acts transitively and freely on B_M from right. The action is given by $\phi((m, e_1, ..., e_n), g) := (m, \sum_{i=1}^n g_{i1} e_i, ..., \sum_{i=1}^n g_{in} e_i)$, where $g = ((g_{ij}))_{1 \leq i,j \leq n} \in GL(n, \mathbb{R})$. Then G acts freely on the right of B_M. Let $m \in M$ and $m \in U$ such that U is a coordinate neighborhood of M with local coordinates $(x_1, ..., x_n)$. Then we define F_U on $\pi^{-1}(U)$ by $F_U(m', f_1, ..., f_n) = ((dx_j(f_i)))_{1 \leq i,j \leq n} \in GL(n, \mathbb{R})$ for $m' \in U$. Then the functions $y_i := x_i \pi$ and $y_{ij} := g_{ij} F_U$ where g_{ij}'s are standard coordinates of $GL(n, \mathbb{R})$ give local coordinate system on $\pi^{-1}(U)$ of B_M. It is convenient to

identify the bundle of bases as a subbundle of the bundle $E = Hom(M \times \mathbb{R}^n, \Omega^1(M))$ with fibres at m isomorphic to non singular linear transformations from $\mathbb{R}^n \to T^*_m(M)$. If we further demand that the linear transformations to be inner product preserving with respect to the canonical Euclidean inner products of $\mathbb{R}^n$ and the Riemannian inner product of $T^*_m(M)$ we get what is called the orthonormal frame bundle and denoted by O_M. This is a principal fibre bundle with structure group $O(n)$. The total space of the orthonormal frame bundle is always an orientable, parallelizable smooth manifold.

Definition 1.1.18. *Let (P, G, M) be a principal fibre bundle and F be a manifold on which G acts from left. The associated bundle corresponding to (P, G, M) with fibre F is defined as follows:*

Let $B' = P \times F$ with right action of G on B' defined by $\phi((p, f)g) = (\phi(p, g), g^{-1}f)$. Let $B = B'//G$, the quotient space with respect to the action of G. Then if we define $\pi' : B \to M$ by $\pi'((p, f)G) = \pi(p)$, then B is the total space of the associated fibre bundle. If $m \in M$, take U to be the coordinate neighborhood around m with $F_U : \pi^{-1}(U) \to G$, we have $F'_U : \pi'^{-1}(U) \to F$ given by $F'_U((p, f)G) := F_U(p)f$. Then $\pi'^{-1}(U)$ is homeomorphic to $U \times F$. We give B a manifold structure by requiring these homeomorphisms to be diffeomorphisms. With these structures, π' is a smooth map.

Example

The tangent bundle

Consider the bundle of bases $B(M)$. Consider the manifold $\mathbb{R}^n$ and the structure group $GL(n, \mathbb{R})$ acing on it from left. Then the corresponding associated bundle is the Tangent bundle of the manifold M with $\mathbb{R}^n$ as fibre.

1.1.3 Dirac operators

Definition 1.1.19. *Let M be a smooth, oriented Riemannian manifold of dimension n. Then we have the oriented orthonormal frame bundle O_M over M, which is a principal $SO(n)$ bundle. Such a manifold is called a spin manifold if there exists a pair (P, Λ) (called a spin structure) where*
(1) P is a $Spin(n)$ principal bundle over M.
(2) Λ is a map from P to F such that it is a 2-covering map as well as a bundle map (i.e. maps a fibre over a point to the fibre over that same point) over M.
(3) $\Lambda(p\hat{g}) = \Lambda(p)g$ where $\lambda(\hat{g}) = g$.

Given such a spin structure, we consider the associated bundle $S = P \times \Delta_n$ where $Spin(n)$ acts on Δ_n by its representation on Δ_n. This is called the "bundle of spinors" S. Now on the space of smooth sections of spinors $\Lambda(S)$, one can define an inner product by

$$< s_1, s_2 >_S = \int_M < s_1(x), s_2(x) > dvol(x)$$

where *dvol* is the globally defined volume form (which exists on an oriented manifold). The Hilbert space obtained by completing the space of smooth sections with respect to this inner product is denoted by $L^2(S)$ and the members are called the square integrable spinors. We assume the reader's familiarity of theory of connections on Principal fibre bundles and associated bundles. The reader might consult [14]. The Levi Civita connection induces a canonical connection on S which we shall denote by ∇^S.

Definition 1.1.20. *Let M be an oriented, Riemannian spin manifold. Then the Dirac operator D on M is the self adjoint extension of the following operator defined on the smooth sections of S:*

$$(Ds)(m) = \sum_{i=1}^{n} \kappa_n(X_i(m))(\nabla^S_{X_i} s)(m),$$

where $(X_1, ..., X_n)$ are local orthonormal (with respect to the Riemannian structure) defined in a neighborhood of m.

In the above definition, we have viewed $X_i(m)$ belonging to $T_m(M)$ as an element of the Clifford algebra $Cl_{\mathbb{C}}(T_m(M))$, hence $\kappa_n(X_i(m))$ is a map on the fibre of S at m, which is isomorphic to Δ_n. The self adjoint extension of D is again denoted by the same symbol D. We recall three important facts about the Dirac operator:

Proposition 1.1.21. *(1) $C^\infty(M)$ acts on S by multiplication and this action extends to a representation, say π, of the C^* algebra $C(M)$ on the Hilbert space $L^2(S)$.*
(2) For $f \in C^\infty(M)$, $[D, \pi(f)]$ has a bounded extension.
(3) Furthermore, the Dirac operator on a compact manifold has compact resolvents.

The Dirac operator carries a lot of geometric and topological informations. We give two examples:

(a) The Riemannian metric of the manifold is recovered by

$$d(p, q) = sup_{\phi \in C^\infty(M), ||[D, \pi(\phi)]|| \leq 1} |\phi(p) - \phi(q)|.$$

(b) For a compact manifold, the operator e^{-tD^2} is trace class for all $t > 0$. The volume form of the manifold can be recovered by the formula

$$\int_M f \ dvol = c(n) lim_{t \to 0} \frac{Tr(\pi(f) e^{-tD^2})}{Tr(e^{-tD^2})}$$

where $dim \ M = n$, $c(n)$ is a constant depending on the dimension.

1.1.4 Group of orientation preserving Riemannian isometries of a Riemannian spin manifold

We start this subsection with a Lemma.

Lemma 1.1.22. *Let M be a compact metrizable space, B, $\tilde{B}$ Polish spaces (complete separable metric space) such that there is an n-covering map $\Lambda : B \to \tilde{B}$. Then any continuous map $\xi : M \to B$ admits a lifting $\tilde{\xi} : M \to \tilde{B}$ which is Borel measurable and $\Gamma \circ \tilde{\xi} = \xi$. In particular, if $\tilde{B}$ and B are topological bundles over M, with Λ being a bundle map, any continuous section of B admits a lifting which is a measurable section of $\tilde{B}$.*

With this lemma in hand let M be a Riemannaian spin manifold (hence orientable) with a fixed choice of orientation. Let f be a smooth orientation preserving Riemannian isometry of M, and consider the bundles $E = Hom(F, f^*(F))$ and $\tilde{E} = Hom(P, f^*(P))$ (where Hom denotes the set of bundle maps). We view df as a section of the bundle in a natural way. By the previous Lemma, we obtain a measurable lift $\widetilde{df} : M \to \tilde{E}$, which is a measurable section of $\tilde{E}$. Using this, we define a map on the space of measurable section of $S = P \times_{Spin(n)} \Delta_n$ as follows: Given a (measurable) section ξ of S, say of the form $\xi(m) = [p(m), v]$ where $p(m)$ is in P_m, v in Δ_n, we define $U(\xi)$ by $U(\xi)(m) = [\widetilde{df}(f^{-1}(m))(p(f^{-1}(m))), v]$. Note that sections of the above form constitute a total subset in $L^2(S)$, and the map $\xi \to U(\xi)$ is clearly a densely defined linear map on $L^2(S)$, whose fibre wise action is unitary since the $Spin(n)$ action is so on Δ_n. Thus it extends to a unitary U on $\mathcal{H} = L^2(S)$. Any such U induced by the map f, will bw denoted by U_f (it is not unique since the choice of the lifting used in the construction is not unique). With these, below we state a Theorem (whose proof is given in [10]) which will help us to formulate quantum analogue of orientation preserving Riemannian isometry later.

Theorem 1.1.23. *Let M be a compact Riemannian spin manifold (hence orientable and fix a choice of orientation) with the usual Dirac operator D acting as an unbounded self adjoint on the Hilbert space $\mathcal{H}$ of the square integrable spinors, and let S denote the spinor bundle with $\Gamma(S)$ being the $C^\infty(M)$ module of smooth sections of S. Let $f : M \to M$ be a smooth one to one map which is an orientation preserving Riemannian isometry. Then the unitary U_f on $\mathcal{H}$ commutes with D and $U_f M_\phi U_f^* = M_{\phi \circ f}$, for any $\phi \in C(M)$, where M_ϕ denotes the operator of multiplication by ϕ on $L^2(S)$.*

Conversely, suppose that U is a unitary on $\mathcal{H}$ such that $UD = DU$ and the map $\alpha_U = UXU^{-1}$ for $X \in \mathcal{B}(\mathcal{H})$ maps $C(M)$ into $C(M)'' = L^\infty(M)$, then there is a smooth one to one orientation preserving Riemannian isometry f on M such that $U = U_f$.

1.2 Topological $*$-algebras and their tensor products

A topological $*$-algebra is a $*$-algebra such that the algebraic multiplication and the involution operations are continuous with respect to the topology it has. For this thesis purpose we need to go beyond the topological $*$-algebras like C^* and von Neumann algebras and also need to consider their tensor products. This is quite well-known from the literature. But for the sake of completeness and convenience of the reader we plan to discuss them briefly.

1.2.1 C^* algebras and tensor products

Definition 1.2.1. *A C^* algebra $\mathcal{A}$ is a Banach $*$-algebra such that the norm satisfies the C^* identity: $||x^*x|| = ||x||^2$ for all $x \in \mathcal{A}$. A C^* algebra is said to be unital or not according to the fact that the C^*-algebra has a unit or not.*

Given a locally compact Hausdorff space X, the algebra of continuous functions vanishing at infinity denoted by $C_0(X)$, with pointwise multiplication as algebra product, complex conjugation as the involution and usual sup norm as the C^* norm, is an example of a C^* algebra. It is non unital. However if the space X is compact, continuous functions on X denoted by $C(X)$ is a unital C^* algebra. It is remarkable that in fact any non unital commutative C^* algebra is necessarily an algebra of continuous functions vanishing at infinity on some locally compact Hausdorff space whereas any unital C^* algebra is space of continuous functions on some compact Hausdorff space. This is due to Gelfand. In this thesis we shall always be concerned with unital C^* algebras and thus decide to sketch (very briefly) the idea for the unital case.

For a C^* algebra $\mathcal{A}$ one defines the maximal ideal space (denoted by $sp(\mathcal{A})$) as the set of all multiplicative linear functionals on $\mathcal{A}$. Then the map $a \to (\hat{a} : h \to h(a))$, for $a \in \mathcal{A}$ and $h \in sp(\mathcal{A})$ establishes a C^* isomorphism between $\mathcal{A}$ and $C(sp(\mathcal{A}))$. The space $sp(\mathcal{A})$ is a subspace of the dual of $\mathcal{A}$ and endowed with the weak $*$-topology is a compact Hausdorff space. For details of the proof the reader is referred to [19].

In case of a non commutative C^* algebra $\mathcal{A}$, what we can say is that there is a Hilbert space $\mathcal{H}$ such that $\mathcal{A}$ can be faithfully represented as $*$ subalgebra of $\mathcal{B}(\mathcal{H})$. It is a fact that for a C^* homomorphism between two C^* algebras injectivity implies isometry. It follows easily from what is known as continuous functional calculus on a C^* algebra and for details again the reader is referred to first chapter of [19]. Returning to representation of a general noncommutative unital C^* algebra, the existence of such a faithful representation heavily depends on construction of GNS triple. For that recall that any linear functional on a C^* algebra with norm one is called a state. Given such a state ϕ on a C^* algebra $\mathcal{A}$, we have a triple (called GNS triple) $(\mathcal{H}_\phi, \pi_\phi, \xi_\phi)$ consisting of a Hilbert space $\mathcal{H}_\phi$, a $*$-representation π_ϕ of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H}_\phi)$ and a cyclic vector ξ_ϕ in the sense that $\{\pi_\phi(x)\xi_\phi : x \in \mathcal{A}\}$ is total in $\mathcal{H}_\phi$ satisfying

$$\phi(x) = < \xi_\phi, \pi_\phi(x)\xi_\phi > .$$

Let us denote the state space of $\mathcal{A}$ by $\mathcal{S}(\mathcal{A})$. Then taking the representation $\pi := \oplus_{\phi \in \mathcal{S}(\mathcal{A})} \pi_\phi$ of $\mathcal{A}$ on the Hilbert space $\mathcal{H} := \oplus_{\phi \in \mathcal{S}(\mathcal{A})} \mathcal{H}_\phi$, we get a faithful and hence an isometric embedding of $\mathcal{A}$ on the Hilbert space $\mathcal{H}$. For details see [19].

Examples

1. Commutative C^* algebra

For any compact Hausdorff space X, the space of continuous functions on X with the usual $*$-algebra structure and sup norm is a unital C^* algebra and as already noted, any commutative C^* algebra comes this way.

2. Compact operators

Given a separable Hilbert space $\mathcal{H}$, any $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ which is norm closed is an example of a C^*-algebra. It is a well known fact that the algebra of compact operators (which is the norm closure of finite rank operators) denoted by $\mathcal{B}_0(\mathcal{H})$ is an example of such a C^*-algebra. Its an example of a non unital C^* algebra when the Hilbert space is infinite dimensional.

3. Noncommutative two-torus

A large class of C^* algebras is obtained by the following construction. Let $\mathcal{A}_0$ be an associative $*$-algebra without a-priori a norm such that the set $\mathcal{F} = \{\pi : \mathcal{A}_0 \rightarrow \mathcal{B}(\mathcal{H}_\pi) * - \textit{homomorphism},\ \mathcal{H}_\pi \textit{ a Hilbert space}\}$ is non empty and $||.||_u$ defined by $||a||_u = sup\{||\pi(a)|| : \pi \in \mathcal{F}\}$ is finite for all $a \in \mathcal{A}_0$. Then the corresponding completion of $\mathcal{A}_0$ with respect to $||.||_u$ is called the universal C^* algebra corresponding to $\mathcal{A}_0$. Here is an example:

Let θ belongs to $[0, 1]$. Consider the $*$-algebra $\mathcal{A}_0$ generated by two unitary symbols U and V satisfying the relation $UV = e^{2\pi i\theta} VU$. It has a representation π on the Hilbert space $L^2(S^1)$ defined by $\pi(U)(f)(z) = f(e^{2\pi i\theta} z)$, $\pi(V)(f)(z) = zf(z)$ where $f \in L^2(S^1)$ and $z \in S^1$. Then $||a||_u$ is finite for all $a \in \mathcal{A}_0$. The resulting C^* algebra is called the noncommutative two torus and denoted by $\mathcal{A}_\theta$.

4. Group C^* algebra

Let G be a locally compact group with left Haar measure μ. One can make $L^1(G)$ a Banach $*$-algebra by defining

$$(f \star g) = \int_G f(s)g(s^{-1}t)d\mu(s),$$

$$f^*(t) = \Delta(t)^{-1}\overline{f(t^{-1})}.$$

Here $f, g \in L^1(G)$, Δ is the modular homomorphism of G.

$L^1(G)$ has a distinguished representation π_{reg} on $L^2(G)$ defined by $\pi_{reg}(f) = \int f(t)\pi(t)d\mu(t)$ where $\pi(t)$ is the unitary operator on $L^2(G)$ defined by $(\pi(t)f)(g) = f(t^{-1}g)(f \in L^2(G), t, g \in G)$. The reduced group C^* algebra of G is defined to be $C_r^*(G) := \overline{\pi_{reg}(L^1(G))}^{\mathcal{B}(L^2(G))}$.

Remark 1.2.2. *For G Abelian, we have $C_r^*(G) \cong C_0(\hat{G})$ where $\hat{G}$ is the group of characters on G.*

One can also consider the universal C^* algebra as described in example 2 corresponding to the Banach $*$-algebra $L^1(G)$. This is called the free or full group C^* algebra and denoted by $C^*(G)$.

Remark 1.2.3. *For the so-called amenable groups we have $C^*(G) \cong C_r^*(G)$.*

Now we briefly recall how we can form tensor products of two C^* algebras. Given two C^* algebras $\mathcal{A}$ and $\mathcal{B}$, we can form their algebraic tensor product (denoted by $\mathcal{A} \otimes \mathcal{B}$ throughout this thesis) which will again be a $*$-algebra in the usual way. But it turns out that the choice of equipping this algebraic tensor product with a C^* norm is far from unique. For details of this issue we refer the reader to Appendix T of [41]. One such choice is what is called the spatial C^* norm. It is given in the following way:

Given two C^* algebras $\mathcal{A}$ and $\mathcal{B}$, by the earlier discussion we can find two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ such that $\mathcal{A}$ and $\mathcal{B}$ are embedded isometrically in $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ respectively. Then $\mathcal{A} \otimes \mathcal{B}$ has a natural representation on $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ and hence has a C^* norm inherited from $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. This norm is called the spatial C^* norm and it is a fact that it does not depend upon the embeddings of $\mathcal{A}$ and $\mathcal{B}$. Completion with respect to this norm is called the spatial tensor product and throughout this thesis corresponding C^* algebra will always be denoted by $\mathcal{A} \hat{\otimes} \mathcal{B}$. There are other choices of C^* norms on the algebraic tensor product. But since we shall only use the spatial norm we don't discuss the other choices.

For a C^* algebra $\mathcal{A}$ (possibly non unital), its multiplier algebra, denoted by $\mathcal{M}(\mathcal{A})$ is defined as the maximal C^* algebra which contains $\mathcal{A}$ as an essential two sided ideal, that is, $\mathcal{A}$ is an ideal in $\mathcal{M}(\mathcal{A})$ and for $y \in \mathcal{M}(\mathcal{A})$, $ya = 0$ for all $a \in \mathcal{A}$ implies $y = 0$. The norm of $\mathcal{M}(\mathcal{A})$ is given by $||x|| = sup_{a\in\mathcal{A},||a||\leq 1}\{||xa||, ||ax||\}$. There is a locally convex topology called the strict topology on $\mathcal{M}(\mathcal{A})$, which is given by the family of seminorms $\{||.||_a, a \in \mathcal{A}\}$, where $||x||_a = Max(||xa||, ||ax||)$, for $x \in \mathcal{M}(\mathcal{A})$. $\mathcal{M}(\mathcal{A})$ is the completion of $\mathcal{A}$ in the strict topology. For an example the multiplier algebra of the C^* algebra $\mathcal{B}_0(\mathcal{H})$ is $\mathcal{B}(\mathcal{H})$ for a separable Hilbert space $\mathcal{H}$. The corresponding strict topology is the same as SOT (see section on von Neumann algebra for the definition of SOT topology)$*$ topology of $\mathcal{B}(\mathcal{H})$.

Given a separable Hilbert space $\mathcal{H}$ and a C^* algebra $\mathcal{A}$, we can consider the spatial tensor product $\mathcal{B}_0(\mathcal{H})\hat{\otimes}\mathcal{A}$ and its multiplier algebra $\mathcal{M}(\mathcal{B}_0(\mathcal{H})\hat{\otimes}\mathcal{A})$.

Lemma 1.2.4. *For a state ϕ on $\mathcal{A}$ the map $(id \otimes \phi)$ maps $\mathcal{M}(\mathcal{B}_0(\mathcal{H})\hat{\otimes}\mathcal{A})$ into $\mathcal{B}(\mathcal{H})$.*

Proof:
First note that $(id \otimes \phi) : \mathcal{B}_0(\mathcal{H})\hat{\otimes}\mathcal{Q} \to \mathcal{B}_0(\mathcal{H})$. Let $X \in \mathcal{M}(\mathcal{B}_0(\mathcal{H})\hat{\otimes}\mathcal{Q})$. Then there exists $X_n \in (\mathcal{B}_0(\mathcal{H})\hat{\otimes}\mathcal{Q})$ such that $X_n \to X$ in strict topology of $\mathcal{M}(\mathcal{B}_0(\mathcal{H})\hat{\otimes}\mathcal{Q})$, i.e. $X_nA \to XA$ and $AX_n \to AX$ in the C^* algebra $\mathcal{B}_0(\mathcal{H})\hat{\otimes}\mathcal{Q}$ for all $A \in \mathcal{B}_0(\mathcal{H})\hat{\otimes}\mathcal{Q}$.
We will show that $(id \otimes \phi)X_n \in \mathcal{B}_0(\mathcal{H})$ is strictly Cauchy in $\mathcal{B}_0(\mathcal{H})$ and hence defining $(id \otimes \phi)X$ as the strict limit of $(id \otimes \phi)X_n$ we can deduce that $(id \otimes \phi)X \in \mathcal{B}(\mathcal{H})$.
Let $T \in \mathcal{B}_0(\mathcal{H})$. Then $(T \otimes 1) \in \mathcal{B}_0(\mathcal{H})\hat{\otimes}\mathcal{Q}$ and

$$\begin{aligned} & ||((id \otimes \phi)X_n)T - ((id \otimes \phi)X_m)T|| \\ = \; & ||(id \otimes \phi)(X_n(T \otimes 1)) - (id \otimes \phi)(X_m(T \otimes 1)|| \\ \leq \; & ||X_n(T \otimes 1) - X_m(T \otimes 1)| \end{aligned}$$

□

Definition 1.2.5. *Given a family $(\mathcal{A}_i)_{i\in I}$ of unital C^* algebras, their unital C^* algebra free product $*_{i\in I}\mathcal{A}_i$ is the unique C^*-algebra $\mathcal{A}$ together with unital $*$-homomorphisms $\psi_i : \mathcal{A}_i \to \mathcal{A}$ such that given any C^* algebra $\mathcal{B}$ and unital $*$ homomorphism $\phi_i : \mathcal{A}_i \to \mathcal{B}$ there exists a unique unital $*$-homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ such that $\phi_i = \Phi\psi_i$.*

Remark 1.2.6. *It is a direct consequence of the above definition that given a family of C^* homomorphisms ϕ_i from $\mathcal{A}_i$ to $\mathcal{B}$, there exists a C^* homomorphism $*_i\phi_i$ such that $(*_i\phi_i)\psi_i = \phi_i$ for all i.*

1.2.2 von Neumann algebras

We recall that for a Hilbert space $\mathcal{H}$, the **strong operator topology (SOT)** and the **weak operator topology (WOT)** are the locally convex topologies on $\mathcal{B}(\mathcal{H})$ given by the family of seminorms $\mathcal{F}_1$, $\mathcal{F}_2$ respectively where $\mathcal{F}_1 = \{p_\xi : \xi \in \mathcal{H}\}$, $\mathcal{F}_2 = \{p_{\xi,\eta} : \xi, \eta \in \mathcal{H}\}$ and $p_\xi(x) = ||x\xi||$,$p_{\xi,\eta}(x) = | < x\xi, \eta > |$ for $x \in \mathcal{B}(\mathcal{H})$.

Definition 1.2.7. *A C^* subalgebra $\mathcal{B}$ of $\mathcal{B}(\mathcal{H})$ is called a von Neumann algebra if it is closed in the strong operator topology.*

Here we crucially observe that the definition is dependent on the choice of ambient Hilbert space. There is another intrinsic definition , but since we do not need von Neumann algebra in great detail, we don't go in that direction. Also since C^* algebras are convex, by Theorem 16.2 of [57], we can as well take the WOT closure of $\mathcal{B}$ in $\mathcal{B}(\mathcal{H})$. It is also a standard fact that a C^* algebra inside a $\mathcal{B}(\mathcal{H})$ is closed in SOT topology if and only if it is the same as its double commutant in $\mathcal{B}(\mathcal{H})$ (von Neumann's double commutant theorem, see Theorem 18.6 of [57]).

Examples

1. Abelian von Neumann algebras

We start with a compact Hausdorff space X. Then we have a Borel measure on X. We pick one such Borel measure and denote it by μ. Then we have the standard Hilbert space (denoted by $L^2(X, \mu)$) of square integrable functions on X. Then $C(X)$ (space of all continuous functions on X) acts as multiplication operator on $L^2(X, \mu)$ and is a C^* subalgebra of $\mathcal{B}(L^2(X, \mu))$. Its SOT closure in $\mathcal{B}(L^2(X, \mu))$ is $L^\infty(X, \mu)$ acting again as multiplication operator on $L^2(X, \mu)$. It is an example of Abelian von Neumann algebra. Like the C^* case, we have a similar result for von Neumann algebra. More precisely we have the following (Theorem 22.6 of [57])

Theorem 1.2.8. *Every Abelian von Neumann algebra $\mathcal{B}$ acting on a seperable Hilbert space $\mathcal{H}$ is C^* isomorphic to some $L^\infty(K, \mu)$, where K is a compact Hausdorff space and μ is a finite positive Borel measure on K with supp μ=K.*

2. Group von Neumann algebra

Let G be a discrete group. Then we can form a Hilbert space from this group (denoted by $l^2(G)$) where the space is $l^2(G) = \{f : G \to \mathbb{C} | \sum_{g\in G} |f(g)|^2 < \infty\}$ and the inner product is given by $< f, h >= \sum_{g\in G} \overline{f(g)} h(g)$. With respect to this inner product we have an orthonormal basis $\{\chi_g(h) := \delta_{g,h} : g \in G\}$. Then we can define left regular representation λ of G on the Hilbert space $l^2(G)$ given by

$$\begin{aligned} \lambda : \quad & G \to \mathcal{B}(l^2(G)) \\ & g \to (\lambda(g)\chi_h = \chi_{gh}) \end{aligned}$$

Then span$\{\lambda(G)\}$ is a $*$-subalgebra of $\mathcal{B}(l^2(G))$. The SOT closure of span$\{\lambda(G)\}$ in $\mathcal{B}(l^2(G))$ is called the group von Neumann algebra.

1.2.3 Fréchet algebras and their tensor products: Nuclearity

A locally convex space is a vector space together with a locally convex topology given by a family of seminorms. A locally convex space is called a Fréchet space if the corresponding family of seminorms is countable (hence the space is metrizable) and the space is complete with respect to its topology. Let $\mathcal{A}$ and $\mathcal{B}$ be two such locally convex spaces with family of seminorms $\mathcal{S}_\mathcal{A}$ and $\mathcal{S}_\mathcal{B}$ respectively. Then one wants a family of seminorms $\{\gamma_{pq} : p \in \mathcal{S}_\mathcal{A}, q \in \mathcal{S}_\mathcal{B}\}$ (called cross seminorm) on the algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$ such that $\gamma_{pq}(a \otimes b) = p(a)q(b)$ for $a \in \mathcal{A}, b \in \mathcal{B}$. The problem is that this choice is far from unique and there is a maximal and minimal such choice.

Firstly, equip $\mathcal{A}\otimes\mathcal{B}$ with the locally convex topology given by the family of semi norms $\{\gamma_{pq} : p \in \mathcal{S}_\mathcal{A}, q \in \mathcal{S}_\mathcal{B}\}$ where $\gamma_{pq}(\xi) = inf \sum p(a_i)q(b_i)$ and the infimum is taken over all possible expressions of $\xi = \sum a_i \otimes b_i$. This is called the projective tensor product. We denote the completion of $\mathcal{A} \otimes \mathcal{B}$ in this topology by $\mathcal{A}\hat{\otimes}_\pi\mathcal{B}$. The projective seminorm is maximal in the sense that if $\{\theta_{pq} : p \in \mathcal{S}_\mathcal{A}, q \in \mathcal{S}_\mathcal{B}\}$ is another family of cross seminorms then for $\xi \in \mathcal{A} \otimes \mathcal{B}$, $\gamma_{pq}(\xi) > \theta_{pq}(\xi)$.

There is another topology which we can give to the algebraic tensor product. For that for any subspace K of $\mathcal{A}(\mathcal{B})$, we denote its polar by K^0, i.e. $K^0 = \{\phi \in \mathcal{A}^*(\mathcal{B}^*); \phi(y) \leq 1,\ for\ y \in K\}$. Then we define a family of seminorms on $\mathcal{A} \otimes \mathcal{B}$ by

$$\lambda_{pq}(\xi) = sup_{a' \in E_p^0, b' \in E_q^0}(|\sum_{i=1}^{n} a^{'}(a_i)b^{'}(b_i)|),$$

where $\xi = \sum_{i=1}^n a_i \otimes b_i$, $E_p = \{a \in \mathcal{A}|p(a) \leq 1\}$ and $E_q = \{b \in \mathcal{B}|q(b) \leq 1\}$.

The topology induced by this family of seminorms on $\mathcal{A} \otimes \mathcal{B}$ is called the injective topology and corresponding completion is called the injective tensor product of $\mathcal{A}$ and $\mathcal{B}$. We denote the injective tensor product of $\mathcal{A}$ and $\mathcal{B}$ by $\mathcal{A}\hat{\otimes}_\epsilon\mathcal{B}$. The injective seminorm is minimal in the sense that if $\{\theta_{pq} : p \in \mathcal{S}_\mathcal{A}, q \in \mathcal{S}_\mathcal{B}\}$ is another family of cross seminorms then for $\xi \in \mathcal{A} \otimes \mathcal{B}$, $\lambda_{pq}(\xi) < \theta_{pq}(\xi)$.

Definition 1.2.9. *A locally convex space $\mathcal{A}$ is said to be nuclear if for any other locally convex space $\mathcal{B}$, $\mathcal{A}\hat{\otimes}_\epsilon\mathcal{B} \cong \mathcal{A}\hat{\otimes}_\pi\mathcal{B}$.*

Remark 1.2.10. *Since, the projective seminorm is maximal cross seminorm and injective seminorm is minimal cross seminorm, we conclude that for a nuclear locally convex space, there is essentially only one topological tensor product with another locally convex space.*

Furthermore if the Fréchet space is a $*$ algebra then we demand that its $*$ algebraic structure is compatible with its locally convex topology i.e. the adjoint $(*)$ is continuous and multiplication is jointly continuous with respect to the topology. Projective or injective tensor product of two such Fréchet $*$ algebras are again Fréchet $*$ algebra. We shall mostly use unital $*$ algebras. Henceforth all the Fréchet $*$-algebras will be unital unless otherwise mentioned.

Examples

We discuss an example of a nuclear Fréchet algebra which will be needed in this thesis.

Let M be compact smooth n dimensional manifold. Recall from Section (1) the $*$-algebra $C^\infty(M)$. We equip it with a locally convex topology : we say a sequence $f_n \in C^\infty(M)$ converges to an $f \in C^\infty(M)$ if for a compact set K within a single coordinate neighborhood (M being compact, has finitely many such neighborhoods) and a multi index α, $\partial^\alpha f_n \to \partial^\alpha f$ uniformly over K. Equivalently let $U_1, U_2, ..., U_l$ be a finite cover of M. Then it is a locally convex topology described by a countable family of seminorms given by:

$$p_i^{K,\alpha} = \sup_{x \in K} |\partial^\alpha f(x)|,$$

where K is a compact set within U_i, α is any multi index, $i = 1, 2, ...l$. $C^\infty(M)$ is complete with respect to this topology (example 1.46 of [45] with obvious modifications) and hence this makes $C^\infty(M)$ a locally convex Fréchet $*$ algebra. Note that, by choosing a finite C^∞ partition of unity on the compact manifold M, we can obtain finite set $\{\delta_1, ..., \delta_N\}$ for some $N \geq n$ of globally defined vector fields on M which is complete in the sense that $\{\delta_1(m), ..., \delta_N(m)\}$ spans $T_m(M)$ for all m (need not be a basis). This is a $*$-subalgebra of the C^*-algebra $C(M)$.

1.3 Hilbert bimodules

Let $\mathcal{A}$ and $\mathcal{B}$ be two topological $*$-subalgebras of $\mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$. Then a pre Hilbert bimodule $\mathcal{E}$ is an $\mathcal{A}-\mathcal{B}$ bimodule with a $\mathbb{C}$ bilinear map $<<,>>: \mathcal{E} \times \mathcal{E} \to \mathcal{B}$ with the following properties:

(i) $<< x, yb >>=<< x, y >> b$ $(x, y \in \mathcal{E}, b \in \mathcal{B})$

(ii)$<< x, y >>=<< y, x >>^*$ $(x, y \in \mathcal{E})$.

(iii)$<< x, x >>\geq 0$ and $<< x, x >>= 0$ if and only if $x = 0$.

If furthermore $\mathcal{B}$ is a Fréchet $*$-algebra, then we can talk about convergence of a sequence in $\mathcal{E}$. We say a sequence $a_n \in \mathcal{E}$ converges to $a \in \mathcal{E}$ if $<< a_n - a, a_n - a >>\to 0$ in the Fréchet topology of $\mathcal{B}$. The completion of $\mathcal{E}$ in this topology is called a Hilbert $\mathcal{A} - \mathcal{B}$ bimodule. If the second condition of (iii) is dropped then the Hilbert module is called pre Hilbert bimodule.

Examples

1. Any C^* algebra $\mathcal{A}$ is an example of a Hilbert Fréchet bimodule with the bimodule structure coming from the multiplication of the algebra and corresponding $\mathcal{A}$ valued inner product is given by $<< a, b >>:= a^*b$.
2. Consider a separable infinite dimensional Hilbert space $\mathcal{H}$. For a C^* algebra $\mathcal{A}$, the algebraic tensor product $\mathcal{H} \otimes \mathcal{A}$ can be given trivial Hilbert $\mathcal{A}$ bimodule structure. The bimodule structure is given by $a(\sum_{i=1}^k \xi_i \otimes a_i)a' := (\sum_{i=1}^k \xi_i \otimes aa_ia')$ for $a, a_i, a' \in \mathcal{A}$ and the corresponding $\mathcal{A}$-valued sesquilinear form defined on $\mathcal{H} \otimes \mathcal{A}$ by $<< \xi \otimes a, \xi' \otimes a' >>=< \xi, \xi' > a^*a'$. The completion of $\mathcal{H} \otimes \mathcal{A}$ with respect to this pre Hilbert bimodule structure is a Hilbert $\mathcal{A}$ bimodule will be denoted by $\mathcal{H}\bar{\otimes}\mathcal{A}$.

3. Hilbert bimodule structure on exterior bundles

Riemannian structure and $C^\infty(M)$-valued inner product on one-forms

Let M be a compact smooth manifold. Also let $\Lambda^k(C^\infty(M))$ be the space of smooth k forms on the manifold M. We equip $\Lambda^1(C^\infty(M))$ with the natural locally convex topology induced by the locally convex topology of $C^\infty(M)$ given by a family of seminorms $\{p_{(U,(x_1,\ldots,x_n),K,\beta)}\}$, where $(U,(x_1,\ldots,x_n))$ is a local coordinate chart, $\beta = (\beta_1, \beta_2, \ldots, \beta_r)$ is a multi-index with $\alpha_i \in \{1, 2, \ldots, n\}$ as before, K is a compact subset, and $p_{(U,(x_1,\ldots,x_n),K,\beta)}(\omega) := \sup_{x\in K, 1\leq i\leq n} |\partial_\beta f_i(x)|$, where $f_i \in C^\infty(M)$ such that $\omega|_U = \sum_{i=1}^n f_i dx_i|_U$. It is clear from the definition that the differential map $d : C^\infty(M) \to \Omega^1(C^\infty(M))$ is Fréchet continuous.

Lemma 1.3.1. *Let $\mathcal{A}$ be a Fréchet dense subalgebra of $C^\infty(M)$. Then $\Lambda^1(\mathcal{A}) := Sp\ \{fdg : f, g \in \mathcal{A}\}$ is dense in $\Lambda^1(C^\infty(M))$.*

Proof:
It is enough to approximate fdg where $f, g \in C^\infty(M)$ by elements of $\Omega^1(\mathcal{A})$. By Fréchet density of $\mathcal{A}$ in $C^\infty(M)$ we can choose sequences $f_m, g_m \in \mathcal{A}$ such that $f_m \to f$ and $g_m \to g$ in the Fréchet topology, hence by the continuity of d and the $C^\infty(M)$ module multiplication of $\Lambda^1(C^\infty(M))$, we see that $f_m dg_m \to fdg$ in $\Lambda^1(C^\infty(M))$. □

By the universal property $\exists$ a surjective bimodule morphism $\pi \equiv \pi_{(1)} : \Omega^1(C^\infty(M))_u \to \Lambda^1(C^\infty(M))$, such that $\pi(\delta g) = dg$. $\Omega^1(C^\infty(M))_u$ has a $C^\infty(M)$ bimodule structure:

$$f(\sum_{i=1}^{n} g_i \delta h_i) = \sum_{i=1}^{n} f g_i \delta h_i$$

$$(\sum_{i=1}^{n} g_i \delta h_i) f = \sum_{i=1}^{n} (g_i \delta (h_i f) - g_i h_i \delta f)$$

As M is compact, there is a Riemannian structure. Using the Riemannian structure on M we can equip $\Omega^1(C^\infty(M))$ with a $C^\infty(M)$ valued inner product $<< \sum_{i=1}^n f_i dg_i, \sum_{i=1}^n f'_i dg'_i >> \in C^\infty(M)$ by the following prescription:
for $x \in M$ choose a coordinate neighborhood $(U, x_1, x_2, ..., x_n)$ around x such that $dx_1, dx_2, ..., dx_n$ is an orthonormal basis for $T^*_x M$. Note that the topology does not depend upon any particular choice of the Riemannian metric. Then

$$<< \sum_{i=1}^{n} f_i dg_i, \sum_{i=1}^{n} f'_i dg'_i >> (x) = (\sum_{i,j,k,l} \bar{f}_i f'_j (\frac{\partial \bar{g}_i}{\partial x_k} \frac{\partial g'_j}{\partial x_l}))(x)$$

We see that a sequence $\omega_n \to \omega$ in $\Lambda^1(C^\infty(M))$ if $<< \omega_n - \omega, \omega_n - \omega >> \to 0$ in Fréchet topology of $C^\infty(M)$. With this $\Lambda^1(C^\infty(M))$ becomes a Hilbert module.

Hilbert bimodule of higher forms

Let us now recall from [34] (pages 95-108) an algebraic construction of the $C^\infty(M)$ bimodule of k-forms $\Lambda^k(C^\infty(M))$ on a manifold M from the so-called universal forms. $\Omega^2(C^\infty(M))_u = \Omega^1(C^\infty(M))_u \otimes_{C^\infty(M)} \Omega^1(C^\infty(M))_u$ and $\Omega^k(C^\infty(M))_u = \Omega^{k-1}(C^\infty(M))_u \otimes_{C^\infty(M)} \Omega^1(C^\infty(M))_u$.

Also $\Omega^1(C^\infty(M)) \equiv \Lambda^1(C^\infty(M))$. For $k \geq 2$, $\Omega^k(C^\infty(M)) = \Omega^{k-1}(C^\infty(M)) \otimes_{in} \Omega^1(C^\infty(M))$.

$$\dot{\Omega}(C^\infty(M)) = \oplus_{k \geq 0} \Omega^k(C^\infty(M)).$$

By the universality of $\Omega^2(C^\infty(M))_u$, we have a surjective bimodule morphism $\pi_{(2)} : \Omega^2(C^\infty(M))_u \to \Omega^2(C^\infty(M))$.
Let $\mathcal{J}_2$ be a submodule of $\Omega^2(C^\infty(M))$ given by $\mathcal{J}_2 = \{\pi_{(2)}(\delta\omega) | \pi(\omega) = 0 \; for \; \omega \in \Omega^1(C^\infty(M))_u\}$. In fact it is closed. Denote $\frac{\Omega^2(C^\infty(M))}{\mathcal{J}_2}$ by $\Lambda^2(C^\infty(M))$. Similarly $\Lambda^k(C^\infty(M)) = \frac{\Omega^k(C^\infty(M))}{\mathcal{J}_k}$ where $\mathcal{J}_k = \{\pi_{(k)}(\delta\omega) | \pi_{(k-1)}(\omega) = 0 \; for \; \omega \in \Omega^{k-1}(C^\infty(M))_u\}$. If ω and η belong to $\Omega^1(C^\infty(M))$, sometimes we denote the image of $\omega \otimes \eta$ in $\Omega^2(C^\infty(M))$ by $\omega\eta$ and in $\Lambda^2(C^\infty(M))$ by $\omega \wedge \eta$. Similar notations will be used for products in $\Omega^k(C^\infty(M))$ and $\Lambda^k(C^\infty(M))$. With this, the familiar de Rham

differential is given by

$$d : \Lambda^k(C^\infty(M)) \to \Lambda^{k+1}(C^\infty(M))$$
$$[\pi_{(k)}(\omega)] \to [\pi_{(k+1)}(\delta\omega)] \ ([\xi] := \xi + \mathcal{J}_k \ for \ \xi \in \Omega^k(C^\infty(M)))$$

Note that for all k, $\Omega^k(C^\infty(M))$ is the module of smooth sections of a hermitian, smooth, locally trivial vector bundle $E_k = \Lambda^1(M) \underbrace{\otimes ... \otimes}_{k-times} \Lambda^1(M)$ on M, whose fibre at x can be identified with the finite dimensional Hilbert space $E_k^x := (T_x^*M)^{\otimes^k}$, the inner product coming from the Riemannian structure. By construction, the closed submodule $\mathcal{J}_k$ is nothing but the module of smooth sections of a sub bundle (say V_k) of E_k, so that $\frac{E_k}{V_k} \cong \Lambda^k(M)$. At the fibres of $x \in M$, we have the orthogonal decomposition of Hilbert spaces $E_k^x = V_k^x \oplus (V_k^x)^\perp$ identifying the fibre of $\Lambda^k(M)$ at x with $(V_k^x)^\perp$. So we have the following orthogonal decomposition of the Hilbert bimodule $\Omega^k(C^\infty(M))$:

Lemma 1.3.2. $\Omega^k(C^\infty(M)) = \Lambda^k(C^\infty(M)) \oplus \mathcal{J}_k$.

In other words, $\Lambda^k(C^\infty(M))$ is an orthocomplemented closed submodule of $\Omega^k(C^\infty(M))$.

We can also derive the above orthogonal decomposition in a purely algebraic way. For example for $k = 2$, let $\pi_{(2)}(\delta f \otimes \delta g) \in \Omega^2(C^\infty(M))$. Then $\pi_{(2)}(\delta(\delta(g)f)) = -\pi_{(2)}(\delta g \otimes \delta f)$, hence $\pi_{(2)}(\delta(f\delta g - \delta g f)) = \pi_{(2)}(\delta f \otimes \delta g + \delta g \otimes \delta f)$. But $\pi(f\delta g - \delta g f) = 0$ in $\Omega^1(C^\infty(M))$. So $\frac{1}{2}\pi_{(2)}(\delta f \otimes \delta g + \delta g \otimes \delta f) \in \mathcal{J}_2$. Similarly $\frac{1}{2}\pi_{(2)}(\delta f \otimes \delta g - \delta g \otimes \delta f) \in \Lambda^2(C^\infty(M))$.

Thus, $\pi_{(2)}(\eta\delta f \otimes \delta g) = \frac{1}{2}\pi_{(2)}(\eta(\delta f \otimes \delta g + \delta g \otimes \delta f)) + \frac{1}{2}\pi_{(2)}(\eta(\delta f \otimes \delta g - \delta g \otimes \delta f))$. Also by definition $<< \pi_{(2)}(\delta f \otimes \delta g + \delta g \otimes \delta f), \pi_{(2)}(\delta f \otimes \delta g - \delta g \otimes \delta f) >>= 0$

For $k \geq 2$, observe that the permutation group S_k naturally acts on $\Omega^k(M)$, where the action is induced by the obvious S_k-action on the finite dimensional Hilbert space $(T_x^*M)^{\otimes^k}$, which permutes the copies of T_x^*M at $x \in M$. Then we have the orthogonal decomposition of the Hilbert space $(T_x^*M)^{\otimes^k}$ into the spectral subspaces with respect to the action of S_k. Explicitly

$$(T_x^*M)^{\otimes^k} = \oplus_{\chi \in \hat{S_k}} P_\chi^x((T_x^*M)^{\otimes^k}),$$

where P_χ^x is the spectral projection with respect to the character χ. Observe that when $\chi = sgn$, i.e. $\chi(\sigma) = sgn(\sigma)$, then $P_\chi^x((T_x^*M)^{\otimes^k}) = \Lambda^k(T_x^*M)$ and $V_k^x = \oplus_{\chi \neq sgn} P_\chi^x((T_x^*M)^{\otimes^k})$.

Clearly this fibre-wise decomposition induces a similar decomposition at the Hilbert bimodule level, i.e.

$$\begin{aligned}\Omega^k(C^\infty(M)) &= \oplus_{\chi\neq sgn}P_\chi(\Omega^k(C^\infty(M)))\oplus P_{sgn}(\Omega^k(C^\infty(M)))\\ &= \mathcal{J}_k(C^\infty(M))\oplus\Lambda^k(C^\infty(M)),\end{aligned}$$

where P_χ now denotes the spectral projection with respect to χ for the S_k-action on $\Omega^k(C^\infty(M))$ coming from P^x_χ fibre wise.

Also observe that the above arguments go through if we replace $C^\infty(M)$ by any subalgebra $\mathcal{A}$. In fact, if we are given any semi-Riemannian (possibly degenerate) structure on M which gives a nonnegative definite bilinear C^∞-valued form which is faithful (i.e. strictly positive definite) on $\Omega^1(\mathcal{A})$ then the action of the permutation group S_k on the k-fold tensor product $\Omega^k(\mathcal{A})$ is inner product preserving, hence different spectral subspaces for S_k-action are easily seen to be mutually orthogonal w.r.t. the $C^\infty(M)$-valued inner product. Thus, we have the following

Corollary 1.3.3. $\Omega^k(\mathcal{A})=\Lambda^k(\mathcal{A})\oplus\mathcal{J}^{\mathcal{A}}_k$, *where* $\Omega^1(\mathcal{A})=\{\sum f_i dg_i, f_i, g_i\in\mathcal{A}\}$,

$$\Omega^k(\mathcal{A})=\Omega^{k-1}(\mathcal{A})\otimes_{\mathcal{A}}\Omega^1(\mathcal{A}),$$

$$\mathcal{J}^{\mathcal{A}}_k=\{\pi_{(k)}(\delta\omega)|\pi_{(k-1)}(\omega)=0 \ for\ \omega\in\Omega^{k-1}(\mathcal{A})_u\}\ and$$

$$\Lambda^k(\mathcal{A})=\frac{\Omega^k(\mathcal{A})}{\mathcal{J}^{\mathcal{A}}_k},$$

Moreover if $\mathcal{A}$ *is Fréchet dense in* $C^\infty(M)$, $\Omega^k(\mathcal{A})$, $\Lambda^k(\mathcal{A})$ *and* $\mathcal{J}^{\mathcal{A}}_k$ *are dense in the Hilbert modules* $\Omega^k(C^\infty(M))$, $\Lambda^k(C^\infty(M))$ *and* $\mathcal{J}_k$ *respectively.*

Now for a C^* algebra $\mathcal{Q}$, $\Lambda^k(C^\infty(M))\bar{\otimes}\mathcal{Q}$ has a natural $C^\infty(M)\hat{\otimes}\mathcal{Q}$ bimodule structure. The left action is given by

$$(\sum_i f_i\otimes q_i)(\sum_j[\pi_{(k)}(\omega_j)]\otimes q^{'}_j)=(\sum_{i,j}[\pi_{(k)}(f_i\omega_j)]\otimes q_i q^{'}_j)$$

The right action is similarly given. The inner product is given by

$$<<\sum_i\omega_i\otimes q_i,\sum_j\omega^{'}_j\otimes q^{'}_j>>=\sum_{i,j}<<\omega_i,\omega^{'}_j>>\otimes q^*_i q^{'}_j.$$

Topology on $\Lambda^k(C^\infty(M))\bar{\otimes}\mathcal{Q}$ is given by requiring $\omega_n\rightarrow\omega$ if and only if $<<\omega_n-\omega,\omega_n-\omega>>\rightarrow 0$ in $C^\infty(M)\hat{\otimes}\mathcal{Q}$ or $C^\infty(M,\mathcal{Q})$.

1.3.1 Tensor products of Hilbert bimodules: interior and exterior tensor products

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two Hilbert bimodules over two Fréchet $*$ algebras $\mathcal{C}$ and $\mathcal{D}$ respectively where one of them is nuclear (say $\mathcal{D}$) and they are $*$ subalgebras of some $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$ as before. We denote the algebra valued inner product for the Hilbert bimodules by $<<,>>$. When the bimodule is a Hilbert space, we denote the corresponding scalar valued inner product by $<,>$. Then $\mathcal{E}_1 \otimes \mathcal{E}_2$ has an obvious $\mathcal{C} \otimes \mathcal{D}$ bimodule structure, given by $(a \otimes b)(e_1 \otimes e_2)(a' \otimes b') = ae_1a' \otimes be_2b'$ for $a, a' \in \mathcal{C}, b, b' \in \mathcal{D}$ and $e_1 \in \mathcal{E}_1, e_2 \in \mathcal{E}_2$. Also on $\mathcal{E}_1 \otimes \mathcal{E}_2$ define $<< e_1 \otimes e_2, f_1 \otimes f_2 >>=<< e_1, f_1 >> \otimes << e_2, f_2 >>$ for $e_1, f_1 \in \mathcal{E}_1$ and $e_2, f_2 \in \mathcal{E}_2$.

Lemma 1.3.4. *Extending the above definition of $<<,>>$ linearly we get a sesquilinear form on $\mathcal{E}_1 \otimes \mathcal{E}_2$.*

Proof:
Observe that for $z = \sum e_i \otimes f_i$, $<< z, z >>= \sum_{i,j} << e_i, e_j >> \otimes << f_i, f_j >>$ and apply lemmas 4.2 and 4.3 of [33]. □

In fact this semi inner product is actually an inner product on $\mathcal{E}_1 \otimes \mathcal{E}_2$. For a proof consult chapter 6 of [33]. Then taking the double completion (both on the bimodule and the Fréchet algebra level) we get a $\mathcal{C} \hat{\otimes} \mathcal{D}$ Hilbert bimodule which we shall denote by $\mathcal{E}_1 \bar{\otimes} \mathcal{E}_2$. This is called the exterior tensor product of two Hilbert bimodules. Note that when one of the bimodule is a Hilbert space $\mathcal{H}$ (with trivial Hilbert $\mathbb{C}$ bimodule structure) and another is a C^* algebra $\mathcal{A}$, then performing the exterior tensor product we get the usual Hilbert C^* bimodule $\mathcal{H} \bar{\otimes} \mathcal{A}$ as discussed earlier. Also when $\mathcal{H} = \mathbb{C}^N$, we have a natural identification of an element $T = ((T_{ij})) \in M_N(\mathcal{Q})$ with the right $\mathcal{Q}$ linear map of $\mathbb{C}^N \otimes \mathcal{Q}$ given by

$$e_i \mapsto e_j \otimes T_{ji},$$

where $\{e_i\}_{i=1,\ldots,N}$ is a basis for $\mathbb{C}^N$.

Let $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ be three locally convex $*$ algebras. Also let $\mathcal{E}_1$ be an $\mathcal{B} - \mathcal{C}$ Hilbert bimodule $\mathcal{E}_2$ be a $\mathcal{C} - \mathcal{D}$ Hilbert bimodule. Then $\mathcal{E}_1 \otimes_{\mathcal{C}} \mathcal{E}_2$ is an $\mathcal{B} - \mathcal{D}$ bimodule in the usual way. We can define a $\mathcal{D}$ valued inner product that will make $\mathcal{E}_1 \otimes_{\mathcal{C}} \mathcal{E}_2$ a pre-Hilbert $\mathcal{B} - \mathcal{D}$ bimodule. For that take $\omega_1, \omega_2 \in \mathcal{E}_1$ and $\eta_1, \eta_2 \in \mathcal{E}_2$ and define

$$<< \omega_1 \otimes \eta_1, \omega_2 \otimes \eta_2 >>:=<< \eta_1, << \omega_1, \omega_2 >> \eta_2 >> .$$

Again as before this is a sesquilinear form on $\mathcal{E}_1 \otimes_{\mathcal{C}} \mathcal{E}_2$. Let $\mathcal{I} = \{\xi \in \mathcal{E}_1 \otimes_{\mathcal{C}} \mathcal{E}_2$ such that $<< \xi, \xi >>= 0\}$. Then define $\mathcal{E}_1 \otimes_{in} \mathcal{E}_2 = \mathcal{E}_1 \otimes_{\mathcal{C}} \mathcal{E}_2 / \mathcal{I}$. We note that this semi inner product is actually an inner product, so that $\mathcal{I} = \{0\}$ (see proposition 4.5 of [33]). The topological completion of $\mathcal{E}_1 \otimes_{in} \mathcal{E}_2$ is called the interior tensor product and we shall denote it by $\mathcal{E}_1 \bar{\otimes}_{in} \mathcal{E}_2$. We denote the projection map from $\mathcal{E}_1 \otimes_{\mathbb{C}} \mathcal{E}_2$ to $\mathcal{E}_1 \otimes_{in} \mathcal{E}_2$ by π. We also make the convention of calling a Hilbert $\mathcal{A} - \mathcal{A}$ bimodule simply Hilbert $\mathcal{A}$ bimodule.

Example

Recall $\Omega^k(C^\infty(M))$. As a $C^\infty(M)$ bimodule this is nothing but

$$\Omega^1(C^\infty(M)) \underbrace{\otimes_{C^\infty(M)} \ldots \otimes_{C^\infty(M)}}_{k-times} \Omega^1(C^\infty(M)).$$

But for the Hilbert bimodule structure we take the interior tensor products recursively i.e. $\Omega^2(C^\infty(M)) = \Omega^1(C^\infty(M)) \otimes_{in} \Omega^1(C^\infty(M))$ and $\Omega^k(C^\infty(M)) = \Omega^1(C^\infty(M)) \otimes_{in} \Omega^{k-1}(C^\infty(M))$. It is straightforward to verify that this indeed gives the isomorphism as Hilbert bimodules.

1.4 Quantum groups: Hopf $*$-algebras

Definition 1.4.1. *A pair $(\mathcal{A}, \Delta)$ consisting of a unital $*$-algebra $\mathcal{A}$ and a unital $*$-homomorphism $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is called a Hopf $*$-algebra, if $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ and there exists linear maps $\epsilon : \mathcal{A} \to \mathbb{C}$ and $\kappa : \mathcal{A} \to \mathcal{A}$ such that*

$$(\epsilon \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes \epsilon)\Delta(a) = a \text{ and } m \circ (\kappa \otimes \mathrm{id})\Delta(a) = m \circ (\mathrm{id} \otimes \kappa)\Delta(a) = \epsilon(a)1_{\mathcal{A}},$$

where $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the multiplication map of the algebra $\mathcal{A}$.

Examples

1. For any compact group G, the algebra $C(G)$ is a Hopf $*$-algebra with $\Delta(f)(g, h) := f(gh)$, $\epsilon(f) := f(e)$ and $\kappa(f)(g) := f(g^{-1})$ where e is the identity element of G.
2. Assume that Γ is a discrete group and let $G = \hat{\Gamma}$, so $C(G) = C_r^*(\Gamma)$ and $\Delta(\lambda_\gamma) = \lambda_\gamma \otimes \lambda_\gamma$. The elements $\lambda_\gamma \in C_r^*(\Gamma)$ are one dimensional representations of G and since they already span a dense subspace of $C(G)$, from the orthogonality relations we conclude that there are no other irreducible representations. So the dense subspace spanned by the matrix coefficients of irreducible representations is the group algebra on Γ, spanned by the operators λ_γ which is a Hopf $*$-algebra with $\Delta(\lambda_\gamma) = \lambda_\gamma \otimes \lambda_\gamma$, $\epsilon(\lambda_\gamma) = 1$ and $\kappa(\lambda_\gamma) = \lambda_{\gamma^{-1}}$.

Sweedler notation

We introduce the so called Sweedler notation for coproduct. If a is an element of a coalgebra H, the element $\Delta(a) \in H \otimes H$ is a finite sum $\Delta(a) = \sum_i a_{1i} \otimes a_{2i}$ where $a_{1i}, a_{2i} \in H$

for all i. Moreover the representation is not unique. For notational simplicity we shall supress the index i and write the above sum symbolically as $\Delta(a) = a_{(1)} \otimes a_{(2)}$.

Definition 1.4.2. *Let $\mathcal{B}$ be a $*$ algebra. A Hopf $*$ algebra $(\mathcal{A}, \Delta, \epsilon, \kappa)$ is said to act on $\mathcal{B}$ if there is a $*$ homomorphism $\alpha : \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ satisfying*
(i)$(\alpha \otimes \mathrm{id})\alpha = (\mathrm{id} \otimes \Delta)\alpha$.
(ii)$(\mathrm{id} \otimes \epsilon)\alpha = \mathrm{id}$.

1.4.1 Compact Quantum Groups

Definition 1.4.3. *A compact quantum group (CQG in short) is a pair $(\mathcal{Q}, \Delta)$, where $\mathcal{Q}$ is a unital C^* algebra and $\Delta : \mathcal{Q} \to \mathcal{Q}\hat{\otimes}\mathcal{Q}$ is a unital $*$-homomorphism (called the comultiplication), such that*
(i)$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ *as homomorphism $\mathcal{Q} \to \mathcal{Q}\hat{\otimes}\mathcal{Q}\hat{\otimes}\mathcal{Q}$ (coassociativity).*
(ii) the spaces $\Delta(\mathcal{Q})(1 \otimes \mathcal{Q}) = \mathrm{span}\{\Delta(b)(1 \otimes a) | a, b \in \mathcal{Q}\}$ and $(1 \otimes \mathcal{Q})\Delta(\mathcal{Q})$ are dense in $\mathcal{Q}\hat{\otimes}\mathcal{Q}$.

Definition 1.4.4. *A morphism from a CQG $(\mathcal{Q}_1, \Delta_1)$ to another CQG $(\mathcal{Q}_2, \Delta_2)$ is a unital C^* homomorphism $\pi : \mathcal{Q}_1 \to \mathcal{Q}_2$ such that*

$$(\pi \otimes \pi)\Delta_1 = \Delta_2 \pi$$

Definition 1.4.5. *A Woronowicz C^*-subalgebra of a CQG $(\mathcal{Q}_1, \Delta)$ is a C^*-subalgebra $\mathcal{Q}_2$ of $\mathcal{Q}_1$ such that $(\mathcal{Q}_2, \Delta|_{\mathcal{Q}_2})$ is a CQG such that the inclusion map from $\mathcal{Q}_2 \to \mathcal{Q}_1$ is a morphism of CQG's.*

Definition 1.4.6. *A Woronowicz C^* ideal of a CQG $(\mathcal{Q}, \Delta)$ is a C^* ideal J of $\mathcal{Q}$ such that $\Delta(J) \subset Ker(\pi \otimes \pi)$, where π is the quotient map from $\mathcal{Q}$ to $\mathcal{Q}/J$*

We recall the following isomorphism theorem:

Proposition 1.4.7. *The quotient of a CQG $(\mathcal{Q}, \Delta)$ by a Woronowicz C^* ideal $\mathcal{I}$ has a unique CQG structure such that the quotient map π is a morphism of CQGs. More precisely the coproduct $\tilde{\Delta}$ on $\mathcal{Q}/\mathcal{I}$ is given by $\tilde{\Delta}(q + \mathcal{I}) = (\pi \otimes \pi)\Delta(q)$.*

Definition 1.4.8. *A CQG $(\mathcal{Q}', \Delta')$ is called a quantum subgroup of another CQG $(\mathcal{Q}, \Delta)$ if there is a Woronowicz C^*-ideal J of $\mathcal{Q}$ such that $(\mathcal{Q}', \Delta') \cong (\mathcal{Q}, \Delta)/J$.*

Examples

1. Let G be a compact group. Take $\mathcal{Q}$ to be the C^* algebra of continuous functions on G (denoted by $C(G)$). Then it can be proved that $\mathcal{Q}\hat{\otimes}\mathcal{Q} = C(G \times G)$ (see Appendix

of [41]). So we can define Δ by

$$\Delta(f)(g,h) := f(gh) \; for \; all \; g,h \in G.$$

Coassociativity follows from associativity of the group multiplication. For the span density condition note that $\Delta(\mathcal{Q})(1 \otimes \mathcal{Q})$ is a $*$-subalgebra spanned by all functions of the form $(g,h) \to f_1(g)f_2(gh)$. Since such functions separate points of G, applying Stone-Weierstrass theorem, we get the density.

In fact any compact quantum group $(\mathcal{Q}, \Delta)$ with $\mathcal{A}$ abelian is of the form $C(G)$, for some compact group G. Indeed, by Gelfand theorem, $\mathcal{Q} = C(G)$ for some compact Hausdorff space G. Then since $\mathcal{Q}\hat{\otimes}\mathcal{Q} = C(G \times G)$, the unital $*$-homomorphism Δ is defined by a continuous map $G \times G \to G$. Coassociativity means that

$$f((gh)k) = f(g(hk)) \; for \; all \; f \in C(G),$$

whence $((gh)k) = (g(hk))$. So G is a compact semigroup. The span density condition implies the cancellation property and hence G is a compact group.

2. Recall the C^* algebra of example 4 of Subsection 1.2.1. In that example if we take G to be a discrete group Γ with counting measure then an element γ is mapped to an operator λ_γ on $l^2(\Gamma)$ defined by $\lambda_\gamma(\delta_{\gamma'}) = \delta_{\gamma\gamma'}$. If we define $\Delta : C_r^*(\Gamma) \to C_r^*(\Gamma)\hat{\otimes}C_r^*(\Gamma)$ by $\Delta(\lambda_\gamma) = \lambda_\gamma \otimes \lambda_\gamma$, then $(C_r^*(\Gamma), \Delta)$ is a compact quantum group.

3. Assume $q \in [-1,1]$, $q \neq 0$. The quantum group $SU_q(2)$ is defined as follows:
The algebra $C(SU_q(2))$ is the universal C^* algebra generated by elements α and γ such that

$$\alpha^*\alpha + \gamma^*\gamma = 1, \alpha\alpha^* + q^2\gamma^*\gamma = 1, \gamma^*\gamma = \gamma\gamma^*, \alpha\gamma = q\gamma\alpha, \alpha\gamma^* = q\gamma^*\alpha.$$

The comultiplication is defined by

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

4. For $n \in \mathbb{N}$, denote by $A_s(n)$ the universal C^* algebra generated by elements $\{u_{ij} | 1 \leq i,j \leq n\}$, such that

$$U = ((u_{ij})) \; is \; a \; magic \; unitary,$$

meaning that U is unitary, all its entries u_{ij} are projections, and the sum of the entries in every row and column in U is equal to 1. The comultiplication is defined by

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}.$$

This is called the quantum permutation group.
5. For an $n \times n$ positive invertible matrix $Q = (Q_{ij})$, let $A_u(Q)$ the universal C^* algebra generated by $\{u^Q_{kj} : k, j = 1 \ldots n\}$ where $u := ((u^Q_{kj}))$ satisfies

$$uu^* = u^*u = I_n, u'Q\bar{u}Q^{-1} = Q\bar{u}Q^{-1}u'. \tag{1.4.1}$$

Here $u' = ((u^Q{}_{ji}))$ and $\bar{u} = ((u^{Q^*}{}_{ij}))$. The coproduct say $\tilde{\Delta}$ is given by

$$\tilde{\Delta}(u^Q_{ij}) = \sum_{k=1}^{n} u^Q_{ik} \otimes u^Q_{kj}.$$

1.4.2 Representation of CQG over Hilbert spaces:

The existence of Haar measure for a compact group plays a very crucial role in the representation theory of compact groups. In case of compact group, the existence of Haar measure translates into the existence of a convolution invariant state on $C(G)$. It turns out that for a general compact quantum group $\mathcal{A}$ also there is a bi invariant state (to be called the Haar state and denoted by h) on $\mathcal{Q}$ in the sense that $(h \otimes \mathrm{id})\Delta(q) = (\mathrm{id} \otimes h)\Delta(q) = h(q)1_{\mathcal{A}}$ for all $q \in \mathcal{Q}$, where $1_{\mathcal{Q}}$ is the unit of $\mathcal{Q}$. The C^*-completion $\mathcal{Q}_r$ of $\mathcal{Q}_0$ in the norm of $\mathcal{B}(L^2(h))$ (GNS space associated to h) is a CQG and called the reduced quantum group corresponding to $\mathcal{Q}$. If h is faithful then $\mathcal{Q}$ and $\mathcal{Q}_r$ coincide. In general there will be a surjective CQG morphism from $\mathcal{Q}$ to $\mathcal{Q}_r$ identifying the latter as a quantum subgroup of the former.

We would like to mention here that there is also a von Neumann algebraic framework of quantum groups suitable for development of the theory of locally compact quantum groups (see [31], [51] and references therein). In this theory, the von Neumann algebraic version of CQG is a von Neumann algebra $\mathcal{M}$ with a coassociative, normal, injective coproduct map Δ from $\mathcal{M}$ to $\mathcal{M} \otimes_w \mathcal{M}$ and a faithful, normal, bi-invariant state ψ on $\mathcal{M}$. Indeed, given a CQG $\mathcal{Q}$, the weak closure $\mathcal{Q}_r''$ of the reduced quantum group in the GNS space of the Haar state is a von Neumann algebraic compact quantum group.

Definition 1.4.9. *Let $(\mathcal{Q}, \Delta)$ be a CQG. A unitary representation of $\mathcal{Q}$ on a Hilbert space $\mathcal{H}$ is a $\mathbb{C}$-linear map U from $\mathcal{H}$ to the Hilbert module $\mathcal{H}\bar{\otimes}\mathcal{Q}$ such that*
1. $<< U(\xi), U(\eta) >>=< \xi, \eta > 1_{\mathcal{Q}}$, where $\xi, \eta \in \mathcal{H}$.
2. $(U \otimes \mathrm{id})U = (\mathrm{id} \otimes \Delta)U$.
3. Sp $\{U(\mathcal{H})\mathcal{Q}\}$ is dense in $\mathcal{H}\bar{\otimes}\mathcal{Q}$.

Given such a unitary representation we have a unitary element $\widetilde{U}$ belonging to $\mathcal{M}(\mathcal{B}_0(\mathcal{H})\hat{\otimes}\mathcal{Q})$ given by $\widetilde{U}(\xi \otimes b) = U(\xi)b, (\xi \in \mathcal{H}, b \in \mathcal{Q})$ satisfying

$(id \otimes \Delta)\widetilde{U} = \widetilde{U}^{12}\widetilde{U}^{13}$.

Definition 1.4.10. *A closed subspace $\mathcal{H}_1$ of $\mathcal{H}$ is said to be invariant if $U(\mathcal{H}_1) \subset \mathcal{H}_1 \bar{\otimes} \mathcal{Q}$. A unitary representation U of a CQG is said to be irreducible if there is no proper invariant subspace.*

It is a well known fact that every irreducible unitary representation is finite dimensional.

We denote by $Rep(\mathcal{Q})$ the set of inequivalent irreducible unitary representations of $\mathcal{Q}$. For $\pi \in Rep(\mathcal{Q})$, let d_π and $\{q^\pi_{jk} : j,k = 1,...,d_\pi\}$ be the dimension and matrix coefficients of the corresponding finite dimensional representation, say U_π respectively. Corresponding to $\pi \in Rep(\mathcal{Q})$, let ρ^π_{sm} be the linear functional on $\mathcal{Q}$ given by $\rho^\pi_{sm}(x) = h(x^\pi_{sm}x), s, m = 1,...,d_\pi$ for $x \in \mathcal{Q}$, where $x^\pi_{sm} = (M_\pi)q^{\pi *}_{km}(F_\pi(k,s))$. Also let $\rho^\pi = \sum_{s=1}^{d_\pi} \rho^\pi_{ss}$. Given a unitary representation V on a Hilbert space $\mathcal{H}$, we get a decomposition of $\mathcal{H}$ as

$$\mathcal{H} = \oplus_{\pi \in Rep(\mathcal{Q}), 1 \leq i \leq m_\pi} \mathcal{H}^\pi_i,$$

where m_π is the multiplicity of π in the representation V and $V|_{\mathcal{H}^\pi_i}$ is same as the representation U_π. The subspace $\mathcal{H}^\pi = \oplus_i \mathcal{H}^\pi_i$ is called the spectral subspace of type π corresponding to the irreducible representation π. It is nothing but the image of the spectral projection given by $(\mathrm{id} \otimes \rho_\pi)V$.
For each $\pi \in Rep(\mathcal{Q})$, we have a unique $d_\pi \times d_\pi$ complex matrix F_π such that
(1) F_π is positive and invertible with $Tr(F_\pi) = Tr(F_\pi^{-1}) = M_\pi > 0$(say).
(2) $h(q^\pi_{ij} q^{\pi^*}_{kl}) = \frac{1}{M_\pi}\delta_{ik}F_\pi(j,l)$.

Recall from [43], the modular operator $\Phi = S^*S$, where S is the anti unitary acting on the $L^2(h)$ (where $L^2(h)$ is the GNS space of $\mathcal{Q}$ corresponding to the Haar state on which $\mathcal{Q}$ has left regular representation) given by $S(a.1) := a^*.1$ for $a \in \mathcal{Q}$. The one parameter modular automorphism group (see [43]) say Θ_t, corresponding to the state h is given by $\Theta_t(a) = \Phi^{it}a\Phi^{-it}$. Note that here we have used the symbol Φ for the modular operator as Δ has been used for the coproduct. From (2), we see that

$$\Phi|_{L^2(h)^\pi_i} = F^\pi, \text{ for all } \pi \text{ and } i. \tag{1.4.2}$$

In particular Φ maps $L^2(h)^\pi_i$ into $L^2(h)^\pi_i$ for all i.

Let us discuss in some details a few facts about algebraic representation of $\mathcal{Q}$ on a vector space without any (apriori) topology i.e. $\Gamma : \mathcal{K} \to \mathcal{K} \otimes \mathcal{Q}_0$ and $(\Gamma \otimes \mathrm{id})\Gamma = (\mathrm{id} \otimes \Delta)\Gamma$, where $\mathcal{K}$ is some vector space. In this case the non degeneracy condition $Sp\ \Gamma(\mathcal{K})(1 \otimes \mathcal{Q}_0) = \mathcal{K} \otimes \mathcal{Q}_0$ is equivalent to the condition $\Gamma_\epsilon := (\mathrm{id} \otimes \epsilon)\Gamma$ is identity on $\mathcal{K}$. Then we have the following:

Proposition 1.4.11. *Given two algebraic non degenerate representations Γ_1, Γ_2 of $\mathcal{Q}$ on two vector spaces $\mathcal{K}$ and $\mathcal{L}$ respectively, we can define the tensor product and direct sum of the representations (to be denoted by $\Gamma_1 \otimes \Gamma_2$ and $\Gamma_1 \oplus \Gamma_2$ respectively) by $(\Gamma_1 \otimes \Gamma_2)(k \otimes l) := k_{(0)} \otimes l_{(0)} \otimes k_{(1)} l_{(1)})$ and $(\Gamma_1 \oplus \Gamma_2)(k, l) := (k_{(0)}, l_{(0)}) \otimes k_{(1)} + l_{(1)}$. Then $\Gamma_1 \otimes \Gamma_2$ and $\Gamma_1 \oplus \Gamma_2$ are algebraic non degenerate representations of $\mathcal{Q}$ on the vector spaces $\mathcal{K} \otimes \mathcal{L}$ and $\mathcal{K} \oplus \mathcal{L}$ respectively.*

Proof:
They are a consequence of the simple observations that $(\Gamma_1 \otimes \Gamma_2)_\epsilon = (\Gamma_{1\epsilon} \otimes \Gamma_{2\epsilon})$ and $(\Gamma_1 \oplus \Gamma_2)_\epsilon = (\Gamma_{1\epsilon} \oplus \Gamma_{2\epsilon})$. □

The subspace spanned by the matrix coefficients of inequivalent irreducible (hence finite dimensional) representations of a CQG $\mathcal{Q}$ is denoted by $\mathcal{Q}_0$. Firstly, $\mathcal{Q}_0$ is a subalgebra as the product of two matrix elements of finite dimensional unitary representations is a matrix element of the tensor product of these representations. Moreover, as the adjoint of a finite dimensional unitary representation is equivalent with a unitary representation, $\mathcal{Q}_0$ is $*$ invariant. For the definitions of tensor product of two representations and adjoint of a unitary representation the reader is again refereed to [35]. Below we state the analogue of Peter-Weyl theory for the representation theory of CQG. In the following h stands for the Haar state of $\mathcal{Q}$.

Proposition 1.4.12. *(1) $\mathcal{Q}_0$ is a dense $*$-subalgebra of $\mathcal{Q}$.*
(2) Let $\{U^\alpha : \alpha \in I\}$ be a complete set of mutually inequivalent, irreducible unitary representations. We will denote the representation space and dimension of U^α by $\mathcal{H}_\alpha$ and n_α respectively. Then the Schur's orthogonality relation takes the following form:

For any $\alpha \in I$, there is a positive invertible operator F^α acting on $\mathcal{H}_\alpha$ such that for any $\alpha, \beta \in I$ and $1 \leq j, q \leq d_\alpha$, $1 \leq i, p \leq d_\beta$

$$h(u^\alpha_{ij} u^{\beta^*}_{kl}) = \frac{1}{M_\alpha} \delta_{\alpha\beta} \delta_{ik} F_\alpha(j, l)$$

(3) $\{u^\alpha_{pq} : \alpha \in I, 1 \leq p, q \leq d_\alpha\}$ form a basis for $\mathcal{Q}_0$. (4) $\mathcal{Q}_0$ is a Hopf $*$-algebra with $\{\Delta(u^\alpha_{ij}) = \sum_{k=1}^{n_\alpha} u^\alpha_{ik} \otimes u^\alpha_{kj} : 1 \leq i, j \leq n_\alpha\}$ for all $\alpha \in I$, the counit ϵ and the antipode κ are defined on $\mathcal{Q}_0$ respectively by the formulae,

$$\epsilon(u^\alpha_{ij}) = \delta_{ij}, \kappa(u^\alpha_{pq}) = (u^\alpha_{qp})^*,$$

for all $\alpha \in I$ and $1 \leq p, q \leq n_\alpha$.

Here we would like to mention that corresponding to $\mathcal{Q}_0$ there might be several CQG's containing $\mathcal{Q}_0$ as a Hopf $*$-algebra and there is a universal compact quantum group containing $\mathcal{Q}_0$ as a Hopf $*$-algebra. It is denoted by $\mathcal{Q}^u$. It is obtained as the universal enveloping C^* algebra of $\mathcal{Q}_0$. We also say that a CQG $\mathcal{Q}$ is universal if $\mathcal{Q} = \mathcal{Q}^u$. For details the reader is referred to [31]. Examples (4) and (5) of CQG's are such examples of universal CQG's.

Now we discuss free product of CQG's which was developed in [53]. Let $(\mathcal{Q}_1, \Delta_1)$ and $(\mathcal{Q}_2, \Delta_2)$ be two CQG's and i_1,i_2 denote the canonical injections of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ respectively into the C^* algebra $\mathcal{Q}_1 * \mathcal{Q}_2$. Put $\rho_1 = (i_1 \otimes i_1)\Delta_1$ and $\rho_2 = (i_2 \otimes i_2)\Delta_2$. By the universal property of $\mathcal{Q}_1 * \mathcal{Q}_2$, there exists a map $\Delta : \mathcal{Q}_1 * \mathcal{Q}_2 \to (\mathcal{Q}_1 * \mathcal{Q}_2)\hat{\otimes}(\mathcal{Q}_1 * \mathcal{Q}_2)$ such that $\Delta i_1 = \rho_1$ and $\Delta i_2 = \rho_2$. It can indeed be shown that Δ has the required properties so that $(\mathcal{Q}_1 * \mathcal{Q}_2, \Delta)$ is a CQG.

Let $\{\mathcal{Q}_n\}_{n \in \mathbb{N}}$ be an inductive sequence of CQGs, where the connecting morphisms π_{mn} from $\mathcal{Q}_n$ to $\mathcal{Q}_m$ $(n < m)$ are injective morphisms of CQGs. Then from Proposition 3.1 of [53], we have that the inductive limit $\mathcal{Q}$ of $\mathcal{Q}_n$ s has a unique CQG structure with the following property: for any CQG $\mathcal{Q}^{'}$ and any family of CQG morphisms $\phi_n : \mathcal{Q}_n \to \mathcal{Q}^{'}$ such that $\phi_m \pi_{mn} = \phi_n$, the uniquely defined morphism $\lim_n \phi_n$ in the category of unital C^* algebras is a morphism in the category of CQGs as well.

Combining the above two results, it follows that the free product C^* algebra of an arbitrary sequence of CQGs has a natural CQG stucture. Moreover the following result was derived in [53].

Proposition 1.4.13. *Let* Γ_1,Γ_2 *be two discrete abelain groups. Then the natural isomorphisms* $C^*(\Gamma_1) \cong C(\hat{\Gamma_1})$ *and* $C^*(\Gamma_1) * C^*(\Gamma_2) \cong C^*(\Gamma_1 * \Gamma_2)$ *are isomorphisms of CQG's*

Let i_1,i_2 be the canonical injections of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ respectively into the C^* algebra $\mathcal{Q}_1 * \mathcal{Q}_2$. If U_1 and U_2 are unitary representations of CQGs $\mathcal{Q}_1$ and $\mathcal{Q}_2$ on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, then the free product representation of U_1 and U_2 is a representation of the CQG $\mathcal{Q}_1 * \mathcal{Q}_2$ on the Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_2$ given by the $\mathcal{Q}_1 * \mathcal{Q}_2$ valued matrix

$$\begin{pmatrix} (\mathrm{id} \otimes i_1)U_1 & 0 \\ 0 & (\mathrm{id} \otimes i_2)U_2. \end{pmatrix}$$

Similarly, the free product representation of an arbitrary sequence of CQG representation can be defined.

1.4.3 Hopf $*$-algebra of matrix coefficients: Dual of a CQG

Let $\mathcal{A}$ be a finite dimensional Hopf $*$-algebra, then the algebraic dual $\mathcal{U}$ of $\mathcal{A}$ is again a unital $*$-algebra with the product (called the convolution and we denote it by $*$) and involution defined by

$$\omega * \nu := (\omega \otimes \nu)\Delta, \omega^* = \bar{\omega}\kappa,$$

with ϵ as the unit, where $\bar{\omega}$ is defined by $\bar{\omega}(a) = \overline{\omega(a^*)}$ for $a \in \mathcal{A}$. Moreover it is a Hopf $*$-algebra with coproduct

$$\hat{\Delta}(\omega)(a \otimes b) = \omega(ab) \ for \ a, b \in \mathcal{A}.$$

The antipode is given by $\hat{\kappa}(\omega) = \omega\kappa$ and the counit by $\hat{\epsilon} = \omega(1)$. The Hopf $*$-algebra $(\mathcal{U}, \hat{\Delta})$ is called the dual of $(\mathcal{A}, \Delta)$.

However we see that the Hopf $*$-algebra $\mathcal{Q}_0$ associated to a CQG $\mathcal{Q}$ is no longer finite dimensional. When the Hopf $*$-algebra is not finite dimensional, then by the prescription above we do not get a Hopf $*$-algebra, but we do get what is called a multiplier Hopf $*$-algebra which is very similar to a Hopf $*$-algebra. Although the general dual of an infinite dimensional Hopf $*$-algebra can be considered, for our purpose we will consider only the dual of $\mathcal{Q}_0$ in this thesis. We denote the dual of $\mathcal{Q}_0$ by $\hat{\mathcal{Q}_0}$.

For every finite dimensional representation U of $\mathcal{Q}$, we can define a representation π_U of $\hat{\mathcal{Q}_0}$ on $\mathcal{H}$ by $\pi_U(\omega) = (\mathrm{id} \otimes \omega)U$. It is a $*$-representation if U is unitary. Fix a representative U^α of an inequivalent irreducible representation of $\mathcal{Q}$. Then it follows that the homomorphisms π_{U^α} define a $*$-isomorphism

$$\hat{\mathcal{Q}_0} \cong \Pi_{\alpha \in I} \mathcal{B}(\mathcal{H}_\alpha).$$

The dual of $(\widehat{\mathcal{Q}_0 \otimes \mathcal{Q}_0})$ is isomorphic to $\Pi_{\alpha,\beta \in I} \mathcal{B}(\mathcal{H}_\alpha \otimes \mathcal{H}_\beta)$. Define a map $\hat{\Delta} : \hat{\mathcal{Q}_0} \to (\widehat{\mathcal{Q}_0 \otimes \mathcal{Q}_0})$ by $\hat{\Delta(\omega)}(a \otimes b) := \omega(ab)$ for $a, b \in \mathcal{Q}_0$. Note that in general the image of $\hat{\Delta}$ does not lie in the algebraic tensor product $\hat{\mathcal{Q}_0} \otimes \hat{\mathcal{Q}_0}$. But it is contained in $\mathcal{M}(\hat{\mathcal{Q}_0} \otimes \hat{\mathcal{Q}_0})$. Defining $\hat{\epsilon}(\omega) := \omega(1)$ and $\hat{\kappa}(\omega) = \omega\kappa$ we can verify that the pair $(\hat{\mathcal{Q}_0}, \hat{\Delta})$ satisfies the axioms of what is called a multiplier Hopf $*$-algebra. Since $\hat{\mathcal{Q}_0}$ is nothing but a direct sum of matrix algebras, we can give a unique C^* norm, we can complete it with respect to that C^* norm. We denote the corresponding completion by $\hat{\mathcal{Q}}$ and $(\hat{\mathcal{Q}}, \hat{\Delta})$ is called the dual discrete quantum group (on the C^* level). For details about discrete quantum group see section 8 of [35]. We record the following proposition which will be useful later.

Proposition 1.4.14. *In the following we denote the matrix coefficients of* $\mathcal{Q}$ *by* $\{q^\alpha_{ij} : 1 \leq i, j \leq n_\alpha, \alpha \in I$. *Let* U *be a unitary representation of a CQG* $\mathcal{Q}$ *on a Hilbert space* $\mathcal{H}$. *Then* $\Pi_U : \hat{\mathcal{Q}} \to \mathcal{B}(\mathcal{H})$ *defined by* $\Pi_U(\omega)(h) := (\mathrm{id} \otimes \omega)U(h)$ *is a non degenerate* $*$*-homomorphism and hence extends as a* $*$*-homomorphism from* $\mathcal{M}(\hat{\mathcal{Q}})$ *to* $\mathcal{B}(\mathcal{H})$.

Proof:
Consider the spectral decomposition $\mathcal{H} = \oplus_{\pi \in \mathcal{I}, 1 \leq i \leq m_\pi} \mathcal{H}^\pi_i$, $U|_{\mathcal{H}^\pi_i}$, $i = 1, ..., m_\pi$ is equivalent to the irreducible representation of type π. Moreover fix orthonormal ba-

sis $e^{\pi}_{ij}, j = 1, ..., d_\pi, i = 1, ..., m_\pi$ for $\mathcal{H}^{\pi}_i$ such that

$$U(e^{\pi}_{ij}) = \sum_k e^{\pi}_{ik} \otimes q^{\pi}_{kj}$$

for all $\pi \in Rep(\mathcal{Q})$. Now for a fixed $\pi \in Rep(\mathcal{Q})$, $p, r = 1, ..., d_\pi$ observe that $\Pi_U(\rho^{\pi}_{pr})(\xi) = 0$ for all $\xi \in \mathcal{H}^{\pi'}_i$ and for $\pi \neq \pi'$. Also $\Pi_U(\rho^{\pi}_{pr})(e^{\pi}_{ij}) = \delta_{jr} e^{\pi}_{ip}$, i.e. $\Pi_U(\rho^{\pi}_{pr})|_{\mathcal{H}^{\pi}_i}$ is nothing but the rank one operator $|e^{\pi}_{ip} >< e^{\pi}_{ir}|$. This proves that $\Pi_U(\omega)$ is bounded for $\omega \in \hat{\mathcal{Q}}_0$, and moreover identifying $\hat{\mathcal{Q}}_0$ with the direct sum of matrix algebras $\oplus_{\pi \in Rep(\mathcal{Q})} M_{d_\pi}$, we see that Π_U is nothing but the map which sends $X \in M_{d_\pi}$ to $X \otimes 1_{\mathbb{C}^{m_\pi}}$ in $\mathcal{B}(\mathcal{H})$. This proves that Π_U extends to a non-degenerate $*$-homomorphism. □

We return to the Hop $*$-algebra $\mathcal{Q}_0$ of a compact quantum group $(\mathcal{Q}, \Delta)$. In general the counit ϵ is not bounded on $\mathcal{Q}_0$ and can not be extended to the whole of $\mathcal{Q}$, so is the antipode κ. Now let $U : \mathcal{H} \to \mathcal{H} \bar{\otimes} \mathcal{Q}$ be a unitary representation of $\mathcal{Q}$ on $\mathcal{H}$. U decomposes into finite dimensional irreducible representations and $\mathcal{H}$ decomposes into finite dimensional Hilbert spaces $\mathcal{H}_\alpha$. Let the irreducible representations be $\{U^\alpha : \alpha \in I\}$ and the corresponding Hilbert spaces be $\mathcal{H}_\alpha$. Let $\{e^{\alpha}_i : 1 \leq i \leq n_\alpha\}$ (where dim $\mathcal{H}_\alpha = n_\alpha$) be orthonormal basis for $\mathcal{H}_\alpha$. If we denote $Sp\ \{e^{\alpha}_i : 1 \leq i \leq n_\alpha, \alpha \in I\}$ by $\mathcal{H}_0$ then from the condition (3) of a unitary representation, we can deduce that $\mathcal{H}_0$ is dense in the Hilbert space $\mathcal{H}$ and $U(\mathcal{H}_0) \subset \mathcal{H}_0 \otimes \mathcal{Q}_0$ such that $Sp\ \{U(\mathcal{H}_0)\mathcal{Q}_0\} = \mathcal{H}_0 \otimes \mathcal{Q}_0$.

1.5 Noncommutative Geometry

In this section we recall some basic notions of noncommutative geometry. For a detailed discussion, the reader is referred to [16].

1.5.1 Spectral triples

Motivated by the facts in Proposition 1.1.21, Alain Connes defined his formulation of noncommutative manifold, based on the idea of a spectral triple:

Definition 1.5.1. *A spectral triple is a triple $(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ where $\mathcal{H}$ is a separable Hilbert space, $\mathcal{A}^\infty$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, (not necessarily norm closed) and $\mathcal{D}$ is a self adjoint (typically unbounded) operator such that for all $a \in \mathcal{A}^\infty$, the operator $[\mathcal{D}, a]$ has a bounded extension. Such spectral triple is also called an odd spectral triple. If in addition, we have $\gamma \in \mathcal{B}(\mathcal{H})$ satisfying $\gamma = \gamma^* = \gamma^{-1}$, $\mathcal{D}\gamma = -\gamma\mathcal{D}$ and $[a, \gamma] = 0$ for all $a \in \mathcal{A}^\infty$, then we say the quadruplet $(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D}, \gamma)$ is an even spectral triple. The operator $\mathcal{D}$ is called the Dirac operator corresponding to the spectral triple.*

Since in the classical case, the Dirac operator has compact resolvent if the manifold is compact, we say the spectral triple is of compact type if $\mathcal{A}^\infty$ is unital and $\mathcal{D}$ has a compact resolvent.

Examples

1. Let M be a smooth spin manifold. Then from proposition 1.1.21, we see that $(C^\infty(M), L^2(S), D)$ is a spectral triple. It is of compact type when the manifold is compact.

We recall that when the dimension of the manifold is even, $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$. An L^2 section s has a decomposition $s = s_1 + s_2$, where $s_1(m), s_2(m)$ belong to $\Delta_n^+(m)$ and $\Delta_n^-(m)$ respectively where $\Delta_n^{+-}(m)$ denotes the subspace of the fibre over m. This decomposition of $L^2(S)$ induces a grading operator γ on $L^2(S)$. It can be seen that D anticommutes with γ.

2. This example comes from the classical Hilbert space of forms . One considers the self adjoint extension of the operator $d + d^*$ on $\mathcal{H} = \oplus_k \mathcal{H}^k(M)$ which is again denoted by $d + d^*$. $C^\infty(M)$ has a representation on each $\mathcal{H}^k(M)$, which gives a representation say π on $\mathcal{H}$. Then it can be shown that $(C^\infty(M), \mathcal{H}, d + d^*)$ is a spectral triple. The operator $d + d^*$ is called the Hodge Dirac operator.

3. **The noncommutative torus**

We recall the noncommutative 2-torus $\mathcal{A}_\theta$ is the universal C^*-algebra generated by two unitaries U and V satisfying $UV = e^{2\pi i\theta} VU$ where θ is a number in $[0, 1]$.

There are two derivations d_1 and d_2 on $\mathcal{A}_\theta$ obtained by extending linearly the rule:

$$d_1(U) = U, d_2(V) = 0$$

$$d_1(U) = 0, d_2(V) = 0.$$

Then d_1, d_2 are well defined on the dense $*$-algebra of $\mathcal{A}^\infty$:

$$\mathcal{A}_\theta^\infty = \{ \sum_{m,n \in \mathbb{Z}} a_{mn} U^m V^n : sup_{m,n} |m^k n^l a_{mn}| < \infty \ for \ all \ k, l \ in \mathbb{N} \}.$$

There is a unique faithful trace on $\mathcal{A}_\theta$ defined as follows:

$$\tau(\sum a_{mn} U^m V^n) = a_{00}.$$

Let $\mathcal{H} = L^2(\tau) \oplus L^2(\tau)$ where $L^2(\tau)$ denotes the GNS Hilbert space of $\mathcal{A}_\theta$ with respect to the state τ. We note that $\mathcal{A}_\theta^\infty$ is embedded as a subalgebra of $\mathcal{B}(\mathcal{H})$ by $a \rightarrow diag(a, a)$.

We now define $\mathcal{D}$ as

$$\begin{pmatrix} 0 & d_1 + id_2 \\ d_1 - id_2 & 0. \end{pmatrix}$$

Then $(\mathcal{A}_\theta^\infty, \mathcal{H}, \mathcal{D})$ is a spectral triple of compact type.

1.5.2 The space of forms in noncommutative geometry

We start this subsection by recalling the universal space of one forms corresponding to an algebra.

Proposition 1.5.2. *Given an algebra $\mathcal{B}$, there is a (unique upto isomorphism) $\mathcal{B}-\mathcal{B}$ bimodule $\Omega^1(\mathcal{B})_u$ and a derivation $\delta : \mathcal{B} \to \Omega^1(\mathcal{B})_u$ (i.e. $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{B}$), satisfying the following properties:*

(i) $\Omega^1(\mathcal{B})_u$ is spanned as a vector space by elements of the form $a\delta(b)$ with $a, b \in \mathcal{B}$.

(ii) For any $\mathcal{B}-\mathcal{B}$ bimodule E and a derivation $d : \mathcal{B} \to E$, there is a unique $\mathcal{B}-\mathcal{B}$ linear map $\eta : \Omega^1(\mathcal{B}) \to E$ such that $d = \eta\delta$.

The bimodule $\Omega^1(\mathcal{B})$ is called the space of universal one forms on $\mathcal{B}$ and the δ is called the universal derivation. We can also introduce universal space of higher forms on $\mathcal{B}$, $\Omega^k(\mathcal{B})$ for $k = 2, 3, ..$ by defining them recursively as follows: $\Omega^{k+1}(\mathcal{B}) = \Omega^k(\mathcal{B}) \otimes_{\mathcal{B}} \Omega^1(\mathcal{B})$ and also set $\Omega^0(\mathcal{B}) = \mathcal{B}$.

Now we briefly discuss the notion of the noncommutative Hilbert space of forms which will need noncommutative volume form for a spectral triple of compact type. We refer to [20] (page 124 -127) and the references therein for more details.

Definition 1.5.3. *A spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ of compact type is said to be Θ-summable if e^{-tD^2} is of trace class for all $t > 0$. A Θ-summable spectral triple is called finitely summable when there is some $p > 0$ such that $t^{\frac{p}{2}}\ \mathrm{Tr}(e^{-tD^2})$ is bounded on $(0, \delta]$ for some $\delta > 0$. The infimum of all such p, say p' is called the dimension of the spectral triple and the spectral triple is called p'-summable.*

Remark 1.5.4. *We remark that the definition of Θ-summability to be used in this thesis is stronger than the one in [16] (page 390, definition 1.) in which a spectral triple is called Θ-summable if $\mathrm{Tr}(e^{-tD^2}) < \infty$.*

For a Θ-summable spectral triple, let $\sigma_\lambda(T) = \frac{\mathrm{Tr}(Te^{-\frac{1}{\lambda}D^2})}{\mathrm{Tr}(e^{-\frac{1}{\lambda}D^2})}$ for $\lambda > 0$. We note that $\lambda \mapsto \sigma_\lambda(T)$ is bounded.

Let

$$\tau_\lambda(T) = \frac{1}{\log \lambda} \int_a^\lambda \sigma_u(T) \frac{du}{u} \text{for } \lambda \geq a \geq e.$$

For any state ω on the C^* algebra B_∞, $Tr_\omega(T) = \omega(\tau(T))$ for all T in $\mathcal{B}(\mathcal{H})$ defines a functional on $\mathcal{B}(\mathcal{H})$. As we are not going to need the choice of ω in this thesis, we will suppress the suffix ω and simply write $\mathrm{Lim}_{t\to 0^+} \frac{\mathrm{Tr}(Te^{-tD^2})}{\mathrm{Tr}(e^{-tD^2})}$ for $Tr_\omega(T)$. This is a kind of Banach limit because if $\lim_{t\to 0^+} \frac{\mathrm{Tr}(Te^{-tD^2})}{\mathrm{Tr}(e^{-tD^2})}$ exists, then it agrees with the functional $\mathrm{Lim}_{t\to 0^+}$. Moreover, $\mathrm{Tr}_\omega(T)$ coincides (upto a constant) with the Dixmier trace (see chapter IV, [16]) of the operator $T|D|^{-p}$ when the spectral triple has a finite dimension $p > 0$, where $|D|^{-p}$ is to be interpreted as the inverse of the restriction of $|D|^p$ on the closure of its range. In particular, this functional gives back the volume form for the classical spectral triple on a compact Riemannian manifold.

Let $\Omega^k(\mathcal{A}^\infty)$ be the space of universal k-forms on the algebra $\mathcal{A}^\infty$ which is spanned by $a_0\delta(a_1)\cdots\delta(a_k)$, a_i belonging to $\mathcal{A}^\infty$, where δ is as in Proposition 1.5.2. There is a natural graded algebra structure on $\Omega \equiv \bigoplus_{k\geq 0}\Omega^k(\mathcal{A}^\infty)$, which also has a natural involution given by $(\delta(a))^* = -\delta(a^*)$, and using the spectral triple, we get a $*$-representation $\Pi : \Omega \to \mathcal{B}(\mathcal{H})$ which sends $a_0\delta(a_1)\cdots\delta(a_k)$ to $a_0 d_D(a_1)\cdots d_D(a_k)$, where $d_D(a) = [D, a]$. Consider the state τ on $\mathcal{B}(\mathcal{H})$ given by, $\tau(X) = \mathrm{Lim}_{t\to 0^+} \frac{\mathrm{Tr}(Xe^{-tD^2})}{\mathrm{Tr}(e^{-tD^2})}$, where Lim is as above. Using τ, we define a positive semi definite sesquilinear form on $\Omega^k(\mathcal{A}^\infty)$ by setting $\langle w, \eta\rangle = \tau(\Pi(w)^*\Pi(\eta))$. Let $K^k = \{w \in \Omega^k(\mathcal{A}^\infty) : \langle w,\ w\rangle = 0\}$, for $k \geq 0$, and $K^{-1} := (0)$. Let $\overline{\Omega^k_D}$ be the Hilbert space obtained by completing the quotient $\Omega^k(\mathcal{A}^\infty)/K^k$ with respect to the inner product mentioned above, and we define $\mathcal{H}^k_D := P_k^\perp\overline{\Omega^k_D}$, where P_k denotes the projection onto the closed subspace generated by $\delta(K^{k-1})$. Clearly, $\mathcal{H}^k_D$ has a total set consisting of elements of the form $[a_0\delta(a_1)\cdots\delta(a_k)]$, with a_i in $\mathcal{A}^\infty$ and $[\omega]$ denoting the equivalence class $P_k^\perp(w + K^k)$ for ω belonging to $\Omega^k(\mathcal{A}^\infty)$. Then we can extend the map d_D to the total subspace of $\mathcal{H}^k_D$ by $d_D[a_0\delta(a_1)\cdots\delta(a_k)] := [\delta(a_0)\delta(a_1)\cdots\delta(a_k)]$ and hence it maps into $\mathcal{H}^{k+1}_D$. Then d_D is a densely defined unbounded operator on $\mathcal{H}_{d+d^*} := \bigoplus_{k\geq 0}\mathcal{H}^k_D$. If $\mathrm{dom}(d_D^*)$ contains $\Omega^k_D(\mathcal{A}^\infty)$ for all k, we can consider the operator $D' := d + d^* \equiv d_D + d_D^*$ as a closable densely defined operator. Assume it has a self adjoint extension. Then we denote the extension again by $d + d^*$. There is a $*$-representation $\pi_{d+d^*} : \mathcal{A} \to \mathcal{B}(\mathcal{H}_{d+d^*})$, given by $\pi_{d+d^*}(a)([a_0\delta(a_1)\cdots\delta(a_k)]) = [aa_0\delta(a_1)\cdots\delta(a_k)]$. Then it is easy to see that

Proposition 1.5.5. *Under the above assumptions,* $(\mathcal{A}^\infty, \mathcal{H}_{d+d^*}, d + d^*)$ *is a spectral triple.*

We reamark that the assumptions about the domain and closability of d_D^* and $d_D + d_D^*$ on $\mathcal{H}_{d+d^*}$ are valid for a large class of spectral triples including all classical ones and their Rieffel deformations. Moreover, we shall see in the next subsection that it is easy to prove closability of at least $d_D|_{\mathcal{H}^0_D}$ under mild condition.

1.5.3 Laplacian in Noncommutative geometry

Now, we discuss the notion of Laplacian in noncommutative geometry as introduced in [22]. Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a spectral triple. Consider the operator $d_D : \mathcal{H}_D^0 \to \mathcal{H}_D^1$. Then the operator $-d_D^* d_D : \mathcal{H}_D^0 \to \mathcal{H}_D^0$ is the natural candidate for the noncommutative Laplacian. However to have a reasonable definition, one at least needs d_D to be closable. We now give some natural sufficient condition for closability of d_D. Let us consider the locally convex space $\mathcal{B}(\mathcal{H})$ with its ultra-weak topology. Then on $\mathcal{B}(\mathcal{H})$, $d_D(.) := [D, .]$ is an unbounded derivation. The strongly continuous one parameter group generated by d_D is given by $\sigma_t(X) = \exp(itD)(X)\exp(-itD)$ for $X \in \mathcal{B}(\mathcal{H})$. The following result is proved in [22] (Lemma 2.6).

Lemma 1.5.6. *Suppose that for every element $a \in \mathcal{A}^\infty$, the map $t(\in \mathbb{R}) \mapsto \sigma_t(X) := \exp(itD)X\exp(-itD)$ is differentiable at $t = 0$ in the norm topology of $\mathcal{B}(\mathcal{H})$, where $X = a$ or $X = [D, a]$. Then d_D is closable. In this case, the densely defined unbounded operator $d_D^* d_D$ maps $\mathcal{A}^\infty$ into its weak closure in $\mathcal{B}(\mathcal{H})$.*

Now we state and prove a sufficient condition for the norm differentiability condition in the previous Lemma.

Lemma 1.5.7. *Suppose that $a \in \mathcal{A}^\infty$ belongs to the domain of repeated commutator d_D^n given by $d_D^n(a) := [D, [D, [...[D, a]]]...]$ for $n = 1, 2, 3$. Then the condition of the Lemma 1.5.6 is satisfied. In particualr d_D is closable.*

Proof:
Since $a \in \mathrm{dom}(d_D)$, $\frac{1}{t}(\sigma_t(a) - a) - [D, a] = \frac{1}{t}\int_0^t (\sigma_s'(a) - [D, a])ds$. Since $a \in \mathrm{dom}(d_D)$, $\sigma_s'(a) = \sigma_s([D, a])$. Also using the fact that $a \in \mathrm{dom} d_D^2$, we get

$$\begin{aligned}
& (\sigma_s([D, a]) - [D, a]) \\
= & \int_0^s \sigma_u'([D, a])du \\
= & \int_0^s \sigma_u([D, [D, a]])du
\end{aligned}$$

Using the face that $\{\sigma_t\}$ is contractive, we get the estimate $||\frac{1}{t}(\sigma_t(a) - a) - [D, a]|| \leq \frac{t}{2}||[D, [D, a]]||$ establishing that as $t \to 0$, $\frac{1}{t}(\sigma_t(a) - a)$ goes to $[D, a]$ in norm topology. Hence it proves the norm differentiability of $\sigma_t(a)$ for all $a \in \mathcal{A}^\infty$. Similarly using the higher domain condition we can prove the norm differentiability of $\sigma_t(X)$ for $X = [D, a]$. □

Now we impose some regularity conditions on the spectral triple so that we can define a noncommutative Laplacian.

Assumptions

1. $(\mathcal{A}^\infty, \mathcal{H}, D)$ is a compact type, p-summable (for some $p > 0$) spectral triple.

2. $a \in \mathcal{A}^\infty$ belongs to the domains of repeated commutator d_D^n, where $d_D(X) := [D, X]$ is considered as an unbounded derivation on the locally convex space $\mathcal{B}(\mathcal{H})$ with its ultra-weak topology. Also assume that it is QC^∞, that is, $\mathcal{A}^\infty$ and $\{[D,a] : a \in \mathcal{A}^\infty\}$ is contained in the domains of all powers of the derivation $[|D|, \cdot]$.

We can adapt the proof of Proposition 3.4 of [23] (with $R = I$) to deduce

Lemma 1.5.8. *τ defined by $\tau(X) = \mathrm{Lim}_{t \to 0} \frac{\mathrm{Tr}(Xe^{-tD^2})}{\mathrm{Tr}(e^{-tD^2})}$ is a positive trace on the C^*-subalgebra generated by $\mathcal{A}^\infty$ and $\{[D,a] : \ a \in \mathcal{A}^\infty\}$.*

We further assume that

3. τ is faithful on the C^* subalgebra generated by $\mathcal{A}^\infty$ and $\{[D,a] : \ a \in \mathcal{A}^\infty\}$.

Now we see that the assumption (**2**) on the spectral triple implies that it satisfies the condition of the Lemma 1.5.7. Hence in particular d_D is closable.

Definition 1.5.9. *Under the stated assumptions on the spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ the non commutative Laplacian $\mathcal{L}$ is defined to be the densely defined operator $-d_D^* d_D$. It maps $\mathcal{A}^\infty$ into $(\mathcal{A}^\infty)''$.*

Chapter 2

Quantum isometry groups

There are two approaches in the formulation of quantum isometry groups of a spectral triple. One is based on a Laplacian on a spectral triple (developed by Goswami in [22]) and the other is based on the Dirac operator of the spectral triple (developed by Goswami and Bhowmick in [10]). Before discussing these two notions we shall first recall what is meant by a C^* action of a CQG on a C^* algebra.

2.1 Action of a compact quantum group on a C^* algebra

Definition 2.1.1. *We say that the compact quantum group $(\mathcal{Q}, \Delta)$ (co)-acts on a unital C^* algebra $\mathcal{B}$, if there is a unital C^*-homomorphism (called an action) $\alpha : \mathcal{B} \to \mathcal{B}\hat{\otimes}\mathcal{Q}$ satisfying the following :*
(i) $(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \Delta) \circ \alpha$, *and*
(ii) the linear span of $\alpha(\mathcal{B})(1 \otimes \mathcal{Q})$ is norm-dense in $\mathcal{B}\hat{\otimes}\mathcal{Q}$.

It is known (see, for example [42]) that (ii) is equivalent to the existence of a norm-dense, unital $*$-subalgebra B of $\mathcal{B}$ such that $\alpha(B) \subseteq B \otimes \mathcal{Q}_0$ and on B, $(\mathrm{id} \ \otimes \epsilon) \circ \alpha = \mathrm{id}$ i.e. α is a Hopf $*$-algebraic action of $\mathcal{Q}_0$ on B in the sense of Definition 1.4.2. Given a C^* action α of a CQG $\mathcal{Q}$ on a C^* algebra $\mathcal{B}_0$ of $\mathcal{B}$ which is the maximal $*$-subalgebra of $\mathcal{B}$ over which the action α is algebraic i.e. $\mathcal{B}_0 = \{b \in \mathcal{B} | \alpha(b) \subset \mathcal{B} \otimes \mathcal{Q}_0\}$. From now on given any action of a CQG $\mathcal{Q}$ on a C^* algebra $\mathcal{B}$, by $\mathcal{B}_0$ we shall always mean this maximal $*$-subalgebra over which the action is algebraic. Later when we shall extend the notion of action on Fréchet algebras, we shall use the same notation. Note that this maximal subalgebras are dense in the corresponding topologies.

Proposition 2.1.2. *(i) For any $b \in \mathcal{B}_0$, $\alpha(b) \subset B \otimes \mathcal{Q}_0$.*
(ii) $\mathcal{B}_0 = B \oplus ker(\alpha)$.

Proof:
Recall from 1.4.12, $\{u^{\beta}_{ij} : \beta \in \hat{\mathcal{Q}} : i, j = 1, ..., d_\beta\}$ is a basis for $\mathcal{Q}_0$. Hence we can write

$\alpha(b)$ in terms of this basis as $\alpha(b) = \sum_{\beta\in\mathcal{J}}\sum_{i,j=1}^{d_\beta} b_{ij}^\beta \otimes u_{ij}^\beta$, where $\mathcal{J}$ is a finite subset of $\hat{\mathcal{Q}}$. Using the coassociativity of α, we get

$$\sum_{\beta\in\mathcal{J}}\sum_{i,j=1}^{d_\beta} \alpha(b_{ij}^\beta)\otimes u_{ij}^\beta = \sum_{\beta\in\mathcal{J}}\sum_{i,j,s=1}^{d_\beta} b_{ij}^\beta \otimes u_{is}^\beta \otimes u_{sj}^\beta.$$

Now recall the linear functionals ρ_{kl}^γ from chapter 1. Applying $(\mathrm{id}\otimes\mathrm{id}\otimes\rho_{kl}^\gamma)$ for some $\gamma\in\hat{\mathcal{Q}}$, to both sides of the above identity, we get

$$\alpha(b_{kl}^\gamma) = \sum_{i=1}^{d_\gamma} b_{il}^\gamma \otimes u_{ik}^\gamma.$$

This means $\alpha(b)\subset \mathcal{B}_0\otimes\mathcal{Q}_0$.

For the second statement, write $\alpha(b) = \sum_{\beta\in\mathcal{J}}\sum_{i,j=1}^{d_\beta} b_{ij}^\beta\otimes u_{ij}^\beta$. If we take $b' = \sum_{\beta\in\mathcal{J}}\sum_{i=1}^{d_\beta} b_{ii}^\beta$,

$$\alpha(b') = \sum_{\beta\in\mathcal{J}}\sum_{i,k=1}^{d_\beta} b_{ki}^\beta\otimes u_{ki}^\beta.$$

It follows that $\alpha(b)=\alpha(b')$ i.e. $(b-b')\in ker(\alpha)$ and $b'\in B$. □

Definition 2.1.3. *Let $(\mathcal{Q},\alpha)$ has a C^* action α on the C^* algebra $\mathcal{B}$. We say that the action α is* **faithful** *if there is no proper Woronowicz C^*-subalgebra $\mathcal{Q}_1$ of $\mathcal{Q}$ such that α is a C^* action of $\mathcal{Q}_1$ on $\mathcal{B}$.*

Definition 2.1.4. *Let $(\mathcal{Q},\alpha)$ has a C^* action α on the C^* algebra $\mathcal{B}$. A continuous linear functional ϕ on $\mathcal{B}$ is said to be* **invariant under** *α if*

$$(\phi\otimes\mathrm{id})\alpha(b) = \phi(b).1_{\mathcal{Q}}.$$

Now, we recall the work of Shuzhou Wang done in [55]. One can also see [5], [7].

Recall from chapter 1, the **quantum permutation group** $A_s(n)$ which is defined to be the C^* algebra generated by a_{ij} ($i,j=1,2,...,n$) satisfying the following relations:

$$a_{ij}^2 = a_{ij} = a_{ij}^*,\ i,j=1,2,...n,$$

$$\sum_{j=1}^{n} a_{ij} = 1,\ i=1,2,...n,$$

$$\sum_{i=1}^{n} a_{ij} = 1,\ i=1,2,...n,$$

with the coproduct on the generators is given by $\Delta(a_{ij}) = \sum_{k=1}^n a_{ik}\otimes a_{kj}$. Also the antipode κ on the canonical dense Hopf-$*$ algebra is given by $\kappa(a_{ij}) = a_{ji}$, whereas the counit is given by $\epsilon(a_{ij}) = \delta_{ij}$. The name comes from the fact that the universal commutative C^* algebra generated by the above set of relations is isomorphic to $C(S_n)$ where S_n denotes the permutation group on n symbols.

Let us consider the category with objects as compact groups acting on on a n-point set $X_n = \{x_1, x_2, ..., x_n\}$. If two groups G_1 and G_2 have actions α_1 and α_2 respectively, then a morphism from G_1 to G_2 is a group homomorphism ϕ such that $\alpha_2(\phi \times \mathrm{id}) = \alpha_1$. Then $C(S_n)$ is the universal object in this category. It is proved in [55] that the quantum permutation group enjoys a similar property.

We have that $C(X_n) = C^*\{e_i : e_i^2 = e_i = e_i^*,\ \sum_{r=1}^{n} e_r = 1, i = 1, 2, ..., n\}$. Then $\mathcal{QU}_n$ has a C^* action on $C(X_n)$ via the formula:

$$\alpha(e_j) = \sum_{i=1}^{n} e_i \otimes a_{ij}, j = 1, 2, ...n.$$

Proposition 2.1.5. *Consider the category with objects as CQG s having a C^* action on $C(X_n)$ and morphisms as CQG morphisms intertwining the actions as above. Then $A_s(n)$ is the universal object in this category.*

Now we note down a simple fact for future use.

Lemma 2.1.6. *Let α be an action of a CQG $\mathcal{S}$ on $C(X)$ where X is a finite set. Then α automatically preserves the functional τ corresponding to the counting measure:*

$$(\tau \otimes \mathrm{id})(\alpha(f)) = \tau(f).1_{\mathcal{S}}.$$

Proof:
Let $X = \{1, ..., n\}$ for some $n \in I\!N$ and denote by δ_i the characteristic function of the point i. Let $\alpha(\delta_i) = \sum_j \delta_j \otimes q_{ij}$ where $\{q_{ij} : i, j = 1 \dots n\}$ are the images of the canonical generators of the quantum permutation group as above. Then τ-preservation of α follows from the properties of the generators of the quantum permutation group, which in particular imply that $\sum_j q_{ij} = 1 = \sum_i q_{ij}$. □

Wang also identified the universal object in the category of all CQG s having a C^* action α_1 on $M_n(\mathbb{C})$ (with morphisms as before) such that the functional $\frac{1}{n}\mathrm{Tr}$ is kept invariant under α_1. However, no such universal object exists if the invariance of the functional is not assumed. The precise statement is contained in the following theorem.

Before that, we recall that $M_n(\mathbb{C}) = C^*\{e_{ij} : e_{ij}e_{kl} = \delta_{jk}e_{il},\ e_{ij}^* = e_{ji},\ \sum_{r=1}^{n} e_{rr} = 1, i, j, k, l = 1, 2, ...n\}$.

Proposition 2.1.7. *Let $\mathcal{QU}_{M_n(\mathbb{C}),\frac{1}{n}\mathrm{Tr}}$ be the C^* algebra with generators a_{ij}^{kl} and the following defining relations:*

$$\sum_{v=1}^{n} a_{ij}^{kv} a_{rs}^{vl} = \delta_{jr} a_{is}^{kl},\ \ i,j,k,l,r,s = 1,2,...,n,$$

$$\sum_{v=1}^{n} a_{lv}^{sr} a_{vk}^{ji} = \delta_{jr} a_{lk}^{si}, i,j,k,l,r,s = 1,2,...,n,$$

$$a_{ij}^{kl*} = a_{ji}^{lk},\ \ i,j,k,l = 1,2,...,n,$$

$$\sum_{r=1}^{n} a_{rr}^{kl} = \delta_{kl},\ \ k,l = 1,2,...,n,$$

$$\sum_{r=1}^{n} a_{kl}^{rr} = \delta_{kl},\ \ k,l = 1,...,n.$$

Then,

(1) $\mathcal{QU}_{M_n(\mathbb{C}),\frac{1}{n}\mathrm{Tr}}$ is a CQG with coproduct Δ defined by $\Delta(a_{ij}^{kl}) = \sum_{r,s=1}^{n} a_{rs}^{kl} \otimes a_{ij}^{rs},\ i,j,k,l = 1,2,...,n.$

(2) $\mathcal{QU}_{M_n(\mathbb{C}),\frac{1}{n}\mathrm{Tr}}$ has a C^* action α_1 on $M_n(\mathbb{C})$ given by $\alpha_1(e_{ij}) = \sum_{k,l=1}^{n} e_{kl} \otimes a_{ij}^{kl},\ i,j = 1,2,...,n$. Moreover, $\mathcal{QU}_{M_n(\mathbb{C}),\frac{1}{n}\mathrm{Tr}}$ is the universal object in the category of all CQG s having C^* action on $M_n(\mathbb{C})$ such that the functional $\frac{1}{n}\mathrm{Tr}$ is kept invariant under the action.

(3) There does not exist any universal object in the category of all CQG s having C^* action on $M_n(\mathbb{C})$.

Proposition 2.1.8. *Since, any faithful state on a finite dimensional C^* algebra $\mathcal{A}$ is of the form $\mathrm{Tr}(Rx)$ for some operator R, it follows from Theorem 6.1, (2) of [55] that the universal CQG acting on any finite dimensional C^* algebra preserving a faithful state ϕ exists and is going to be denoted by $\mathcal{QU}_{\mathcal{A},\phi}$.*

Now let X be a compact, Hausdorff space. Then $C(X)$ is a C^* algebra. We say a CQG $\mathcal{Q}$ acts continuously and faithfully on the space X if $\mathcal{Q}$ has a faithful C^* action on the C^* algebra $C(X)$. We show in the following example (which is due to Huang [29]) that a genuine CQG (i.e. non commutative as a C^* algebra) can act continuously and faithfully on a compact Hausdorff space.

Example:
Recall the quantum permutation group $A_s(n)$. It acts continuously and faithfully on $C(X_n)$ where X_n is the finite space with n points $\{x_1, ..., x_n\}$. If e_i for $1 \leq i \leq n$ be the functions on X_n such that $e_i(x_j) = \delta_{ij}$ and Y be a compact Hausdorff space then (Lemma 3.1 of [29])

Lemma 2.1.9. *There exists an action α of the quantum permutation group $A_s(n)$ on $X_n \times Y$ given by $\alpha(e_i \otimes f) = \sum_{k=1}^{n} e_k \otimes f \otimes a_{ki}$ for all $1 \leq i \leq n$ and $f \in C(Y)$.*

Now Y_1 be a closed subset of Y. We define an equivalence relation $\sim$ on $X_n \times Y$ as follows: For $y', y'' \in Y$ and $x', x'' \in X_n$, two points (x', y') and (x'', y'') are equivalent if one of the followings is true:

(1)$y' = y'' \in Y_1$.

(2)$y' = y''$ and $x' = x''$.

With this, we have (Lemma 3.2 and Proposition 3.7 of [29])

Lemma 2.1.10. *The quotient space $X_n \times Y/\sim$ is compact and Hausdorff. Moreover if Y is connected and Y_1 is non empty, $X_n \times Y/\sim$ is also connected.*

Also note that $C(X_n \times Y/\sim)$ is a C^* subalgebra of $C(X_n \times Y)$. Then we have (Proposition 3.4 and Theorem 3.5 of [29])

Theorem 2.1.11. *If $Y_1 \neq Y$, the restriction $\widetilde{\alpha}$ of the action α on $C(X_n \times Y/\sim)$ is a faithful action of $A_s(n)$ on the compact, connected, Hausdorff space $X_n \times Y/\sim$.*

For a concrete example, take $Y = [0, 1]$ and $Y_1 = \{0\}$, then $X_n \times Y/\sim$ is a wedge sum of n unit intervals by identifying $(x_i, 0)$ to a single point for all $1 \leq i \leq n$. The quantum permutation group $A_s(n)$ which is non commutative for $n \geq 4$ acts faithfully on this space.

Now we turn to the formulation of quantum isometry groups based on Laplacian and the Dirac operator. First we start with the formulation of quantum isometry group based on Laplacian.

2.2 Formulation of the quantum isometry group based on Laplacian

2.2.1 Characterization of isometry group for a compact Riemannian manifold

Let M be a compact Riemannian manifold. Consider the category with objects being the pairs (G, α) where G is a compact metrizable group acting on M by the smooth and isometric action α. If (G_1, α) and (G_2, β) are two objects in this category, $\mathrm{Mor}((\mathrm{G}_1, \alpha), (\mathrm{G}_2, \beta))$ consists of group homomorphisms π from G_1 to G_2 such that $\beta \circ \pi = \alpha$. Then the isometry group of M is the universal object in this category.

More generally, the isometry group of a classical compact Riemannian manifold, viewed as a compact metrizable space (forgetting the group structure), can be seen to be the universal object of a category whose object class consists of subsets (not generally subgroups) of the set of smooth isometries of the manifold. Then it can be proved that this universal compact set has a canonical group structure. Thus, motivated by the ideas of Woronowicz and Soltan Goswami considered in [22] a bigger category with objects as the pair (S, f) where S is a compact metrizable space and $f : S \times M \to M$ such that the map from M to itself defined by $m \mapsto f(s, m)$ is a smooth isometry for all s in S. The morphism set is defined as above (replacing group homomorphisms by continuous set maps).

Therefore, to define the quantum isometry group, it is reasonable to consider a category of compact quantum groups which act on the manifold (or more generally, on a noncommutative manifold given by spectral triple) in a 'nice' way, preserving the Riemannian structure in some suitable sense, which is precisely formulated in [22], where it is also proven that a universal object in the category of such quantum groups does exist if one makes some natural regularity assumptions on the spectral triple.

2.2.2 The definition and existence of the quantum isometry group

Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a Θ-summable spectral triple of compact type. We recall from the first chapter the Hilbert spaces of k-forms $\mathcal{H}^k_D, k = 0, 1, 2, ...$ and also the Laplacian $\mathcal{L} = -d^*_D d_D$.

To define the quantum isometry group, we need the following assumptions:

Assumptions

1. d_D is closable and $\mathcal{A}^\infty \subseteq \mathrm{Dom}(\mathcal{L})$ where $\mathcal{A}^\infty$ is viewed as a dense subspace of $\mathcal{H}^0_D$.

2. $\mathcal{L}$ has compact resolvents.

3. $\mathcal{L}(\mathcal{A}^\infty) \subseteq \mathcal{A}^\infty$.

4. Each eigenvector of $\mathcal{L}$ (which has a discrete spectrum, hence a complete set of eigenvectors) belongs to $\mathcal{A}^\infty$.

5. (connectedness assumption) The kernel of $\mathcal{L}$ is one dimensional, spanned by the identity 1 of $\mathcal{A}^\infty$, viewed as a unit vector in $\mathcal{H}^0_D$.

6. The complex linear span of the eigenvectors of $\mathcal{L}$, denoted by $\mathcal{A}^\infty_0$ is norm dense in $\mathcal{A}^\infty$.

Definition 2.2.1. *We say that a spectral triple satisfying the assumptions* **1.** - **6.** *admissible. Note that in the last section of the first chapter we had to only assume the regularity of the spectral triple so that a noncommutative Laplacian can be defined. But for the formulation of the quantum isometry group based on laplacian we need to assume that the spectral triple is admissible.*

The following result is contained in Remark 2.16 of [22].

Proposition 2.2.2. *If an admissible spectral triple* $(\mathcal{A}^\infty, \mathcal{H}, D)$ *satisfies the condition* $\bigcap \mathrm{Dom}(\mathcal{L}^{\mathrm{n}}) = \mathcal{A}^\infty$, *and if* $\alpha : \bar{\mathcal{A}} \to \bar{\mathcal{A}} \otimes S$ *is a smooth isometric action on* $\mathcal{A}^\infty$ *by a CQG* S, *then for all state* ϕ *on* S, $\alpha_\phi (= (\mathrm{id} \otimes \phi)\alpha)$ *keeps* $\mathcal{A}^\infty$ *invariant.*

In view of the characterization of smooth isometric action on a classical compact manifold (Proposition 1.1.13 in Chapter 1), Goswami gave the following definition in [22].

Definition 2.2.3. *A quantum family of smooth isometries of the noncommutative manifold* $\mathcal{A}^\infty$ *(or more precisely on the corresponding spectral triple) is a pair* $(\mathcal{S}, \alpha)$ *where* $\mathcal{S}$ *is a separable unital* C^* *algebra,* $\alpha : \overline{\mathcal{A}} \to \overline{\mathcal{A}} \otimes \mathcal{S}$ *(where* $\overline{\mathcal{A}}$ *denotes the* C^* *algebra obtained by completing* $\mathcal{A}^\infty$ *in the norm of* $\mathcal{B}(\mathcal{H}_D^0)$*) is a unital* C^* *homomorphism, satisfying the following:*

a. $\overline{\mathrm{Sp}}(\alpha(\overline{\mathcal{A}})(1 \otimes \mathcal{S}) = \overline{\mathcal{A}} \otimes \mathcal{S}$

b. $\alpha_\phi = (\mathrm{id} \otimes \phi)\alpha$ *maps* $\mathcal{A}_0^\infty$ *into itself and commutes with* $\mathcal{L}$ *on* $\mathcal{A}_0^\infty$, *for every state* ϕ *on* $\mathcal{S}$.

In case, the C^* *algebra has a coproduct* Δ *such that* $(\mathcal{S}, \Delta)$ *is a compact quantum group and* α *is an action of* $(\mathcal{S}, \Delta)$ *on* $\overline{\mathcal{A}}$, *we say that* $(\mathcal{S}, \Delta)$ *acts smoothly and isometrically on the noncommutative manifold.*

Notations

1. We will denote by $\mathbf{Q}^{\mathcal{L}}$ the category with the object class consisting of all quantum families of isometries $(\mathcal{S}, \alpha)$ of the given noncommutative manifold, and the set of morphisms $\mathrm{Mor}((\mathcal{S}, \alpha), (\mathcal{S}', \alpha'))$ being the set of unital C^* homomorphisms $\phi : \mathcal{S} \to \mathcal{S}'$ satisfying $(\mathrm{id} \otimes \phi)\alpha = \alpha'$.

2. We will denote by $\mathbf{Q}'_{\mathcal{L}}$ the category whose objects are triplets $(\mathcal{S}, \Delta, \alpha)$ where $(\mathcal{S}, \Delta)$ is a CQG acting smoothly and isometrically on the given noncommutative manifold, with α being the corresponding action. The morphisms are the homomorphisms of compact quantum groups which are also morphisms of the underlying quantum families.

Let $\{\lambda_1, \lambda_2, ...\}$ be the set of eigenvalues of $\mathcal{L}$, with V_i being the corresponding (finite dimensional) eigenspace. We will denote by $\mathcal{U}_i$ the Wang algebra $A_{u,d_i}(I)$ (as introduced in the chapter 1) where d_i is the dimension of the subspace V_i. We fix a representation $\beta_i : V_i \to V_i \otimes \mathcal{U}_i$ on the Hilbert space V_i, given by $\beta_i(e_{ij}) = \sum_k e_{ik} \otimes u_{kj}^{(i)}$ for $j = 1, 2, ...d_i$, where $\{e_{ij}\}$ is an orthonormal basis for V_i, and $u^{(i)} \equiv u_{kj}^{(i)}$ are the generators of $\mathcal{U}_i$. Thus, both $u^{(i)}$ and $\overline{u^{(i)}}$ are unitaries. The representations β_i canonically induce the free product representation $\beta = *_i \beta_i$ of the free product CQG $\mathcal{U} = *_i \mathcal{U}_i$ on the Hilbert space $\mathcal{H}_D^0$ such that the restriction of β on V_i coincides with β_i for all i.

The following Lemma (Lemma 2.12 of [22]) will be needed later and hence we record it.

Lemma 2.2.4. *Consider an admissible spectral triple* $(\mathcal{A}^\infty,\ \mathcal{H},\ D)$ *and let* $(\mathcal{S}, \alpha)$ *be a quantum family of smooth isometries of the spectral triple. Moreover, assume that the action is faithful in the sense that there is no proper* C^* *subalgebra* $\mathcal{S}_1$ *of* $\mathcal{S}$ *such that* $\alpha(\mathcal{A}^\infty) \subseteq \mathcal{A}^\infty \otimes \mathcal{S}_1$. *Then* $\widetilde{\alpha} : \mathcal{A}^\infty \otimes \mathcal{S} \to \mathcal{A}^\infty \otimes \mathcal{S}$ *defined by* $\widetilde{\alpha}(a \otimes b) = \alpha(a)(1 \otimes b)$ *extends to an* $\mathcal{S}$ *linear unitary on the Hilbert* $\mathcal{S}$ *module* $\mathcal{H}^0_D \otimes \mathcal{S}$, *denoted again by* $\widetilde{\alpha}$. *Moreover, we can find a* C^* *isomorphism* $\phi : \mathcal{U}/\mathcal{I} \to \mathcal{S}$ *between* $\mathcal{S}$ *and a quotient of* $\mathcal{U}$ *by a* C^* *ideal* $\mathcal{I}$ *of* $\mathcal{U}$, *such that* $\alpha = (\mathrm{id} \otimes \phi) \circ (\mathrm{id} \otimes \Pi_{\mathcal{I}}) \circ \beta$ *on* $\mathcal{A}^\infty \subseteq \mathcal{H}^0_D$, *where* $\Pi_{\mathcal{I}}$ *denotes the quotient map from* $\mathcal{U}$ *to* $\mathcal{U}/\mathcal{I}$.

If furthermore, there is a CQG structure on $\mathcal{S}$ *given by a coproduct* Δ *such that* α *is a* C^* *action of a CQG on* $\overline{\mathcal{A}}$, *then the map* $\alpha : \mathcal{A}^\infty \to \mathcal{A}^\infty \otimes \mathcal{S}$ *extends to a unitary representation (denoted again by* α *) of the CQG* $(\mathcal{S}, \Delta)$ *on* $\mathcal{H}^0_D$. *In this case, the ideal* $\mathcal{I}$ *is a Woronowicz* C^* *ideal and the* C^* *isomorphism* $\phi : \mathcal{U}/\mathcal{I} \to \mathcal{S}$ *is a morphism of CQG s.*

Using this, the following result has been proved in [22], which defines and gives the existence of $QISO^{\mathcal{L}}$.

Theorem 2.2.5. *For any admissible spectral triple* $(\mathcal{A}^\infty, \mathcal{H}, D)$, *the category* $\mathbf{Q}^{\mathcal{L}}$ *has a universal object denoted by* $(QISO^{\mathcal{L}}, \alpha_0)$. *Moreover,* $QISO^{\mathcal{L}}$ *has a coproduct* Δ_0 *such that* $(QISO^{\mathcal{L}}, \Delta_0)$ *is a CQG and* $(QISO^{\mathcal{L}}, \Delta_0, \alpha_0)$ *is a universal object in the category* $\mathbf{Q}'_{\mathcal{L}}$. *The action* α_0 *is faithful.*

We very briefly outline the main ideas of the proof. The universal object $QISO^{\mathcal{L}}$ is constructed as a suitable quotient of $\mathcal{U}$. Let $\mathcal{F}$ be the collection of all those C^*-ideals $\mathcal{I}$ of $\mathcal{U}$ such that the composition $\Gamma_{\mathcal{I}} = (\mathrm{id} \otimes \Pi_{\mathcal{I}}) \circ \beta : \mathcal{A}^\infty_0 \to \mathcal{A}^\infty_0 \otimes_{\mathrm{alg}} (\mathcal{U}/\mathcal{I})$ extends to a C^*-homomorphism from $\overline{\mathcal{A}}$ to $\overline{\mathcal{A}} \otimes (\mathcal{U}/\mathcal{I})$. Then it can be shown that $\mathcal{I}_0$ $(= \cap_{\mathcal{I} \in \mathcal{F}} \mathcal{I}$) is again a member of $\mathcal{F}$ and $(\mathcal{U}/\mathcal{I}_0,\ \Gamma_{\mathcal{I}_0})$ is the required universal object. Thus,

Remark 2.2.6. $QISO^{\mathcal{L}}$ *is a quantum subgroup of the CQG* $\mathcal{U} = *_i A_{u,d_i}(I)$. *As* $A_{u,d_i}(I)$ *satisfies* $\kappa^2 = \mathrm{id}$, $QISO^{\mathcal{L}}$ *has tracial Haar state.*

Remark 2.2.7. *It is proved in [22] that to ensure the existence of* $QISO^{\mathcal{L}}$, *the assumption (***5***) can be replaced by the condition that the action* α *is* τ *preserving, that is,* $(\tau \otimes \mathrm{id})\alpha(a) = \tau(a).1$. *In [22] it was also shown (Lemma 2.5, b* $\Rightarrow$ *a) that for an isometric group action on a not necessarily connected classical manifold, the volume functional is automatically preserved. It can be easily seen that the proof goes verbatim for a quantum group action, and consequently we get the existence of* $QISO^{\mathcal{L}}$ *for a (not necessarily connected) compact Riemannian manifold.*

Unitary representation of $QISO^{\mathcal{L}}$ on a spectral triple

We shall also need the following result proved in section 2.4 of [22].

Proposition 2.2.8. *$QISO^{\mathcal{L}}$ has a unitary representation $U \equiv U_{\mathcal{L}}$ on $\mathcal{H}_D$ such that U commutes with $d + d^*$. Let δ be as in subsection 1.6.2. On the Hilbert space of k-forms, that is. $\mathcal{H}_D^k$, U is defined by:*

$$U([a_0\delta(a_1)\cdots\delta(a_k)] \otimes q) = [a_0^{(1)}\delta(a_1^{(1)})\cdots\delta(a_k^{(1)})] \otimes (a_0^{(2)}a_1^{(2)}\cdots a_k^{(2)})q,$$

where q belongs to $QISO^{\mathcal{L}}$, a_i belongs to $\mathcal{A}_0^\infty$, and for x in $\mathcal{A}_0$, (the $*$-subalgebra generated by the eigenvectors of $\mathcal{L}$) we write in Sweedler notation $\alpha(x) = x^{(1)} \otimes x^{(2)} \in \mathcal{A}_0 \otimes (QISO^{\mathcal{L}})_0$ (α denotes the action of $QISO^{\mathcal{L}}$).

2.2.3 $QISO^{\mathcal{L}}$ for S^n and $\mathbb{T}^n$

In the example toward the end of Subsection 2.1 we saw that a genuine CQG (i.e. non commutative as a C^* algebra) can act faithfully on a compact, connected, Hausdorff space. But in [11] and [9], it was observed that the quantum isometry group of n sphere and n tori turn out to be classical i.e.
(1) $QISO^{\mathcal{L}}(S^n) \equiv C(O(n+1))$.
(2) $QISO^{\mathcal{L}}(\mathbb{T}^n) \equiv C(ISO(\mathbb{T}^n))$. For computation we refer the reader to [11] and [9].

2.3 Formulation of the quantum isometry group based on the Dirac operator

The approach of formulation of quantum isometry group for a spectral triple had a major draw back as it needed the existence of a "good" Laplacian on the spectral triple. A direct approach based on the Dirac operator of the spectral triple was called for and that was successfully achieved by Goswami, Bhowmick in their paper [10]. In consistency with the classical case, this is called quantum group of orientation preserving Riemannian isometry. First recall Theorem 1.57 of chapter 1. Motivated by this we give the following operator theoretic characterization of "set of orientation preserving isometries ". For the proof of the theorem see [10].

Theorem 2.3.1. *Let X be a compact metrizable space and $\psi : X \times M \to M$ is a map such that ψ_x defined by $\psi_x(m) = \psi(x, m)$ is a smooth orientation preserving Riemannian isometry and $x \mapsto \psi_x \in C^\infty(M, M)$ is continuous with respect to the locally convex topology of $C^\infty(M, M)$ mentioned before.*

Then there exists a ($C(X)$-linear) unitary U_ψ on the Hilbert $C(X)$-module $\mathcal{H} \otimes C(X)$ (where $\mathcal{H} = L^2(S)$) such that for all x belonging to X, $U_x := (id \otimes ev_x)U_\psi$ is a unitary of the form U_{ψ_x} on the Hilbert space $\mathcal{H}$ commuting with D and $U_x M_\phi U_x^{-1} = M_{\phi\circ\psi_x^{-1}}$. If in addition, the manifold is even dimensional, then U_{ψ_x} commutes with the grading operator γ.

Conversely, if there exists a $C(X)$-linear unitary U on $\mathcal{H} \otimes C(X)$ such that $U_x := (\mathrm{id} \otimes \mathrm{ev}_x)(U)$ is a unitary commuting with D for all x, (and U_x commutes with the grading operator γ if the manifold is even dimensional) and $(\mathrm{id} \otimes \mathrm{ev}_x)\alpha_U(L^\infty(M)) \subseteq L^\infty(M)$ for all x in X, then there exists a map $\psi : X \times M \to M$ satisfying the conditions mentioned above such that $U = U_\psi$.

2.3.1 Quantum group of orientation-preserving isometries of an R-twisted spectral triple

In view of the characterization of orientation-preserving isometric action on a classical manifold (Theorem 2.3.1), we give the following definitions.

Definition 2.3.2. *A quantum family of orientation preserving isometries for the (odd, compact type) spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ is given by a pair $(\mathcal{S}, U)$ where $\mathcal{S}$ is a separable unital C^*-algebra and U is a linear map from $\mathcal{H}$ to $\mathcal{H} \otimes \mathcal{S}$ such that $\widetilde{U}$ given by $\widetilde{U}(\xi \otimes b) = U(\xi)(1 \otimes b)$ (ξ in $\mathcal{H}$, b in $\mathcal{S}$) extends to a unitary element of $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{S})$ satisfying the following:*

(i) for every state ϕ on $\mathcal{S}$ we have $U_\phi D = DU_\phi$, where $U_\phi := (\mathrm{id} \otimes \phi)(\widetilde{U})$;

(ii) $(\mathrm{id} \otimes \phi) \circ \mathrm{ad}_{\widetilde{U}}(a) \in (\mathcal{A}^\infty)''$ for all a in $\mathcal{A}^\infty$ and state ϕ on $\mathcal{S}$, where $\mathrm{ad}_{\widetilde{U}}(x) := \widetilde{U}(x \otimes 1)\widetilde{U}^*$ for x belonging to $\mathcal{B}(\mathcal{H})$. In case the C^*-algebra $\mathcal{S}$ has a coproduct Δ such that $(\mathcal{S}, \Delta)$ is a compact quantum group and U is a unitary representation of $(\mathcal{S}, \Delta)$ on $\mathcal{H}$, we say that $(\mathcal{S}, \Delta)$ acts by orientation-preserving isometries on the spectral triple. In case the spectral triple is even with the grading operator γ, a quantum family of orientation preserving isometries $(\mathcal{A}^\infty, \mathcal{H}, D, \gamma)$ will be defined exactly as above, with the only extra condition being that U commutes with γ.

From now on, we will mostly consider odd spectral triples. However let us remark that in the even case, all the definitions and results obtained by us will go through with some obvious modifications. We also remark that all our spectral triples are of compact type.

Consider the category $\mathbf{Q} \equiv \mathbf{Q}(\mathcal{A}^\infty, \mathcal{H}, D) \equiv \mathbf{Q}(D)$ with the object-class consisting of all quantum families of orientation preserving isometries $(\mathcal{S}, U)$ of the given spectral triple, and the set of morphisms $\mathrm{Mor}((\mathcal{S}, U), (\mathcal{S}', U'))$ being the set of unital C^*-homomorphisms $\Phi : \mathcal{S} \to \mathcal{S}'$ satisfying $(\mathrm{id} \otimes \Phi)(U) = U'$. We also consider another category $\mathbf{Q}' \equiv \mathbf{Q}'(\mathcal{A}^\infty, \mathcal{H}, D) \equiv \mathbf{Q}'(D)$ whose objects are triplets $(\mathcal{S}, \Delta, U)$, where $(\mathcal{S}, \Delta)$ is a compact quantum group acting by orientation preserving isometries on the given spectral triple, with U being the corresponding unitary representation. The morphisms are the homomorphisms of compact quantum groups which are also morphisms of the underlying quantum families of orientation preserving isometries. The forgetful functor $F : \mathbf{Q}' \to \mathbf{Q}$ is clearly faithful, and we can view $F(\mathbf{Q}')$ as a subcategory of $\mathbf{Q}$.

Unfortunately, in general $\mathbf{Q}'$ or $\mathbf{Q}$ will not have a universal object. It is easily seen by taking the standard example $\mathcal{A}^\infty = M_n(\mathbb{C})$, $\mathcal{H} = \mathbb{C}^n$, $D = I$. Any CQG having a unitary representation on $\mathbb{C}^n$ is an object of $\mathbf{Q}'(M_n(\mathbb{C}), \mathbb{C}^n, I)$. But by Proposition 2.1.7, there is no universal object in this category. However, the fact that comes to our rescue is that a universal object exists in each of the subcategories which correspond to the CQG actions preserving a given faithful functional on M_n.

On the other hand, given any equivariant spectral triple, it has been shown in [23] that there is a (not necessarily unique) canonical faithful functional which is preserved by the CQG action. For readers' convenience, we state this result (in a form suitable to us) briefly here. Before that, let us recall the definition of an R-twisted spectral data from [23].

Definition 2.3.3. *An R-twisted spectral data (of compact type) is given by a quadruplet $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ where*

1. ($\mathcal{A}^\infty, \mathcal{H}, D$) is a spectral triple of compact type.

2. R a positive (possibly unbounded) invertible operator such that R commutes with D.

3. For all $s \in \mathbb{R}$, the map $a \mapsto \sigma_s(a) := R^{-s} a R^s$ gives an automorphism of $\mathcal{A}^\infty$ (not necessarily $*$-preserving) satisfying $\sup_{s \in [-n,n]} \|\sigma_s(a)\| < \infty$ for all positive integer n.

We shall also sometimes refer to $(\mathcal{A}^\infty, \mathcal{H}, D)$ as an R-twisted spectral triple.

Proposition 2.3.4. *Given a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ (of compact type) which is $\mathcal{Q}$-equivariant with respect to a representation of a CQG $\mathcal{Q}$ on $\mathcal{H}$, we can construct a positive (possibly unbounded) invertible operator R on $\mathcal{H}$ such that $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ is a twisted spectral data, and moreover, we have*

α_U preserves the functional τ_R defined at least on a weakly dense $*$-subalgebra $\mathcal{E}_D$ of $\mathcal{B}(\mathcal{H})$ generated by the rank-one operators of the form $|\xi >< \eta|$ where ξ, η are eigenvectors of D, given by

$$\tau_R(x) = Tr(Rx), \quad x \in \mathcal{E}_D.$$

Remark 2.3.5. *When the Haar state of $\mathcal{Q}$ is tracial, then it follows from the definition of R in Lemma 3.1 of [23] that R can be chosen to be I.*

Remark 2.3.6. *If V_λ denotes the eigenspace of D corresponding to the eigenvalue, say λ, it is clear that $\tau_R(X) = e^{t\lambda^2} \mathrm{Tr}(Re^{-tD^2} X)$ for all $X = |\xi >< \eta|$ with ξ, η belonging to V_λ and for any $t > 0$. Thus, the α_U-invariance of the functional τ_R on $\mathcal{E}_D$ is equivalent to the α_U-invariance of the functional $X \mapsto \mathrm{Tr}(XRe^{-tD^2})$ on $\mathcal{E}_D$ for each $t > 0$. $\mathcal{E}_D$, that is, for all $|\xi >< \eta|$ with ξ, η belonging to V_λ, $(\tau_R \otimes \mathrm{id})\mathrm{ad}_{\widetilde{U}}(|\xi >< \eta|) = \tau_R(|\xi >< \eta|).1$ Therefore, $(\tau_R \otimes \mathrm{id})\mathrm{ad}_{\widetilde{U}}(|\xi >< \eta|) = \tau_R(|\xi >< \eta|).1 = e^{t\lambda^2} Tr(Re^{-tD^2} |\xi >< \eta|)$. On the other hand, $(\tau_R \otimes \mathrm{id})\mathrm{ad}_{\widetilde{U}}(|\xi >< \eta|) = e^{t\lambda^2}(Tr(Re^{tD^2}.) \otimes \mathrm{id})\alpha_U(|\xi >< \eta|)$. (by* *writing the formula of trace in terms of an orthonormal basis)) If, furthermore, the R-twisted spectral triple is* Θ*-summable in the sense that* Re^{-tD^2} *is trace class for every* $t > 0$*, the above is also equivalent to the* α_U*-invariance of the bounded normal functional* $X \mapsto \mathrm{Tr}(XRe^{-tD^2})$ *on the whole of* $\mathcal{B}(\mathcal{H})$*. In particular, this implies that* α_U *preserves the functional* $\mathcal{B}(\mathcal{H}) \ni x \mapsto \mathrm{Lim}_{t\to 0+} \frac{\mathrm{Tr}(xRe^{-tD^2})}{\mathrm{Tr}(Re^{-tD^2})}$*, where* Lim *is as defined in subsection 1.6.2.*

This motivates the following definition:

Definition 2.3.7. *Given an R-twisted spectral data* $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ *of compact type, a quantum family of orientation preserving isometries* $(\mathcal{S}, U)$ *of* $(\mathcal{A}^\infty, \mathcal{H}, D)$ *is said to preserve the R-twisted volume, (simply said to be volume-preserving if R is understood) if one has* $(\tau_R \otimes \mathrm{id})(\mathrm{ad}_{\widetilde{U}}(x)) = \tau_R(x).1_{\mathcal{S}}$ *for all* x *in* $\mathcal{E}_D$*, where* $\mathcal{E}_D$ *and* τ_R *are as in Proposition 2.3.4. We shall also call* $(\mathcal{S}, U)$ *a quantum family of orientation-preserving isometries of the R-twisted spectral triple.*

If, furthermore, the C^**-algebra* $\mathcal{S}$ *has a coproduct* Δ *such that* $(\mathcal{S}, \Delta)$ *is a CQG and* U *is a unitary representation of* $(\mathcal{S}, \Delta)$ *on* $\mathcal{H}$*, we say that* $(\mathcal{S}, \Delta)$ *acts by volume and orientation-preserving isometries on the R-twisted spectral triple.*

We shall consider the categories $\mathbf{Q}_R \equiv \mathbf{Q}_R(D)$ *and* $\mathbf{Q}'_R \equiv \mathbf{Q}'_R(D)$ *which are the full subcategories of* $\mathbf{Q}$ *and* $\mathbf{Q}'$ *respectively, obtained by restricting the object-classes to the volume-preserving quantum families.*

Remark 2.3.8. *We shall not need the full strength of the definition of twisted spectral data here; in particular the third condition in the definition 2.3.3. However, we shall continue to work with the usual definition of R-twisted spectral data, keeping in mind that all our results are valid without assuming the third condition.*

Let us now fix a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ which is of compact type. The C^*-algebra generated by $\mathcal{A}^\infty$ in $\mathcal{B}(\mathcal{H})$ will be denoted by $\mathcal{A}$. Let $\lambda_0 = 0, \lambda_1, \lambda_2, \cdots$ be the eigenvalues of D with V_i denoting the (d_i-dimensional, $0 \leq d_i < \infty$) eigenspace for λ_i. Let $\{e_{ij}, j = 1, ..., d_i\}$ be an orthonormal basis of V_i. We also assume that there is a positive invertible R on $\mathcal{H}$ such that $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ is an R-twisted spectral data. The operator R must have the form $R|_{V_i} = R_i$, say, with R_i positive invertible $d_i \times d_i$ matrix. Let us denote the CQG $A_{u,d_i}(R_i^T)$ by $\mathcal{U}_i$, with its canonical unitary representation β_i on $V_i \cong \mathbb{C}^{d_i}$, given by $\beta_i(e_{ij}) = \sum_k e_{ik} \otimes u_{kj}^{R_i^T}$. Let $\mathcal{U}$ be the free product of $\mathcal{U}_i, i = 1, 2, ...$ and $\beta = *_i \beta_i$ be the corresponding free product representation of $\mathcal{U}$ on $\mathcal{H}$. We shall also consider the corresponding unitary element $\tilde{\beta}$ in $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{U})$. For the proofs of the following Lemma and following Theorem the reader is referred to [10].

Lemma 2.3.9. *Consider the R-twisted spectral triple* $(\mathcal{A}^\infty, \mathcal{H}, D)$ *and let* $(\mathcal{S}, U)$ *be a quantum family of volume and orientation preserving isometries of the given spectral* *triple. Moreover, assume that the map U is faithful in the sense that there is no proper C^*-subalgebra $\mathcal{S}_1$ of $\mathcal{S}$ such that $\widetilde{U}$ belongs to $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{S}_1)$. Then we can find a C^*-isomorphism $\phi : \mathcal{U}/\mathcal{I} \to \mathcal{S}$ between $\mathcal{S}$ and a quotient of $\mathcal{U}$ by a C^*-ideal $\mathcal{I}$ of $\mathcal{U}$, such that $U = (\mathrm{id} \otimes \phi) \circ (\mathrm{id} \otimes \Pi_{\mathcal{I}}) \circ \beta$, where $\Pi_{\mathcal{I}}$ denotes the quotient map from $\mathcal{U}$ to $\mathcal{U}/\mathcal{I}$.*

If, furthermore, there is a compact quantum group structure on $\mathcal{S}$ given by a coproduct Δ such that $(\mathcal{S}, \Delta, U)$ is an object in $\mathbf{Q}'_R$, the ideal $\mathcal{I}$ is a Woronowicz C^*-ideal and the C^*-isomorphism $\phi : \mathcal{U}/\mathcal{I} \to \mathcal{S}$ is a morphism of compact quantum groups.

Theorem 2.3.10. *For any R-twisted spectral triple of compact type $(\mathcal{A}^\infty, \mathcal{H}, D)$, the category $\mathbf{Q}_R$ of quantum families of volume and orientation preserving isometries has a universal (initial) object, say $(\widetilde{\mathcal{G}}, U_0)$. Moreover, $\widetilde{\mathcal{G}}$ has a coproduct Δ_0 such that $(\widetilde{\mathcal{G}}, \Delta_0)$ is a compact quantum group and $(\widetilde{\mathcal{G}}, \Delta_0, U_0)$ is a universal object in the category $\mathbf{Q}'_R$. The representation U_0 is faithful.*

Consider the $*$-homomorphism $\alpha_0 : \mathrm{ad}_{\widetilde{U_0}}$, where $(\widetilde{\mathcal{G}}, U_0)$ is the universal object obtained in the previous theorem. For every state ϕ on $\widetilde{\mathcal{G}}$, $(\mathrm{id} \otimes \phi) \circ \alpha_0$ maps $\mathcal{A}$ into $\mathcal{A}''$. However, in general α_0 may not be faithful even if U_0 is so, and let $\mathcal{G}$ denote the C^*-subalgebra of $\widetilde{\mathcal{G}}$ generated by the elements $\{(f \otimes \mathrm{id}) \circ \alpha_0(a) : f \in \mathcal{A}^*, a \in \mathcal{A}\}$.

Definition 2.3.11. *We shall call $\mathcal{G}$ the quantum group of orientation-preserving isometries of R-twisted spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ and denote it by $QISO^+_R(\mathcal{A}^\infty, \mathcal{H}, D, R)$ or even simply as $QISO^+_R(D)$. The quantum group $\widetilde{\mathcal{G}}$ is denoted by $\widetilde{QISO^+_R}(D)$.*

If the spectral triple is even, then we will denote $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ by $QISO^+_R(D, \gamma)$ and $\widetilde{QISO^+_R}(D, \gamma)$ respectively.

Let $(\mathcal{Q}, V)$ be an object in the category $\mathbf{Q}'_R(\mathcal{D})$. We would like to give a necessary and sufficient condition on the unbounded operator R so that $\mathrm{ad}_{\widetilde{V}}$ preserves the R-twisted volume. For that break the Hilbert space $\mathcal{H}$ (on which $\mathcal{D}$ acts) into finite dimensional eigen spaces of the operator $\mathcal{D}$ i.e. let $\mathcal{H} = \oplus_k \mathcal{H}_k$ where each $\mathcal{H}_k$ is a finite dimensional eigen space for $\mathcal{D}$. Since $\mathcal{D}$ commutes with V, V preserves each of the $\mathcal{H}_k$'s and on each $\mathcal{H}_k$, V is a unitary representation of the compact quantum group $\mathcal{Q}$. Then we have the decomposition of each $\mathcal{H}_k$ into the irreducibles, say

$$\mathcal{H}_k = \oplus_{\pi \in \mathcal{I}_k \subset Rep(\mathcal{Q})} \mathbb{C}^{d_\pi} \otimes \mathbb{C}^{m_{\pi,k}},$$

where $m_{\pi,k}$ is the multiplicity of the irreducible representation of type π on $\mathcal{H}_k$ and $\mathcal{I}_k$ is some finite subset of $Rep(\mathcal{Q})$. Since R commutes with V, R preserves direct summands of $\mathcal{H}_k$. Then we have the following

Theorem 2.3.12. $\mathrm{ad}_{\widetilde{V}}$ *preserves the R-twisted volume if and only if*

$$R|_{\mathcal{H}_k} = \oplus_{\pi \in \mathcal{I}_k} F_\pi \otimes T_{\pi,k},$$

for some $T_{\pi,k} \in \mathcal{B}(\mathbb{C}^{m_{\pi,k}})$, where F_π's are as in Subsection 2.1.

Proof:
Only if $\Rightarrow$: let $\{e_i\}_{i=1}^{d_\pi}$ and $\{f_j\}_{j=1}^{m_{\pi,k}}$ be orthonormal bases for $\mathbb{C}^{d_\pi}$ and $\mathbb{C}^{m_{\pi,k}}$ respectively. Also let $R(e_i \otimes f_j) = \sum_{s,t} R(s,t,i,j) e_s \otimes f_t$. We have $\widetilde{V}^*(e_i \otimes f_j \otimes 1_{\mathcal{Q}}) = \sum_k e_k \otimes f_j \otimes q_{ik}^*$. We denote the restriction of the trace of $\mathcal{B}(\mathcal{H})$ on $\mathbb{C}^{d_\pi} \otimes \mathbb{C}^{m_{\pi,k}}$ again by Tr. Let $a \in \mathcal{B}(\mathbb{C}^{d_\pi} \otimes \mathbb{C}^{m_{\pi,k}})$ and $\chi(a) := Tr(a.R)$. Then we have

$$
\begin{aligned}
& (\chi \otimes h)\mathrm{ad}_{\widetilde{V}}(a) \\
= & \sum_{i,j} < \widetilde{V}^*(e_i \otimes f_j \otimes 1_{\mathcal{Q}}), (a \otimes 1)\widetilde{V}^* R(e_i \otimes f_j) > \\
= & \sum_{i,j,k,s,t,u} < e_k \otimes f_j \otimes q_{ik}^*, R(s,t,i,j) a(e_u \otimes f_t) \otimes q_{su}^* > \\
= & \sum_{i,j,k,s,t,u} \frac{R(s,t,i,j)}{M_\pi} < e_k \otimes f_j, a(e_u \otimes f_t) > \delta_{is} F_\pi(k,u) \\
= & \sum_{i,j,k,t,u} \frac{R(i,t,i,j)}{M_\pi} < e_k \otimes f_j, a(e_u \otimes f_t) > F_\pi(k,u)
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\chi(a) &= Tr(a.R) \\
&= \sum_{i,j} < e_i \otimes f_j, aR(e_i \otimes f_j) > \\
&= \sum_{k,j,u,t} R(u,t,k,j) < e_k \otimes f_j, a(e_u \otimes f_t) >
\end{aligned}
$$

Hence $(\chi \otimes h)\mathrm{ad}_{\widetilde{V}}(a) = \chi(a) \Rightarrow$:

$$
\sum_{i,j,k,t,u} \frac{R(i,t,i,j)}{M_\pi} < e_k \otimes f_j, a(e_u \otimes f_t) > F_\pi(k,u) \tag{2.3.1}
$$

$$
= \sum_{k,j,u,t} R(u,t,k,j) < e_k \otimes f_j, a(e_u \otimes f_t) > \tag{2.3.2}
$$

Now fix u_0, t_0 and consider $a \in \mathcal{B}(\mathcal{H})$ such that $a(e_{u_0} \otimes f_{t_0}) = e_p \otimes f_q$ and zero on the other basis elements. Then from (2), we get

$$\sum_{i,j,k} \frac{R(i,t_0,i,j)}{M_\pi} < e_k \otimes f_j, e_p \otimes f_q > F_\pi(k,u_0) = \sum_{k,j} R(u_0,t_0,k,j) < e_k \otimes f_j, e_p \otimes f_q >$$

$$\Rightarrow \quad \sum_{i} \frac{R(i,t_0,i,q)}{M_\pi} F_\pi(p,u_0) = R(u_0,t_0,p,q)$$

This establishes that $R|_{\mathcal{H}_k} = \oplus_{\pi \in \mathcal{I}_k} F_\pi \otimes T_{\pi,k}$ with $T_{\pi,k} \in \mathcal{B}(\mathbb{C}^{m_{\pi,k}})$.
The if part is straightforward and was essentially done in [23]. □

Chapter 3

Smooth and inner product preserving action

3.1 Introduction

In the 2nd chapter we have defined continuous and faithful action of a compact quantum group on a compact, Hausdorff space. If the compact space has a smooth manifold structure then we can consider the notion of a smooth action of a CQG on that smooth, compact manifold. In this chapter we shall define a smooth action of a CQG on a smooth, compact manifold and see the connection between a smooth and a continuous action of a CQG on a compact manifold. Also for such a smooth action α, we shall deduce a necessary and sufficient condition for extending the smooth action as a well defined bimodule morphism on the $C^{\infty}(M)$ bimodule of smooth one forms on the manifold. Then we shall define an inner product preserving smooth action of a CQG and we shall show that an inner product preserving smooth action will lift to unitary representations of the CQG on the bimodules of k-forms. We start with defining a smooth action. For this chapter M will stand for a compact n-dimensional orientable manifold (possibly with boundary).

3.2 Smooth action of a CQG

Definition 3.2.1. *We call a Fréchet $*$ algebra $\mathcal{A}$ a nice algebra if it is a dense $*$-subalgebra of a C^* algebra $\mathcal{A}_1$ such that*
1. There are finitely many densely defined closed $*$ derivations $\delta_1, ..., \delta_N$ on $\mathcal{A}_1$.
2. $\mathcal{A} \subset \mathcal{D}(\delta_i)$ and $\mathcal{A}$ is stable under δ_i for all i.
3. The topology of $\mathcal{A}$ is given by the family of seminorms $\{||x||_{\underline{\alpha}} = ||\delta_{\underline{\alpha}}(x)||\}$, where $\{\underline{\alpha} = (i_1, ..., i_k) : 1 \leq i_j \leq k, k \geq 1$ is a multi index or $\underline{\alpha} = \phi(null\ index)$, $\delta_{\underline{\alpha}} = \delta_{i_1}..\delta_{i_k}$,

$\delta_\phi = \mathrm{id}$ *and* $||.||$ *is the* C^* *norm of* $\mathcal{A}_1$.

Given two such nice algebras $\mathcal{A}(\subset \mathcal{A}_1)$ and $\mathcal{B}(\subset \mathcal{B}_1)$ with finitely many derivations $\{\delta_1, ..., \delta_n\}$ and $\{\eta_1, ..., \eta_m\}$ respectively, we have $\mathcal{A}\otimes\mathcal{B} \subset \mathcal{A}_1\hat{\otimes}\mathcal{B}_1$ and $\mathcal{A}\otimes\mathcal{B}$ has finitely many derivations $\{\delta_1 \otimes \mathrm{id}, ..., \delta_n \otimes \mathrm{id}, \mathrm{id} \otimes \eta_1, ..., \mathrm{id} \otimes \eta_m\}$. We topologize $\mathcal{A} \otimes \mathcal{B}$ by the family of seminorms $\{||.||_{\underline{\alpha\beta}}\}$ as before where this time the norm $||.||$ is the spatial norm of $\mathcal{A}_1\hat{\otimes}\mathcal{B}_1$. We denote the completion with respect to this topology by $\mathcal{A}\hat{\otimes}\mathcal{B}$. It is clearly again a nice algebra with finitely many derivations. It should be mentioned that the construction of $\mathcal{A}\hat{\otimes}\mathcal{B}$ may depend on the choice of derivations and the ambient C^* algebra. However if $\mathcal{A}$ is a nuclear algebra(for example $C^\infty(M)$ with its canonical locally convex topology), $\mathcal{A}\hat{\otimes}\mathcal{B}$ does not depend upon such choices.

Example

From the discussions in chapter 1, it is clear that $C^\infty(M)$ is a 'nice' algebra in the sense described above with $C(M)$ being the ambient C^* algebra. Also for a C^* algebra $\mathcal{Q}$, $C^\infty(M)\hat{\otimes}\mathcal{Q}$ is again a nice algebra with the ambient C^* algebra being $C(M)\hat{\otimes}\mathcal{Q}$ and the derivations being $\{\eta_1 \otimes \mathrm{id}, ..., \eta_m \otimes \mathrm{id}\}$ where $\{\eta_1, ..., \eta_m\}$ are the derivations of $C^\infty(M)$. We could have taken any other set of derivations on $\mathcal{Q}$. But for a C^* algebra all derivations being bounded, the topology would not have changed. We make the convention of choosing the above derivations for a topological tensor product between a nice algebra and a C^* algebra. $C^\infty(M)$ being a nuclear algebra the topology on $C^\infty(M)\hat{\otimes}\mathcal{Q}$ does not depend upon the choice of derivations on $C^\infty(M)$.

Let E be any locally convex space. Then we can define the space of E valued smooth functions on a compact manifold M. Take a centered coordinate chart (U, ψ) around a point $x \in M$. Then an E valued function f on M is said to be smooth at x if $f \circ \psi^{-1}$ is smooth E valued function at $0 \in \mathbb{R}^n$ in the sense of [50](definition 40.1). We denote the space of E valued smooth functions on M by $C^\infty(M, E)$. We can give a locally convex topology on $C^\infty(M, E)$ by the family of seminorms given by $p_i^{K,\alpha}(f) := sup_{x\in K}||\partial^\alpha f(x)||$, where i, K, α are as before. Then we have the following

Proposition 3.2.2. *1. If E is complete, then so is* $C^\infty(M, E)$.
2. Suppose E *is a complete locally convex space. Then we have* $C^\infty(M)\hat{\otimes}E \cong C^\infty(M, E)$.
3. Let M *and* N *be two smooth compact manifolds with boundary. Then* $C^\infty(M)\hat{\otimes}C^\infty(N) \cong C^\infty(M, C^\infty(N)) \cong C^\infty(M \times N)$ *and contains* $C^\infty(M) \otimes C^\infty(N)$ *as Fréchet dense subalgebra.*
4. Let $\mathcal{A}$ *be a* C^* *algebra. Then* $C^\infty(M)\hat{\otimes}\mathcal{A} \cong C^\infty(M, \mathcal{A})$ *as Fréchet* $*$ *algebras.*

For the proof of the first statement see 44.1 of [50]. The second statement also follows from 44.1 of [50] and the fact that $C^\infty(M)$ is nuclear. In particular the isomorphism does not depend on the choice of derivations.The third and fourth statements follow from the second statement (replacing E suitably).

It is worth mentioning that what we call a 'nice' algebra is an example of a Fréchet (D^*_∞)-subalgebra of a C^* algebra in the sense of [8] (See definition 1.2 and example 1.5 of [8]). Also see [15], where such algebras were studied under the name of smooth subalgebras of a C^* algebra. We state the following (Proposition 3.6) from [8]

Proposition 3.2.3. *For $i = 1, 2$, let $\mathcal{B}_i$ be a Fréchet (D^*_∞)-subalgebra of a C^* algebra $\mathcal{A}_i$. Let $\phi : \mathcal{B}_1 \to \mathcal{B}_2$ be a $*$- homomorphism. Then the following hold:*
(1) ϕ is C^* norm decreasing.
(2) ϕ extends uniquely to a C^* algebra homomorphism from $\mathcal{A}_1$ to $\mathcal{A}_2$.
(3) If ϕ is injective, then ϕ is an isometry for the respective C^* norms.

Using the above proposition we can prove that a nice algebra is independent of its embedding in a C^* algebra. More precisely, we have the following

Lemma 3.2.4. *If $\mathcal{A}$ is a nice algebra embedded in two C^* algebras $\mathcal{A}_1$ and $\mathcal{A}_2$. Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic as C^* algebras.*

Proof:
Consider the identity map $i : \mathcal{A} \to \mathcal{A}$. By (2) of Proposition 3.2.3, we have two extensions of this map as C^* algebra homomorphisms, namely $i_{12} : \mathcal{A}_1 \to \mathcal{A}_2$ and $i_{21} : \mathcal{A}_2 \to \mathcal{A}_1$. Also as the identity map is injective, by (3) of Proposition 3.2.3, both i_{12} and i_{21} are isometric. Now we shall prove that $i_{12} \circ i_{21} : \mathcal{A}_2 \to \mathcal{A}_2$ is the identity map. For that let $a_2 \in \mathcal{A}_2$. Then by density of $\mathcal{A}$ in $\mathcal{A}_2$, we have a sequence $a_2^i (\in \mathcal{A}) \to a_2$ in $\mathcal{A}_2$ in its C^* norm. Now

$$\begin{aligned} & ||i_{12} \circ i_{21}(a_2) - a_2|| \\ = \; & ||i_{12} \circ i_{21}(a_2) - a_2^i + a_2^i - a_2|| \\ \leq \; & ||i_{12} \circ i_{21}(a_2) - i_{12} \circ i_{21}(a_2^i)|| + ||a_2^i - a_2|| \end{aligned}$$

Now since both i_{12} and i_{21} are norm preserving, we have $||i_{12} \circ i_{21}(a_2) - i_{12} \circ i_{21}(a_2^i)|| = ||a_2^i - a_2||$ proving that $i_{12} \circ i_{21}(a_2) = a_2$ i.e. $i_{12} \circ i_{21}$ is the identity map on $\mathcal{A}_2$. Similarly, it can be shown that $i_{21} \circ i_{12}$ is the identity map on $\mathcal{A}_1$ proving that i_{12} and i_{21} are inverses of each other. So $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic as C^* algebras. □

Proposition 3.2.5. *If $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ are nice algebras as above and $\Phi : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}_3$ is a bilinear map which is separately continuous in each of the arguments. Then Φ extends to a continuous linear map from the projective tensor product of $\mathcal{A}_1$ with $\mathcal{A}_2$ to $\mathcal{A}_3$. If furthermore, $\mathcal{A}_1$ is nuclear, Φ extends to a continuous map from $\mathcal{A}_1 \hat{\otimes} \mathcal{A}_2$ to $\mathcal{A}_3$.*

As a special case, suppose that $\mathcal{A}_1, \mathcal{A}_2$ are subalgebras of a nice algebra $\mathcal{A}$ and also that $\mathcal{A}_1$ is isomorphic as a Fréchet space to some nuclear space. Then the multiplication map say m of $\mathcal{A}$ extends to a continuous map from $\mathcal{A}_1 \hat{\otimes} \mathcal{A}_2$ to $\mathcal{A}$.

Proof:
By the universal property of tensor product of algebras, Φ extends as a linear map from $\mathcal{A}_1 \otimes \mathcal{A}_2$ to $\mathcal{A}_3$. We continue to denote the extension by Φ itself. It is only left to show that Φ is continuous with respect to the projective tensor product of $\mathcal{A}_1$ and $\mathcal{A}_2$. For that let us choose a seminorm r for $\mathcal{A}_3$. Then since Φ is separately continuous, we have two seminorms p on $\mathcal{A}_1$ and q on $\mathcal{A}_2$ such that there is a positive constant C with $r(\Phi(a \otimes b)) \leq Cp(a)q(b)$ for all $a \in \mathcal{A}_1$ and $b \in \mathcal{A}_2$. Then for $\xi = \sum a_i \otimes b_i$, $r(\Phi(\xi)) \leq \sum Cp(a_i)q(b_i)$. So by definition of the seminorms of projective tensor product, we have $r(\Phi(\xi)) \leq C(p \otimes q)(\xi)$ proving the continuity of Φ. □

Let M be a smooth, compact n-dimensional manifold and $\mathcal{A}$, $\mathcal{B}$ be two C^* algebras. We take a set of derivations $\{\mu_1, ..., \mu_N\}$ for some $N \in \mathbb{N}$ such that they generate the locally convex topology of $C^\infty(M)$. $C^\infty(M)$ is a nice algebra and by Lemma 3.2.4 we can take the ambient C^* algebra to be $C(M)$ without loss of generality. A similar fact holds for $C^\infty(M, \mathcal{A})$ as well. Thus we can treat $C^\infty(M, \mathcal{A})$ as embedded in $C(M, \mathcal{A})$ and the set of derivations $\{\mu_1 \otimes \mathrm{id}, ..., \mu_N \otimes \mathrm{id}\}$ generating the topology of $C^\infty(M, \mathcal{A})$. $C^\infty(M, \mathcal{A})$ is nothing but the completion of $C^\infty(M) \otimes \mathcal{A}$ with respect to this topology. Now let $\{\eta_1, ..., \eta_l\}$ be any set of closed $*$ derivations of $C(M, \mathcal{A})$ generating the locally convex topology of $C^\infty(M, \mathcal{A})$. Then by definition of topological tensor product of two nice algebras in our sense, $C^\infty(M, \mathcal{A}) \hat{\otimes} \mathcal{B}$ is the completion of $C^\infty(M, \mathcal{A}) \otimes \mathcal{B}$ with respect to the family of seminorms given by the set of closed $*$ derivations $\{\eta_1 \otimes \mathrm{id}, ..., \eta_l \otimes \mathrm{id}\}$, as $\mathcal{B}$ is a C^* algebra. From now on for any derivation D on a nice algebra $\mathcal{C}$, we shall denote the corresponding derivation $D \otimes \mathrm{id}$ on another nice algebra $\mathcal{C} \hat{\otimes} \mathcal{D}$ by $\hat{D}$. In the following we shall show that the tensor product $C^\infty(M, \mathcal{A}) \hat{\otimes} \mathcal{B}$ does not depend on the choice of closed $*$ derivations on $C(M, \mathcal{A})$. In fact we shall show that for any choice of derivations on $C^\infty(M, \mathcal{A})$, $C^\infty(M, \mathcal{A}) \hat{\otimes} \mathcal{B} = C^\infty(M, \mathcal{A} \hat{\otimes} \mathcal{B})$, where $C^\infty(M, \mathcal{A} \hat{\otimes} \mathcal{B})$ is the completion of $C^\infty(M) \otimes \mathcal{A} \otimes \mathcal{B}$ with respect to the topology generated by the seminorms given by the closed $*$ derivations $\{\mu_1 \otimes \mathrm{id}_{\mathcal{A} \hat{\otimes} \mathcal{B}}, ..., \mu_N \otimes \mathrm{id}_{\mathcal{A} \hat{\otimes} \mathcal{B}}\}$ on the C^* algebra $C(M, \mathcal{A} \hat{\otimes} \mathcal{B})$. Moreover $C^\infty(M)$ being nuclear, the tensor product does not depend upon the choice of derivations on $C^\infty(M)$. First we deduce a few results.

Lemma 3.2.6. *Consider $C^\infty(M, \mathcal{A})$ as a $C^\infty(M)$ bimodule using the algebra inclusion $C^\infty(M) \cong C^\infty(M) \otimes 1 \subset C^\infty(M, \mathcal{A})$. Let $D : C^\infty(M) \mapsto C^\infty(M, \mathcal{A})$ be a derivation. Then given any coordinate neighborhood $(U, x_1, ..., x_n)$, there exists*

$a_1, ..., a_j \in C^\infty(M, \mathcal{A})$ such that for any $m \in U$,

$$D(f)(m) = \sum_{i=1}^{n} a_i(m) \frac{\partial f}{\partial x_i}(m).$$

Proof:
It follows by standard arguments similar to those used in the proof of the fact that any derivation on $C^\infty(M)$ is a vector field. □

Corollary 3.2.7. *Let η be a derivation on $C^\infty(M, \mathcal{A})$. Then there exists a norm bounded derivation $\eta^{\mathcal{A}} : \mathcal{A} \to C^\infty(M, \mathcal{A})$ such that for all coordinate neighborhoods $(U, x_1, ..., x_n)$ and $F \in C^\infty(M, \mathcal{A})$ and for all $m \in U$,*

$$(\eta F)(m) = \sum_{i=1}^{n} a_i(m) \frac{\partial F}{\partial x_i}(m) + \eta^{\mathcal{A}}(F(m))(m),$$

for some $a_i \in C^\infty(M, \mathcal{A})$.

Proof:
Define $\eta^{\mathcal{A}}(q) := \eta(1 \otimes q)$. As any closed $*$ derivation on a C^* algebra is norm bounded, the result follows from the Lemma 3.2.6 and the observation that $\eta(f \otimes q) = \eta(f \otimes 1)(1 \otimes q) + (f \otimes 1)\eta(1 \otimes q)$. □

Lemma 3.2.8. *Let $F \in C(M, \mathcal{A}\hat{\otimes}\mathcal{B})$ such that for all $\omega \in \mathcal{B}^*$, $(\mathrm{id} \otimes \mathrm{id} \otimes \omega)F \in C^\infty(M, \mathcal{A})$. Then $F \in C^\infty(M, \mathcal{A}\hat{\otimes}\mathcal{B})$.*

Proof:
We first prove it when M is an open subset U of $\mathbb{R}^n$ with compact closure say K. We denote the standard coordinates of $\mathbb{R}^n$ by $\{x_1, ..., x_n\}$. Let us choose a point $x_0 = (x_0^1, ..., x_0^n)$ on the manifold and $h, h' > 0$ such that $(x_0^1, ..., x_0^i + h, ..., x_0^n)$ and $(x_0^1, ..., x_0^i + h^{'}, ..., x_0^n)$ both belong to the open set U for a fixed $i \in \{1, ..., n\}$. We shall show that $\frac{\partial F}{\partial x_i}(x_0)$ exists. That is we have to show that

$$\Omega^F(x_0; h) := \frac{F(x_0^1, ..., x_0^i + h, ..., x_0^n) - F(x_0^1, ..., x_0^i, ..., x_0^n)}{h}$$

is Cauchy in $\mathcal{A}\hat{\otimes}\mathcal{B}$ as $h \to 0$. For that first observe that $((\mathrm{id} \otimes \mathrm{id} \otimes \omega)F)(x) = (\mathrm{id} \otimes$

$\omega)(F(x))$ for all $x \in M$ and $\omega \in \mathcal{B}^*$, the space of bounded linear functionals on $\mathcal{B}$. Now

$$\begin{aligned}
& (\mathrm{id} \otimes \omega)(\Omega^F(x_0; h) - \Omega^F(x_0; h')) \\
= \; & \frac{h'(((\mathrm{id} \otimes \mathrm{id} \otimes \omega)F)(x_0^1, ..., x_0^i + h, ..., x_0^n) - ((\mathrm{id} \otimes \mathrm{id} \otimes \omega)F)(x_0^1, ..., x_0^i, ..., x_0^n))}{hh'} \\
& - \frac{h(((\mathrm{id} \otimes \mathrm{id} \otimes \omega)F)(x_0^1, ..., x_0^i + h', ..., x_0^n) - ((\mathrm{id} \otimes \mathrm{id} \otimes \omega)F)(x_0^1, ..., x_0^i, ..., x_0^n))}{hh'} \\
= \; & \frac{h' \int_0^h \frac{\partial}{\partial x_i}((\mathrm{id} \otimes \mathrm{id} \otimes \omega)F)(x_0^1, ..., x_0^i + u, ..., x_0^n) du}{hh'} \\
& - \frac{h \int_0^{h'} \frac{\partial}{\partial x_i}((\mathrm{id} \otimes \mathrm{id} \otimes \omega)F)(x_0^1, ..., x_0^i + v, ..., x_0^n) dv}{hh'} \\
= \; & \frac{\int_0^h \int_0^{h'} du dv \int_u^v \frac{\partial^2}{\partial x_i^2}((\mathrm{id} \otimes \mathrm{id} \otimes \omega)F)(x_0^1, ..., x_0^i + s, ..., x_0^n) ds}{hh'},
\end{aligned}$$

where all the integrals involved above are Banach space valued Bochner integrals. Let $\sup_{x \in K} || \frac{\partial^2}{\partial x_i^2}((\mathrm{id} \otimes \mathrm{id} \otimes \omega)F)(x)|| = M_\omega$. Then using the face that for a measure μ and a Banach space valued function F, $|| \int F d\mu|| \leq \int ||F|| d\mu$, we get

$$||(\mathrm{id} \otimes \omega)(\Omega^F(x_0; h) - \Omega^F(x_0; h'))|| \leq M_\omega \epsilon,$$

where $\epsilon = \min\{h, h'\}$. Now consider the family $\beta^\phi_{x_0;h,h'} = (\phi \otimes \mathrm{id})(\Omega^F(x_0; h) - \Omega^F(x_0; h'))$ for $\phi \in \mathcal{A}^*$ with $||\phi|| \leq 1$. Then for $\omega \in \mathcal{B}^*$

$$\omega(\beta^\phi_{x_0;h,h'}) = (\phi \otimes \mathrm{id})(\mathrm{id} \otimes \omega)(\Omega^F(x_0; h) - \Omega^F(x_0; h')).$$

So

$$\begin{aligned}
& |\omega(\beta^\phi_{x_0;h,h'})| \\
\leq \; & ||(\mathrm{id} \otimes \omega)(\Omega^F(x_0; h) - \Omega^F(x_0; h'))|| \\
\leq \; & M_\omega \epsilon.
\end{aligned}$$

Hence by the uniform boundedness principle we get a constant $M > 0$ such that $||(\beta^\phi_{x_0;h,h'})|| \leq M\epsilon$. But $||(\Omega^F(x_0; h) - \Omega^F(x_0; h'))|| = \sup_{||\phi|| \leq 1} ||\beta^\phi_{x_0;h,h'}||$. Therefore we get

$$||(\Omega^F(x_0; h) - \Omega^F(x_0; h'))|| \leq M\epsilon \; for \; all \; h, h' < \epsilon.$$

Hence $\Omega^F(x_0; h)$ is Cauchy as h goes to zero i.e. $\frac{\partial F}{\partial x_i}(x_0)$ exists. By similar arguments we can show the existence of higher order partial derivatives. So, for a general smooth, compact manifold M, going to the coordinate neighborhood and applying the above result we can show that $F \in C^{\infty}(M, \mathcal{A}\hat{\otimes}\mathcal{B})$. □

Applying the above Lemma for $\mathcal{A} = \mathbb{C}$, we get

Corollary 3.2.9. *For $f \in C(M, \mathcal{B})$, if $(\mathrm{id} \otimes \phi)f \in C^{\infty}(M)$ for all $\phi \in \mathcal{B}^*$, then $f \in C^{\infty}(M, \mathcal{B})$.*

Lemma 3.2.10. *The locally convex topology on $C^{\infty}(M, \mathcal{A})\hat{\otimes}\mathcal{B}$ does not depend upon the choice of derivations on $C^{\infty}(M, \mathcal{A})$, hence $C^{\infty}(M, \mathcal{A})\hat{\otimes}\mathcal{B} \cong C^{\infty}(M, \mathcal{A}\hat{\otimes}\mathcal{B})$.*

Proof:
Let us fix a choice of closed $*$ derivations $\{\eta_1, ..., \eta_l\}$ of $C(M, \mathcal{A})$ such that they generate the locally convex topolgy of $C^{\infty}(M, \mathcal{A})$. Our aim is to prove that the locally convex space obtained by completing $C^{\infty}(M, \mathcal{A}) \otimes \mathcal{B}$ with respect to the topolgy given by the set of derivations $\{\hat{\eta_1}, ..., \hat{\eta_l}\}$ of $C(M, \mathcal{A})\hat{\otimes}\mathcal{B}$ is equal to $C^{\infty}(M, \mathcal{A}\hat{\otimes}\mathcal{B})$ i.e. $C^{\infty}(M, \mathcal{A})\hat{\otimes}\mathcal{B} = C^{\infty}(M, \mathcal{A}\hat{\otimes}\mathcal{B})$. First we show that

$$C^{\infty}(M, \mathcal{A}\hat{\otimes}\mathcal{B}) \subseteq C^{\infty}(M, \mathcal{A})\hat{\otimes}\mathcal{B},$$

and the inclusion map is Fréchet continuous. To prove the above inclusion it is enough to show that if a sequence in $C^{\infty}(M) \otimes \mathcal{A} \otimes \mathcal{B}$ is Cauchy in the topology of the L.H.S., it is also Cauchy in the topology of R.H.S.. This follows from the descriptions of η_j's given in the Lemma 3.2.7. Moreover, observe that for any $\omega \in \mathcal{B}^*$, $(\mathrm{id} \otimes \omega)F \in C^{\infty}(M, \mathcal{A})$ for any $F \in C^{\infty}(M, \mathcal{A})\hat{\otimes}\mathcal{B}$. Hence by Lemma 3.2.8, we get $C^{\infty}(M, \mathcal{A})\hat{\otimes}\mathcal{B} \subseteq C^{\infty}(M, \mathcal{A}\hat{\otimes}\mathcal{B})$ as well i.e. the two spaces coincide as sets. So by closed graph theorem we conclude that they are isomorphic as Fréchet spaces as well.

Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}$ be three nice algebras in our sense embedded in C^* algebras $\widehat{\mathcal{A}_1}, \widehat{\mathcal{A}_2}, \widehat{\mathcal{B}}$ respectively. Also we choose derivations $\{\zeta_1, ..., \zeta_n\}$, $\{\eta_1, ..., \eta_m\}$ and $\{\xi_1, ..., \xi_k\}$ giving the locally convex topologies of $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}$ respectively. Then we have the nice algebras $\mathcal{A}_1\hat{\otimes}\mathcal{B}$ and $\mathcal{A}_2\hat{\otimes}\mathcal{B}$ respectively. The seminorms on $\mathcal{A}_i\hat{\otimes}\mathcal{B}$ be $\{\delta^i_{\bar{\alpha}\bar{\beta}}\}$ for $i = 1, 2$ and multiindices $\bar{\alpha}$ and $\bar{\beta}$ as discussed earlier. Now let $u : \mathcal{A}_1 \to \mathcal{A}_2$ be a continuous homomorphism. Then we have the following

Lemma 3.2.11. *the mapping $u \otimes \mathrm{id}$ is continuous with respect to the locally convex topology on the nice algebras. We denote the continuous extension again by $u \otimes \mathrm{id}$.*

Proof:
Fix a seminorm $\{\delta^2_{\bar{\alpha}\bar{\beta}}\}$ for some multiindices $\bar{\alpha}, \bar{\beta}$ for $\mathcal{A}_2\hat{\otimes}\mathcal{B}$. Now let $X \in \mathcal{A}_2 \otimes \mathcal{B}$. Then by definition $||\delta^2_{\bar{\alpha}\bar{\beta}}(u \otimes \mathrm{id})(X)|| = \sup_{||\omega|| \leq 1, \omega \in \mathcal{B}^*} ||(\mathrm{id} \otimes \omega)(\delta^2_{\bar{\alpha}\bar{\beta}}(u \otimes \mathrm{id})(X))||$. So

$$\begin{aligned} ||\delta^2_{\bar{\alpha}\bar{\beta}}(u \otimes \mathrm{id})(X)|| &= \sup_{||\omega|| \leq 1, \omega \in \mathcal{B}^*} ||(\mathrm{id} \otimes \omega)((\eta_{\bar{\alpha}} \otimes \mathrm{id})(\mathrm{id} \otimes \xi_{\bar{\beta}})(u \otimes \mathrm{id})(X))|| \\ &= \sup_{||\omega|| \leq 1, \omega \in \mathcal{B}^*} ||\eta_{\bar{\alpha}}(u((\mathrm{id} \otimes \omega)(\mathrm{id} \otimes \xi_{\bar{\beta}})X))||. \end{aligned}$$

So by continuity of u we have a multi index α' and a constant $C > 0$ such that $||\eta_{\bar{\alpha}}(u((\mathrm{id} \otimes \omega)(\mathrm{id} \otimes \xi_{\bar{\beta}})X))|| \leq C||\zeta_{\bar{\alpha'}}(((\mathrm{id} \otimes \omega)(\mathrm{id} \otimes \xi_{\bar{\beta}})X))||$. Hence

$$||\delta^2_{\bar{\alpha}\bar{\beta}}(u \otimes \mathrm{id})(X)|| \leq \sup_{||\omega|| \leq 1, \omega \in \mathcal{B}^*} C||(\mathrm{id} \otimes \omega)(\delta^1_{\bar{\alpha'}\bar{\beta}}(X))||.$$

That is $||\delta^2_{\bar{\alpha}\bar{\beta}}(u \otimes \mathrm{id})(X)|| \leq C||\delta^1_{\bar{\alpha'}\bar{\beta}}(X)||$ for some constant $C > 0$ and some multi index $\alpha^{'}$, proving the continuity of $(u \otimes \mathrm{id})$. □

Now we are ready to define a smooth action of a CQG $\mathcal{Q}$ on a compact manifold M (with or without boundary).

Definition 3.2.12. *Let $\mathcal{Q}$ be a compact quantum group. A $\mathbb{C}$ linear map $\alpha : C^\infty(M) \to C^\infty(M)\hat{\otimes}\mathcal{Q}$ is said to be a smooth action of $\mathcal{Q}$ on M if*
1. α is a continuous $*$ algebra homomorphism.
2. $(\alpha \otimes \mathrm{id})\alpha = (\mathrm{id} \otimes \Delta)\alpha$ (co-associativity).
3. Sp $\alpha(C^\infty(M))(1 \otimes \mathcal{Q})$ is dense in $C^\infty(M)\hat{\otimes}\mathcal{Q}$ in the corresponding Fréchet topology.

Again proceeding along the same lines of [48], we can have the maximal dense subspace say $\mathcal{A}$, over which α is algebraic. in fact $\mathcal{A}$ is a subalgebra. From now on we shall use this maximal algebra and denote it by $\mathcal{A}$.

Lemma 3.2.13. *A smooth action α of $\mathcal{Q}$ on M extends to a C^* action on $C(M)$ which is denoted by α again.*

Proof:
It follows from (1) of Proposition 3.2.3. □

Lemma 3.2.14. *Given a C^* action $\alpha : C(M) \to C(M)\hat{\otimes}\mathcal{Q}$, $\alpha(C^\infty(M)) \subset C^\infty(M, \mathcal{Q})$ if and only if $(\mathrm{id} \otimes \phi)(\alpha(C^\infty(M))) \subset C^\infty(M)$ for all bounded linear functionals ϕ on $\mathcal{Q}$.*

Proof:
For the only if part see discussion in Subsection 4.2.
The converse part follows from Corollary 3.2.9.

Theorem 3.2.15. *Suppose we are given a C^* action α of $\mathcal{Q}$ on M. Then following are equivalent:*
1) $\alpha(C^\infty(M)) \subset C^\infty(M, \mathcal{Q})$ and $\overline{Sp}\ \alpha(C^\infty(M))(1 \otimes \mathcal{Q}) = C^\infty(M, \mathcal{Q})$.
2) α is smooth.
3) $(\mathrm{id} \otimes \phi)\alpha(C^\infty(M)) \subset C^\infty(M)$ for all state ϕ on $\mathcal{Q}$, and there is a Fréchet dense subalgebra $\mathcal{A}$ of $C^\infty(M)$ over which α is algebraic.

Proof:
(1)$\Rightarrow$ (2):

Observe that it is enough to show that α is Fréchet continuous. Let $f_n \to f$ in Fréchet topology of $C^\infty(M)$ and $\alpha(f_n) \to \xi$ in Fréchet topology of $C^\infty(M, \mathcal{Q})$. Then $f_n \to f$ in norm topology of $C(M)$. So by the C^* continuity of α, $\alpha(f_n) \to \alpha(f)$. Similarly, $\alpha(f_n) \to \xi$ in the norm topology of $C(M, \mathcal{Q})$. So $\alpha(f) = \xi$ and by the closed graph theorem α is Fréchet continuous.
(2)$\Rightarrow$ (3):
Follows from the remark after definition 3.2.4. and Lemma 3.2.14.
(3)$\Rightarrow$ (1):
From Lemma 3.2.14, it follows that $\alpha(C^\infty(M)) \subset C^\infty(M, \mathcal{Q})$. The density condition follows from densities of $\mathcal{A}$ and $\mathcal{A} \otimes \mathcal{Q}_0$ in $C^\infty(M)$ and $C^\infty(M, \mathcal{Q})$ respectively. □

As we are interested in quantum group symmetries of certain Fréchet algebras, we may wonder whether it is possible to allow some topological structures on quantum groups (e.g. some kind of Hopf-Fréchet algebras) which are more general than the C^* algebraic structure. Indeed, any compact group G acting smoothly on a compact smooth manifold M is necessarily a Lie group and one may consider $C^\infty(G)$ (with its Fréchet topology) as the symmetry object. However the action of G is completely determined by the Hopf-algebraic coaction of the $*$-algebra $C(G)_0$ on a suitable Fréchet dense subalgebra of $C^\infty(M)$. Thus, no greater generality is really achieved by considering $C^\infty(G)$ instead of $C(G)$. This is one reason for which we restricted the category of quantum group to the usual C^* algebraic compact quantum groups. The other reason of not exploring any kind of Fréchet-algebraic compact quantum groups is the absence of any general notion or theory of such structures in the noncommutative world. For quantum groups coming from q-deformation of classical Lie groups one can possibly consider some analogue of Fréchet topology, but we have no idea how, if at all, this can be done for more general compact quantum groups acting smoothly on $C^\infty(M)$.

Recall the definition of an isometric action of a compact quantum group $\mathcal{Q}$ on a smooth, compact, Riemannian manifold M.

Theorem 3.2.16. *$QISO^{\mathcal{L}}$ (and hence any subobject in the category $\mathcal{Q}^{\mathcal{L}}$) has a smooth action on $C^\infty(M)$.*

Proof:
We denote the C^* action of $QISO^{\mathcal{L}}$ on $C(M)$ by α. By Sobolev embedding theorem, for any state ϕ on $QISO^{\mathcal{L}}$, $(\mathrm{id} \otimes \phi)(C^\infty(M)) \subset C^\infty(M)$. Let $\{e_{ij} : j = 1, ..., d_i\}$ be the orthonormal eigenvectors of $\mathcal{L}$ forming a basis for the eigen space corresponding to the eigenvector λ_i. We denote the linear span of $\{e_{ij} : 1 \leq j \leq d_i, i \geq 1\}$ by $\mathcal{A}_0^\infty$. Then this is a subspace of $C^\infty(M)$. Furthermore, it is easy to see that α is algebraic over $\mathcal{A}_0^\infty$ and hence total. The proof of the theorem will be complete by applying Lemma 3.2.15,

if we can show that $\mathcal{A}_0^\infty$ is Fréchet dense in $C^\infty(M)$ which is a consequence of Sobolev theorem. However, we include a proof for the sake of completeness. The idea is similar to that of Lemma 2.3 of [22].

By Theorem 1.2 of [21] There are constants C and C' such that $||e_{ij}||_\infty \leq C|\lambda_i|^{\frac{n-1}{4}}$ and $d_i \leq C'|\lambda_i|^{\frac{n-1}{2}}$, where n is the dimension of the manifold. For $f \in C^\infty(M)$ there are complex numbers f_{ij} such that $\sum_{ij} f_{ij}e_{ij}$ converges to f in L^2 norm. Since $f \in dom(\mathcal{L}^k)$ for all $k \geq 1$, $\sum_{ij} |\lambda_i|^{2k}|f_{ij}|^2 < \infty$ for all k. Choose and fix sufficiently large k such that $\sum_{i\geq 0} |\lambda_i|^{n-2k} < \infty$. This is possible by the well-known Weyl asymptotics of the eigenvalues of Laplacian.

$\mathcal{L}(\sum_{ij} f_{ij}e_{ij}) = \sum_{ij} \lambda_i f_{ij}e_{ij}$ converges to $\mathcal{L}(f)$ in the L^2 norm. By Cauchy-Scwartz inequality,

$$\sum_{ij} |\lambda_i f_{ij}|||e_{ij}||_\infty \leq C(C')^{\frac{1}{2}}(\sum_{ij} |f_{ij}|^2|\lambda_i|^{2k})^{\frac{1}{2}}(\sum_{i\geq 0} |\lambda_i|^{n-2k}) < \infty.$$

Hence $\mathcal{L}(\sum_{ij} f_{ij}e_{ij}) = \sum_{ij} \lambda_i f_{ij}e_{ij}$ converges to $\mathcal{L}(f)$ in the sup norm of $C(M)$. Similarly we can show that $\mathcal{L}^k(\sum_{ij} f_{ij}e_{ij})$ converges in the sup norm of $C(M)$ for any k. So $\mathcal{A}_0^\infty$ is Fréchet dense in $C^\infty(M)$.

□

3.2.1 Defining $d\alpha$ for a smooth action α

Let $\alpha : C^\infty(M) \to C^\infty(M, \mathcal{Q})$ be a smooth action and set $d\alpha(df) := (d \otimes \mathrm{id})\alpha(f)$ for all $f \in C^\infty(M)$.

Theorem 3.2.17. *$d\alpha$ extends to a well defined continuous map from $\Omega^1(C^\infty(M))$ to $\Omega^1(C^\infty(M))\bar{\otimes}\mathcal{Q}$ satisfying $d\alpha(df) = (d \otimes \mathrm{id})\alpha(f)$, if and only if*

$$(\nu \otimes \mathrm{id})\alpha(f)\alpha(g) = \alpha(g)(\nu \otimes \mathrm{id})\alpha(f) \tag{3.2.1}$$

for all $f, g \in C^\infty(M)$ and all smooth vector fields ν on M.

Proof:
Only if part ⇒ We have $d\alpha(df.g) = (d\otimes\mathrm{id})\alpha(f).\alpha(g)$, $d\alpha(g.df) = \alpha(g).(d\otimes\mathrm{id})\alpha(f)$. But $df.g = g.df$ in $\Omega^1(C^\infty(M))$, which gives $(d \otimes \mathrm{id})\alpha(f).\alpha(g) = \alpha(g).(d \otimes \mathrm{id})\alpha(f)$, $\forall f, g \in C^\infty(M)$. Observe that as ν is a smooth vector field, ν is a Fréchet continuous map from $C^\infty(M)$ to $C^\infty(M)$. Thus it is enough to prove (3.2.1) for f, g belonging to the Fréchet dense subalgebra $\mathcal{A}$ as in Theorem 3.2.15 . Let $\alpha(f) = f_{(0)} \otimes f_{(1)}$ and $\alpha(g) = g_{(0)} \otimes g_{(1)}$(Sweedler's notation). Let $x \in M$ and $(U, x_1, ..., x_n)$ be a coordinate neighbourhood around x. Then $[(d \otimes \mathrm{id})\alpha(f)\alpha(g)](x) = \sum_{i=1}^n g_{(0)}(x)\frac{\partial f_{(0)}}{\partial x_i}(x)f_{(1)}g_{(1)}dx_i|_x$.

So

$$\begin{aligned} & [(d \otimes \mathrm{id})\alpha(f)\alpha(g)](x) = [\alpha(g)(d \otimes \mathrm{id})\alpha(f))](x) \\ \Rightarrow \quad & g_{(0)}(x)\frac{\partial f_{(0)}}{\partial x_i}(x)f_{(1)}g_{(1)} = g_{(0)}(x)\frac{\partial f_{(0)}}{\partial x_i}(x)g_{(1)}f_{(1)} \end{aligned} \tag{3.2.2}$$

for all $i = 1, ..., n$. Now let $a_i \in C^\infty(M)$ for $i = 1, ..., n$ such that $\nu(x) = \sum_{i=1}^n a_i(x)\frac{\partial}{\partial x_i}|_x$ for all $x \in U$.
So

$$\begin{aligned} & [(\nu \otimes \mathrm{id})\alpha(f)\alpha(g)](x) \\ = \quad & \sum_{i=1}^n a_i(x)\frac{\partial f_{(0)}}{\partial x_i}(x)g_{(0)}(x)f_{(1)}g_{(1)} \end{aligned}$$

and

$$\begin{aligned} & [\alpha(g)(\nu \otimes \mathrm{id})\alpha(f)](x) \\ = \quad & \sum_{i=1}^n a_i(x)\frac{\partial f_{(0)}}{\partial x_i}(x)g_{(0)}(x)g_{(1)}f_{(1)} \end{aligned}$$

Hence by (3.2.2) $[\alpha(g)(\nu \otimes \mathrm{id})\alpha(f)](x) = [(\nu \otimes \mathrm{id})\alpha(f)\alpha(g)](x)$ for all $x \in M$ i.e. $[\alpha(g)(\nu \otimes \mathrm{id})\alpha(f)] = [(\nu \otimes \mathrm{id})\alpha(f)\alpha(g)]$ for all $f, g \in \mathcal{A}$.

Proof of the if part $\Rightarrow$ This needs a number of intermediate lemmas. Let $x \in M$ and $(U, x_1, ..., x_n)$ be a coordinate neighbourhood around it. Choose smooth vector fields ν_i's on M which are $\frac{\partial}{\partial x_i}$ on U. So $[\alpha(g)(\nu_i \otimes \mathrm{id})\alpha(f)](x) = \frac{\partial f_{(0)}}{\partial x_i}(x)g_{(0)}(x)g_{(1)}f_{(1)}$ and $[(\nu_i \otimes \mathrm{id})\alpha(f)\alpha(g)](x) = \frac{\partial f_{(0)}}{\partial x_i}(x)g_{(0)}(x)f_{(1)}g_{(1)}$. Hence by the assumption

$$\begin{aligned} & \sum_i \frac{\partial f_{(0)}}{\partial x_i}(x)g_{(0)}(x)g_{(1)}f_{(1)}dx_i|_x = \sum_i \frac{\partial f_{(0)}}{\partial x_i}(x)g_{(0)}(x)f_{(1)}g_{(1)}dx_i|_x \\ \Rightarrow \quad & [(d \otimes \mathrm{id})\alpha(f)\alpha(g)](x) = [\alpha(g)(d \otimes \mathrm{id})\alpha(f)](x) \end{aligned}$$

Since x is arbitrary, we conclude that $[\alpha(g)(d \otimes \mathrm{id})\alpha(f)] = [(d \otimes \mathrm{id})\alpha(f)\alpha(g)]$ for all $f, g \in \mathcal{A}$. So by Fréchet continuity of d and α we can prove the result for $f, g \in C^\infty(M)$. $\square$

We use the commutativity to deduce the following:

Lemma 3.2.18. *For $F \in C^\infty(\mathbb{R}^n)$ and $g_1, g_2, .., g_n \in C^\infty(M)$*

$$(d \otimes \mathrm{id})\alpha(F(g_1, ..., g_n)) = \sum_{i=1}^{n} \alpha(\partial_i F(g_1, ..., g_n))(d \otimes \mathrm{id})\alpha(g_i), \tag{3.2.3}$$

where $\partial_i F$ denotes the partial derivative of F with respect to the ith coordinate of $\mathbb{R}^n$.

Proof:
As $\{(g_1(x) \ldots g_n(x)) | x \in M\}$ is a compact subset of $\mathbb{R}^n$, for $F \in C^\infty(\mathbb{R}^n)$, we get a sequence of polynomials P_m in $\mathbb{R}^n$ such that $P_m(g_1, ..., g_n)$ converges to $F(g_1, ..., g_n)$ in the Fréchet topology of $C^\infty(M)$. We see that for P_m,

$$\begin{aligned} & (d \otimes \mathrm{id})\alpha(P_m(g_1, ..g_n)) \\ = \; & (d \otimes \mathrm{id})P_m(\alpha(g_1, ..., g_n)) \\ = \; & \sum_{i=1}^{n} \alpha(\partial_i P_m(g_1, ..., g_n))(d \otimes \mathrm{id})\alpha(g_i), \end{aligned}$$

using $(d \otimes \mathrm{id})\alpha(f)\alpha(g) = \alpha(g)(d \otimes \mathrm{id})\alpha(f)$ as well as the Leibnitz rule for $(d \otimes \mathrm{id})$. The lemma now follows from Fréchet continuity of α and $(d \otimes \mathrm{id})$. □

Lemma 3.2.19. *Let U be a coordinate neighborhood. Also let $g_1, g_2, ..., g_n \in C^\infty(M)$ be such that $(g_1|_U, \ldots g_n|_U)$ gives a local coordinate system on U. Then*

$$(d \otimes \mathrm{id})\alpha(f) = \sum_{j=1}^{n} \alpha(\partial_{g_j} f)(d \otimes \mathrm{id})\alpha(g_j),$$

for all $f \in C^\infty(M)$ supported in U.

Proof:
Let $F \in C^\infty(\mathbb{R}^n) \to \mathbb{R}$ be a smooth function such that $f(m) = F(g_1(m), ..., g_n(m)) \; \forall m \in U$. Choose $\chi \in C^\infty(M)$ with $\chi \equiv 1$ on $K = supp(f)$ and $supp(\chi) \subset U$. Then $\chi f = f$ as $\chi \equiv 1$ on K. Hence $\chi F(g_1, ..., g_n) = f(\chi F = \chi f = f \; on \; U, \; \chi F = 0 \; outside \; U)$. Also $\chi^2 F(g_1, ..., g_n) = \chi F(g_1, ..., g_n)$, since on K, $\chi^2 = \chi = 1$ and outside K, $\chi^2 F(g_1, ..., g_n) = \chi F(g_1, ..., g_n) = 0$. Let $T := \alpha(\chi)$ and $S := \alpha(F(g_1, ..., g_n))$. Also denote $(d \otimes \mathrm{id})\alpha(F(g_1, ..., g_n))$ by S' and $(d \otimes \mathrm{id})\alpha(\chi)$ by T'.

So we have $T^2 S = TS$ and by (3.2.1) we have $T' T = T T'$ and $S' S = S S'$.

$$\begin{aligned} T^2S^{'} &= \alpha(\chi^2)(d\otimes \mathrm{id})\alpha(F(g_1,...,g_n)) \\ &= \alpha(\chi^2)\sum_{i=1}^{n}\alpha(\partial_i F(g_1,...,g_n))(d\otimes \mathrm{id})\alpha(g_i) \ (by\ (3.2.3)) \\ &= \alpha(\chi)\sum_{i=1}^{n}\alpha(\chi\partial_i F(g_1,...,g_n))(d\otimes \mathrm{id})\alpha(g_i) \\ &= \alpha(\chi)\sum_{i=1}^{n}\alpha(\partial_{g_i} f)(d\otimes \mathrm{id})\alpha(g_i) \ (as\ supp(\partial_{g_i} f)\subset K). \end{aligned} \tag{3.2.4}$$

$$\begin{aligned} TS^{'} &= \alpha(\chi)(d\otimes \mathrm{id})\alpha(F(g_1,...,g_n)) \\ &= \sum_{i=1}^{n}\alpha(\chi\partial_i F(g_1,...,g_n))(d\otimes \mathrm{id})\alpha(g_i) \\ &= \sum_{i=1}^{n}\alpha(\chi^2\partial_i F(g_1,...,g_n))(d\otimes \mathrm{id})\alpha(g_i) \\ &= \alpha(\chi)\sum_{i=1}^{n}\alpha(\partial_{g_i} f)(d\otimes \mathrm{id})\alpha(g_i) \end{aligned} \tag{3.2.5}$$

Combining (3.2.4) and (3.2.5) we get

$$T^2S^{'} = TS^{'} \tag{3.2.6}$$

Now

$$\begin{aligned} & T^2S = TS \\ \Rightarrow\ & (d\otimes \mathrm{id})(T^2S) = (d\otimes \mathrm{id})TS \\ \Rightarrow\ & 2TT'S + T^2S' = TS' + T'S (by\ Leibnitz\ rule\ and\ T'T = TT') \\ \Rightarrow\ & 2TT'S = T'S\ (by\ (3.2.6)) \\ \Rightarrow\ & 2\alpha(\chi)(d\otimes \mathrm{id})\alpha(\chi)\alpha(F(g_1,...,g_n)) = (d\otimes \mathrm{id})\alpha(\chi)\alpha(F(g_1,...,g_n)) \\ \Rightarrow\ & 2\alpha(\chi^2)(d\otimes \mathrm{id})\alpha(\chi)\alpha(F(g_1,...,g_n)) = \alpha(\chi)(d\otimes \mathrm{id})\alpha(\chi)\alpha(F(g_1,...,g_n)) \\ \Rightarrow\ & 2(d\otimes \mathrm{id})\alpha(\chi)\alpha(f) = (d\otimes \mathrm{id})\alpha(\chi)\alpha(f)(\ using\ the\ assumption\ and\ \chi^2F = f) \\ \Rightarrow\ & (d\otimes \mathrm{id})\alpha(\chi)\alpha(f) = 0 \end{aligned} \tag{3.2.7}$$

So

$$\begin{aligned}
(d\otimes \mathrm{id})\alpha(f) &= (d\otimes \mathrm{id})\alpha(\chi f)\\
&= (d\otimes \mathrm{id})\alpha(\chi)\alpha(f) + \alpha(\chi)(d\otimes \mathrm{id})\alpha(f)\\
&= \alpha(\chi)(d\otimes \mathrm{id})\alpha(f)(\ \textit{by}\ (3.2.7))\\
&= \alpha(\chi)(d\otimes \mathrm{id})\alpha(\chi F(g_1,...,g_n))\\
&= \alpha(\chi)(d\otimes \mathrm{id})\alpha(\chi)\alpha(F(g_1,...,g_n)) + \alpha(\chi^2)(d\otimes \mathrm{id})\alpha(F(g_1,...,g_n))\\
&= (d\otimes \mathrm{id})\alpha(\chi)\alpha(f) + \alpha(\chi^2)(d\otimes \mathrm{id})\alpha(F(g_1,...,g_n))(\textit{Again by assumption})\\
&= \alpha(\chi^2)\sum_{i=1}^{n}\alpha(\partial_i F(g_1,...,g_n))(d\otimes \mathrm{id})\alpha(g_i)\\
&= \sum_{i=1}^{n}\alpha(\chi^2\partial_i F(g_1,...,g_n))(d\otimes \mathrm{id})\alpha(g_i)\\
&= \sum_{i=1}^{n}\alpha(\partial_{g_i} f)(d\otimes \mathrm{id})\alpha(g_i)
\end{aligned}$$

□

Now to complete the proof of the theorem, we want to first define a bimodule morphism β extending $d\alpha$ locally, i.e. we define $\beta_U(\omega)$ for any coordinate neighborhood U and any smooth 1-form ω supported in U as follows:

Choose C^∞ functions $g_1 \dots g_n$ as before such that they give a local coordinate system on U and ω has the unique expression $\omega = \sum_{j=1}^n \phi_j dg_j$. Then we define $\beta_U(\omega) := \sum_{j=1}^n \alpha(\phi_j)(d\otimes \mathrm{id})\alpha(g_j)$. We show that this definition is independent of the choice of the coordinate function i.e. if $(h_1,\dots,h_n)$ is another such set of coordinate functions on U with $\omega = \sum_{j=1}^n \psi_j dh_j$ for some ψ_j's in $C^\infty(M)$, then
$\sum_{j=1}^n \alpha(\phi_j)(d\otimes \mathrm{id})\alpha(g_j) = \sum_{j=1}^n \alpha(\psi_j)(d\otimes \mathrm{id})\alpha(h_j)$.

To that end Let χ be a smooth function which is 1 on the support of ω and 0 outside U. We have $F_1,\dots,F_n \in C^\infty(\mathbb{R}^N)$ such that $g_j = F_j(h_1,\dots,h_n)$ for all $j = 1,\dots,n$ on U. Then $\chi g_j = \chi F_j(h_1,\dots,h_n)$ for all $j = 1,\dots,n$.
Hence $dg_j = \sum_{k=1}^n \partial_{h_k}(F_j(h_1,\dots,h_n))dh_k$ on U, so that $\omega = \sum_{j,k}\chi\phi_j\partial_{h_k}(F_j(h_1,\dots,h_n))dh_k$.
Hence $\psi_k = \sum_j \chi\phi_j\partial_{h_k}(F_j(h_1,\dots,h_n))$.
Also, note that, as $\chi \equiv 1$ on the support of ϕ_j for all j, we must have $\phi_j\partial_{h_k}(\chi) \equiv 0$, so $\chi\phi_j\partial_{h_k}(F_j(h_1,\dots,h_n)) = \chi\phi_j\partial_{h_k}(\chi F_j(h_1,\dots,h_n))$. Thus

$$
\begin{aligned}
& \sum_k \alpha(\psi_k)(d \otimes \mathrm{id})\alpha(h_k) \\
= & \sum_{k,j} \alpha(\chi\phi_j \partial_{h_k}(F_j(h_1,\ldots,h_n)))(d \otimes \mathrm{id})\alpha(h_k) \\
= & \sum_{k,j} \alpha(\phi_j)\alpha(\partial_{h_k}(\chi F_j(h_1,\ldots,h_n)))(d \otimes \mathrm{id})\alpha(h_k) \\
= & \sum_j \alpha(\phi_j)(d \otimes \mathrm{id})\alpha(\chi F_j(h_1,\ldots,h_n)) \ (\textit{by Lemma } 3.2.18) \\
= & \sum_j \alpha(\phi_j)(d \otimes \mathrm{id})\alpha(\chi g_j) \\
= & \sum_j \alpha(\phi_j)(d \otimes \mathrm{id})\alpha(g_j)
\end{aligned}
$$

Where the last step follows from Leibnitz rule and the fact that

$$
\begin{aligned}
& \alpha(\phi_j)(d \otimes \mathrm{id})(\alpha(\chi)) \\
= & \sum_k \alpha(\phi_j)\alpha(\partial_{h_k}(\chi))(d \otimes \mathrm{id})(\alpha(h_k)) \\
= & \sum_k \alpha(\phi_j \partial_{h_k}(\chi))(d \otimes \mathrm{id})(\alpha(h_k)) \\
= & \ 0 \ (\textit{using } \phi_j \partial_{h_k}(\chi)) \equiv 0).
\end{aligned}
$$

Hence the definition is indeed independent of choice of coordinate system. Then for any two coordinate neighborhoods U and V, $\beta_U(\omega) = \beta_V(\omega)$ for any ω supported in $U \cap V$. It also follows from the definition and Lemma 3.2.19 that β_U is a $C^\infty(M)$ bimodule morphism and $\beta_U(df) = (d \otimes \mathrm{id})\alpha(f)$ for all $f \in C^\infty(M)$ supported in U. Now we define β globally as follows:

Choose (and fix) a smooth partition of unity $\{\chi_1,\ldots,\chi_l\}$ subordinate to a cover $\{U_1,\ldots,U_l\}$ of the manifold M such that each U_i is a coordinate neighborhood. Define β by:

$$
\beta(\omega) := \sum_{i=1}^{l} \beta_{U_i}(\chi_i \omega),
$$

for any smooth one form ω. Then for any $f \in C^\infty(M)$,

$$\begin{aligned}
\beta(df) &= \sum_{i=1}^{l} \beta_{U_i}(\chi_i df) \\
&= \sum_{i=1}^{l} \beta_{U_i}(d(\chi_i f) - f d\chi_i) \\
&= \sum_{i=1}^{l} [(d \otimes \mathrm{id})\alpha(f\chi_i) - \alpha(f)(d \otimes \mathrm{id})\alpha(\chi_i)] \\
&= \sum_{i=1}^{l} (d \otimes \mathrm{id})\alpha(f)\alpha(\chi_i) \text{ (by Leibnitz rule)} \\
&= (d \otimes \mathrm{id})\alpha(f).
\end{aligned}$$

This completes the proof of the Theorem 3.2.17.

□

We end this subsection with an interesting fact which will be used later. For this, we need to recall that $C^\infty(M)$ is a nuclear locally convex space and hence so is any quotient by closed ideals.

Lemma 3.2.20. *If $\mathcal{Q}$ has a faithful smooth action on $C^\infty(M)$, where M is compact manifold, then for every fixed $x \in M$ there is a well-defined extension of the counit map ϵ to the subalgebra $\mathcal{Q}_x^\infty := \{\alpha_r(f)(x) : f \in C^\infty(M)\}$ satisfying $\epsilon(\alpha(f)(x)) = f(x)$, where α_r is the reduced action discussed earlier.*

Proof:
Replacing $\mathcal{Q}$ by $\mathcal{Q}_r$ we can assume without loss of generality that $\mathcal{Q}$ has faithful Haar state and $\alpha = \alpha_r$. In this case $\mathcal{Q}$ will have bounded antipode κ. Let $\alpha_x : C^\infty(M) \to \mathcal{Q}_x^\infty$ be map $\alpha_x(f) = \alpha(f)(x)$. It is clearly continuous w.r.t. the Frechet topology of $C^\infty(M)$ and hence the kernel say $\mathcal{I}_x$ is a closed ideal, so that the quotient which is isomorphic to $\mathcal{Q}_x^\infty$ is a nuclear space. Let us consider $\mathcal{Q}_x^\infty$ with this topology and then by nuclearity, the projective and injective tensor products with $\mathcal{Q}$ (viewed as a separable Banach space, where separability follows from the fact that $\mathcal{Q}$ faithfully acts on the separable C^* algebra $C(M)$) coincide with $\mathcal{Q}_x^\infty \hat{\otimes} \mathcal{Q}$ and the multiplication map $m : \mathcal{Q}_x^\infty \hat{\otimes} \mathcal{Q} \to \mathcal{Q}$ is indeed continuous. Now, observe that $\mathcal{Q}_x^\infty \hat{\otimes} \mathcal{Q}$ is isomorphic as a Fréchet algebra with the quotient of $C^\infty(M) \hat{\otimes} \mathcal{Q}$ by the ideal $\mathrm{Ker}(\alpha_x \otimes \mathrm{id}) = \mathcal{I}_x \hat{\otimes} \mathcal{Q}$. Moreover, it follows from the relation $\Delta \circ \alpha = (\alpha \otimes \mathrm{id}) \circ \alpha$ that Δ maps $\mathcal{I}_x$ to $\mathcal{I}_x \hat{\otimes} \mathcal{Q}$, and in fact it is the restriction of the Fréchet-continuous map $\alpha \otimes \mathrm{id}$ there, hence induces a continuous map from $\mathcal{Q}_x^\infty \cong C^\infty(M)/\mathcal{I}_x$ to $\mathcal{Q}_x^\infty \hat{\otimes} \mathcal{Q} \cong (C^\infty(M) \hat{\otimes} \mathcal{Q})/(\mathcal{I}_x \hat{\otimes} \mathcal{Q})$. Thus, the composite

map $m \circ (\mathrm{id} \otimes \kappa) \circ \Delta : \mathcal{Q}_x^\infty \to \mathcal{Q}_x^\infty \hat{\otimes} \mathcal{Q}$ is continuous and this coincides with $\epsilon(\cdot)1_\mathcal{Q}$ on the Fréchet-dense subalgebra of $\mathcal{Q}_x^\infty$ spanned by elements of the form $\alpha(f)(x)$, with f varying in a Fréchet-dense subalgebra of $C^\infty(M)$ on which the action is algebraic. This completes the proof of the lemma. □

Remark 3.2.21. *It is clear that ϵ extends to the $*$-algebra generated by $\mathcal{Q}_x^\infty$ and $\mathcal{Q}_0$ and the extension is still a $*$-homomorphism. This follows from the facts that (i) ϵ is $*$-homomorphism on $\mathcal{Q}_0$, (ii) $f \mapsto \epsilon(\alpha_r(f)(x)) = f(x)$ is continuous with respect to the Fréchet topology of $C^\infty(M)$, and (iii) $\mathcal{Q}_x^\infty \bigcap \mathcal{Q}_0$ is dense in $\mathcal{Q}_x^\infty$ because it contains the elements of the form $\alpha_r(f)(x)$ for f varying in a Fréchet-dense $*$-algebra on which α is algebraic (so that $\alpha_r(f)(x) \in \mathcal{Q}_0$).*

3.3 Action which preserves a Riemannian inner product

3.3.1 Equivariant representation of a CQG over Hilbert Fréchet bimodules

We generalize the notion of unitary representation on Hilbert spaces to another direction, namely on Hilbert bimodules over unital topological $*$-algebras. Let $\mathcal{E}$ be a Hilbert $\mathcal{C}-\mathcal{D}$ bimodule over topological $*$-algebras $\mathcal{C}$ and $\mathcal{D}$ and let $\mathcal{Q}$ be a compact quantum group. If we consider $\mathcal{Q}$ as a bimodule over itself, then we can form the exterior tensor product $\mathcal{E}\bar{\otimes}\mathcal{Q}$ which is a $\mathcal{C}\hat{\otimes}\mathcal{Q} - \mathcal{D}\hat{\otimes}\mathcal{Q}$ bimodule. Also let $\alpha_\mathcal{C} : \mathcal{C} \to \mathcal{C}\hat{\otimes}\mathcal{Q}$ and $\alpha_\mathcal{D} : \mathcal{D} \to \mathcal{D}\hat{\otimes}\mathcal{Q}$ be topological actions of $\mathcal{C}$ and $\mathcal{D}$ on $\mathcal{Q}$ in the sense discussed earlier. Then using α we can give $\mathcal{E}\bar{\otimes}\mathcal{Q}$ a $\mathcal{C}-\mathcal{D}$ bimodule structure given by $a.\eta.a' = \alpha_\mathcal{C}(a)\eta\alpha_\mathcal{D}(a')$, for $\eta \in \mathcal{E}\bar{\otimes}\mathcal{Q}$ and $a \in \mathcal{C}, a' \in \mathcal{D}$ (but without any $\mathcal{D}$ valued inner product).

Definition 3.3.1. *A $\mathbb{C}$-linear map $\Gamma : \mathcal{E} \to \mathcal{E}\bar{\otimes}\mathcal{Q}$ is said to be an $\alpha_\mathcal{D}$ equivariant unitary representation of $\mathcal{Q}$ on $\mathcal{E}$ if*
1. $\Gamma(\xi.d) = \Gamma(\xi)\alpha_\mathcal{D}(d)$ *and* $\Gamma(c.\xi) = \alpha_\mathcal{C}(c)\Gamma(\xi)$*) for* $c \in \mathcal{C}, d \in \mathcal{D}$.
2. $<< \Gamma(\xi), \Gamma(\xi') >>= \alpha_\mathcal{D}(<< \xi, \xi' >>)$*, for* $\xi, \xi' \in \mathcal{E}$.
3. $(\Gamma \otimes \mathrm{id})\Gamma = (\mathrm{id} \otimes \Delta)\Gamma$ *(co associativity)*
4. $\overline{Sp}\ \Gamma(\mathcal{E})(1 \otimes \mathcal{Q}) = \mathcal{E}\bar{\otimes}\mathcal{Q}$ *(non degeneracy).*

In the definition note that condition (2) allows one to define $(\Gamma \otimes \mathrm{id})$. Given an α equivariant representation Γ of $\mathcal{Q}$ on a Hilbert bimodule $\mathcal{E}$, proceeding as in subsection 4.2, we can get spectral decomposition of $\mathcal{E}$. We have $\mathcal{E}_\pi := Im\ P_\pi (= (\mathrm{id} \otimes \rho^\pi)\Gamma)$. Define $\mathcal{E}_0 := Sp\{\mathcal{E}_\pi : \pi \in \hat{\mathcal{Q}}\} \oplus ker(\Gamma)$. In case Γ is one-one which is equivalent to α being one-one, $\mathcal{E}_0$ coincides with the spectral subspace. Then again it can be shown along the same lines of 2.1.2 that $\mathcal{E}_0$ is the maximal subspace over which Γ is algebraic and $\mathcal{E}_0$ is dense in the Hilbert module $\mathcal{E}$.

Lemma 3.3.2. *Let $\mathcal{E}_1$ be a Hilbert $\mathcal{B}-\mathcal{C}$ bimodule and $\mathcal{E}_2$ be a Hilbert $\mathcal{C}-\mathcal{D}$ bimodule. $\alpha_{\mathcal{B}}$,$\alpha_{\mathcal{C}}$,$\alpha_{\mathcal{D}}$ be topological actions on a compact quantum group $\mathcal{Q}$ of topological $*$-algebras $\mathcal{B},\mathcal{C},\mathcal{D}$ respectively. $\Gamma_1 : \mathcal{E}_1 \to \mathcal{E}_1\bar{\otimes}\mathcal{Q}$ and $\Gamma_2 : \mathcal{E}_2 \to \mathcal{E}_2\bar{\otimes}\mathcal{Q}$ be $\alpha_{\mathcal{C}}$ and $\alpha_{\mathcal{D}}$ equivariant unitary representations as discussed earlier. Then*

$$<< \Gamma_2(\eta), << \Gamma_1(\omega), \Gamma_1(\omega') >> \Gamma_2(\eta') >>= \alpha_{\mathcal{D}} << \eta, << \omega, \omega' >> \eta' >> .$$

Proof:

$$\begin{aligned}
& << \Gamma_2(\eta), << \Gamma_1(\omega), \Gamma_1(\omega') >> \Gamma_2(\eta') >> \\
= \ & << \Gamma_2(\eta), \alpha_{\mathcal{C}}(<< \omega, \omega' >>)\Gamma_2(\eta') >> \\
= \ & << \Gamma_2(\eta), \Gamma_2(<< \omega, \omega' >> \eta') >> \\
= \ & \alpha_{\mathcal{D}} << \eta, << \omega), \omega' >> \eta' >>
\end{aligned}$$

□

Theorem 3.3.3. *Given an $\alpha_{\mathcal{D}}$ equivariant unitary representation $\Gamma : \mathcal{E} \to \mathcal{E}\bar{\otimes}\mathcal{Q}$ of a CQG $\mathcal{Q}$ on a Hilbert $\mathcal{C}-\mathcal{D}$ bimodule $\mathcal{E}$. Let $\mathcal{E}_0$, $\mathcal{C}_0$ and $\mathcal{D}_0$ be as defined earlier. Then $\mathcal{E}_0$ is a Hilbert $\mathcal{C}_0-\mathcal{D}_0$ bimodule.*

Proof:

Note that $\mathcal{E}_0$ is dense in $\mathcal{E}$ and $\mathcal{E}_0$ is the maximal $\mathbb{C}$ linear subspace over which Γ is algebraic. Similarly we have dense subalgebras $\mathcal{C}_0$ of $\mathcal{C}$ and $\mathcal{D}_0$ of $\mathcal{D}$ over which $\alpha_{\mathcal{C}}$ and $\alpha_{\mathcal{D}}$ are algebraic. $\mathcal{C}_0$ and $\mathcal{D}_0$ are also maximal subspaces over which $\alpha_{\mathcal{C}}$ and $\alpha_{\mathcal{D}}$ are algebraic.

Let $e \in \mathcal{E}_0$ and $a \in \mathcal{D}_0$. Then $\Gamma(ea) = \Gamma(e)\alpha_{\mathcal{D}}(a) = e_{(0)}a_{(0)} \otimes e_{(1)}a_{(1)}$, where $\Gamma(e) = e_{(0)} \otimes e_{(1)}$ and $\alpha_{\mathcal{D}}(a) = a_{(0)} \otimes a_{(1)}$.

$e_{(0)} \in \mathcal{E}_0$ *and* $a_{(0)} \in \mathcal{D}_0$ implies $\Gamma(\mathcal{E}_0\mathcal{D}_0) \subset \mathcal{E}_0\mathcal{D}_0 \otimes \mathcal{Q}_0$, that is Γ is algebraic on $\mathbb{C}$-linear subspace and hence by maximality $\mathcal{E}_0\mathcal{D}_0 \subset \mathcal{E}_0$. Similarly we can show that $\mathcal{C}_0\mathcal{E}_0 \subset \mathcal{E}_0$.

Finally let $\mathcal{F} = Sp\ \{<< e, e' >> |e, e' \in \mathcal{E}_0\}$ and for $e, e' \in \mathcal{E}_0$ we have

$$\begin{aligned} & \alpha_{\mathcal{D}}(<< e, e' >>) \\ = \ & << \Gamma(e), \Gamma(e') >> \ (\textit{by } \alpha_{\mathcal{D}} \textit{ equivariance}) \\ = \ & << e_{(0)}, e'_{(0)} >> \otimes e^*_{(1)} e'_{(1)} \\ \subset \ & \mathcal{F} \otimes \mathcal{Q}_0 \end{aligned}$$

Again by maximality we have $\mathcal{F} \subset \mathcal{D}_0$. Hence indeed $\mathcal{E}_0$ is a Hilbert $\mathcal{C}_0 - \mathcal{D}_0$ bimodule. □

We can form direct sum and tensor product of equivariant representations extending the algebraic constructions. We have

Lemma 3.3.4. *Let $\Gamma_1, ..., \Gamma_k$ be α equivariant representations on the Hilbert $\mathcal{A}$ bimodule $\mathcal{E}_1, ..., \mathcal{E}_k$ respectively. Then $\Gamma_1 \oplus ... \oplus \Gamma_k$ is again an α equivariant representation on the direct sum bimodule $\mathcal{E}_1 \oplus ... \oplus \mathcal{E}_k$.*

Proof:
Follows from the definition and 1.4.11.

Lemma 3.3.5. *Let $\mathcal{E}_1, \mathcal{E}_2, \mathcal{B}, \mathcal{C}, \mathcal{D}, \alpha_{\mathcal{B}}, \alpha_{\mathcal{C}}, \alpha_{\mathcal{D}}, \Gamma_1, \Gamma_2, \mathcal{Q}$ be as in Lemma 3.3.2. By Theorem 3.3.3, we have dense subspaces $\mathcal{E}_{i0}$ of $\mathcal{E}_i$ for $i = 1, 2$ and dense $*$-subalgebras $\mathcal{B}_0, \mathcal{C}_0, \mathcal{D}_0$ such that $\mathcal{E}_{10}$ is a Hilbert $\mathcal{B}_0 - \mathcal{C}_0$ bimodule and $\mathcal{E}_{20}$ is a Hilbert $\mathcal{C}_0 - \mathcal{D}_0$ bimodule. Then we have an $\alpha_{\mathcal{D}}$ equivariant representation Γ of $\mathcal{Q}$ on the Hilbert $\mathcal{B} - \mathcal{D}$ bimodule $\mathcal{E}_1 \bar{\otimes}_{in} \mathcal{E}_2$.*

Proof:
Recall from subsection on algebraic representations of CQG on vector spaces, the representation $\Gamma_1 \otimes \Gamma_2$ of $\mathcal{Q}$ on $\mathcal{E}_{10} \otimes \mathcal{E}_{20}$. The non degeneracy of Γ_1 and Γ_2 implies the non degeneracy of $\Gamma_1 \otimes \Gamma_2$. Also applying Lemma 3.3.2, it is easy to see that $(\Gamma_1 \otimes \Gamma_2) \circ \pi = (\pi \otimes \mathrm{id}_{\mathcal{Q}})(\Gamma_1 \otimes \Gamma_2)$ on $\mathcal{E}_{10} \otimes \mathcal{E}_{20}$, where $\pi : \mathcal{E}_{10} \otimes \mathcal{E}_{20} \to \mathcal{E}_{10} \otimes_{in} \mathcal{E}_{20}$ is the projection map as in subsection 2.2. Hence $\Gamma_1 \otimes \Gamma_2$ descends to an algebraic representation of $\mathcal{Q}$ on $\mathcal{E}_{10} \otimes_{in} \mathcal{E}_{20}$. Lemma 3.3.2 also implies the $\alpha_{\mathcal{D}}$ equivariance of $\Gamma_1 \otimes \Gamma_2$. So by density of $\mathcal{E}_{10} \otimes_{in} \mathcal{E}_{20}$ in $\mathcal{E}_1 \bar{\otimes}_{in} \mathcal{E}_2$ and density of $\mathcal{E}_{10} \otimes_{in} \mathcal{E}_{20} \otimes \mathcal{Q}_0$ in $\mathcal{E}_1 \bar{\otimes}_{in} \mathcal{E}_2 \bar{\otimes} \mathcal{Q}$, we get the desired Γ. □

In particular when $\mathcal{E}$ is the trivial $\mathcal{C}$-bimodule of rank N, we have the following:

Lemma 3.3.6. *Given an α equivariant representation Γ of $\mathcal{Q}$ on $\mathbb{C}^N \otimes \mathcal{C}$ such that $\Gamma(e_i \otimes 1_{\mathcal{A}}) = \sum_{j=1}^{N} e_j \otimes b_{ji}$, $b_{ij} \in \mathcal{C} \hat{\otimes} \mathcal{Q}$ for all $i, j = 1, ..., N$, where $\{e_i; i = 1, ..., N\}$ is an orthonormal basis of $\mathbb{C}^N$, then $U = ((b_{ij}))_{i,j=1,....,N}$ is a unitary element of $M_N(\mathcal{C} \hat{\otimes} \mathcal{Q})$.*

Proof:
Since Γ is α-equivariant, we have

$$\begin{aligned} & << \Gamma(e_i \otimes 1), \Gamma(e_j \otimes 1) >>= \alpha(<< e_i \otimes 1, e_j \otimes 1 >>) \\ \Rightarrow & \sum_{k,l=1}^{N} << e_k \otimes b_{ki}, e_l \otimes b_{lj} >>= \alpha(\delta_{ij} 1_{\mathcal{A}}) \\ \Rightarrow & \sum_{k=1}^{N} b_{ki}^* b_{kj} = \delta_{ij} 1_{\mathcal{A}\hat{\otimes}\mathcal{Q}} \end{aligned}$$

Hence $U^*U = 1_{M_N(\mathcal{C}\hat{\otimes}\mathcal{Q})}$, i.e. U is a partial isometry in $M_N(\mathcal{C}\hat{\otimes}\mathcal{Q})$. Viewing U as a right $\mathcal{C}\hat{\otimes}\mathcal{Q}$ linear map on the trivial module $(\mathcal{C}\hat{\otimes}\mathcal{Q})^N$ (as discussed in this section 2.3.3), it is enough to show that range of U is dense in $(\mathcal{C}\hat{\otimes}\mathcal{Q})^N$. This follows from density of $Sp\ \Gamma(\mathbb{C}^N \otimes \mathcal{C})(1 \otimes \mathcal{Q}) = \mathbb{C}^N \otimes \mathcal{C}\hat{\otimes}\mathcal{Q}$ and observing that

$$\begin{aligned} & \Gamma(e_i \otimes a)(1 \otimes q) \\ = & \sum_j e_j \otimes b_{ji} \alpha(a)(1 \otimes q) \\ = & U(e_i \otimes 1_{\mathcal{C}\hat{\otimes}\mathcal{Q}})\alpha(a)(1 \otimes q) \\ \subset & R(U) \end{aligned}$$

for $a \in \mathcal{C}, q \in \mathcal{Q}$. □

3.3.2 Definition of inner product preserving action and its implications

Definition 3.3.7. *We call a smooth action α on a Riemannian manifold M inner product preserving if*

$$<< (d \otimes \text{id})\alpha(f), (d \otimes \text{id})\alpha(g) >>= \alpha << df, dg >> \tag{3.3.1}$$

for all $f, g \in C^\infty(M)$.

Remark 3.3.8. *It is easy to see, by Fréchet continuity of the maps d and α, that it suffices to verify equation (3.3.1) with f, g varying in some dense $*$-subalgebra of $C^\infty(M)$.*

Theorem 3.3.9. *If α is inner product preserving for a Riemannian structure then there is an α equivariant unitary representation (in the sense described earlier) $d\alpha$ on $\Omega^1(C^\infty(M))$ satisfying $d\alpha(df) = (d \otimes \text{id})\alpha(f)$ for all $f \in C^\infty(M)$.*

Proof:
For any $\omega \in \Omega^1(C^\infty(M))$ such that $\omega = \sum_{i=1}^k f_i dg_i$ for $f_i, g_i \in C^\infty(M)$, we define $d\alpha(\omega) := \sum_{i=1}^k (d \otimes \mathrm{id})\alpha(g_i)(\alpha(f_i))$. We need to check that $d\alpha$ is a well defined bimodule morphism. To that end let $\omega = \sum_{i=1}^m f_i dg_i$ be a one form such that $\omega = 0$ i.e. $<< \sum_{i=1}^m f_i dg_i, \sum_{i=1}^m f_i dg_i >>= 0$. By definition

$$
\begin{aligned}
& << d\alpha(\omega), d\alpha(\omega) >> \\
= \; & << \sum_{i=1}^m (d \otimes \mathrm{id})\alpha(g_i)\alpha(f_i), \sum_{i=1}^m (d \otimes \mathrm{id})\alpha(g_i)\alpha(f_i) >> \\
= \; & \sum_{i,j=1}^m \alpha(f_i)^* << (d \otimes \mathrm{id})\alpha(g_i), (d \otimes \mathrm{id})\alpha(g_j) >> \alpha(f_j) \\
= \; & \sum_{i,j=1}^m \alpha(\bar{f}_i << dg_i, dg_j >> f_j) \\
= \; & \alpha(\sum_{i,j=1}^m << f_i dg_i, f_j dg_j >>) \\
= \; & \alpha(<< \omega, \omega >>).
\end{aligned}
$$

So $d\alpha(\omega) = 0$ proving that $d\alpha$ is a well defined bimodule morphism and hence $\alpha(f)(d \otimes \mathrm{id})\alpha(g) = (d \otimes \mathrm{id})\alpha(g)\alpha(f)$ for $f, g \in C^\infty(M)$ by 3.2.17. The α equivariance of $d\alpha$ can be proved by similar computations. The coassociativity of $d\alpha$ follows from that of α. To prove the span density condition first choose the maximal dense subalgebra $\mathcal{A}$ of $C^\infty(M)$ over which α is algebraic and Sp $\alpha(\mathcal{A})(1 \otimes \mathcal{Q}_0) = \mathcal{A} \otimes \mathcal{Q}_0$. We shall show that Sp $d\alpha(\Omega^1(\mathcal{A}))(1 \otimes \mathcal{Q}_0) = \Omega^1(\mathcal{A}) \otimes \mathcal{Q}_0$, for then the span density condition will follow from the density of $\Omega^1(\mathcal{A}) \otimes \mathcal{Q}_0$ in $\Omega^1(C^\infty(M)) \bar{\otimes} \mathcal{Q}$. To that end first observe that it suffices to prove that for $f, g \in \mathcal{A}$, there are $\omega_i \in \Omega^1(\mathcal{A})$ and $q_i \in \mathcal{Q}_0$ such that $\sum_i d\alpha(\omega_i)(1 \otimes q_i) = f dg \otimes 1$. Now since $(f \otimes 1) \in \mathcal{A} \otimes \mathcal{Q}_0$, there are $f_i \in \mathcal{A}$ and $q_i \in \mathcal{Q}_0$ for some $i = 1, ..., m$ such that $\sum_{i=1}^m \alpha(f_i)(1 \otimes q_i) = f \otimes 1$. Similarly there are $g_i \in \mathcal{A}$

and $q_i^{'} \in \mathcal{Q}_0$ for some $i = 1, ..., n$ such that $\sum_{i=1}^{n} \alpha(g_i)(1 \otimes q_i') = g \otimes 1$. Then

$$\begin{aligned}
& \sum_{i=1,j=1}^{i=m,j=n} d\alpha(f_i dg_j)(1 \otimes q_j' q_i) \\
= & \sum_{i=1}^{m} (\alpha(f_i) \sum_{j=1}^{n} (d \otimes \mathrm{id})\alpha(g_j)(1 \otimes q_j^{'})(1 \otimes q_i)) \\
= & \sum_{i=1}^{m} (\alpha(f_i)(d \otimes \mathrm{id})(\sum_{j=1}^{n} \alpha(g_j)(1 \otimes q_j^{'}))(1 \otimes q_i)) \\
= & \sum_{i=1}^{m} (\alpha(f_i)(1 \otimes q_i))(dg \otimes 1) \\
= & (f dg \otimes 1).
\end{aligned}$$

So $d\alpha$ is an α equivariant unitary representation on the $C^\infty(M)$ bimodule of one forms.

□

Let us fix a smooth inner product preserving action α on $C^\infty(M)$ for the rest of this subsection. We have

Lemma 3.3.10. *$d\alpha_{(k)} : \Omega^k(C^\infty(M)) \to \Omega^k(C^\infty(M))\bar{\otimes}\mathcal{Q}$ is an α equivariant unitary representation for all $k = 1, \ldots, n$.*

Proof:
As α is inner product preserving, by the Theorem 3.3.9, we see that $d\alpha$ is an α-equivariant unitary representation on the bimodule $\Omega^1(C^\infty(M))$.For $2 \leq k \leq n$, take $\mathcal{E}_1 = \Omega^{k-1}(C^\infty(M))$, $\mathcal{E}_2 = \Omega^1(C^\infty(M))$, $\Gamma_1 = d\alpha_{(k-1)}$, $\Gamma_2 = d\alpha$, $\mathcal{B} = \mathcal{C} = \mathcal{D} = C^\infty(M)$, $\alpha_{\mathcal{B}} = \alpha_{\mathcal{C}} = \alpha_{\mathcal{D}} = \alpha$ and apply Lemma 3.3.5 to get the desired $d\alpha_{(k)}$. □

Now we want to show that $d\alpha_{(k)}$ actually descends to the $C^\infty(M)$ bi module $\Lambda^k(C^\infty(M))$ of sections of smooth k-forms which is actually a quotient submodule of $\Omega^k(C^\infty(M))$. To this end we recall from first chapter the algebraic construction of the $C^\infty(M)$ bimodule of k-forms $\Lambda^k(C^\infty(M))$ on a manifold M from the so-called universal forms. Now for the smooth action α we pass to the maximal dense subalgebra $\mathcal{A}$ of $C^\infty(M)$ on which α is algebraic. By the Lemma 3.3.10, $d\alpha_{(k)}$ extends to a well defined bimodule morphism from $\Omega^k(C^\infty(M))$ to $\Omega^k(C^\infty(M))\bar{\otimes}\mathcal{Q}$ such that $<< d\alpha(\omega), d\alpha(\omega') >>= \alpha(<< \omega, \omega' >>)$ for all $\omega, \omega' \in \Omega^k(C^\infty(M))$. So in particular $<< d\alpha(\omega), d\alpha(\omega') >>= \alpha(<< \omega, \omega' >>)$ for $\omega, \omega' \in \Omega^k(\mathcal{A})$ and Sp $d\alpha(\Omega^k(\mathcal{A}))(1 \otimes \mathcal{Q}_0) = \Omega^k(\mathcal{A}) \otimes \mathcal{Q}_0$. Also as $d\alpha_{(k)}$ is inner product preserving, by maximality of $\mathcal{A}$, we have $<< \omega, \omega' >>\in \mathcal{A}$ for $\omega, \omega' \in \Omega^k(\mathcal{A})$.

Remark 3.3.11. *Let $\omega \in \Omega^{k-1}(\mathcal{A})$. Then by construction it is easy to see that $d\alpha_{(k)}(d\omega) = (d \otimes id)d\alpha_{(k-1)}(\omega)$ for $\omega \in \Omega^{k-1}(\mathcal{A})$.*

Now Recall that by Corollary 1.3.3, $\Omega^k(\mathcal{A}) = \Lambda^k(\mathcal{A}) \oplus \mathcal{J}_k^{\mathcal{A}}$ and $\mathcal{J}_k^{\mathcal{A}}$ is a complemented submodule of $\Omega^k(\mathcal{A})$. We have the following:

Lemma 3.3.12. *$d\alpha_{(k)}$ leaves $\mathcal{J}_k^{\mathcal{A}}$ invariant.*

Proof:
Let $\omega \in \Omega^{k-1}(\mathcal{A})$ such that $\omega = 0$. Then $d\omega \in \mathcal{J}_k^{\mathcal{A}}$. We have by Remark 3.3.11, $d\alpha_{(k)}(d\omega) = (d \otimes \mathrm{id})d\alpha_{(k-1)}(\omega)$. But by α-equivariance of $d\alpha_{(k-1)}$, we get

$$<< d\alpha_{(k-1)}(\omega), d\alpha_{(k-1)}(\omega) >>= \alpha << \omega, \omega >>= 0.$$

Hence $d\alpha_{(k)}$ leaves $\mathcal{J}_k^{\mathcal{A}}$ invariant. □

Lemma 3.3.13. *$d\alpha_{(k)} : \Lambda^k(\mathcal{A}) \to \Lambda^k(\mathcal{A}) \otimes \mathcal{Q}_0$, is α equivariant and Sp $d\alpha_{(k)}(\Lambda^k(\mathcal{A}))(1 \otimes \mathcal{Q}_0) = \Lambda^k(\mathcal{A}) \otimes \mathcal{Q}_0$.*

Proof:
We have for $\omega, \omega' \in \Omega^k(\mathcal{A})$, $<< \omega, \omega' >> \in \mathcal{A}$. If τ' is the faithful state of $C(M)$ corresponding to the volume, then $\tau := (\tau' \otimes h)\alpha$ is a faithful, α invariant state on $\mathcal{A}$, where h is the Haar state of $\mathcal{Q}$. Using this state, we define a scalar valued inner product on $\Omega^k(\mathcal{A})$ by

$$< \omega, \omega' > := \tau(<< \omega, \omega' >>),$$

for all $\omega, \omega' \in \Omega^k(\mathcal{A})$. We denote the Hilbert space obtained as the completion of $\mathcal{A}$ bimodule $\Omega^k(\mathcal{A})$ with respect to this inner product by $\mathcal{H}$. Also we denote the closed subspace obtained as the completion of the submodule $\mathcal{J}_k^{\mathcal{A}}$ with respect to this inner product inside the Hilbert space $\mathcal{H}$ by $\mathcal{F}$ and we denote the orthogonal projection onto this subspace by p.

For $e, e' \in \Omega^k(\mathcal{A})$,

$$\begin{aligned}
& < d\alpha_{(k)}(e), d\alpha_{(k)}(e') > \\
= \; & (\tau \otimes \mathrm{id}) << d\alpha_{(k)}(e), d\alpha_{(k)}(e') >> \\
= \; & (\tau \otimes \mathrm{id})\alpha(<< e, e' >>).1_{\mathcal{Q}} \; (\textit{by } \alpha \textit{ equivariance of } d\alpha_{(k)}) \\
= \; & \tau(<< e, e' >>).1_{\mathcal{Q}} \; (\textit{by } \alpha \textit{ invariance of} \tau) \\
= \; & < e, e' > 1_{\mathcal{Q}}
\end{aligned}$$

Hence for any $h \in \mathcal{H}$, we can define $U(h) := lim_{n \to \infty} d\alpha(e_n)$ where e_n is a sequence from $\Omega^k(\mathcal{A})$ converging to $\mathcal{H}$ in the Hilbert space sense and the right hand side limit is taken in the Hilbert C^* module $\mathcal{H}\bar{\otimes}\mathcal{Q}$. Then we have $< U(h), U(h') >=< h, h' > .1_{\mathcal{Q}}$. The fact that Sp $U(\mathcal{H})\mathcal{Q}$ is dense in the Hilbert C^* module $\mathcal{H}\bar{\otimes}\mathcal{Q}$ follows from the fact that Sp $d\alpha_{(k)}(\Omega^k(\mathcal{A}))(1 \otimes \mathcal{Q}_0) = \Omega^k(\mathcal{A}) \otimes \mathcal{Q}_0$. Hence U is a unitary representation of the CQG $\mathcal{Q}$ on the Hilbert space $\mathcal{H}$.

Then by Proposition 6.2 of [35], U leaves both $p\mathcal{H}$ and $p^{\perp}\mathcal{H}$ invariant. Let P be the orthogonal projection onto the complemented submodule $\mathcal{J}_k^{\mathcal{A}}$.

Claim $p^{\perp}\mathcal{H} \cap \Omega^k(\mathcal{A}) = P^{\perp}\Omega^k(\mathcal{A}) = \Lambda^k(\mathcal{A})$.

Proof of the claim:

Let $e \in p^{\perp}\mathcal{H} \cap \Omega^k(\mathcal{A})$. Then $< e, Pe >= 0$, since $Pe \in \mathcal{J}_k^{\mathcal{A}} \in \mathcal{F}$. That implies

$$\begin{aligned} & \tau(<< e, Pe >>) = 0 \\ \Rightarrow \ & \tau(<< Pe, Pe >>) = 0 \\ \Rightarrow \ & << Pe, Pe >>= 0 (since\ \tau\ is\ faithful\ on\ \mathcal{A}) \\ \Rightarrow \ & Pe = 0. \end{aligned}$$

Hence $e \in P^{\perp}\Omega^k(\mathcal{A})$. Conversely suppose $f \in P^{\perp}\Omega^k(\mathcal{A})$. Then $< f, Pe >= \tau(<< f, Pe >>) = 0$. But since $P\Omega^k(\mathcal{A})$ is dense in $p\mathcal{H}$, $f \in p^{\perp}\mathcal{H}$. This completes the proof of the claim.

As U agrees with $d\alpha_{(k)}$ on $p^{\perp}\mathcal{H} \cap \Omega^k(\mathcal{A})$, $d\alpha_{(k)}$ leaves both $\mathcal{J}_k^{\mathcal{A}}$ and $\mathcal{J}_k^{\mathcal{A}\perp}$ invariant. Let $\xi \in \mathcal{J}_k^{\mathcal{A}}\bar{\otimes}\mathcal{Q}_0$. So there exists $e_i \in \Omega^k(\mathcal{A})$ and $q_i \in \mathcal{Q}_0$ such that $\sum_{i=1}^k d\alpha_{(k)}(e_i)(1 \otimes q_i) = \xi$ in the Hilbert Fréchet module $\Omega^k(\mathcal{A}) \otimes \mathcal{Q}_0$. Now for any $e \in \Omega^k(\mathcal{A})$,

$$\begin{aligned} & (P \otimes \mathrm{id}) d\alpha_{(k)}(e) \\ = \ & (P \otimes \mathrm{id}) d\alpha_{(k)}(Pe + P^{\perp}e) \\ = \ & (P \otimes \mathrm{id}) d\alpha_{(k)}(Pe)(as\ d\alpha_{(k)}(P^{\perp}e) \in P^{\perp}\Omega^k(\mathcal{A}) \otimes \mathcal{Q}_0) \\ = \ & d\alpha_{(k)}(Pe) \end{aligned}$$

So

$$\begin{aligned} & (P \otimes \mathrm{id})(\sum_{i=1}^{l} d\alpha_{(k)}(e_i)(1 \otimes q_i)) \\ = \ & \sum_{i=1}^{l} d\alpha_{(k)}(Pe_i)(1 \otimes q_i) \end{aligned}$$

Hence Sp $d\alpha_{(k)}(\mathcal{J}_k^{\mathcal{A}})(1 \otimes \mathcal{Q}_0) = \mathcal{J}_k^{\mathcal{A}} \otimes \mathcal{Q}_0$. Similarly considering the projection $P^{\perp}$ we can conclude that Sp $d\alpha_{(k)}(\Lambda^k(\mathcal{A}))(1 \otimes \mathcal{Q}_0) = \Lambda^k(\mathcal{A}) \otimes \mathcal{Q}_0$. The α equivariance follows from that of $d\alpha_{(k)}$ on $\Omega^k(\mathcal{A})$. □

Corollary 3.3.14. *The restriction of the α-equivariant representation $d\alpha_{(k)}$ of $\mathcal{Q}$ on the Hilbert module $\Omega^k(C^\infty(M))$ onto the closed submodule $\Lambda^k(C^\infty(M))$ is again an α-equivariant representation.*

Proof:
Follows from the densities of $\Lambda^k(\mathcal{A})$ and $\mathcal{Q}_0$ in the Hilbert bimodule $\Lambda^k(C^\infty(M))$ and the C^* algebra $\mathcal{Q}$ respectively and the Lemma 3.3.13. □

Here we also make the convention $d\alpha_0 \equiv \alpha$. As $d\alpha$ is a well defined bimodule morphism, by Theorem 3.2.17, $\alpha(f)(x)$ commute among themselves for different f's and also commute with $((\phi \otimes \mathrm{id})\alpha(g))(x)$'s where $f, g \in C^\infty(M)$ and ϕ is any smooth vector field. For $x \in M$ let us denote by $\mathcal{Q}_x$ the unital C^*-subalgebra of $\mathcal{Q}$ generated by elements of the forms $\alpha(f)(x), ((\phi \otimes \mathrm{id})\alpha(g))(x)$, where f, ϕ are as before. Using the lift of $d\alpha_{(2)}$ to $\Lambda^2(C^\infty(M))$, we can show more. Indeed, we now claim that actually $((\phi \otimes \mathrm{id})\alpha(g))(x)$'s commute among themselves too, for different choices of ϕ and g. In other words:

Lemma 3.3.15. *$\mathcal{Q}_x$ is commutative.*

Proof:
The proof is very similar to the proof of Proposition (4) of [37] for the case $q = 1$. The statement of the lemma is clearly equivalent to proving $\tilde{d\alpha}(f)(= (d \otimes \mathrm{id})(\alpha(f)))$ and $\tilde{d\alpha}(g)$ commute for $f, g \in C^\infty(M)$. For $x \in M$, choose smooth one-forms $\{\omega_1, \ldots, \omega_n\}$ such that they form a basis of T^*M at x. Let $F_i(x), G_i(x), i = 1, \ldots, n$ be elements of $\mathcal{Q}$ (actually in $\mathcal{Q}_x$) such that $\tilde{d\alpha}(f)(x) = \sum_i \omega_i(x)F_i(x)$, $\tilde{d\alpha}(g) = \sum_i \omega_i(x)G_i(x)$. Now $d\alpha_{(2)}$ leaves invariant the submodules of symmetric and antisymmetric tensor product of $\Lambda^1(C^\infty(M))$, thus in particular, $C^s_{ij} = C^s_{ji}$, $C^a_{ij} = -C^a_{ji}$ for all i, j, where C^s_{ij} and C^a_{ij} denote the $\mathcal{Q}$-valued coefficient of $w_i(x) \otimes w_j(x)$ in the expression of $d\alpha_{(2)}(df \otimes dg + dg \otimes df)|_x$ and $d\alpha_{(2)}(df \otimes dg - dg \otimes df)|_x$ respectively. By a simple calculation using these relations, we get the commutativity of $F_i(x), G_j(x)$ for all i, j. □

We also have the following observation which follows from the constructions of $d\alpha_{(k)}$'s and the definition of $\mathcal{Q}_x$.

Lemma 3.3.16. *For every $k \geq 0$, $x \in M$ and $\omega, \omega' \in \Lambda^k(C^\infty(M))$, we have $<< \omega' \otimes 1_{\mathcal{Q}}, d\alpha_{(k)}(\omega) >> (x) \in \mathcal{Q}_x$.*

Chapter 4

Characterizations of isometric action

4.1 Introduction

In this chapter first we shall show that an isometric action α of a CQG $\mathcal{Q}$ on an n-dimensional compact Riemannian manifold is inner product preserving. Then we give two sufficient conditions with which the converse also holds. We have already seen in the previous chapter that an isometric action is automatically smooth (Theorem 3.2.16) and hence by the results of the previous chapter it lifts to α-equivariant unitary representation on the bimodule of k-forms for all $k = 1, ..., n$. We say an isometric action is also orientation preserving for an oriented manifold if $d\alpha_{(n)}(\text{dvol}) = \text{dvol}.1_{\mathcal{Q}}$.

4.2 Geometric characterization of an isometric action

In this section first we shall prove that an isometric action is automatically inner product preserving in our sense. Then we will prove a partial converse to this. More precisely we shall prove that an inner product preserving action is isometric provided either of the following holds:
(a) The manifold is orientable and the action is orientation preserving.
(b) The action preserves the functional coming from the Riemannian volume measure.
For that first recall from chapter 3, the definition of an inner product preserving action of a CQG on a compact Riemannian manifold.

As before let $\alpha : C^\infty(M) \to C^\infty(M)\hat{\otimes}\mathcal{Q}$ be a smooth action (as introduced earlier) and let us fix the maximal Fréchet dense subalgebra $\mathcal{A}$ of $C^\infty(M)$ over which the action is algebraic i.e. $\alpha(\mathcal{A}) \subset (\mathcal{A} \otimes \mathcal{Q}_0)$ and $Sp\ \alpha(\mathcal{A})(1 \otimes \mathcal{Q}_0) = \mathcal{A} \otimes \mathcal{Q}_0$. Note that for $f \in C^\infty(M)$, $(d \otimes \text{id})\alpha(f) \in \Omega^1(C^\infty(M))\bar{\otimes}\mathcal{Q}$.

Lemma 4.2.1. *If α commutes with the geometric Laplacian $\mathcal{L}$ on $\mathcal{A}$, i.e. α is isometric, then α is inner product preserving.*

Proof:
Let α be isometric i.e. it commutes with the geometric Laplacian. Then for $f, g \in \mathcal{A}$,

$$\begin{aligned} & << (d \otimes \mathrm{id})\alpha(f), (d \otimes \mathrm{id})\alpha(g) >> \\ = & << df_{(0)}, dg_{(0)} >> \otimes f^*_{(1)} g_{(1)} \\ = & [\mathcal{L}(\overline{f_{(0)}} g_{(0)}) - \mathcal{L}(\overline{f_{(0)}}) g_{(0)} - \overline{f_{(0)}} \mathcal{L}(g_{(0)})] \otimes f^*_{(1)} g_{(1)} \end{aligned}$$

On the other hand

$$\begin{aligned} & \alpha(<< df, dg >>) \\ = & \alpha[\mathcal{L}(\bar{f} g) - \mathcal{L}(\bar{f}) g - \bar{f} \mathcal{L}(g)] \\ = & [\mathcal{L}(\overline{f_{(0)}} g_{(0)}) - \mathcal{L}(\overline{f_{(0)}}) g_{(0)} - \overline{f_{(0)}} \mathcal{L}(g_{(0)})] \otimes f^*_{(1)} g_{(1)} (\textit{ since } \alpha \textit{ commutes with } \mathcal{L}). \end{aligned}$$

So by Remark 3.3.8 of chapter 3, we conclude that α is inner product preserving. □

Now we proceed to prove the sufficient conditions (a) and (b) for a smooth action to be isometric made in the beginning of this section. The condition (b) is actually both necessary and sufficient.

Theorem 4.2.2. *A smooth action of a CQG on a compact Riemannian manifold is isometric if and only if it preserves the Riemannian inner product and the functional coming from Riemannian volume measure.*

Proof:
Let $\alpha : C^\infty(M) \to C^\infty(M) \hat{\otimes} \mathcal{Q}$ be an isometric action. Then by Lemma 4.2.1, the action is inner product preserving. Also by [22] (see Lemma 2.10), the action preserves the functional coming from the Riemannian volume measure. Conversely let the action preserves both the Riemannian inner product and the functional coming from the Riemannian volume measure. As the action is inner product preserving we have an α equivariant representation $d\alpha$ of $\mathcal{Q}$ over the $C^\infty(M)$ bimodule $\Omega^1(C^\infty(M))$ satisfying $d\alpha(df) = (d \otimes \mathrm{id})\alpha(f)$. Let us denote the Riemannian volume measure by μ. We define the inner product on $C^\infty(M)$ by

$$< f, g >= \int_M \overline{f} g d\mu.$$

for $f, g \in C^\infty(M)$. We denote the completed Hilbert space by $\mathcal{H}_0$. Similarly defining inner product on $\Omega^1(C^\infty(M))$ we denote the completed Hilbert space by $\mathcal{H}_1$. As α is

Riemannian volume preserving and $d\alpha$ is α equivariant, both α and $d\alpha$ extend as unitary representations of $\mathcal{Q}$ over the Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ respectively. We denote the corresponding unitaries by U_0 and U_1 respectively. The de-Rham differential operator d can be viewed as a closable densely defined unbounded operator between the Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$. Then $(d \otimes \mathrm{id})$ is a densely defined unbounded operator between Hilbert modules $\mathcal{H}_0 \bar{\otimes} \mathcal{Q}$ and $\mathcal{H}_1 \bar{\otimes} \mathcal{Q}$. We have $U_1(df \otimes 1) = (d \otimes \mathrm{id})U_0(f \otimes 1)$ for $f \in C^\infty(M)$. Let $\omega \in \Omega^1(C^\infty(M))$ be such that $\omega \in \mathrm{dom}(d^*)$. Then we claim that $U_1(\omega \otimes 1) \in \mathrm{dom}(d^* \otimes 1)$ and $(d^* \otimes \mathrm{id})U_1(\omega \otimes 1) = U_0(d^* \otimes \mathrm{id})(\omega \otimes 1)$. As $\omega \in \mathrm{dom}(d^*)$, $\omega \otimes 1 \in \mathrm{dom}(d^* \otimes 1)$. Let $\sum f_i \otimes q_i \in C^\infty(M) \otimes \mathcal{Q}$. Then

$$\begin{aligned} & < U_1(\omega \otimes 1), (d \otimes \mathrm{id})U_0(\sum f_i \otimes q_i) > \\ = & < U_1(\omega \otimes 1), U_1(d \otimes \mathrm{id})(\sum f_i \otimes q_i) > \\ = & < \omega \otimes 1, (d \otimes 1)(\sum f_i \otimes q_i) > . \end{aligned}$$

Now since $(\omega \otimes 1) \in \mathrm{dom}(d^* \otimes \mathrm{id})$, there is a constant $C > 0$, such that $| < \omega \otimes 1, (d \otimes 1)(\sum f_i \otimes q_i) > | < C|\sum f_i \otimes q_i|$. That implies that $U_1(\omega \otimes 1) \in \mathrm{dom}(d^* \otimes 1)$ and $(d^* \otimes \mathrm{id})U_1(\omega \otimes 1) = U_0(d^* \otimes \mathrm{id})(\omega \otimes 1)$. Now for $f \in C^\infty(M)$, $df \in \mathrm{dom}(d^*)$. So we have

$$(d^* \otimes \mathrm{id})U_1(df \otimes 1) = U_0(d^* \otimes \mathrm{id})(df \otimes 1).$$

That is $\alpha(d^*d(f)) = (d^*d \otimes \mathrm{id})\alpha(f)$ for $f \in C^\infty(M)$. Hence α commutes with the Laplacian. □

To prove the sufficient condition (a), first we recall the Hodge $*$ operator. For that we assume the manifold to be orientable and fix a choice of orientation. We introduce the Hodge star operator, which is a point wise isometry $* = *_x : \Lambda^k T_x^* M \to \Lambda^{n-k} T_x^* M$. Choose a positively oriented orthonormal basis $\{\theta^1, \theta^2, ..., \theta^n\}$ of T_x^*M. Since $*$ is a linear transformation it is enough to define $*$ on a basis element $\theta^{i_1} \wedge \theta^{i_2} \wedge ... \wedge \theta^{i_k} (i_1 < i_2 < ... < i_k)$ of $\Lambda^k T_x^* M$. Note that

$$\begin{aligned} \mathrm{dvol}(x) &= \sqrt{det(< \theta^i, \theta^j >)}\theta^1 \wedge \theta^2 \wedge ... \wedge \theta^n \\ &= \theta^1 \wedge \theta^2 \wedge ... \wedge \theta^n \end{aligned}$$

Definition 4.2.3. $*(\theta^{i_1} \wedge \theta^{i_2} \wedge ... \wedge \theta^{i_k}) = \theta^{j_1} \wedge \theta^{j_2} \wedge ... \wedge \theta^{j_{n-k}}$ *where* $\theta^{i_1} \wedge \theta^{i_2} \wedge ... \wedge \theta^{i_k} \wedge \theta^{j_1} .. \wedge \theta^{j_{n-k}} = \mathrm{dvol}(x)$.

Since we are using $\mathbb{C}$ as the scalar field, we would like to define $\bar{\omega}$ for a k form ω. In the set-up introduced just before the definition we have some scalars $c_{i_1,\ldots,i_k}$ such that $\omega(x) = \sum c_{i_1,\ldots,i_k}\theta^{i_1}\wedge\theta^{i_2}\wedge\ldots\wedge\theta^{i_k}$. Then define $\bar{\omega}$ to be $\bar{\omega}(x) = \sum \bar{c}_{i_1,\ldots,i_k}\theta^{i_1}\wedge\theta^{i_2}\wedge\ldots\wedge\theta^{i_k}$. Then the equation $<<\omega,\eta>>= *(\bar{\omega}\wedge *\eta)$ defines an inner product on the Hilbert module $\Lambda^k(C^\infty(M))$ for all $k = 1,\ldots,n$ which is the same as the $C^\infty(M)$ valued inner product defined earlier. Then the Hodge star operator is a unitary between two Hilbert modules $\Lambda^k(C^\infty(M))$ and $\Lambda^{n-k}(C^\infty(M))$ i.e. $<< *\omega, *\eta >>=<< \omega,\eta >>$. For further details about the Hodge star operator we refer the reader to [44].

We have

$$(*\otimes \mathrm{id}) : \Lambda^k(C^\infty(M)) \otimes \mathcal{Q} \to \Lambda^{n-k}(C^\infty(M)) \otimes \mathcal{Q}.$$

Since Hodge $*$ operator is an isometry, $(*\otimes \mathrm{id})$ is continuous with respect to the Hilbert module structure of $\dot{\Lambda}(C^\infty(M))\hat{\otimes}\mathcal{Q}$. So we have

$$(*\otimes \mathrm{id}) : \Lambda^k(C^\infty(M))\bar{\otimes}\mathcal{Q} \to \Lambda^{n-k}(C^\infty(M))\bar{\otimes}\mathcal{Q}.$$

We derive a characterization for $(*\otimes \mathrm{id}) : \Lambda^k(C^\infty(M))\bar{\otimes}\mathcal{Q} \to \Lambda^{n-k}(C^\infty(M))\bar{\otimes}\mathcal{Q}$ for all $k = 1,\ldots,n$.

Lemma 4.2.4. *Let $\xi \in \Lambda^{n-k}(C^\infty(M))\bar{\otimes}\mathcal{Q}$ and $X \in \Lambda^k(C^\infty(M))\bar{\otimes}\mathcal{Q}$. Then the following are equivalent:*
(i) For all $Y \in \Lambda^k(C^\infty(M))\bar{\otimes}\mathcal{Q}$,

$$\xi \wedge Y =<< \bar{X}, Y >> (\mathrm{dvol} \otimes 1_{\mathcal{Q}}) \tag{4.2.1}$$

(ii) $\xi = (*\otimes \mathrm{id})X$.

Proof:
(i) $\Rightarrow$ *(ii)*:
Let $m \in M$. Choose a coordinate neighborhood $(U, x_1, x_2, \ldots., x_n)$ around x in M such that $\{dx_1(m),\ldots,dx_n(m)\}$ is an orthonormal basis for $T^*_m(M)$ for all $m \in U$. Now for any $l \in \{1,\ldots,n\}$, let Σ_l be the set consisting of l tuples $(i_1,\ldots,i_l)$ such that $i_1 < i_2 < \ldots < i_l$ and $i_j \in \{1,\ldots,n\}$ for $j = 1,\ldots,l$. For $I = (i_1,\ldots,i_l) \in \Sigma_l$, we write $dx_I(m)$ for $dx_{i_1}\wedge\ldots\wedge dx_{i_l}(m)$. Also for $I(= (i_1,\ldots,i_p)) \in \Sigma_p$, $J(= (j_1,\ldots,j_q)) \in \Sigma_q$, we write (I, J) for $(i_1,\ldots,i_p,j_1,\ldots,j_q)$.

Now fix $I \in \Sigma_k$. Then we have a unique $I' \in \Sigma_{n-k}$ such that

$$(*(dx_I))(m) = \epsilon(I)dx_{I'}(m),$$

where $\epsilon(I)$ is the sign of the permutation (I, I'). As $X \in \Lambda^k(C^\infty(M))\bar{\otimes}\mathcal{Q}$, for $m \in M$, we have $q_I(m) \in \mathcal{Q}$ such that

$$X(m) = \sum_{I \in \Sigma_k} dx_I(m) q_I(m).$$

Also for $\xi \in \Lambda^{n-k}(C^\infty(M)) \bar{\otimes} \mathcal{Q}$, we have $w_J(m) \in \mathcal{Q}$ such that

$$\xi(m) = \sum_{J \in \Sigma_{n-k}} dx_J(m) w_J(m).$$

Hence

$$((* \otimes \mathrm{id})X)(m) = \sum_{I \in \Sigma_k} \epsilon(I) dx_{I'}(m) q_I(m),$$

where $I' \in \Sigma_{n-k}$ is as mentioned before.

Now we fix some $L \in \Sigma_k$ and choose $Y \in \Lambda^k(C^\infty(M)) \bar{\otimes} \mathcal{Q}$ such that $Y(m) = dx_L(m) 1_{\mathcal{Q}}$. Hence

$$(\xi \wedge Y)(m) = \sum_{J \in \Sigma_{n-k}} dx_J \wedge dx_L w_J(m).$$

But for a fixed $L \in \Sigma_k$, there is a unique $J' \in \Sigma_{n-k}$ such that

$$dx_{J'}(m) \wedge dx_L(m) = \epsilon(L) \mathrm{dvol}(m).$$

Hence

$$(\xi \wedge Y)(m) = \epsilon(L) w_{J'}(m) \mathrm{dvol}(m).$$

On the other hand

$$\begin{aligned} & << \bar{X}, Y >> (m) \mathrm{dvol}(m) \\ = & \sum_{I \in \Sigma_k} < dx_I(m) q_I(m)^*, dx_L(m) 1_{\mathcal{Q}} > \mathrm{dvol}(m) \\ = & \; q_L(m) \mathrm{dvol}(m). \end{aligned}$$

Hence we have $q_L(m) = \epsilon(L) w_{J'}$. So varying Y, we have

$$\xi(m) = \sum_{L \in \Sigma_k} \epsilon(L) dx_J(m) q_L(m),$$

implying that $(* \otimes \mathrm{id})X = \xi$.

The other direction of the proof is trivial. $\square$.

Lemma 4.2.5. *Let N be an m-dimensional compact, oriented, Riemannian manifold (possibly with boundary) with* dvol $\in \Lambda^m(C^\infty(N))$ *being a globally defined nonzero* *form. Moreover, let η be a smooth inner product preserving action on N such that $d\eta_{(m)}(\mathrm{dvol}) = \mathrm{dvol} \otimes 1$. Then η commutes with the geometric Laplacian.*

Proof:
First we note that as η is an inner product preserving smooth action, by 3.3.14 it lifts to an α-equivariant unitary representations $d\eta_{(k)} : \Lambda^k(C^\infty(N)) \to \Lambda^k(C^\infty(N))\bar{\otimes}\mathcal{Q}$ for all $k = 1, ..., m$. First we claim that We have $\forall\ k = 1, ..., m,\ d\eta_{(m-k)}(*\omega) \wedge \beta =<< d\eta_{(k)}(\omega), \beta >> (\mathrm{dvol} \otimes 1_{\mathcal{Q}})\ \forall\ \beta \in \Lambda^k(C^\infty(N))\bar{\otimes}\mathcal{Q}$.

For that let $\beta = d\eta_{(k)}(\omega')(1 \otimes q')$. Then

$$\begin{aligned} & d\eta_{(m-k)}(*\omega) \wedge \beta \\ = \ & d\eta_{(m-k)}(*\omega) \wedge d\eta_{(k)}(\omega')(1 \otimes q') \\ = \ & \eta << \omega, \omega' >> (\mathrm{dvol} \otimes q')\ (by\ Lemma\ 3.3.16) \end{aligned}$$

On the other hand from unitarity of $d\eta_{(k)}$,

$$\begin{aligned} & << d\eta_{(k)}\omega, d\eta_{(k)}(\omega^{'})(1 \otimes q^{'}) >> \\ = \ & \eta << \omega, \omega^{'} >> (1 \otimes q^{'}). \end{aligned}$$

So by replacing β by finite sums of the type $\sum_i d\eta_{(k)}(\omega_i)(1 \otimes q_i)$, we can show that $\omega \in \Lambda^k(C^\infty(N))$ and $\beta \in Sp\ d\eta_{(k)}\Lambda^k(C^\infty(N))(1 \otimes \mathcal{Q})$,

$$d\eta_{(m-k)}(*\omega) \wedge \beta =<< d\eta_{(k)}(\omega), \beta >> (\mathrm{dvol} \otimes 1_{\mathcal{Q}}).$$

Now, since Sp $d\eta_{(k)}(\Lambda^k(C^\infty(N))(1 \otimes \mathcal{Q})$ is dense in $\Lambda^k(C^\infty(N))\bar{\otimes}\mathcal{Q}$, we get a sequence β_n belonging to Sp $d\eta_{(k)}(\Lambda^k(C^\infty(N)))(1 \otimes \mathcal{Q})$ such that $\beta_n \to \beta$ in the Hilbert module $\Lambda^k(C^\infty(N))\bar{\otimes}\mathcal{Q}$.
But we have

$$d\eta_{(m-k)}(*\omega) \wedge \beta_n =<< d\eta_{(k)}(\omega), \beta_n >> (\mathrm{dvol} \otimes 1_{\mathcal{Q}}).$$

Hence the claim follows from the continuity of $<<,>>$ and $\wedge$ in the Hilbert module $\dot{\Lambda}(C^\infty(N))\bar{\otimes}\mathcal{Q}$. □

Now combining Lemma 4.2.4 and the previous result we immediately conclude the following:

$$d\eta_{(m-k)}(*\omega) = (* \otimes \mathrm{id})d\eta_{(k)}(\omega)\ for\ k \geq 0. \tag{4.2.2}$$

Now we can prove that η commutes with the geometric Laplacian of N. For $\phi \in C^\infty(N))$,

$$
\begin{aligned}
& \eta(*d*d\phi) \\
= \quad & (*\otimes \mathrm{id})d\eta_{(m)}(d*d\phi) \text{ (\textit{by equation} 4.3.2)} \\
= \quad & (*d\otimes \mathrm{id})d\eta_{(m-1)}(*d\phi) \\
= \quad & (*d\otimes \mathrm{id})(*\otimes \mathrm{id})d\eta(d\phi) \text{ (\textit{again by equation} 4.3.2)} \\
= \quad & (*d\otimes \mathrm{id})(*\otimes \mathrm{id})\tilde{d}\eta(\phi) \\
= \quad & (*d\otimes \mathrm{id})(*d\otimes \mathrm{id})\eta(\phi) \\
= \quad & (*d*d\otimes \mathrm{id})\eta(\phi).
\end{aligned}
$$

□

Chapter 5

Quantum Isometry Group of a Stably Parallelizable, Compact, Connected Riemannian Manifold

5.1 Introduction

In this chapter we shall show that if a CQG $\mathcal{Q}$ acts isometrically on a compact, connected, stably parallelizable Riemannian manifold M, then $\mathcal{Q}$ must be commutative as C^* algebra i.e. $\mathcal{Q} \cong C(G)$ for some compact group G. Using this we can conclude that the quantum isometry group of a compact, connected, stably parallelizable manifold is $C(\mathrm{ISO}(M))$.

5.2 Basics of normal bundle

First we recall the basics of normal bundle of a compact Riemannian manifold. We state some basic definitions and facts about the normal bundle of a manifold without boundary embedded isometrically in some Euclidian space. For details of the topic we refer to [46]. Let $M \subseteq \mathbb{R}^N$ be a smooth embedded submanifold of $\mathbb{R}^N$ such that the embedding say j is an isometry. For each point $x \in M$ define the space of normals to M at x to be

$$N_x(M) = \{v \in \mathbb{R}^N : v \perp T_x(M)\}.$$

The total space $\mathcal{N}(M)$ of the normal bundle is defined to be

$$\mathcal{N}(M) = \{(x, v) \in M \times \mathbb{R}^N ; v \perp T_x(M)\}$$

with the projection π on the first coordinate. Then define $\mathcal{N}_\epsilon(M) = \{(x,v) \in \mathcal{N}(M); ||v|| \leq \epsilon\}$. With the introduced notations we have
Fact: $\mathcal{N}(M)$ is a manifold of dimension N. (see page no. 153 of [46]).

Lemma 5.2.1. *(i) Let $B_\epsilon^{N-n}(0)$ be a closed euclidean $(N-n)$ ball of radius ϵ centered at 0. If M is a compact n-manifold without boundary embedded isometrically in some Euclidean space $\mathbb{R}^N$ such that it has trivial normal bundle, then there exists an $\epsilon > 0$ and a global diffeomorphism $F : M \times B_\epsilon^{N-n}(0) \to \mathcal{N}_\epsilon(M) \subseteq \mathbb{R}^N$ given by*

$$F(x, u_1, u_2, ..., u_{N-n}) = j(x) + \sum_{i=1}^{N-n} \xi_i(x) u_i$$

where $(\xi_1(x), ..., \xi_{N-n}(x))$ is an orthonormal basis of $N_x(M)$ for all x, and $x \mapsto \xi_i(x)$ is smooth $\forall\ i = 1, ..., (N-n)$.
(ii) With the diffeomorphism F as above we get an algebra isomorphism π_F : $C^\infty(\mathcal{N}_\epsilon(M)) \to C^\infty(M \times B_\epsilon^{N-n}(0))$ given by $\pi_F(f)(x, u_1, u_2, ..., u_{N-n}) = f(j(x) + \sum_{i=1}^{N-n} \xi_i(x) u_i)$.
(iii) F is actually a Riemannian isometry between the product Riemannian manifold $M \times B_\epsilon^{N-n}(0)$ and $\mathcal{N}_\epsilon(M) \subset \mathbb{R}^N$, where $B_\epsilon^{N-n}(0)$ is equipped with the usual Euclidean Riemannian structure inherited from $\mathbb{R}^{N-n}$.
(iv) Under the diffeomorphism F, the Riemannian volume measure of $M \times B_\epsilon^{N-n}(0)$ is carried to the Lebesgue measure of $\mathbb{R}^N$.

Proof:
(i) is a consequence of the tubular neighborhood lemma. For the proof see [46].
(ii) Let (U_i, ϕ_i) be a coordinate chart for M. So $(F(U_i), \phi_i F^{-1})$ is a coordinate chart for $\mathcal{N}_\epsilon(M)$. F is smooth $\Rightarrow (\phi_i F \phi_i^{-1}) : \phi_i(U_i) \subset H^n \to H^n$ is continuous, smooth in $\phi_i(U_i) \cap int\ H^n$ and all the partial derivatives extend continuously on $\phi_i(U_i) \cap \partial H^n$.

Let $f_m \to f$ in τ topology of $C^\infty(\mathcal{N}_\epsilon(M))$. We need to show $\pi_F f_m \to \pi_F f$ in τ topology of $C^\infty(M)$ i.e. for a compact set K within U_i, and a multiindex α, $\exists$ an n_0 such that

$$\sup_{x \in K} |\partial^\alpha(\pi_F(f_m)) - \partial^\alpha(\pi_F(f))| < \epsilon\ \forall\ m \geq n_0.$$

We compute for $\alpha = 1$. Let y_i's be coordinate functions for $\mathcal{N}_\epsilon(M)$ on $F(U_i)$

$$|\partial_1(\pi_F(f_m))(x) - \partial_1(\pi_F(f))(x)| = |\sum_{i=1}^{n} \frac{\partial F_i}{\partial x_1}(x)(\frac{\partial f_n}{\partial y_i} - \frac{\partial f}{\partial y_i})F(x)|$$

By definition $\frac{\partial F_i}{\partial x_1}$ is bounded on K $\forall$ i and $f_m \to f$ in N. Hence $\exists$ $n_0 \in \mathbb{N}$ such that

$$\sup_{x\in K} |(\partial_1(\pi_F f_m) - \partial_1(\pi_F f))(x)| < \epsilon \ \forall \ m \geq n_0$$

For any multi index α we can do similar calculation. So by closed graph theorem π_F is continuous.

Using the same urguement for $\pi_{F^{-1}}$ which is π_F^{-1}, we get

$$\pi_F : C^\infty(\mathcal{N}_\epsilon(M)) \cong C^\infty(M).$$

(iii) Let $(U, x_1, ..., x_n)$ be a coordinate neighborhood of M. Then $(U \times B_\epsilon^{N-n}(0), x_1, ..., x_n, u_1, ..., u_{N-n})$ is a coordinate neighborhood of $M \times B_\epsilon^{N-n}(0)$, where $(u_1, ..., u_{N-n})$ is the standard coordinates of $\mathbb{R}^{N-n}$. So $F(U \times B_\epsilon^{N-n}(0))$ is a coordinate neighborhood for $\mathcal{N}_\epsilon(M)$. We denote the corresponding coordinates for $\mathcal{N}_\epsilon(M)$ by $(y_1, ..., y_n, v_1, ..., v_{N-n})$. As $(\xi_1(x), \xi_2(x), ..., \xi_{N-n}(x))$ form an onb of the normal space at $j(x)$. Then the above coordinate functions are given by

$$G : \mathcal{N}_\epsilon(M) \overset{F^{-1}}{\to} M \times B_\epsilon^{(N-n)}(0) \overset{\xi\times id}{\to} \mathbb{R}^N \ (\xi \text{ is a coordinate map for } M):$$

$$y \to (\pi(y), \mathcal{U}(y)) \to (x_1, ..., x_n, u_1, ..., u_{N-n}).$$

First observe that $F|_{M\times 0}$ is nothing but the embedding j of M in $\mathbb{R}^N$. We have $\mathcal{N}_\epsilon(M) \subset \mathbb{R}^N$ and let $(p_1, p_2, ..., p_N)$ be the usual coordinate functions for $\mathbb{R}^N$. Without loss of generality let $\phi \in C^\infty(\mathcal{N}_\epsilon(M))$ and $y \in \mathcal{N}_\epsilon(M)$ be an interior point (for points on the boundary the proof will be similar) and $\phi \in C^\infty(\mathcal{N}_\epsilon M)$ and $y \in \mathcal{N}_\epsilon M$ such that $G^{-1}(0, ...0) = y$. Let $\xi_i(y) = (\xi_i^1(y), ...\xi_i^N(y))$ for all $i = 1, ..., N-n$. Then

$$\begin{aligned}(\tfrac{\partial}{\partial v_i}\phi)(y) &= \tfrac{d}{dt}|_{t=0}\phi(G^{-1}(0, ...t, ...0)) \ (t \text{ in } ith \text{ position}) \\ &= \tfrac{d}{dt}|_{t=0}\phi(\xi^{-1}(0) + t\xi_i) \\ &= \textstyle\sum_{k=1}^N \xi_i^k(y) \tfrac{\partial\phi}{\partial p_k}|_y,\end{aligned}$$

where p_j's are coordinate functions for $\mathbb{R}^N$. Therefore we have

$$\tfrac{\partial}{\partial v_i} = \textstyle\sum_{k=1}^N \xi_i^k \tfrac{\partial}{\partial p_k}.$$

That is, $\frac{\partial}{\partial v_i}|_y$ is nothing but the vector $\xi_i = (\xi_k^i(y); k = 1, \ldots, N)$ under the canonical identification of $\mathbb{R}^N$ with $T_y\mathbb{R}^N$. As $\xi_i(y) \in N_{\pi(y)}(M)$ and $\frac{\partial}{\partial y_k} \in dj(T_{\pi(y)}(M))$ for every

y, $< \frac{\partial}{\partial v_i}|_y, \frac{\partial}{\partial y_k}|_y >= 0$. Also this implies that $\{\frac{\partial}{\partial v_k}\}_{k=1}^{N-n}$ is a set of orthnormal vectors of $\mathbb{R}^N$. Hence $< \frac{\partial}{\partial v_k}, \frac{\partial}{\partial v_l} >=< \frac{\partial}{\partial u_k}, \frac{\partial}{\partial u_l} >$. Also $dF(\frac{\partial}{\partial x_k}) = \frac{\partial}{\partial y_k}$. As dF is nothing but dj on M and j is an isometry, we have $< \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l} >=< \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l} >$. Hence F is an isometry.
(iv) It is an easy consequence of the definitions of the Riemannian volume measure and (iii).

We now introduce the notion of stably parallelizable manifolds.

Definition 5.2.2. *A manifold M is said to be stably parallelizable if its tangent bundle is stably trivial.*

We recall the following from [47]:

Proposition 5.2.3. *A manifold M is stably parallelizable if and only if it has trivial normal bundle when embedded in a Euclidean space of dimension higher than twice the dimension of M.*

Proof:
see discussion following the Theorem (7.2) of [30]. □

We note that parallelizable manifolds (i.e. which has trivial tangent bundles) are in particular stably parallelizable. Moreover, given any compact Riemannian manifold M, its orthonormal frame bundle O_M is parallelizable. Also given any stably parallelizable manifold M, the total space of its cotangent bundle is again stably parallelizable.

5.3 Lifting an action to the tubular neighborhood of a stably parallelizable manifold

Now as before let M be a compact, oriented Riemannain n-manifold . Assume furthermore that M is stably parallelizable and let $M \subset \mathbb{R}^N$ be an isometric embedding with trivial normal bundle, for sufficiently large $N \geq n$. Let $\mathcal{Q}$ be a CQG which acts faithfully on M as in the sense mentioned earlier. Now as in subsection 4.2, we have the maximal Fréchet dense subalgebra β_0 of $C^\infty(M)$ over which α is algebraic and Sp $(\alpha(\beta_0))(1 \otimes \mathcal{Q}_0) = \beta_0 \otimes \mathcal{Q}_0$.

Now since M is a manifold with a trivial normal bundle, Recall from Lemma 5.2.1, the global diffeomorphism F and corresponding isomorphism

$$\pi_F : C^\infty(\mathcal{N}_\epsilon M) \to C^\infty(M \times B_\epsilon^{N-n}(0)).$$

Define

$$\widehat{\alpha}:\beta_0\otimes C^\infty(B_\epsilon^{N-n}(0))\to\beta_0\otimes C^\infty(B_\epsilon^{N-n}(0))\otimes\mathcal{Q}_0 \textit{ by } \widehat{\alpha}=\sigma_{23}\circ(\alpha\otimes\mathrm{id})$$

and extend $\hat{\alpha}$ as a Fréchet continuous map by Lemma 3.2.11.

Now we have $\pi_F : C^\infty(\mathcal{N}_\epsilon M)\to C^\infty(M\times B_\epsilon(0))$, which implies that $\pi_{F^{-1}} : C^\infty(M\times B_\epsilon(0))\to C^\infty(\mathcal{N}_\epsilon M)$. Hence

$$(\pi_{F^{-1}}\otimes\mathrm{id}_{\mathcal{Q}}) : C^\infty(M\times B_\epsilon(0))\widehat{\otimes}\mathcal{Q}\to C^\infty(\mathcal{N}_\epsilon M)\widehat{\otimes}\mathcal{Q}.$$

Set $\mathcal{A}_0 := \pi_{F^{-1}}(\beta_0\otimes C^\infty(B_\epsilon(0)))$. Then $\mathcal{A}_0$ is a Fréchet dense subalgebra of $C^\infty(\mathcal{N}_\epsilon(M))$. So, defining

$$\Phi := (\pi_{F^{-1}}\otimes\mathrm{id})\circ\widehat{\alpha}\circ\pi_F : C^\infty(\mathcal{N}_\epsilon M)\to C^\infty(\mathcal{N}_\epsilon M)\widehat{\otimes}\mathcal{Q},$$

we see that by construction, Φ is algebraic over $\mathcal{A}_0$ and moreover, $Sp\ \Phi(\mathcal{A}_0)(1\otimes\mathcal{Q}_0) = \mathcal{A}_0\otimes\mathcal{Q}_0$. Φ is also Fréchet continuous by Lemma 3.2.11. Hence Φ is a smooth action of $\mathcal{Q}$ on $\mathcal{N}_\epsilon M$.

Lemma 5.3.1. *If α is inner product presreving, so is Φ.*

Proof:
Note that by Remark 3.3.8 of Chapter 3, it suffices to show that $<<(d\otimes\mathrm{id})\Phi(\phi), d\otimes\mathrm{id})\Phi(\psi)>> = \Phi(<<d\phi, d\psi>>)$ for $\phi,\psi\in\mathcal{A}_0$. Now Consider $\phi,\psi\in\mathcal{A}_0$ of the form $\phi(y)=\xi\circ\pi(y)$ and $\psi(y)=\eta\circ\mathcal{U}(y)$, where ξ,η are smooth functions. Then $\Phi(\phi)(y) = \alpha(\xi)(\pi(y))$, $\Phi(\psi)=\psi\otimes 1$, and moreover it is easy to observe that $<<d\phi, d\psi>> = 0$ and

$$\begin{aligned}
& <<d\Phi(d\phi), d\Phi(d\psi)>> \\
= \ & <<(d\otimes\mathrm{id})\Phi(\phi), (d\otimes\mathrm{id})\Phi(\psi)>> \\
= \ & 0
\end{aligned}$$

Using this and Leibnitz formula we get $<<d\Phi(df_1), d\Phi(df_2)>> = \Phi(<<df_1, df_2>>)$ for f_1, f_2 of the form $f_i = (\xi_i\circ\pi)\eta_i\circ\mathcal{U}$, $i=1,2$.

As a general element of $\mathcal{A}_0$ is a finite sum of product functions of the form $(\xi\circ\pi).(\eta\circ\mathcal{U})$. □

Lemma 5.3.2. *If α preserves the Riemannian volume measure of M, then Φ preserves the Riemannian volume measure of $\mathcal{N}_\epsilon(M)$, which is the restriction of Lebesgue measure of $\mathbb{R}^N$.*

Proof:
We denote the Lebesgue measure of $\mathbb{R}^N$, restricted to $\mathcal{N}_\epsilon(M)$ by μ, the Riemannian volume measure of M by ν and the Lebesgue measure of $B_\epsilon^{N-n}(0)$ by ν'. Then by (iv) of lemma 5.2.1, we have for any $G \in C^\infty(\mathcal{N}_\epsilon(M))$,

$$\int_{M\times B_\epsilon^{N-n}(0)} \pi_F(G)d\nu d\nu' = \int_{\mathcal{N}_\epsilon(M)} Gd\mu.$$

Also for any $f \in C^\infty(M \times B_\epsilon^{N-n}(0))\hat{\otimes}\mathcal{Q}$,

$$\int_{\mathcal{N}_\epsilon(M)} (\pi_{F^{-1}} \otimes \mathrm{id})fd\mu = \int_{M\times B_\epsilon^{N-n}(0)} fd\nu d\nu'.$$

Hence it is enough to prove that $\hat{\alpha}$ preserves the Riemannian volume measure of $M \times B_\epsilon^{N-n}(0)$. For that let $\sum_{i=1}^k f_i \otimes q_i \in C^\infty(M) \otimes C^\infty(B_\epsilon^{N-n}(0))$. As α preserves the Riemannian volume measure of M, we have

$$\begin{aligned} &\int_{M\times B_\epsilon^{N-n}(0)} \hat{\alpha}(\sum_{i=1}^k f_i \otimes g_i)(x, u_1, ..., u_{N-n})d\nu d\nu' \\ = &\sum_{i=1}^k \int_M \alpha(f_i)(x)d\nu \int_{B_\epsilon^{N-n}(0)} g_i(u_1, ..., u_{N-n})d\nu' \\ = &\int_{M\times B_\epsilon^{N-n}(0)} (\sum_{i=1}^k f_i \otimes g_i)d\nu d\nu'. \end{aligned}$$

As $M \times B_\epsilon^{N-n}(0)$ is a compact manifold, we can conclude that $\int_{M\times B_\epsilon^{N-n}(0)} \hat{\alpha}(f)d\nu d\nu' = \int_{M\times B_\epsilon^{N-n}(0)} fd\nu d\nu'$ for any $f \in C^\infty(M \times B_\epsilon^{N-n}(0))$. □

5.4 Nonexistence of genuine quantum group action

Let $\{y_i : i = 1, .., N\}$ be the standard coordinates for $\mathbb{R}^N$. We will also use the same notation for the restrictions of y_i's if no confusion arises.

Definition 5.4.1. *A twice continuously differentiable, complex-valued function Ψ defined on a non empty, open set $\Omega \subset \mathbb{R}^N$ is said to be harmonic on Ω if*

$$\mathcal{L}_{\mathbb{R}^N}\Psi \equiv 0,$$

where $\mathcal{L}_{\mathbb{R}^N} \equiv \sum_{i=1}^N \frac{\partial^2}{\partial y_i^2}$.

We note the following result.

Lemma 5.4.2. *Let $\mathcal{C}$ be a unital commutative C^* algebra and $x_1, x_2, \ldots, x_N$ be self adjoint elements of $\mathcal{C}$ such that $\{x_i x_j : 1 \leq i \leq j \leq N\}$ are linearly independent and $\mathcal{C}$ be a unital C^* algebra generated by $\{x_1, x_2, ..., x_N\}$. Let $\mathcal{Q}$ be a compact quantum group acting faithfully on $\mathcal{C}$ such that the action leaves the span of $\{x_1, x_2, ..., x_N\}$ invariant. Then $\mathcal{Q}$ must be commutative as a C^* algebra, i.e. $\mathcal{Q} \cong C(G)$ for some compact group G.*

Proof:
Let us introduce a convenient terminology: call a finite dimensional vector subspace V of a commutative algebra quadratically independent if the dimension of the subspace $\{vw, v, w \in V\}$ is equal to the square of the dimension of V. Clearly, this is equivalent to the following: for any basis $\{v_1, v_2, \ldots, v_k\}$ of V, $\{v_i v_j, i \leq j \leq k\}$ will be linearly independent, hence a basis of $V \otimes^{\text{sym}} V$. From this definition we also see that any nonzero subspace of a quadratically independent space V is again quadratically independent.

Let us now denote the action of $\mathcal{Q}$ by α and set $V := \text{Sp}\{x_i, i = 1, \ldots, N\}$ which is a quadratically independent subspace of dimension N in $\mathcal{C}$. We claim that without loss of generality we can assume the existence of an inner product on V for which $\alpha|_V$ is a unitary representation. Indeed, we recall the maximal algebraic subspace $\mathcal{C}_0$ which is a direct sum of $\ker(\alpha)$ and the spectral subspace for the action. In fact, as α is $*$-preserving, we can have a similar decomposition of the real algebra $\mathcal{C}^{\text{s.a.}}$ consisting of self-adjoint elements of $\mathcal{C}$. Thus, we can decompose V into real subspaces $V_0 \oplus V_1$, where $V_0 \subseteq \text{Ker}(\alpha)$ and $V_1 \subseteq \mathcal{C}^{\text{s.a.}}$ is contained in the spectral subspace for the action. In particular, α is injective on the algebra generated by V_1. Moreover, V_1 must be nonzero, because otherwise $\mathcal{Q}$ will be 0, and thus V_1 is quadratically independent. Replacing V by V_1 if necessary, we can assume that V is contained in the spectral subspace for the action α, so in particular, α is a non-degenerate algebraic representation on V and moreover, both α and the Haar state (say h) of $\mathcal{Q}$ are faithful on the $*$-algebra generated by the elements of V. Choose some faithful positive functional ϕ on the unital separable C^* algebra $\mathcal{C}$ and consider the convolved functional $\overline{\phi} = (\phi \otimes h) \circ \alpha$ which is clearly faithful on the $*$-algebra generated by $\{x_i\}$'s and also $\mathcal{Q}$-invariant, so that α gives a unitary representation w.r.t. the inner product say $< \cdot, \cdot >_{\overline{\phi}}$ coming from $\overline{\phi}$ on V. As $\mathcal{A}$ is commutative and x_i are self-adjoint, so are $x_i x_j$ for all i, j, and hence $< x_i, x_j >_{\overline{\phi}}$'s are real numbers. Thus, Gram-Schmidt orthogonalization on $\{x_1, ... x_N\}$ will give an orthonormal set $\{y_1, ..., y_N\}$ consisting of self-adjoint elements, with the same span as $V = \text{Span}\{x_1, ..., x_N\}$. Replacing x_i's by y_i's, let us assume for the rest of the proof that $\{x_1, ..., x_N\}$ is an orthonormal set, there are $Q_{ij} \in \mathcal{Q}$, $i, j = 1, \ldots, N$ such that

$\mathcal{Q} = C^*(Q_{ij}, i, j = 1, \ldots, N)$ and

$$\alpha(x_i) = \sum_{j=1}^{N} x_j \otimes Q_{ij}, \quad \forall i = 1, \ldots, N.$$

Since $x_i^* = x_i$ for each i and α is a $*$-homomorphism, we must have $Q_{ij}^* = Q_{ij}$ $\forall i, j = 1, 2, ..., N$.

The condition that x_i,x_j commute $\forall i, j$ gives

$$Q_{ij}Q_{kj} = Q_{kj}Q_{ij} \forall i, j, k, \tag{5.4.1}$$

$$Q_{ik}Q_{jl} + Q_{il}Q_{jk} = Q_{jk}Q_{il} + Q_{jl}Q_{ik}. \tag{5.4.2}$$

As $Q = ((Q_{ij})) \in M_N(\mathcal{Q})$ is a unitary, $Q^{-1} = Q^* = Q^T := ((Q_{ji}))$, since in this case entries of Q are self-adjoint elements.

Clearly, the matrix Q is an N-dimensional unitary representation of $\mathcal{Q}$, so $Q^{-1} = (\mathrm{id} \otimes \kappa)(Q)$, where κ is the antipode map.

So we obtain

$$\kappa(Q_{ij}) = Q_{ij}^{-1} = Q_{ij}^T = Q_{ji}. \tag{5.4.3}$$

Now from (5.4.1) , we have $Q_{ij}Q_{kj} = Q_{kj}Q_{ij}$. Applying κ on this equation and using the fact that κ is an antihomomorphism along with (5.4.3) , we have $Q_{jk}Q_{ji} = Q_{ji}Q_{jk}$ Similarly , applying κ on (5.4.2), we get

$$Q_{lj}Q_{ki} + Q_{kj}Q_{li} = Q_{li}Q_{kj} + Q_{ki}Q_{lj} \ \forall i, j, k, l.$$

Interchanging between k and i and also between l, j gives

$$Q_{jl}Q_{ik} + Q_{il}Q_{jk} = Q_{jk}Q_{il} + Q_{ik}Q_{jl} \ \forall i, j, k, l. \tag{5.4.4}$$

Now, by (5.4.2)-(5.4.4) , we have

$$[Q_{ik}, Q_{jl}] = [Q_{jl}, Q_{ik}],$$

hence

$$[Q_{ik}, Q_{jl}] = 0.$$

Therefore the entries of the matrix Q commute among themselves. However, by faithfulness of the action of $\mathcal{Q}$, it is clear that the C^*-subalgebra generated by entries of Q must be the same as $\mathcal{Q}$, so $\mathcal{Q}$ is commutative. □

Lemma 5.4.3. *Let W be a manifold (possibly with boundary) embedded in some $\mathbb{R}^N$*

and $\{y_i\}$'s for $i = 1, ..., N$, be the coordinate functions for $\mathbb{R}^N$ restricted to W. If W has non empty interior in $\mathbb{R}^N$, then $\{1, y_i y_j, y_i : 1 \leq i, j \leq N\}$ are linearly independent.

Proof:
If possible let on W $c.1+\sum c_{ij}y_iy_j+\sum_k d_k y_k = 0$ for some c_{ij}, d_k . Pick an interior point $y \in W$. Then at y, we can take partial derivatives in any direction. Hence applying $\frac{\partial}{\partial y_i}|_y \frac{\partial}{\partial y_j}|_y$ to $c.1 + \sum c_{ij}y_iy_j + \sum_k d_k y_k = 0$, we conclude that $c_{ij} = 0 \; \forall \; i, j$. Similarly we can prove d_k's are 0 and hence $c = 0$. □

Lemma 5.4.4. *Let Φ be a smooth action of a CQG on a compact subset of $\mathbb{R}^N$ which commutes with $\mathcal{L}_{\mathbb{R}^N}$, Then Φ is affine i.e.*

$$\Phi(y_i) = 1 \otimes q_i + \sum_{j=1}^{N} y_j \otimes q_{ij}, \; for \; some \; q_{ij}, q_i \in \mathcal{Q}$$

for all $i = 1, ..., N$, where y_i's coordinates of $\mathbb{R}^N$.

Proof:
As Φ commutes with the geometric Laplacian and $\mathcal{L}_{\mathbb{R}^N}\frac{\partial}{\partial y_j} = \frac{\partial}{\partial y_j}\mathcal{L}_{\mathbb{R}^N}$, $\mathcal{L}_{\mathbb{R}^N} y_j = 0$ for all j, we get

$$\begin{aligned} & (\mathcal{L}_{\mathbb{R}^N} \otimes \mathrm{id})(\frac{\partial}{\partial y_j} \otimes id)\Phi(y_i) \\ = \; & (\frac{\partial}{\partial y_j} \otimes \mathrm{id})\Phi(\mathcal{L}_{\mathbb{R}^N} y_i) \\ = \; & 0. \end{aligned}$$

Let $D_{ij}(y) = ((\frac{\partial}{\partial y_i} \otimes \mathrm{id})\Phi(y_j))(y)$. Note that as $d\Phi$ is an Φ-equivariant unitary representation, by Lemma 3.3.6 $((D_{ij}(y)))_{i,j=1,...,N}$ is unitary for all $y \in \mathcal{N}_\epsilon(M)$. Pick y_0 in the interior of $\mathcal{N}_\epsilon M$(which is non empty). Then the new $\mathcal{Q}$ valued matrix $((G_{ij}(y))) = ((D_{ij}(y)))((D_{ij}(y_0)))^{-1}$ is unitary (since $D_{ij}(y)$ is so). $G_{ij}(y)$ is unitary for all $y \Rightarrow |\psi(G_{ij}(y))| \leq 1$ and $|\psi(G_{ii}(y_0))| = 1$. $\psi(G_{ii}(y))$ is a harmonic function on an open *connected* set $Int(\mathcal{N}_\epsilon M)$ which attains its supremum at an interior point. Hence by corollary 1.9 of [1] we conclude that $\psi(G_{ii}(y)) = \psi(G_{ii}(y_0))$. $((G_{ij}(y)))$ being unitary for all y, $G_{ij} = \delta_{ij}.1_{\mathcal{Q}}$. Then $((D_{ij}(y)))((D_{ij}(y_0)))^{-1} = 1_{M_N(\mathcal{Q})}$. So $((D_{ij}(y))) = ((D_{ij}(y_0)))$ for all $y \in \mathcal{N}_\epsilon(M)$. Hence Φ is affine with $q_{ij} = D_{ij}(y_0)$ □

Remark 5.4.5. *This is the only place where we have made use of the assumption that the manifold is connected.*

Corollary 5.4.6. *Let M be a smooth, compact, orientable, connected, stably parallelizable manifold. Then if α is an isometric action of a CQG $\mathcal{Q}$ on M, then $\mathcal{Q}$ must be commutative as a C^* algebra i.e. $\mathcal{Q} \cong C(G)$ for some compact group G.*

Proof:
As the action is isometric, it preserves the Riemannian inner product and the functional coming from the Riemannian volume measure. As the manifold is stably parallelizable, we can embed it isometrically into some $\mathbb{R}^N$ such that it has trivial normal bundle. Then we lift the action to the tubular neighborhood of the manifold such that it preserves the Riemannian inner product of the tubular neighborhood and the functional corresponding to the Riemannian volume measure by Lemmas 5.3.1 and 5.3.2. So by Theorem 4.2.2, it commutes with the geometric Laplacian of the tubular neighborhood, which is a subset of $\mathbb{R}^N$ for some N. As the tubular neighborhood has non empty interior in $\mathbb{R}^N$, applying Lemma 5.4.4, Lemma 5.4.3 and Lemma 5.4.2, we complete the proof. □

Corollary 5.4.7. *The quantum isometry group of a compact, connected, stably parallelizable manifold M is commutative as a C^* algebra i.e. isomorphic to $C(\mathrm{ISO}(M))$.*

Chapter 6

QISO of cocycle twisted manifolds

6.1 Introduction

Most of the examples of noncommutative manifolds are obtained by deforming classical spectral triples. It was shown in [10] that the quantum isometry group of a Rieffel-deformed noncommutative manifold can be obtained by a similar deformation (Rieffel-Wang, see [54]) of the quantum isometry group of the original (undeformed) noncommutative manifold. In this chapter our goal is to generalize Bhowmick-Goswami's results about Rieffel-deformation to any cocycle twisted spectral triple. Combining this with the fact (proved in the previous chapter) that the quantum isometry group of a classical compact, connected, stably parallelizable Riemannian manifold is the same as the classical isometry group of such manifolds (i.e. there is no genuine quantum isometry for such manifold), we shall be able to compute the quantum isometry group of non commutative manifolds obtained from classical (compact, connected, stably parallelizable) manifolds using unitary 2-cocycle. We shall also relax the assumption of existence of a dense $*$-algebra for which the action is algebraic in case of Rieffel deformed manifolds.

6.2 Discrete quantum group

Let $\mathcal{Q}$ be a compact quantum group. Recall the dense Hopf $*$-algebra $\mathcal{Q}_0$ spanned by the matrix coefficients of its inequivalent irreducible representations. Also recall the dual discrete quantum group $\hat{\mathcal{Q}}$ of $\mathcal{Q}$. With these notations we have the following

To define the cocycle twist of a CQG $\mathcal{Q}$, we first briefly discuss about discrete quantum group from [52].

Let $\mathcal{A}_0 = \oplus_{\alpha \in I} M_{n_\alpha}$ for some index set I, where M_{n_α} is a complex $n_\alpha \times n_\alpha$ matrix.

Any element in $\mathcal{A}_0$ looks like $\sum_{\alpha\in I} a_\alpha$ for $a_\alpha \in M_{n_\alpha}$ and for all but finitely many indices α, a_α is non zero. Then $\mathcal{A}_0 \otimes \mathcal{A}_0 = \oplus_{\alpha,\beta} M_{n_\alpha} \otimes M_{n_\beta}$, algebraic multipliers of $\mathcal{A}_0$ and $\mathcal{A}_0 \otimes \mathcal{A}_0$ are $\Pi_{\alpha\in I} M_{n_\alpha}$ and $\Pi_{\alpha,\beta\in I} M_{n_\alpha} \otimes M_{n_\beta}$ respectively.

A comultiplication on $\mathcal{A}_0$ is a $*$- homomorphism $\Delta : \mathcal{A}_0 \to M_{alg}(\mathcal{A}_0 \otimes \mathcal{A}_0)$ such that it extends to a unital $*$-homomorphism from $M_{alg}(\mathcal{A}_0)$ to $M_{alg}(\mathcal{A}_0 \otimes \mathcal{A}_0)$ such that it is coassociative meaning $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ and both $\Delta(a)(1 \otimes b)$ and $(a \otimes 1)\Delta(b) \in \mathcal{A}_0 \otimes \mathcal{A}_0$ for $a, b \in \mathcal{A}_0$.

Definition 6.2.1. *A discrete quantum group is a pair* $(\mathcal{A}_0, \Delta)$ *where* $\mathcal{A}_0 = \oplus_{\alpha\in I} M_{n_\alpha}$ *and* Δ *is a comultiplication as above with the following conditions:*
(1) $T_1 : \mathcal{A}_0 \otimes \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{A}_0$ *given by* $T_1(a \otimes b) := \Delta(a)(1 \otimes b)$ *is bijective on* $\mathcal{A}_0 \otimes \mathcal{A}_0$.
(2) $T_2 : \mathcal{A}_0 \otimes \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{A}_0$ *given by* $T_2(a \otimes b) := (a \otimes 1)\Delta(b)$ *is also bijective on* $\mathcal{A}_0 \otimes \mathcal{A}_0$.

Let e_α denote the minimal central projection which is identity on M_{n_α} and 0 elsewhere. Also let $p_\gamma := \sum_{\alpha\leq\gamma} e_\alpha$. Then p_γ is also a central projection on $\mathcal{A}_0$. If $b = \sum_{\alpha\leq\gamma} b_\alpha \in \mathcal{A}_0$, $bp_\gamma = b$.

Now given a discrete quantum group $(\mathcal{A}_0, \Delta)$, recall from [52], we have an antipode $\kappa : \mathcal{A}_0 \to \mathcal{A}_0$ which is an antihomomorphism and invertible on $\mathcal{A}_0$. We also have the counit $\epsilon : \mathcal{A}_0 \to \mathbb{C}$ with $m \circ (\mathrm{id} \otimes \kappa)(\Delta(a)(1 \otimes b)) = \epsilon(a)b$ for all $a, b \in \mathcal{A}_0$.

Also given a discrete quantum group $(\mathcal{A}_0, \Delta)$, for fixed $\alpha, \beta \in I$, the set $\{\gamma \in I; \Delta(e_\gamma)(e_\alpha \otimes e_\beta) \neq 0\}$ is finite and hence we can define

$$\Delta : \Pi M_{n_\alpha} \to \Pi(M_{n_\alpha} \otimes M_{n_\beta})$$

by prescribing the $\beta\gamma$ th coordinate of $\Delta((a_\alpha)_\alpha)$ by $\sum \Delta(a_{\gamma'})$ such that $\Delta(a_{\gamma'})(e_\beta \otimes e_\gamma) \neq 0$. Recall that there is a bijection of the index set I such that $\kappa(M_{n_\alpha}) = M_{n_{\alpha'}}$. So define $\kappa : \Pi M_{n_\alpha} \to \Pi M_{n_\alpha}$ by

$$\kappa((a_\alpha)_\alpha) = ((a_{\alpha'})_{\alpha'})$$

where $\kappa(a_\alpha) = a_{\alpha'}$ for some α and α'. For similar reasons we can define $\epsilon(\sum_{\alpha\in I} a_\alpha) = \sum_{\alpha\in I} \epsilon(a_\alpha)$ and $m : \Pi(M_{n_\alpha} \times M_{n_\beta}) \to M_{n_\alpha}$ by

$$m(\sum a_\alpha \otimes b_\beta) = ((a_\alpha b_\alpha)).$$

Then we have $(\epsilon \otimes \mathrm{id})\Delta((a_\alpha)_\alpha) = ((a_\alpha)_\alpha)$ for all $((a_\alpha)_\alpha) \in \Pi M_{n_\alpha}$. To see that let $\eta \in I$ and $\Delta((a_\alpha)_\alpha) = ((\sum_{\gamma\in I} \Delta(a_\gamma)))_{\beta\gamma'}$ where γ's are such that $\Delta(a_\gamma)(e_\beta \otimes e_{\gamma'}) \neq 0$ and hence

$$\begin{aligned}
& (\epsilon \otimes \mathrm{id})\Delta((a_\alpha)_\alpha).e_\eta \\
= & \sum (\epsilon \otimes \mathrm{id})\Delta(a_\gamma)(1 \otimes e_\eta) \\
= & a_\eta.
\end{aligned}$$

The last line follows since $(\epsilon \otimes \mathrm{id})\Delta(a_\gamma)(1 \otimes e_\eta) = a_\gamma e_\eta$. So $(\epsilon \otimes \mathrm{id})\Delta((a_\alpha)_\alpha) = (a_\alpha)_\alpha$.

Also

$$\begin{aligned}
& m \circ (\kappa \otimes \mathrm{id})\Delta((a_\alpha)_\alpha).e_\beta \\
= & \sum_{\gamma \in I} m \circ (\kappa \otimes \mathrm{id})(\Delta(a_\gamma)(1 \otimes e_\beta)) \\
= & \sum_{\gamma \in I} \epsilon(a_\gamma) e_\beta \\
= & \epsilon((a_\alpha)_\alpha) e_\beta
\end{aligned}$$

Hence $m \circ (\kappa \otimes \mathrm{id})\Delta((a_\alpha)_\alpha) = \epsilon((a_\alpha)_\alpha) = \epsilon((a_\alpha)_\alpha).1$ on ΠM_{n_α}. Conversely, let $(\mathcal{A}_0, \Delta)$ be as above with an invertible map $\kappa : \mathcal{A}_0 \to \mathcal{A}_0$ and $\epsilon : \mathcal{A}_0 \to \mathbb{C}$ defined as above with

$$m \circ (\kappa \otimes \mathrm{id})(\Delta(a)(1 \otimes b)) = \epsilon(a)b,$$

for $a, b \in \mathcal{A}_0$. So with similar arguments as above we can extend the maps to $M_{alg}(\mathcal{A}_0)$ and $M_{alg}(\mathcal{A}_0 \otimes \mathcal{A}_0)$ and get the identities

$$(\epsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \epsilon)\Delta \textit{ and } m \circ (\kappa \otimes \mathrm{id})\Delta(.) = \epsilon(.)1,$$

on ΠM_{n_α}.

So we can define $R : \mathcal{A}_0 \otimes \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{A}_0$ by

$$R(a \otimes b) := (\mathrm{id} \otimes m)[(\mathrm{id} \otimes \kappa \otimes \mathrm{id})(((\Delta \otimes \mathrm{id})(a \otimes 1))(1 \otimes 1 \otimes b)].$$

Then

$$\begin{aligned}
R \circ T_1(a \otimes b) &= (\mathrm{id} \otimes m)[(\mathrm{id} \otimes \kappa \otimes \mathrm{id})((\Delta \otimes \mathrm{id})\Delta(a))(1 \otimes 1 \otimes b)] \\
&= (\mathrm{id} \otimes \epsilon)\Delta(a) \otimes b \\
&= a \otimes b
\end{aligned}$$

Similarly we can show that $T_1 \circ R(a \otimes b) = a \otimes b$ and hence T_1 is invertible on $\mathcal{A}_0 \otimes \mathcal{A}_0$. Similarly we can show that T_2 is invertible on $\mathcal{A}_0 \otimes \mathcal{A}_0$. So combining all these we have

Proposition 6.2.2. *Let $\mathcal{A}_0$ be a direct sum of matrix algebras with Δ being the comulti-*

plication on $\mathcal{A}_0$. If there is an invertible map $\kappa : M_{alg}(\mathcal{A}_0) \to M_{alg}(\mathcal{A}_0)$ which maps $\mathcal{A}_0$ onto $\mathcal{A}_0$ such that $m \circ (\kappa \otimes \mathrm{id})(\Delta(a)) = m \circ (\mathrm{id} \otimes \kappa)(\Delta(a)) = \epsilon(a)$ for all $a \in M_{alg}(\mathcal{A}_0)$ and $(\epsilon \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes \epsilon)(\Delta(a)) = a$ for all $a \in M_{alg}(\mathcal{A}_0)$. Then $(\mathcal{A}_0, \Delta)$ is a discrete quantum group.

6.3 Cocycle twist of a CQG

Definition 6.3.1. *By a unitary 2-cocycle σ of a compact quantum group $\mathcal{Q}$, we mean a unitary element of $\mathcal{M}(\widehat{\mathcal{Q}}\hat{\otimes}\widehat{\mathcal{Q}})$ satisfying*

$$(1 \otimes \sigma)(\mathrm{id} \otimes \widehat{\Delta})\sigma = (\sigma \otimes 1)(\widehat{\Delta} \otimes \mathrm{id})\sigma.$$

Let $(\mathcal{Q}, \Delta)$ is a CQG with a unitary 2-cocycle $\sigma \in M(\widehat{\mathcal{Q}} \otimes \widehat{\mathcal{Q}})$. σ viewed as a linear functional on $\mathcal{Q}_0 \otimes \mathcal{Q}_0$ satisfies the cocycle condition (see page 64 of [36])

$$\sigma(b_{(1)}, c_{(1)})\sigma(a, b_{(2)}c_{(2)}) = \sigma(a_{(1)}, b_{(1)})\sigma(a_{(2)}b_{(2)}, c),$$

for $a, b, c \in \mathcal{Q}_0$ (Sweedler's notation). We can deform $\mathcal{Q}_0$ using σ to obtain a new Hopf $*$-algebra $\mathcal{Q}_0^\sigma$. Then $\mathcal{Q}_0^\sigma$ and $\hat{\mathcal{Q}_{0\sigma}}$ again form a non degenerate pairing. We twist the product of the algebra $\mathcal{Q}_0^\sigma$ by the following formula:

$$a._\sigma b := \sigma^{-1}(a_{(1)}, b_{(1)})a_{(2)}b_{(2)}\sigma(a_{(3)}, b_{(3)}),$$

for $a, b \in \mathcal{Q}_0$. The coproduct remains unchanged. The $*$ structure and κ gets changed by the formulae:

$$a^{*_\sigma} := \sum v^{-1}(a_{(1)})a^*_{(2)}v(a_{(3)}),$$
$$\kappa_\sigma(a) := U(a_{(1)})\kappa(a_{(2)})U^{-1}(a_{(3)}).$$

(see page 65 of [36]). All the proofs are done in [36], although for completeness, we include a proof (of the associativity of the new twisted product) here. The other proofs can be done with similar computations.

Lemma 6.3.2. *The new twisted product $._\sigma$ is associative.*

Proof:
Let $\alpha : A_0 \to A_0 \otimes H$ be an action of a Hopf $*$-algebra H on an algebra A_0. If we define $a * b := a_{(0)}b_{(0)}\sigma^{-1}(a_{(1)}, b_{(1)})$, then $*$ is an associative product on A_0. For that observe

that

$$\begin{aligned}(a*b)*c &= a_{(0)(0)}b_{(0)(0)}c_{(0)}\sigma^{-1}(a_{(1)},b_{(1)})\sigma^{-1}(a_{(0)(1)}b_{(0)(1)},c_{(1)})\\ &= a_{(0)}b_{(0)}c_{(0)}\sigma^{-1}(a_{(1)(2)},b_{(1)(2)})\sigma^{-1}(a_{(1)(1)}b_{(1)(1)},c_{(1)})\end{aligned}$$

and

$$\begin{aligned}a*(b*c) &= a_{(0)}b_{(0)(0)}c_{(0)(0)}\sigma^{-1}(b_{(1)},c_{(1)})\sigma^{-1}(a_{(1)},b_{(0)(1)}c_{(0)(1)})\\ &= a_{(0)}b_{(0)}c_{(0)}\sigma^{-1}(a_{(1)},b_{(1)(1)}c_{(1)(1)})\sigma^{-1}(b_{(1)(2)},c_{(1)(2)})\end{aligned}$$

Now using the cocycle condition, we get

$$(a*b)*c = a*(b*c) \tag{6.3.1}$$

Define $\Delta^f : H \to H \otimes H$ by $\Delta^f(a) = a_{(2)} \otimes a_{(1)}$. Then we define $\bar{\Delta} : H \otimes H \to H \otimes H \otimes H \otimes H$ by

$$\bar{\Delta} := f_{23}(\Delta^f \otimes \Delta),$$

where f_{23} is the flip map between 2nd and 3rd copy. Then using the coassociativity of Δ, it is easy to see that $\bar{\Delta}$ is also coassociative. Now define $\bar{\alpha} : H \to H \otimes H \otimes H$ by $\bar{\alpha} := (\Delta^f \otimes \mathrm{id})\Delta$. Then we claim that $(\mathrm{id} \otimes \bar{\Delta})\bar{\alpha} = (\bar{\alpha} \otimes \mathrm{id})\bar{\alpha}$. For that we observe that, for $q \in H$,

$$\begin{aligned}(\bar{\alpha} \otimes \mathrm{id})\bar{\alpha}(q) &= (\bar{\alpha} \otimes \mathrm{id})(\Delta^f \otimes \mathrm{id})\Delta(q)\\ &= q_{(1)(2)(1)(2)} \otimes q_{(1)(2)(1)(1)} \otimes q_{(1)(2)(2)} \otimes q_{(1)(1)} \otimes q_{(2)}\end{aligned}$$

Similarly

$$(\mathrm{id} \otimes \bar{\Delta})\bar{\alpha} = q_{(1)(2)} \otimes q_{(1)(1)(2)} \otimes q_{(2)(1)} \otimes q_{(1)(1)(1)} \otimes q_{(2)(2)}$$

Now using the coassociativity of Δ, we can show that $(\mathrm{id} \otimes \bar{\Delta})\bar{\alpha} = (\bar{\alpha} \otimes \mathrm{id})\bar{\alpha}$. Now we proceed to define a 2-cocycle ψ on $H \otimes H$, that is a convolution invertible linear map $\psi : H \otimes H \otimes H \otimes H \to \mathbb{C}$ by

$$\psi(p \otimes q \otimes r \otimes s) = \sigma(p,r)\sigma^{-1}(q,s).$$

We claim that ψ satisfies

$$\psi(x_{(1)}y_{(1)}, z)\psi(x_{(2)}, y_{(2)}) = \psi(x, y_{(1)}z_{(1)})\psi(y_{(2)}, z_{(2)}),$$

for $x, y, z \in H \otimes H$.
Let $x = p \otimes q,\ y = r \otimes s,\ z = t \otimes u$. Then $\bar{\Delta}(p \otimes q) = p_{(2)} \otimes q_{(1)} \otimes p_{(1)} \otimes q_{(2)}$. Hence $x_{(1)} = p_{(2)} \otimes q_{(1)}$ and $x_{(2)} = p_{(1)} \otimes q_{(2)}$. Similarly $y_{(1)} = r_{(2)} \otimes s_{(1)}$, $y_{(2)} = r_{(1)} \otimes s_{(2)}$, $z_{(1)} = t_{(2)} \otimes u_{(1)}$ and $z_{(2)} = t_{(1)} \otimes u_{(2)}$. Then

$$\begin{aligned}
& \psi(x_{(1)}y_{(1)}, z)\psi(x_{(2)}, y_{(2)}) \\
= \ & \psi(p_{(2)}r_{(2)} \otimes q_{(1)}s_{(1)}, t \otimes u)\psi(p_{(1)} \otimes q_{(2)}, r_{(1)} \otimes s_{(2)}) \\
= \ & \sigma(p_{(2)}r_{(2)}, t)\sigma^{-1}(q_{(1)}s_{(1)}, u)\sigma(p_{(1)}, r_{(1)})\sigma^{-1}(q_{(2)}, s_{(2)}) \\
= \ & \sigma(r_{(1)}, t_{(1)})\sigma(p, r_{(2)}t_{(2)})\sigma^{-1}(q, s_{(1)}u_{(1)})\sigma^{-1}(s_{(2)}, u_{(2)})
\end{aligned}$$

On the other hand ,

$$\begin{aligned}
& \psi(x, y_{(1)}z_{(1)})\psi(y_{(2)}, z_{(2)}) \\
= \ & \psi(p \otimes q, r_{(2)}t_{(2)} \otimes s_{(1)}u_{(1)})\psi(r_{(1)} \otimes s_{(2)}, t_{(1)} \otimes u_{(2)}) \\
= \ & \sigma(p, r_{(2)}t_{(2)})\sigma^{-1}(q, s_{(1)}u_{(1)})\sigma(r_{(1)}, t_{(1)})\sigma^{-1}(s_{(2)}, u_{(2)})
\end{aligned}$$

Now using the action $\bar{\alpha}$ of $H \otimes H$ on H, we define a new twisted product $*_\psi$ on H by

$$a *_\psi b = a_{(0)}b_{(0)}\psi(a_{(1)}, b_{(1)}),$$

where $\bar{\alpha}(a) = a_{(0)} \otimes a_{(1)}$. Then we can easily see that $*_\psi$ is nothing but $._\sigma$ and hence by (2), we conclude that $._\sigma$ is associative. □

We can prove that $(\mathcal{Q}_0^\sigma, \Delta_\sigma, \kappa, \epsilon)$ with the deformed algebra structure is again a unital Hopf $*$ algebra. But we don't know yet whether there is any compact quantum group containing $(\mathcal{Q}_0^\sigma, \Delta, \kappa_\sigma, \epsilon)$ as a Hopf $*$ algebra. Now we turn to prove the existence of such a CQG by duality.

Recall the dual $(\widehat{\mathcal{Q}_0}, \widehat{\Delta})$ which is a discrete quantum group. Hence we have $\widehat{\Delta} : M_{alg}(\widehat{\mathcal{Q}_0}) \to: M_{alg}(\widehat{\mathcal{Q}_0} \otimes \widehat{\mathcal{Q}_0})$, $\widehat{\kappa} : M_{alg}(\hat{\mathcal{Q}}_0) \to M_{alg}(\widehat{\mathcal{Q}_0})$ such that $\widehat{\kappa}$ is invertible and maps $\widehat{\mathcal{Q}_0}$ onto $\widehat{\mathcal{Q}_0}$, $\widehat{\epsilon} : M_{alg}(\widehat{\mathcal{Q}_0}) \to \mathbb{C}$ such that $(\widehat{\epsilon} \otimes \mathrm{id})\widehat{\Delta}(a) = (\mathrm{id} \otimes \widehat{\epsilon})\widehat{\Delta}(a) = a$ for all $a \in M_{alg}(\widehat{\mathcal{Q}_0})$, $m \circ (\widehat{\kappa} \otimes \mathrm{id})\widehat{\Delta}(a) = m \circ (\mathrm{id} \otimes \widehat{\kappa})\widehat{\Delta}(a) = \widehat{\epsilon}(a).1$ for all $a \in M_{alg}(\hat{\mathcal{Q}}_0)$. We can deform the coproduct $\widehat{\Delta_\sigma} : \widehat{\mathcal{Q}_0}^\sigma \to M_{alg}(\widehat{\mathcal{Q}_0}^\sigma \otimes \widehat{\mathcal{Q}_0}^\sigma)$ defined by $\widehat{\Delta}_\sigma(a) := \sigma.\widehat{\Delta}(a).\sigma^{-1}$. Then it is easy to see that $\widehat{\Delta}_\sigma$ extends to $M_{alg}(\widehat{\mathcal{Q}_0}^\sigma) \to M_{alg}(\widehat{\mathcal{Q}_0}^\sigma \otimes \widehat{\mathcal{Q}_0}^\sigma)$ such that

$(\mathrm{id} \otimes \widehat{\Delta}_\sigma)\widehat{\Delta}_\sigma = (\widehat{\Delta}_\sigma \otimes \mathrm{id})\widehat{\Delta}_\sigma$. This easily follows from the cocycle condition. We do not change the algebra structure of $\widehat{\mathcal{Q}_0}$. Also $\widehat{\epsilon}$ is not changed.

Let $U = m \circ (\mathrm{id} \otimes \widehat{\kappa})(\sigma)$. It can be easily shown that $U \in M_{alg}(\widehat{\mathcal{Q}_0^\sigma})$ and U is invertible with $U^{-1} \in M_{alg}(\widehat{\mathcal{Q}_0^\sigma})$. Now we define

$$\widehat{\kappa_\sigma} : M_{alg}(\widehat{\mathcal{Q}_0^\sigma}) \to M_{alg}(\widehat{\mathcal{Q}_0^\sigma}),$$

by $\widehat{\kappa_\sigma}(a) := U\widehat{\kappa}(a)U^{-1}$. Note that $\widehat{\mathcal{Q}_0^\sigma}$ being an ideal in $M_{alg}(\widehat{\mathcal{Q}_0^\sigma})$, $\widehat{\kappa_\sigma}(a) \in \widehat{\mathcal{Q}_0^\sigma}$ for all $a \in \widehat{\mathcal{Q}_0^\sigma}$.

Also we can show that $\widehat{\kappa_\sigma}$ is invertible and $\widehat{\kappa_\sigma^{-1}}$ maps $\widehat{\mathcal{Q}_0^\sigma}$ into $\widehat{\mathcal{Q}_0^\sigma}$. Using similar computations as in [36], we can show that on $M_{alg}(\widehat{\mathcal{Q}_0^\sigma})$,

$$(\widehat{\epsilon} \otimes \mathrm{id})\widehat{\Delta_\sigma} = (\mathrm{id} \otimes \widehat{\epsilon})\widehat{\Delta_\sigma} = \mathrm{id},$$

and

$$m \circ (\widehat{\kappa_\sigma} \otimes \mathrm{id})\widehat{\Delta_\sigma}(.) = m \circ (\mathrm{id} \otimes \widehat{\kappa_\sigma})\widehat{\Delta_\sigma}(.) = \widehat{\epsilon}(.)1.$$

Now let $a, b \in \widehat{\mathcal{Q}_0^\sigma}$. Then $\widehat{\Delta_\sigma}(a)(1 \otimes b) = \sigma\widehat{\Delta}\sigma^{-1}(1 \otimes b)$. Let p_γ be the central projection where $b = \sum_{\alpha \leq \gamma} b_\alpha \in \widehat{\mathcal{Q}_0}$. So

$$\begin{aligned}\widehat{\Delta_\sigma}(a)(1 \otimes b) &= \sigma\widehat{\Delta}(a)\sigma^{-1}(1 \otimes p_\gamma)(1 \otimes b) \\ &= \sigma\widehat{\Delta}(a)(1 \otimes p_\gamma)\sigma^{-1}(1 \otimes b)(\textit{since } p_\sigma \textit{ is central}) \\ &\in \widehat{\mathcal{Q}_0^\sigma} \otimes \widehat{\mathcal{Q}_0^\sigma}\end{aligned}$$

The last line follows since $\widehat{\Delta}(a)(1 \otimes p_\gamma) \in \widehat{\mathcal{Q}_0^\sigma} \otimes \widehat{\mathcal{Q}_0^\sigma}$. So T_1^σ maps $\widehat{\mathcal{Q}_0^\sigma} \otimes \widehat{\mathcal{Q}_0^\sigma}$ into $\widehat{\mathcal{Q}_0^\sigma} \otimes \widehat{\mathcal{Q}_0^\sigma}$. If we define R_1^σ by $R_1^\sigma(a \otimes b) := (1 \otimes \widehat{\kappa_\sigma})((1 \otimes \widehat{\kappa_\sigma}^{-1}(b))\widehat{\Delta_\sigma}(a))$. But $((1 \otimes \widehat{\kappa_\sigma}^{-1}(b))\widehat{\Delta_\sigma}(a)) \in \widehat{\mathcal{Q}_0^\sigma} \otimes \widehat{\mathcal{Q}_0^\sigma}$ and $\widehat{\kappa_\sigma}$ maps $\widehat{\mathcal{Q}_0^\sigma}$ into $\widehat{\mathcal{Q}_0^\sigma}$ proving that T_1^σ is a bijection. Similarly defining R_2^σ, we can prove that T_2^σ is a bijection. Hence applying Proposition 6.2.2, we can conclude that $(\widehat{\mathcal{Q}_0^\sigma}, \widehat{\Delta_\sigma})$ is again a discrete quantum group. Hence $\mathcal{Q}_0^\sigma$ is a Hopf $*$ algebra of the CQG dual to $(\widehat{\mathcal{Q}_0^\sigma}, \widehat{\Delta_\sigma})$.

Definition 6.3.3. *The cocycle twist of a CQG* $(\mathcal{Q}, \Delta)$ *by a unitary 2-cocycle* σ *on* $\mathcal{Q}$ *is defined to be the universal CQG containing* $(\mathcal{Q}_0^\sigma, \Delta, \kappa_\sigma, \epsilon)$ *as a Hopf* $*$ *algebra.*

Let us now discuss how one gets a unitary 2-cocycle on a CQG from such a unitary 2-cocycle on its quantum subgroup. Given two CQG's $\mathcal{Q}_1$, $\mathcal{Q}_2$ and a surjective CQG morphism $\pi : \mathcal{Q}_1 \to \mathcal{Q}_2$ which identifies $\mathcal{Q}_2$ as a quantum subgroup of $\mathcal{Q}_1$ (we shall use the notation $\mathcal{Q}_2 \leq \mathcal{Q}_1$ to mean that $\mathcal{Q}_2$ is a quantum subgroup of $\mathcal{Q}_1$), it can be shown that π maps the Hopf $*$-algebra $(\mathcal{Q}_1)_0$ onto $(\mathcal{Q}_2)_0$. By duality we get a map say $\hat{\pi}$ from $(\mathcal{Q}_2)_0'$ to $(\mathcal{Q}_1)_0'$ and it is easy to check that this indeed maps the dense multiplier Hopf $*$-algebra $\widehat{(\mathcal{Q}_2)_0} \subset \widehat{\mathcal{Q}_2}$ to $\widehat{(\mathcal{Q}_1)_0}$. Indeed $\hat{\pi}$ lifts to a non degenerate $*$-homomorphism from $\mathcal{M}(\widehat{\mathcal{Q}_2})$ to $\mathcal{M}(\widehat{\mathcal{Q}_1})$. So given a unitary 2-cocycle σ on $\mathcal{Q}_2$, we get a unitary 2-cocycle $\sigma' := (\hat{\pi} \otimes \hat{\pi})(\sigma) \in \mathcal{M}(\widehat{\mathcal{Q}_1} \hat{\otimes} \widehat{\mathcal{Q}_1})$. It is easy to check that σ' is again a unitary 2-cocycle on $\mathcal{Q}_1$. We shall often use the same notation for both σ' and σ i.e. denote σ' by σ under slight abuse of notation for convenience.

Lemma 6.3.4. *$\mathcal{Q}_2^\sigma$ is a quantum subgroup of $\mathcal{Q}_1^{\sigma'}$.*

Proof:
First we claim that $\pi : (\mathcal{Q}_1)_0^{\sigma'} \to (\mathcal{Q}_2)_0^\sigma$ is a surjective Hopf $*$-algebra morphism. Since the coproducts remain unchanged , we only need to check that π is again a $*$-algebra homomorphism. For that observe that for $a, b \in (\mathcal{Q}_1)_0^{\sigma'}$,

$$\begin{aligned}
\pi(a ._{\sigma'} b) &= \pi[\sigma'(a_{(1)}, b_{(1)}) a_{(2)} b_{(2)} (\sigma')^{-1}(a_{(3)}, b_{(3)})] \\
&= \sigma'(a_{(1)}, b_{(1)}) \pi(a_{(2)}, b_{(2)}) (\sigma')^{-1}(a_{(3)}, b_{(3)}) \\
&= \sigma(\pi(a_{(1)}), \pi(b_{(1)})) \pi(a_{(2)}) \pi(b_{(2)}) \sigma^{-1}(\pi(a_{(3)}), \pi(b_{(3)})) \\
&= \pi(a) ._{\sigma} \pi(b)
\end{aligned}$$

Similarly we can show that $\pi(a^{*_{\sigma'}}) = (\pi(a))^{*_\sigma}$. Hence $\mathcal{Q}_1^{\sigma'}$ contains $(\mathcal{Q}_2)_0^\sigma$ as a Hopf $*$-algebra and hence by the universality of $\mathcal{Q}_1^{\sigma'}$ we conclude that there is a surjective CQG morphism from $\mathcal{Q}_1^{\sigma'}$ onto $\mathcal{Q}_2^\sigma$.

□

Lemma 6.3.5. *For a universal CQG $\mathcal{Q}$ with a unitary 2-cocycle σ, $(\mathcal{Q}^\sigma)^{\sigma^{-1}} \cong \mathcal{Q}$.*

Proof:
First we claim that as Hopf $*$-algebra $(\mathcal{Q}_0^\sigma)^{\sigma^{-1}} \cong \mathcal{Q}_0$. Again for that it is enough to

check the $*$-algebra structure. For that let $a, b \in \mathcal{Q}_0$.

$$\begin{aligned}
a(._\sigma)_{\sigma^{-1}}b &= \sigma^{-1}(a_{(1)}, b_{(1)})a_{(2)}._\sigma b_{(2)}\sigma(a_{(3)}, b_{(3)}) \\
&= \sigma^{-1}(a_{(1)(1)}, b_{(1)(1)})a_{(1)(2)}._\sigma b_{(1)(2)}\sigma(a_{(2)}, b_{(2)}) \\
&= \sigma^{-1}(a_{(1)(1)}, b_{(1)(1)})\sigma(a_{(1)(2)(1)(1)}, b_{(1)(2)(1)(1)})a_{(1)(2)(1)(2)}b_{(1)(2)(1)(2)} \\
&\quad \sigma^{-1}(a_{(1)(2)(2)}, b_{(1)(2)(2)})\sigma(a_{(2)}, b_{(2)}) \\
&= \sigma^{-1}(a_{(1)(1)}, b_{(1)(1)})\sigma(a_{(1)(2)}, b_{(1)(2)})a_{(2)(1)}b_{(2)(1)} \\
&\quad \sigma^{-1}(a_{(2)(2)(1)}, b_{(2)(2)(1)})\sigma(a_{(2)(2)(2)}, b_{(2)(2)(2)}) \\
&= \epsilon(a_{(1)})\epsilon(b_{(1)})a_{(2)(1)}b_{(2)(1)}\epsilon(a_{(2)(2)})\epsilon(b_{(2)(2)}) \\
&= \epsilon(a_{(1)})\epsilon(b_{(1)})a_{(2)}b_{(2)} \\
&= ab
\end{aligned}$$

Similarly we can show for $*$-structure. So $(\mathcal{Q}^\sigma)^{\sigma^{-1}}$ contains $\mathcal{Q}_0$ as a Hopf $*$-algebra. Hence by the definition, $\mathcal{Q}$ is a quantum subgroup of $(\mathcal{Q}^\sigma)^{\sigma^{-1}}$. Now by universality of $\mathcal{Q}$, $(\mathcal{Q}^\sigma)^{\sigma^{-1}} \cong \mathcal{Q}$. □

6.3.1 Unitary representations of a twisted compact quantum group

Let $\mathcal{Q}$ be a universal compact quantum group (as in the sense of 2.1) with a dual unitary 2-cocycle σ. Then our goal of this brief subsection is to prove that there is a bijective correspondence between the sets of inequivalent irreducible representations of $\mathcal{Q}$ and $\mathcal{Q}^\sigma$ given by,

$$Rep(\mathcal{Q}^\sigma) = \{\pi_\sigma | \pi \in Rep(\mathcal{Q})\},$$

$\dim(\pi_\sigma)$=$\dim(\pi)$ for all π, where the corresponding representation being the same as a linear map. To this end let us begin with a unitary representation U of $\mathcal{Q}$ on a Hilbert space $\mathcal{H}$. Write $\mathcal{H} = \oplus_{k\geq 1}\mathcal{H}_k$, where on $\mathcal{H}_k$, U is irreducible and equivalent to some $\pi \in Rep(\mathcal{Q})$. Hence each $\mathcal{H}_k$ is finite dimensional. Let $\{e_i^k\}_{i=1}^{d_\pi}$ be a basis of $\mathcal{H}_k$ (π is the irreducible type which is equivalent to U on $\mathcal{H}_k$) and we write $U(e_i^k) = \sum_{j=1}^{d_\pi} e_j^k \otimes q_{ji}^\pi$.

Theorem 6.3.6. *U viewed as a same linear map is again a unitary representation of the cocycle twisted compact quantum group $\mathcal{Q}^\sigma$ (note that for this we don't need the CQG to be universal).*

Proof:
Recall the discrete quantum group $\hat{\mathcal{Q}}_0$ and its corresponding twisted discrete quantum group $(\hat{\mathcal{Q}}_0)_\sigma$. As noted previously $\mathcal{Q}_0^\sigma$ and $(\hat{\mathcal{Q}}_0)_\sigma$ again form a non-degenerate dual

pairing. Since the $*$-algebra structure of $\hat{\mathcal{Q}_0}$ does not change, $(\hat{\mathcal{Q}_0})_\sigma$ has the same matrix units as $\hat{\mathcal{Q}_0}$. Let m^π_{pq} be one such matrix unit. Then by the definition of twisted antipode and twisted $*$ on $\mathcal{Q}_0^\sigma$, we get (denoting the dual pairing by $<>$)

$$\begin{aligned} & < \kappa_\sigma(q^\pi_{ij})^{*_\sigma}, m^\pi_{pq} > \\ = & \overline{< q^\pi_{ij}, m^{\pi *}_{pq} >} \\ = & \overline{< q^\pi_{ij}, m^\pi_{qp} >} (\textit{since the } * \textit{ structure does not change}) \\ = & \delta_{iq}\delta_{jp} \end{aligned}$$

But we know $< q^\pi_{ji}, m^\pi_{pq} >= \delta_{iqjp}$. Hence by non-degeneracy of the pairing $\kappa_\sigma(q_{ij}) = q^{*_\sigma}_{ji}$. So U is again a unitary representation of the twisted CQG. □

We shall denote the same U by U_σ when viewed as a unitary representation of $\mathcal{Q}^\sigma$. If we denote an element $a \in \mathcal{Q}$ viewed as an element of $\mathcal{Q}^\sigma$ by $[a]$, then $U_\sigma = (\mathrm{id} \otimes [.])U$. As $\mathcal{Q}$ is a universal CQG, noting the fact that for a universal CQG, $(\mathcal{Q}^\sigma)^{\sigma^{-1}} \cong \mathcal{Q}$ (by Lemma 6.3.5), we can conclude that for a universal CQG $\mathcal{Q}$,

$$Rep(\mathcal{Q}^\sigma) = \{\pi_\sigma | \pi \in Rep(\mathcal{Q})\},$$

dim(π_σ)=dim(π) for all π, where the corresponding representation being the same as a linear map.

Proposition 6.3.7. *(i) The Haar state for the deformed compact quantum group stays the same as in the undeformed compact quantum group.*
(ii) The operator F^σ_π corresponding to the twisted CQG $\mathcal{Q}^\sigma$ given by $\delta_{ik}F^\sigma_\pi(j,l) = M_{d_\pi} h(q^\pi_{ij} \cdot_\sigma q^{\pi *_\sigma}_{kl})$ is related to F_π by the following.

$$F^\sigma_\pi = c_\pi A^*_\pi F_\pi A_\pi,$$

where c_π is some positive constant and $A_\pi \xi := (\mathrm{id} \otimes v)\pi_\sigma \xi$, where v is as in Subsection 5.2.1.

Proof:
(i) By Theorem 6.3.6, the matrix coefficients of irreducible representations do not change for $\mathcal{Q}^\sigma$. The Haar state for $\mathcal{Q}^\sigma$ say h_σ is uniquely determined by $h_\sigma(q^\pi_{ij}) = 0$ for all i, j and non trivial representations π and $h_\sigma(1) = 1$. But since the Haar state h of $\mathcal{Q}$ satisfies the above properties, we conclude that $h_\sigma = h$.
(ii) By equation (1), we see that the modular operator $\Phi|_{L^2(h)^\pi_i} = F^\pi$ for all π and i. Let Φ^σ be the modular operator for the CQG $\mathcal{Q}^\sigma$ and S^σ be the corresponding anti unitary operator. Then recalling the definition of the deformed $*$ of $\mathcal{Q}^\sigma$ (equation (4)), we see

that $\Phi^\sigma = SC$, where $C = AB = BA$, A is the operator given by $A(a) = (\mathrm{id} \otimes v)\Delta(a)$ and B is given by $B(a) = (v^{-1} \otimes \mathrm{id})\Delta(a)$ for $a \in \mathcal{Q}_0$. Then the modular operator $\Phi^\sigma = S^{\sigma *} S^\sigma = A^* B^* \Phi B A$. We know $\Phi^\sigma|_{L^2(h)_i^\pi} = F_\pi^\sigma$ for all π and i. Since A and A^* both map $L^2(h)_i^\pi$ into itself for all π and i, $B^*\Phi B$ also does so. Fix some i and let P_i^π be the projection onto Sp $\{q_{ij}^\pi : j = 1, ..., d_\pi\}$. It is clear from the definition of B that $B(q_{kl}^\pi) = \sum_m b_{km}^\pi q_{ml}^\pi$ and $B^*(q_{kl}^\pi) = \sum_m d_{km}^\pi q_{ml}^\pi$, for some constants b_{km} and d_{km}'s. Then

$$\begin{aligned} & P_i^\pi(B^*\Phi B(q_{ij}^\pi)) \\ = \ & (\sum_k b_{ik} d_{ki})(\sum_m F_\pi(j, m) q_{im}^\pi), \end{aligned}$$

where $\Phi(q_{ij}^\pi) = \sum_m F_\pi(j, m) q_{im}^\pi$. Now if we denote $(\sum_k b_{ik} d_{ki})$ by $c_{\pi,i}$, we note that

$$B^*\Phi B|_{L^2(h)_i^\pi} = c_{\pi,i} \Phi|_{L^2(h)_i^\pi}.$$

In particular taking $c_\pi = c_{\pi,1}$ and denoting the restrictions of A and A^* on Sp $\{q_{1j}^\pi : 1 \leq j \leq d_\pi\}$ by A_π and A_π^* respectively we write $F_\pi^\sigma = c_\pi A_\pi^* F_\pi A_\pi$ and as F_π^σ is positive, invertible, c_π must be a positive constant. □

6.4 Action on von Neumann algebras by conjugation of unitary representation

We now discuss an analogue of "action" (as in [42]) in the context of von Neumann algebra implemented by a unitary representation of the CQG. Given a unitary representation V of a CQG $\mathcal{Q}$ on a Hilbert space $\mathcal{H}$, often we consider the $*$ homomorphism $\mathrm{ad}_{\widetilde{V}}$ on $\mathcal{B}(\mathcal{H})$ or on some suitable von Neumann subalgebra $\mathcal{M}$ of it. We say $\mathrm{ad}_{\widetilde{V}}$ leaves $\mathcal{M}$ invariant if $(\mathrm{id} \otimes \phi)\mathrm{ad}_{\widetilde{V}}(\mathcal{M}) \subset \mathcal{M}$ for every state ϕ of $\mathcal{Q}$. Then taking ρ^π as above, we define $\mathcal{M}^\pi = P_\pi(\mathcal{M})$, where $P_\pi = (\mathrm{id} \otimes \rho^\pi)\mathrm{ad}_{\widetilde{V}} : \mathcal{M} \to \mathcal{M}$ is the spectral projection corresponding to the representation π. We define $\mathcal{M}_0 := Sp\ \{\mathcal{M}^\pi; \pi \in Rep(\mathcal{Q})\}$, which is called the spectral subalgebra. Then we have the following:

Proposition 6.4.1. *$\mathcal{M}_0$ is dense in $\mathcal{M}$ in any of the natural locally convex topologies of $\mathcal{M}$, i.e. $\mathcal{M}_0'' = \mathcal{M}$.*

Proof:
This result must be quite well-known and available in the literature but we could not find it written in this form, so we give a very brief sketch. The proof is basically

almost verbatim adaptation of some arguments in [32] and [51]. First, observe that the spectral algebra $\mathcal{M}_0$ remains unchanged if we replace $\mathcal{Q}$ by the reduced quantum group $\mathcal{Q}_r$ which has the same irreducible representations and dense Hopf-$*$ algebra $\mathcal{Q}_0$. This means we can assume without loss of generality that the Haar state is faithful. The injective normal map $\beta := \mathrm{ad}_V$ restricted to $\mathcal{M}$ can be thought of as an action of the quantum group (in the von Neumann algebra setting as in [32], [51]) $\mathcal{Q}_r''$ (where the double commutant is taken in the GNS space of the Haar state) and it follows from the results in [51] about the implementability of locally compact quantum group actions that there is a faithful normal state, say ϕ, on $\mathcal{M}$ such that β is $\mathcal{Q}_r''$-invariant, i.e. $(\phi \otimes \mathrm{id})(\beta(a)) = \phi(a)1 \; \forall a \in \mathcal{M}$. We can replace $\mathcal{M}$, originally imbedded in $\mathcal{B}(\mathcal{H})$, by its isomorphic image (to be denoted by $\mathcal{M}$ again) in $\mathcal{B}(L^2(\phi))$ and as the ultra-weak topology is intrinsic to a von Neumann algebra, it will suffice to argue the ultra-weak density of $\mathcal{M}_0$ in $\mathcal{M} \subset \mathcal{B}(L^2(\phi))$. To this end, note that Vaes has shown in [51] that $\beta : \mathcal{M} \to \mathcal{M} \otimes \mathcal{Q}_r''$ extends to a unitary representation on $L^2(\phi)$, which implies in particular that $\mathcal{M}_0$ is dense in the Hilbert space $L^2(\phi)$. From this the ultra-weak density follows by standard arguments very similar to those used in the proof of Proposition 1.5 of [32], applying Takesaki's theorem about existence of conditional expectation. For the sake of completeness let us sketch it briefly. Using the notations of [51] and noting that $\delta = 1$ for a CQG, we get from Proposition 2.4 of [51] that V_ϕ commutes with the positive self adjoint operator $\nabla_\phi \otimes Q$ where ∇_ϕ denotes the modular operator i.e. generator of the modular automorphism group σ_t^ϕ of the normal state ϕ. Clearly, this implies that $\beta := \mathrm{ad}_{V_\phi}$ satisfies the following:

$$\beta \circ \sigma_t^\phi = \sigma_t^\phi \otimes \tau_{-t},$$

where τ_t is the automorphism group generated by Q^{-1}. Next, as in Proposition 1.5 of [32], consider the ultra-strong $*$ closure $\mathcal{M}_l$ of the subspace spanned by the elements of the form $(\mathrm{id} \otimes \omega)(\beta(x))$, $x \in M$, ω is a bounded normal functional on $\mathcal{Q}_r''$. It is enough to prove that $\mathcal{M}_l = \mathcal{M}$, as that will prove the ultra-strong $*$ density (now also the ultra-weak density) of $\mathcal{M}_l$ in $\mathcal{M}$. This is clearly a von Neumann subalgebra as β is coassociative, and $\sigma_t^\phi(\mathrm{id} \otimes \omega)(\beta(x)) = (\mathrm{id} \otimes \omega \circ \tau_t)(\beta(\sigma_t^\phi(x)))$. Then by Takesaki's theorem ([49], 10.1) there exists a unique normal faithful conditional expectation E from $\mathcal{M}$ to $\mathcal{M}_l$ satisfying $E(x)P = PxP$ where P is the orthogonal projection as in [32]. Clearly, the range of P contains elements of the form $(\mathrm{id} \otimes \omega)(\beta(x))$. So in particular it contains $\mathcal{M}_0$, which is dense in $L^2(\phi)$. Thus $P = 1$ and $E(x) = x$ proving $\mathcal{M}_l = \mathcal{M}$.
□

From this, it is easy to conclude that

Lemma 6.4.2. *1.* $\mathrm{ad}_{\widetilde{V}}|_{\mathcal{M}_0}$ *is algebraic i.e.* $\mathrm{ad}_{\widetilde{V}}(\mathcal{M}_0) \subset \mathcal{M}_0 \otimes \mathcal{Q}_0$.
2. $\mathcal{M}_0$ *is the maximal subspace over which* $\mathrm{ad}_{\widetilde{V}}$ *is algebraic i.e.* $\mathcal{M}_0 = \{x \in \mathcal{M} | \mathrm{ad}_{\widetilde{V}}(x) \in \mathcal{M} \otimes \mathcal{Q}_0\}$.
3. If $\mathcal{M}_1 \subset \mathcal{M}$ *is SOT dense* $*$*-subalgebra such that* $\mathrm{ad}_{\widetilde{V}}$ *leaves* $\mathcal{M}_1$ *invariant, then* $\mathrm{ad}_{\widetilde{V}}$ *is algebraic over* $Sp\{P_\pi(\mathcal{M}_1) | \pi \in Rep(\mathcal{Q})\}$ *and* $Sp\{P_\pi(\mathcal{M}_1) | \pi \in Rep(\mathcal{Q})\}$ *is SOT dense in* $\mathcal{M}_0$.

Proof:
The statements (1) follows from the lines of argument as in the Theorem 1.5 of [42] whereas (2) follows by arguing along the lines of Proposition 2.2 of [48] and noting that $\mathrm{ad}_{\widetilde{V}}$ is 1-1. (3) follows from (2) and the obvious SOT continuity of each P_π. □

For $x \in \mathcal{M}_0$, we shall use the natural analogue of Swedler's notation, i.e. write $\mathrm{ad}_{\widetilde{V}}(x) = x_{(0)} \otimes x_{(1)}$.

6.4.1 Deformation of a von Neumann algebra by dual unitary 2-cocycles

Let $\mathcal{Q}$ be a CQG with a dual unitary 2-cocycle σ. Also assume that it has a unitary representation V on a Hilbert space $\mathcal{H}$ and choose a dense subspace $\mathcal{N} \subset \mathcal{H}$ on which V is algebraic i.e. $V(\mathcal{N}) \subset \mathcal{N} \otimes \mathcal{Q}_0$. Then the spectral subalgebra $\mathcal{M}_0$ is SOT dense in $\mathcal{M}$ and $\mathrm{ad}_{\widetilde{V}}(\mathcal{M}_0) \subset \mathcal{M}_0 \otimes \mathcal{Q}_0$ by Proposition 6.4.1. Now using the dual unitary 2-cocycle $\sigma \in \mathcal{M}(\hat{\mathcal{Q}} \hat{\otimes} \hat{\mathcal{Q}})$, we can define a new representation of $\mathcal{M}_0$ on $\mathcal{N}$ by,

$$\rho_\sigma(b)(\xi) := b_{(0)} \xi_{(0)} \sigma^{-1}(b_{(1)}, \xi_{(1)}), \ for \ \xi \in \mathcal{N},$$

where $\mathrm{ad}_{\widetilde{V}}(b) = b_{(0)} \otimes b_{(1)}$ and $V(\xi) = \xi_{(0)} \otimes \xi_{(1)}$.

Lemma 6.4.3. *1.* $\rho_\sigma(b)$ *extends to an element of* $\mathcal{B}(\mathcal{H})$ *for all* $b \in \mathcal{M}_0$.
2. $\rho_\sigma P_\pi$ *is SOT continuous, where* P_π *is the spectral projection corresponding to* $\pi \in Rep(\mathcal{Q})$.

Proof:
1. Let $\mathrm{ad}_{\widetilde{V}}(b) = \sum_{i=1}^k b^i_{(0)} \otimes b^i_{(1)}$ and $V(\xi) = \sum_{j=1}^l \xi^j_{(0)} \otimes \xi^j_{(1)}$. Define $\sigma_i \in \mathcal{M}(\hat{\mathcal{Q}})$ by $\sigma_i(q_0) := \sigma^{-1}(b^i_{(1)}, q_0)$ for all $i = 1, ..., k$. So by Theorem 1.4.14, we have $\Pi_V(\sigma_i) \in \mathcal{B}(\mathcal{H})$ for all $i = 1, ..., k$. By definition of ρ_σ on $\mathcal{N}$,

$$\rho_\sigma(b)\xi = \sum_{i=1}^k b^i_{(0)} \Pi_V(\sigma_i)(\xi).$$

Using the facts that $\Pi_V(\sigma_i)$ and $b^i_{(0)}$'s are bounded operators we can conclude that $\rho_\sigma(b) \in \mathcal{B}(\mathcal{H})$ for all $b \in \mathcal{M}_0$.

2. With similar reasoning we can prove 2. □

Definition 6.4.4. *We call* $(\rho_\sigma(\mathcal{M}_0))''$ *the deformation of* $\mathcal{M}$ *by* σ *and denote it by* $\mathcal{M}^\sigma$.

Now in [40], Neshveyev has given a notion of cocycle twist of a C^* algebra with respect to unitarily implemented action of general locally compact quantum groups. As we work with compact quantum groups only, so every dual unitary 2-cocycle is regular. Let σ be a dual unitary 2-cocycle on a CQG $\mathcal{Q}$. For this subsection we shall not need cocycle twisted compact quantum groups. Rather we shall need the reduced twisted group C^* algebra which we shall denote by $C^*_r(\hat{G};\sigma)$ where $C(G)$ will stand for the compact quantum group $\mathcal{Q}$. $L^\infty(G;\sigma)$ will denote the weak closure of the twisted group C^* algebra in $\mathcal{B}(L^2(G))$, where $L^2(G)$ is the GNS Hilbert space of $\mathcal{Q}$ corresponding to the Haar state h. Suppose furthermore that there is a unitary representation V of $\mathcal{Q}$ on a Hilbert space $\mathcal{H}$ and $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a unital C^* algebra such that $\alpha := \mathrm{ad}_V$ gives a C^* action of $\mathcal{Q}$ on $\mathcal{A}$. Then we have the deformed C^* algebra $\mathcal{A}_\sigma$ constructed in [40], which is viewed there as a subalgebra of $\mathcal{B}(\mathcal{H} \otimes L^2(G))$. We also denote by $\mathcal{M}$ the weak closure of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$ and let $\mathcal{M}_\sigma$ be the weak closure of $\mathcal{A}_\sigma$ in $\mathcal{B}(\mathcal{H} \otimes L^2(G))$. Recall $P^{G,r}_\pi$ for $\pi \in Rep(G)$. Then it is clear that $(\mathrm{id} \otimes P^{G,r}_\pi)X \in \mathcal{B}(\mathcal{H}) \otimes_{\mathrm{alg}} \mathcal{Q}^\sigma_0$ for any $X \in (\mathcal{B}(\mathcal{H})\bar{\otimes}\mathcal{Q}'')$. Also we denote by $p^{G,l}_\pi$ and $p^{G,r}_\pi$ the Hilbert space projections on $L^2(G)$ corresponding to the spectral projections $P^{G,l}_\pi$ and $P^{G,r}_\pi$ respectively. Then both $p^{G,r}_\pi$'s and $p^{G,l}_\pi$'s are mutually orthogonal projections whose sum converges to the identity operator on $L^2(G)$ strongly.

Lemma 6.4.5. *Given any* C^* *(or von Neumann algebraic) action* β *on a* C^* *(or von Neumann) algebra* $\mathcal{C} \subset \mathcal{B}(\mathcal{K})$ *and* $X \in \mathcal{C}$ *such that* $X_\pi := P^{\mathcal{C}}_\pi(X) = 0$ *for all* $\pi \in Rep(\mathcal{Q})$ *and* β *is one-one (for example of the form* ad_V *for some unitary* V*), then* $X = 0$.

Proof:
We use γ for the ad_W action. We have $X_\pi = 0$ i.e. $(\mathrm{id} \otimes \rho^\pi)\beta(X) = 0$. So $\beta(\mathrm{id} \otimes \rho^\pi)\beta(X) = 0$. Hence by using the fact that $(\beta \otimes \mathrm{id})\beta = (\mathrm{id} \otimes \gamma)\beta$, we obtain,

$$(\mathrm{id} \otimes \mathrm{id} \otimes \rho^\pi)(\mathrm{id} \otimes \gamma)\beta(X) = 0.$$

But then $(\mathrm{id} \otimes \rho^\pi)\gamma := P^{G,r}_\pi$. So $(\mathrm{id} \otimes P^{G,r}_\pi)\beta(X) = 0$. $\beta(X) \in (\mathcal{B}(\mathcal{K})\bar{\otimes}\mathcal{Q}'')$. Then for any $u \in \mathcal{K}$, $((\mathrm{id} \otimes P^{G,r}_\pi)\beta(X))(u \otimes 1_{\mathcal{Q}}) = 0$, i.e. $(\mathrm{id} \otimes p^{G,r}_\pi)(\beta(X)(u \otimes 1_{\mathcal{Q}}) = 0$. But since $\sum_{\pi \in Rep(\mathcal{Q})} p^{G,r}_\pi$ converges strongly to identity operator on $L^2(G)$, we get $\beta(X)(u \otimes 1_{\mathcal{Q}}) = 0$ for all $u \in \mathcal{H}$. So $1_{\mathcal{Q}}$ being a separating vector for $\mathcal{Q}'' \in \mathcal{B}(L^2(G))$, we conclude that $\beta(X) = 0$ and hence $X = 0$ as β is one one. □

Remark 6.4.6. *The above Lemma clearly holds true if the right action is replaced by the left action.*

Now we shall show that these two frameworks of deformation are in deed equivalent.

For a dual unitary 2-cocycle σ on $C(G)$, we define $\mathcal{Q}_0^\sigma \in \mathcal{B}(L^2(G))$ by the following: As a vector space $\mathcal{Q}_0$ and $\mathcal{Q}_0^\sigma$ are same. But the new representation of $\mathcal{Q}_0^\sigma$ in $\mathcal{B}(L^2(G))$ is given by $q * \xi := q_{(1)}\xi_{(1)}\sigma^{-1}(q_{(2)}, \xi_{(2)})$, where $q, \xi \in \mathcal{Q}_0$ (observe that $\mathcal{Q}_0$ is dense in the Hilbert space $L^2(G)$) and $\Delta(q) = q_{(1)} \otimes q_{(2)}$, $\Delta(\xi) = \xi_{(1)} \otimes \xi_{(2)}$. Here Δ is the coproduct of $C(G)$. That $\mathcal{Q}_0^\sigma \subset \mathcal{B}(L^2(G))$ can be shown along the same way as Lemma 4.7 of [24].

Identifying $\mathcal{Q}_0^\sigma$ with $\mathcal{Q}_0$ as vector space, we have the same counit map $\epsilon : \mathcal{Q}_0^\sigma \to \mathbb{C}$. There is a canonical left action, say Δ_σ, of G on $C_r^*(G;\sigma)$, which coincides with the coproduct Δ as a linear map on the dense $*$-subalgebra $\mathcal{Q}_0^\sigma$ identified with $\mathcal{Q}_0$ as vector spaces. The counit map ϵ satisfies $(\mathrm{id} \otimes \epsilon)\Delta_\sigma = (\epsilon \otimes \mathrm{id})\Delta_\sigma = \mathrm{id}$. However, we should mention that ϵ on $\mathcal{Q}_0^\sigma$ need not be a homomorphism.

Now we introduce the generalised fixed point subalgebras and spaces (we use the notations V and W to denote the unitary representation of $C(G)$ on Hilbert spaces $\mathcal{H}$ and $L^2(G)$ respectively as before):

Definition 6.4.7. *For a subalgebra (not necessarily closed) $\mathcal{B}$ of $\mathcal{B}(\mathcal{H} \otimes L^2(G))$ the generalised fixed point subalgebra $\mathcal{B}^f$ corresponding to $\mathcal{B}$ is given by*

$$\mathcal{B}^f := \{X \in \mathcal{B} : \ (\mathrm{ad}_V \otimes \mathrm{id})(X) = (\mathrm{id} \otimes \mathrm{ad}_W)(X)\}.$$

Similarly, the generalised fixed point subspace $\mathcal{W}^f$ for a Hilbert subspace $\mathcal{W} \subseteq \mathcal{H} \otimes L^2(G)$ is defined to be

$$\mathcal{W}^f := \{\xi \in \mathcal{W} : \ (V \otimes \mathrm{id})(\xi) = (\mathrm{id} \otimes W)(\xi)\}.$$

There are two possible G-actions of $\mathcal{B}(\mathcal{H})\bar{\otimes}\mathcal{B}(L^2(G))$ given by $\sigma_{23}(\mathrm{ad}_V \otimes \mathrm{id})$ and $\sigma_{23}(\mathrm{id} \otimes \mathrm{ad}_W)$, where σ_{23} flips the second and the third tensor copies and by definition these two actions coincide on $\mathcal{B}^f$. For $\mathcal{B}$ and $\pi \in Rep(G)$ the corresponding spectral projection $P_\pi^{\mathcal{B}}$ is given by the restriction of $(P_\pi^{\mathcal{B}(\mathcal{H})} \otimes \mathrm{id})$ or $(\mathrm{id} \otimes P_\pi^{G,l})$ on $\mathcal{B}^f$.

The following observation will be crucial.

Lemma 6.4.8. *Let $\mathcal{L}$ be the range of V viewed as a Hilbert space isometry from $\mathcal{H}$ to $\mathcal{H} \otimes L^2(G)$. Then we have the following:*
(i) $V : \mathcal{H} \to (\mathcal{H} \otimes L^2(G))^f$ is a unitary operator.
(ii) Any X in $(\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(L^2(G)))^f$ leaves $\mathcal{L}$ invariant.

Proof:
(i) That V is an isometric operator has been observed previously. So we need to show that $R(V) = (\mathcal{H} \otimes L^2(G))^f$. To that end first note that $\mathcal{H}$ decomposes into spectral subspaces corresponding to the unitary representation V of $C(G)$. More precisely $\mathcal{H}$ is the norm closure of $\mathrm{Sp}\{\mathcal{H}^\pi : \pi \in Rep(G)\}$, where $\mathcal{H}^\pi = \overline{\mathrm{Sp}}\{e^{\pi,i} : i = 1, ..., d_\pi\}$. d_π is the dimension of the π-th spectral subspace of $C(G)$. Also $V(e^{\pi,i}) = \sum_j e^{\pi,j} \otimes q^\pi_{ji}$, where q^π_{ji}'s are matrix coefficients (see [35]). So we have a basis for the Hilbert space $\mathcal{H} \otimes L^2(G)$ given by the set

$$\{e^{\pi,i} \otimes q^{\pi'}_{i'j} : \pi, \pi' \in Rep(G); i \in \{1, ..., d_\pi\}; i', j \in \{1, ..., d_{\pi'}\}\}.$$

So we can write a vector $\xi \in (\mathcal{H} \otimes L^2(G))^f$ as

$$\sum_{\pi,\pi',i,i',j} c^{\pi,\pi'}_{i,i',j} e^{\pi,i} \otimes q^{\pi'}_{i'j}.$$

Then

$$(V \otimes \mathrm{id})(\xi) = \sum_{\pi,\pi',i,i',j,k} c^{\pi,\pi'}_{i,i',j} e^{\pi,k} \otimes q^\pi_{ki} \otimes q^{\pi'}_{i'j},$$

$$(\mathrm{id} \otimes W)(\xi) = \sum_{\pi,\pi',i,i',j,l} c^{\pi,\pi'}_{i,i',j} e^{\pi,i} \otimes q^{\pi'}_{i'l} \otimes q^{\pi'}_{lj}.$$

Now for $\pi \neq \pi'$, and for all l, k, i, i', j, $e^{\pi,k} \otimes q^\pi_{ki} \otimes q^{\pi'}_{i'j}$ and $e^{\pi,i} \otimes q^{\pi'}_{i'l} \otimes q^{\pi'}_{lj}$ are different basis vectors. Hence $c^{\pi,\pi'}_{i,i',j} = 0$ for all i, i', j whenever $\pi \neq \pi'$. So $\xi = \sum_{\pi,i,i',j} c^\pi_{i,i',j} e^{\pi,i} \otimes q^\pi_{i',j}$. Then

$$(V \otimes \mathrm{id})(\xi) = \sum_{\pi,i,i',j,k} c^\pi_{i,i',j} e^{\pi,k} \otimes q^\pi_{ki} \otimes q^\pi_{i'j}. \tag{6.4.1}$$

$$(\mathrm{id} \otimes W)(\xi) = \sum_{\pi,i,i',j,l} c^\pi_{i,i',j} e^{\pi,i} \otimes q^\pi_{i'l} \otimes q^\pi_{lj}. \tag{6.4.2}$$

Fixing $k = n$ in (1) and $i = n$ in (2)

$$\sum_{\pi,i,i',j} c^\pi_{i,i',j} q^\pi_{ni} \otimes q^\pi_{i'j} = \sum_{\pi,i',j,l} c^\pi_{n,i',j} q^\pi_{i'l} \otimes q^\pi_{lj}$$

Fixing $i = i_0$ on L.H.S. and $i' = n, l = i_0$ on R.H.S. of the above expression we get,

$$\sum_{\pi,i',j} c^\pi_{i_0,i',j} q^\pi_{i'j} = \sum_{\pi,j} c^\pi_{n,n,j} q^\pi_{i_0 j}.$$

So for a fixed i_0 if $i^{'} \neq i_0$, $c^{\pi}_{i_0,i^{'},j} = 0$. Hence the vector ξ can be written as

$$\xi = \sum_{\pi,i,j} c^{\pi}_{i,j} e^{\pi,i} \otimes q^{\pi}_{ij}.$$

Again

$$(V \otimes \mathrm{id})(\xi) = \sum_{\pi,i,j,k} c^{\pi}_{i,j} e^{\pi,k} \otimes q^{\pi}_{ki} \otimes q^{\pi}_{ij}. \tag{6.4.3}$$

$$(\mathrm{id} \otimes W)(\xi) = \sum_{\pi,i,j,l} c^{\pi}_{i,j} e^{\pi,i} \otimes q^{\pi}_{il} \otimes q^{\pi}_{lj}. \tag{6.4.4}$$

Fix $k = m, i = n$ in (3) and $i = m, l = n$ in (4) to obtain

$$\sum_{\pi,j} c^{\pi}_{n,j} e^{\pi,m} \otimes q^{\pi}_{nj} = \sum_{\pi,j} c^{\pi}_{m,j} e^{\pi,m} \otimes q^{\pi}_{nj}.$$

Hence $c^{\pi}_{m,j} = c^{\pi}_{n,j}$ for all m, n i.e. ξ takes the form $\sum_{\pi,i,j} c^{\pi}_{j} e^{\pi,i} \otimes q^{\pi}_{ij}$. So $\xi = V(\sum_{\pi,j} c^{\pi}_{j} e^{\pi,j})$.

(ii) Let $\xi \in \mathcal{L}$. Then by definition of $\mathcal{L}$, $(V \otimes \mathrm{id})(\xi) = (\mathrm{id} \otimes W)(\xi)$. As $X \in (\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(L^2(G)))^f$, we also have by defintion,

$$(\mathrm{ad}_V \otimes \mathrm{id})(X) = (\mathrm{id} \otimes \mathrm{ad}_W)(X).$$

So

$$\begin{aligned}
(V \otimes \mathrm{id})(X\xi) &= [(\mathrm{ad}_V \otimes \mathrm{id})(X)](V \otimes \mathrm{id})(\xi) \\
&= [(\mathrm{id} \otimes \mathrm{ad}_W)(X)](\mathrm{id} \otimes W)(\xi) \\
&= (\mathrm{id} \otimes W)(X\xi).
\end{aligned}$$

Hence by definition $X\xi \in \mathcal{L}$. □

Recall the definition of X_π for all $\pi \in Rep(G)$. For $X \in (\mathcal{B}(\mathcal{H}) \bar{\otimes} L^{\infty}(G;\sigma))^f$, $X_\pi \in (\mathcal{B}(\mathcal{H}) \otimes_{\mathrm{id}} \mathcal{Q}^{\sigma}_0)$. Also X leaves $\mathcal{L}$ invariant. Then we have

Lemma 6.4.9. *If $X \in (\mathcal{B}(\mathcal{H}) \bar{\otimes} L^{\infty}(G;\sigma))^f$ and $X|_{\mathcal{L}} = 0$, then $X_\pi|_{\mathcal{L}} = 0$, where $X_\pi = (\mathrm{id} \otimes \rho^{\pi} \otimes \mathrm{id})((\mathrm{ad}_V \otimes \mathrm{id})(X)) = (\mathrm{id} \otimes \rho^{\pi} \otimes \mathrm{id})((\mathrm{id} \otimes \mathrm{ad}_W)(X))$.*

Proof:
Let $\xi \in \mathcal{H}_0$. Then using Sweedler's notation we shall write $V(\xi) = \xi_{(0)} \otimes \xi_{(1)}$. Also we

denote the operator $\xi(\in \mathcal{H}) \mapsto \xi_{(0)} \otimes \kappa(\xi_{(1)}) \in \mathcal{H} \otimes \mathcal{Q}$ by V'. Then

$$\begin{aligned} X_\pi(V\xi) &= [(\mathrm{id} \otimes P_\pi^{G,l})X](V\xi) \\ &= [(\mathrm{id} \otimes \rho^\pi \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{ad}_W)(X)](V\xi) \\ &= [(\mathrm{id} \otimes \rho^\pi \otimes \mathrm{id})\widetilde{W}_{23}(X \otimes 1)\widetilde{W}_{23}^*](V\xi) \\ &= (\mathrm{id} \otimes \rho^\pi \otimes \mathrm{id})[\widetilde{W}_{23}(X \otimes 1)\widetilde{W}_{23}^*(V\xi \otimes 1)] \\ &= (\mathrm{id} \otimes \rho^\pi \otimes \mathrm{id})[\widetilde{W}_{23}(X \otimes 1)\widetilde{W}_{23}^*(\xi_{(0)} \otimes \xi_{(1)} \otimes 1)] \\ &= (\mathrm{id} \otimes \rho^\pi \otimes \mathrm{id})[\widetilde{W}_{23}(X \otimes 1)(\xi_{(0)} \otimes \xi_{(1)(1)} \otimes \kappa(\xi_{(1)(2)}))] \\ &= (\mathrm{id} \otimes \rho^\pi \otimes \mathrm{id})[\widetilde{W}_{23}(X \otimes 1)(\xi_{(0)(0)} \otimes \xi_{(0)(1)} \otimes \kappa(\xi_{(1)}))] \\ &= (\mathrm{id} \otimes \rho^\pi \otimes \mathrm{id})[\widetilde{W}_{23}(X \otimes 1)(V \otimes \mathrm{id})(V'(\xi))], \end{aligned}$$

where $\widetilde{W}_{23}$ is the corresponding unitary to $(\mathrm{id} \otimes W)$. Since $X|_{\mathcal{L}} = 0$, the above computation implies that $X_\pi(V\xi) = 0$ for all $\xi \in \mathcal{H}_0$. By density of $\mathcal{H}_0$ in $\mathcal{H}$ we can argue that $X_\pi|_{\mathcal{L}} = 0$ for all $\pi \in Rep(G)$. □

It can be easily seen that $(\mathcal{B}(\mathcal{H})\bar{\otimes}L^\infty(G;\sigma))^f$ is a von Neumann subalgebra of the operator algebra $\mathcal{B}(\mathcal{H}\otimes(L^2(G))$. Also by (ii) of the Lemma 6.4.8, any $X \in (\mathcal{B}(\mathcal{H})\otimes L^\infty(G;\sigma))^f$ leaves $\mathcal{L}$ invariant. So we can consider the restriction of X on $\mathcal{L}$ and let $\Phi(X) = X|_{\mathcal{L}}$, which is clearly a normal $*$- homomorphism. With these we have

Lemma 6.4.10. *The map Φ is a C^* or von Neumann algebraic isomorphism.*

Proof:
Observe that as Φ is a normal $*$- homomorphism, it suffices to show that the map Φ is both one-one and onto. We first prove that Φ is surjective. Fix some operator $T \in \mathcal{B}(\mathcal{L})$. Then $V^*TV \in \mathcal{B}(\mathcal{H})$. Then $\widetilde{V}(V^*TV \otimes 1)\widetilde{V}^* \in \mathcal{B}(\mathcal{H} \otimes L^2(G))$. Let $\eta \in \mathcal{L}$. So we have some $\xi \in \mathcal{H}$ such that $V(\xi) = \eta$. Also we observe that $\widetilde{V}^*V(\xi) = \xi \otimes 1$. Then $TV(\xi) = VV^*TV(\xi) = \widetilde{V}(V^*TV(\xi) \otimes 1)$. That is

$$\widetilde{V}(V^*TV \otimes 1)\widetilde{V}^*V(\xi) = TV(\xi).$$

Hence

$$\widetilde{V}(V^*TV \otimes 1)\widetilde{V}^*|_{\mathcal{L}} = T.$$

It is obvious that $\widetilde{V}(V^*TV \otimes 1)\widetilde{V}^* \in (\mathcal{B}(\mathcal{H})\bar{\otimes}\mathcal{B}(L^2(G))^f$. So the map Φ is onto.

To show Φ is one-one, we have to show that for any $X \in (\mathcal{B}(\mathcal{H})\bar{\otimes}L^\infty(G;\sigma))^f$ such that $X|_{\mathcal{L}} = 0$, $X = 0$ on the Hilbert space $(\mathcal{H} \otimes L^2(G))$. By Remark 6.4.6 after Lemma 6.4.5, it suffices to show that $X_\pi = 0$ as an operator on $(\mathcal{H} \otimes L^2(G))$ for all $\pi \in Rep(G)$, where X_π's are as in Lemma 6.4.9. As $X|_{\mathcal{L}} = 0$, by Lemma 6.4.9, $X_\pi|_{\mathcal{L}} = 0$ for all $\pi \in Rep(G)$. As for $X \in (\mathcal{B}(\mathcal{H})\bar{\otimes}\mathcal{B}(L^2(G))^f$, $X_\pi \in \mathcal{B}(\mathcal{H}) \otimes_{\mathrm{alg}} \mathcal{Q}_0^\sigma$, $(\mathrm{id} \otimes \epsilon)X_\pi$ makes sense for all $\pi \in Rep(G)$. Then

$$\begin{aligned} V((\mathrm{id} \otimes \epsilon)X_\pi(\xi)) &= (\mathrm{ad}_V(\mathrm{id} \otimes \epsilon)X_\pi)V(\xi) \\ &= X_\pi(V\xi) \\ &= 0(as\ X_\pi|_{\mathcal{L}} = 0) \end{aligned}$$

Since V is an isometry, $(\mathrm{id} \otimes \epsilon)X_\pi(\xi) = 0$ for all ξ, i.e. $(\mathrm{id} \otimes \epsilon)X_\pi = 0$. Again applying ad_V, we conclude that $X_\pi = 0$ as an element of $\mathcal{B}(\mathcal{H} \otimes L^2(G))$. □

Recall the definition of $\mathcal{Q}_0^\sigma$. If $C(G)$ has a C^* action on $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ implemented by a unitary V, then as before we have a norm dense subalgebra $\mathcal{A}_0\mathcal{A}$ such that $\alpha(= \mathrm{id}_V)$ is algebraic over $\mathcal{A}_0$. We denote the vector space isomorphism between $\mathcal{Q}_0$ and $\mathcal{Q}_0^\sigma$ by π^σ. Define $\alpha^\sigma : \mathcal{A}_0 \to \mathcal{A}_0 \otimes_{\mathrm{alg}} \mathcal{Q}_0^\sigma$ by $(\mathrm{id} \otimes \pi^\sigma)\alpha$. Then we have

Lemma 6.4.11. *$V^*\alpha^\sigma(a)V = \rho_\sigma(a)$ for all $a \in \mathcal{A}_0$.*

Proof:
Let $\xi \in \mathcal{H}_0$ and $V(\xi) = \xi_{(0)} \otimes \xi_{(1)}$ (Sweedler's notation). also let $\alpha(a) = a_{(0)} \otimes a_{(1)}$. Then we have

$$\begin{aligned} & \alpha^\sigma(a)V(\xi) \\ = \; & a_{(0)}\xi_{(0)} \otimes a_{(1)} * \xi_{(1)} \\ = \; & a_{(0)}\xi_{(0)} \otimes a_{(1)(1)}\xi_{(1)(1)}\sigma^{-1}(a_{(1)(2)}, \xi_{(1)(2)}). \end{aligned}$$

On the other hand

$$\begin{aligned} V(\rho_\sigma(a)(\xi)) &= V(a_{(0)}\xi_{(0)}\sigma^{-1}(a_{(1)}, \xi_{(1)})) \\ &= a_{(0)(0)}\xi_{(0)(0)} \otimes a_{(0)(1)}\xi_{(0)(1)}\sigma^{-1}(a_{(1)}, \xi_{(1)}) \\ &= a_{(0)}\xi_{(0)} \otimes a_{(1)(1)}\xi_{(1)(1)}\sigma^{-1}(a_{(1)(2)}, \xi_{(1)(2)}). \end{aligned}$$

Hence $V^*\alpha^\sigma(a)V = \rho_\sigma(a)$ for all $a \in \mathcal{A}_0$. □

Now we are ready to prove the main result of this section.

Theorem 6.4.12. *$\mathcal{A}^\sigma$ and $\mathcal{M}^\sigma$ are isomorphic with $\mathcal{A}_\sigma$ and $\mathcal{M}_\sigma$ (respectively) as a C^* (von Neumann respectively) algebras.*

Proof:
First observe that according to the definition of $\mathcal{A}_\sigma$ (Definition 3.3 of [40]) for the special case where G ($\mathcal{Q}$) is a CQG, treating $1 \in C(G)$ as a unit vector in $L^2(G)$, we can choose ν (according to the notation of Proposition 3.1 of [40]) to be the element of $\mathcal{K}(L^2(G))^*$

given by $y \to < 1, y1 >$ i.e. the vector state given by the vector 1 to show that the map T_ν of Definition 3.3 of [40] is nothing but our map π^σ on the dense subspace $\mathcal{Q}_0$. Hence $\mathcal{A}_\sigma$ ($\mathcal{M}_\sigma$ respectively) is the C^* (von Neumann respectively) closure of $\alpha^\sigma(\mathcal{A}_0)$ in $\mathcal{B}(\mathcal{H} \otimes L^2(G))$, which is isomorphic to C^* (or von Neumann respectively) closure of $\alpha^\sigma(\mathcal{A}_0)|_{\mathcal{L}}$ by Lemma 6.4.10, which by Lemma 6.4.11, is isomorphic to corresponding C^* (or von Neumann respectively) closure of $\rho_\sigma(\mathcal{A}_0)$ in $\mathcal{B}(\mathcal{H})$ i.e. $\mathcal{A}^\sigma$ ($\mathcal{M}^\sigma$ respectively).
□

6.4.2 Identifying $\mathcal{M}^\sigma$ with the generalised fixed point subalgebra

We now give a partial answer to the question asked in [40] by identifying $\mathcal{M}_\sigma$ with the generalised fixed point subalgebra $(\mathcal{M}\bar{\otimes}L^\infty(G;\sigma))^f$ when $\mathcal{M}$ is a von Neumann algebra.

Lemma 6.4.13. *The map $\rho_\sigma(a) \to \alpha^\sigma(a)$ for $a \in \mathcal{M}_0$ is an isomorphism between $\mathcal{M}_0^\sigma$ and $(\mathcal{M}_0 \otimes_{\rm alg} \mathcal{Q}_0^\sigma)^f$, where α^σ is as in the previous Subsection.*

Proof:
It suffices to check that the map is one-one and onto. For injectivity of the map, note that $\rho_\sigma(a) = V^*(\Phi(\alpha^\sigma(a)))V$ in the notation of Lemma 6.4.11. For any $X \in (\mathcal{M}_0 \otimes_{\rm alg} \mathcal{Q}_0^\sigma)^f$, $\alpha((\mathrm{id} \otimes \epsilon)X) = (\mathrm{id} \otimes \epsilon \otimes \mathrm{id})(\alpha \otimes \mathrm{id})X = (\mathrm{id} \otimes \epsilon \otimes \mathrm{id})(\mathrm{id} \otimes \Delta_\sigma)X = X$. Hence α is onto.
□

Theorem 6.4.14. $\mathcal{M}^\sigma = (\mathcal{M}\bar{\otimes}L^\infty(G;\sigma))^f$.

Proof:
We note that it suffices to show the inclusion

$$(\mathcal{M}\bar{\otimes}L^\infty(G;\sigma))^f \subset \mathcal{M}^\sigma.$$

Let $\mathcal{C} = (\mathcal{M}\bar{\otimes}L^\infty(G;\sigma))^f$. Then by Proposition 6.4.1, the spectral subalgebra $\mathcal{C}_0$ is weakly dense in $\mathcal{C}$. Moreover $\alpha^\sigma(\mathcal{M}_0) = (\mathcal{M}_0 \otimes_{\rm alg} \mathcal{Q}_0^\sigma)^f$ is weakly dense in $\mathcal{M}_\sigma$. Hence it is enough to argue that $\mathcal{C}_0 \subset (\mathcal{M}_0 \otimes_{\rm alg} \mathcal{Q}_0^\sigma)^f$. From the definition of the spectral projections $\{P_\pi^{\mathcal{C}} : \pi \in Rep(C(G))\}$ corresponding to $\mathcal{C} \in \mathcal{B}(\mathcal{H} \otimes L^2(G))$ and the fact that $(\mathrm{ad}_V \otimes \mathrm{id}) = (\mathrm{id} \otimes \mathrm{ad}_W)$ gives the von Neumann algebraic action on $\mathcal{C}$, we have $P_\pi^{\mathcal{C}} = (P_\pi^{\mathcal{M}} \otimes \mathrm{id}) = (\mathrm{id} \otimes P_\pi^{G,l})$.

Thus on one hand

$$\begin{aligned}(P_\pi^{\mathcal{C}}(\mathcal{C})) &\subset (P_\pi^{\mathcal{M}} \otimes \mathrm{id})(\mathcal{M}\bar{\otimes}L^\infty(G;\sigma)) \\ &= \mathcal{M}_0 \otimes_{\rm alg} L^\infty(G;\sigma).\end{aligned}$$

And on the other hand,

$$\begin{aligned}(P^{\mathcal{C}}_{\pi}(\mathcal{C})) &\subset (\mathrm{id}\otimes P^{G,l}_{\pi})(\mathcal{M}\bar{\otimes}L^{\infty}(G;\sigma))\\ &= \mathcal{M}\otimes_{\mathrm{alg}}\mathcal{Q}^{\sigma}_{0},\end{aligned}$$

i.e.

$$\begin{aligned}(P^{\mathcal{C}}_{\pi}(\mathcal{C})) &\subset (\mathcal{M}_0\otimes_{\mathrm{alg}}L^{\infty}(G;\sigma))\cap(\mathcal{M}\otimes_{\mathrm{alg}}\mathcal{Q}^{\sigma}_{0})\\ &= (\mathcal{M}_0\otimes_{\mathrm{alg}}\mathcal{Q}^{\sigma}_{0}).\end{aligned}$$

So clearly $P^{\mathcal{C}}_{\pi}(\mathcal{C})\subset(\mathcal{M}_0\otimes_{\mathrm{alg}}\mathcal{Q}^{\sigma}_{0})^f$ for all π. Hence $\mathcal{C}_0\subset(\mathcal{M}_0\otimes_{\mathrm{alg}}\mathcal{Q}^{\sigma}_{0})^f$. □

6.4.3 Deformation of a spectral triple by dual unitary 2-cocycles

Let $(\mathcal{A}^{\infty},\mathcal{H},\mathcal{D})$ be a spectral triple of compact type. Also let R be a positive unbounded operator on $\mathcal{H}$ commuting with $\mathcal{D}$. Then we have R twisted spectral triple $(\mathcal{A}^{\infty},\mathcal{H},\mathcal{D},R)$ as discussed earlier. Let σ be a dual unitary 2-cocycle for $(\mathcal{A}^{\infty},\mathcal{H},\mathcal{D},R)$ on some object in the category $\mathbf{Q}'_{\mathbf{R}}(\mathcal{D})$.

Given such a dual unitary 2-cocycle we get corresponding induced dual unitary 2-cocycle on $\widetilde{QISO}^{+}_{R}(\mathcal{D})$, which will again be denoted by σ with a slight abuse of notation. Let U be a unitary representation of $\widetilde{QISO}^{+}_{R}(\mathcal{D})$ on $\mathcal{H}$. Then U commutes with $\mathcal{D}$. From now onwards we shall denote the von Neumann algebra $(\mathcal{A}^{\infty})''$ inside $\mathcal{B}(\mathcal{H})$ by $\mathcal{M}$. $\mathcal{H}=\oplus_{k\geq 1}\mathcal{H}_k$, where each $\mathcal{H}_k$ is an eigen space for $\mathcal{D}$ and let $\mathcal{N}$ be the dense subspace $\mathcal{H}$ spanned by $\mathcal{H}_k$'s. Since U commutes with $\mathcal{D}$, U also preserves each $\mathcal{H}_k$. So $U(\mathcal{N})\subset\mathcal{N}\otimes\widetilde{QISO}^{+}_{R}(\mathcal{D})_0$ and we have the following decomposition of $\mathcal{H}_k$.

$$\mathcal{H}_k=\oplus_{\pi\in\mathcal{I}_k\subset Rep(\widetilde{QISO}^{+}_{R}(\mathcal{D}))}\mathbb{C}^{d_{\pi}}\otimes\mathbb{C}^{m_{\pi,k}},$$

where $m_{\pi,k}$ is the multiplicity of the irreducible representation π on $\mathcal{H}_k$, and $\mathcal{I}_k$ is some finite subset of $Rep(\widetilde{QISO}^{+}_{R}(\mathcal{D}))$. As $(\mathrm{id}\otimes\phi)\mathrm{ad}_{\widetilde{U}}(\mathcal{M})\subset\mathcal{M}$ for all bounded linear functionals ϕ on $\widetilde{QISO}^{+}_{R}(\mathcal{D})$. Then we have the corresponding spectral subalgebra which is SOT dense. Also for any subalgebra $\mathcal{A}_0$ of $\mathcal{M}$, on which $\mathrm{ad}_{\widetilde{U}}$ is algebraic, we can deform it by a dual unitary 2-cocycle as in Subsection 4.3 to get a new subalgebra in $\mathcal{B}(\mathcal{H})$ which we denote by $\mathcal{A}^{\sigma}_{0}$. Then we have

Theorem 6.4.15. *There is a SOT dense $*$-subalgebra $\mathcal{A}_0$ in $\mathcal{M}$ such that*
(1) $\mathrm{ad}_{\widetilde{U}}$ is algebraic over $\mathcal{A}_0$,
(2) $[\mathcal{D},a]\in\mathcal{B}(\mathcal{H})$ for all $a\in\mathcal{A}_0$,

(3) $(\mathcal{A}_0^\sigma)'' = \mathcal{M}^\sigma$,
(4) $(\mathcal{A}_0^\sigma, \mathcal{H}, \mathcal{D})$ *is again a spectral triple.*

Proof:
We consider the SOT-dense subalgebra $\mathcal{G} = \{a \in \mathcal{M} : [\mathcal{D}, a] \in \mathcal{B}(\mathcal{H})\}$ of $\mathcal{M}$. Let $b \in \mathcal{G}$. Then for a state ϕ of $\widetilde{QISO_R^+}(\mathcal{D})$, we have by definition $(\mathrm{id} \otimes \phi)\mathrm{ad}_{\widetilde{U}}(b) \in \mathcal{M}$. Also

$$\begin{aligned} & [\mathcal{D}, ((\mathrm{id} \otimes \phi)\mathrm{ad}_{\widetilde{U}}(b))] \\ = \; & (\mathrm{id} \otimes \phi)\mathrm{ad}_{\widetilde{U}}([\mathcal{D}, b]), \end{aligned}$$

using the commutativity of $\mathcal{D}$ and $\widetilde{U}$. Hence $(\mathrm{id} \otimes \phi)\mathrm{ad}_{\widetilde{U}}(b) \in \mathcal{G}$ for all bounded linear functional ϕ of $\widetilde{QISO_R^+}(\mathcal{D})$ i.e. $\mathcal{G}$ is $\mathrm{ad}_{\widetilde{U}}$ invariant SOT-dense subalgebra of $\mathcal{M}$. Now (1),(2),(3) follow from part (3) of Lemma 6.4.2, taking $\mathcal{A}_0$ to be the span of the subspaces $P_\pi(\mathcal{G})$, $\pi \in \mathrm{Rep}(\mathcal{Q})$. To prove (4), observe that $\forall a \in \mathcal{A}_0$, $\rho_\sigma(a) \in \mathcal{B}(\mathcal{H})$ by the proof of (1) of the Lemma 6.4.3. So we only need to check that $[\mathcal{D}, \rho_\sigma(a)] \in \mathcal{B}(\mathcal{H})$ for all $a \in \mathcal{A}_0$. Again with similar notations used as before, for $\xi \in \mathcal{N}$ and $a \in \mathcal{A}_0$ we have for some $k \in \mathbb{N}$,

$$\begin{aligned} & [\mathcal{D}, \rho_\sigma(a)](\xi) \\ = \; & \mathcal{D}\rho_\sigma(a)(\xi) - \rho_\sigma(a)\mathcal{D}(\xi) \\ = \; & \sum_{i=1}^{k} (\mathcal{D}a^i_{(0)}\Pi_V(\sigma_i)(\xi) - a^i_{(0)}\Pi_V(\sigma_i)\mathcal{D})(\xi) \\ = \; & \sum_{i=1}^{k} [\mathcal{D}, a^i_{(0)}]\Pi_V(\sigma_i)(\xi) \text{ (using the commutativity of } \mathcal{D} \text{ and } V) \end{aligned}$$

Since $[\mathcal{D}, a^i_{(0)}]$ is bounded for all $i = 1, ..., k$, we have that $[\mathcal{D}, \rho_\sigma(a)]$ is bounded for all $a \in \mathcal{A}_0$. □

Remark 6.4.16. *We can re-cast the deformed spectral triple* $(\mathcal{A}_0^\sigma, \mathcal{H}, \mathcal{D})$ *in the framework of Neshveyev-Tuset ([40]). Consider the image* $\mathcal{K}$ *of the representation* V *viewed as an isometric Hilbert space operator from* $\mathcal{H}$ *to* $\mathcal{H} \otimes L^2(\mathcal{Q}, h)$ *(where* h *is the Haar state of* $\mathcal{Q}$*). It can be shown that* $V\mathcal{A}_0^\sigma V^* \in \mathcal{B}(\mathcal{H} \otimes L^2(\mathcal{Q}, h))$ *is* $*$*-isomorphic with* $\mathcal{A}_0^\sigma$ *and* $(V\mathcal{A}_0^\sigma V^*, \mathcal{K}, \mathcal{D} \otimes 1_{L^2(\mathcal{Q},h)}|_\mathcal{K})$ *is the realisation of the deformed spectral triple in the Neshveyev-Tuset framework.*

6.5 QISO of deformed spectral triple

Fix as in previous section a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ of compact type and a positive unbounded operator R on the Hilbert space $\mathcal{H}$ commuting with the Dirac operator $\mathcal{D}$. Assume that there is a dual unitary 2-cocycle σ, say on $\mathcal{Q}$ (with $\mathcal{Q}$ being an object in the category $\mathbf{Q'_R}(\mathcal{D})$). Then $\mathcal{Q}$ is a quantum subgroup of $\widetilde{QISO_R^+}(\mathcal{D})$. Let U, V be corresponding unitary representations of $\widetilde{QISO_R^+}(\mathcal{D})$ and $\mathcal{Q}$ respectively on $\mathcal{H}$. Let $\mathcal{A}_0$ be any SOT dense subalgebra of $(\mathcal{A}^\infty)'' = \mathcal{M}$ satisfying the conditions of (1) of Theorem 6.4.15. Recall from Subsection 4.1, the induced dual unitary 2-cocycle σ' on $\widetilde{QISO_R^+}(\mathcal{D})$. Since $\mathrm{ad}_{\widetilde{U}}$ is algebraic over $\mathcal{A}_0$, so is $\mathrm{ad}_{\widetilde{V}}$ and it is easy to see that $\mathcal{A}_0^{\sigma'} = \mathcal{A}_0^{\sigma}$. Now the category $\mathbf{Q'_{R^\sigma}}(\mathcal{A}_0^\sigma, \mathcal{H}, \mathcal{D})$ for some unbounded operator R^σ does not depend on the choice of the SOT dense subalgebra $\mathcal{A}_0$. Let us abbreviate it as $\mathbf{Q'_{R^\sigma}}(\mathcal{D}_\sigma)$. We have the following:

Lemma 6.5.1. *$(\mathcal{Q}^\sigma, V_\sigma)$ is an object in the category $\mathbf{Q'_{R^\sigma}}(\mathcal{D}_\sigma)$, where $R^\sigma = \Pi_V(v)^* R \Pi_V(v)$ and v is as in equation (7).*

Proof:
By Theorem 6.3.6, V_σ is again a unitary representation of $\mathcal{Q}^\sigma$ on the Hilbert space $\mathcal{H}$. Also note that $\mathrm{ad}_{\widetilde{V}}$ is algebraic over $\mathcal{A}_0$. Let $a \in \mathcal{A}_0$ and $\xi \in \mathcal{N}$, where $\mathcal{N}$ is the subspace of $\mathcal{H}$ given by span of $\mathcal{H}_k$'s. Then we have

$$\begin{aligned} V_\sigma(\rho_\sigma(a)(\xi)) &= V_\sigma(a_{(0)}\xi_{(0)}\sigma^{-1}(a_{(1)}, \xi_{(1)})) \\ &= a_{(0)(0)}\xi_{(0)(0)}\sigma^{-1}(a_{(1)}, \xi_{(1)}) \otimes a_{(0)(1)}\xi_{(0)(1)} \\ &= a_{(0)}\xi_{(0)}\sigma^{-1}(a_{(1)(2)}, \xi_{(1)(2)}) \otimes a_{(1)(1)}\xi_{(1)(1)} \end{aligned}$$

On the other hand

$$\begin{aligned} & (\rho_\sigma \otimes [.])\mathrm{ad}_{\widetilde{V}}(V_\sigma(\xi)) \\ =\; & (\rho_\sigma(a_{(0)})\xi_{(0)}) \otimes a_{(1)} \cdot_\sigma \xi_{(1)} \\ =\; & a_{(0)(0)}\xi_{(0)(0)}\sigma^{-1}(a_{(0)(1)}, \xi_{(0)(1)}) \otimes \sigma(a_{(1)(1)(1)}, \xi_{(1)(1)(1)})a_{(1)(1)(2)}\xi_{(1)(1)(2)}\sigma^{-1}(a_{(1)(2)}, \xi_{(1)(2)}) \\ =\; & a_{(0)}\xi_{(0)}\sigma^{-1}(a_{(1)(1)(1)}, \xi_{(1)(1)(1)})\sigma(a_{(1)(1)(2)}, \xi_{(1)(1)(2)}) \otimes a_{(1)(2)(1)}\xi_{(1)(2)(1)}\sigma^{-1}(a_{(1)(2)(2)}, \xi_{(1)(2)(2)}) \\ =\; & a_{(0)}\xi_{(0)}\epsilon(a_{(1)(1)})\epsilon(\xi_{(1)(1)})a_{(1)(2)(1)}\xi_{(1)(2)(1)}\sigma^{-1}(a_{(1)(2)(2)}, \xi_{(1)(2)(2)}) \\ =\; & a_{(0)}\xi_{(0)}\epsilon(a_{(1)(1)(1)})\epsilon(\xi_{(1)(1)(1)}) \otimes a_{(1)(1)(2)}\xi_{(1)(1)(2)}\sigma^{-1}(a_{(1)(2)}, \xi_{(1)(2)}) \\ =\; & a_{(0)}\xi_{(0)}\sigma^{-1}(a_{(1)(2)}, \xi_{(1)(2)}) \otimes a_{(1)(1)}\xi_{(1)(1)} \end{aligned}$$

So $\mathrm{ad}_{\widetilde{V_\sigma}}(\rho_\sigma(a))(V_\sigma\xi) = V_\sigma(\rho_\sigma(a)\xi) = (\rho_\sigma \otimes [.])\mathrm{ad}_{\widetilde{V}}(a)(V_\sigma(\xi))$. By density of $\mathcal{N}$ in $\mathcal{H}$ we conclude that

$$\mathrm{ad}_{\widetilde{V_\sigma}}(\rho_\sigma(a)) = (\rho_\sigma \otimes [.])\mathrm{ad}_{\widetilde{V}}(a),$$

for all $a \in \mathcal{A}_0$. Also for $\phi \in (\mathcal{Q}^\sigma)^*$,

$$(\mathrm{id} \otimes \phi)\mathrm{ad}_{\widetilde{V_\sigma}}(\rho_\sigma(a)) = \rho_\sigma(a_{(0)})\phi(a_{(1)}) \in \mathcal{A}_0^\sigma.$$

So in particular $(\mathrm{id} \otimes \phi)\mathrm{ad}_{\widetilde{V_\sigma}}(\mathcal{A}_0) \subset (\mathcal{A}_0)''$.

V_σ commutes with $\mathcal{D}$ since as a linear map V_σ is same as V.

Recall the decomposition (Section 3) of the Hilbert space $\mathcal{H} = \oplus_{k\geq 1 \pi\in\mathcal{I}_k \subset Rep(\mathcal{Q})}\mathbb{C}^{d_\pi} \otimes \mathbb{C}^{m_{\pi,k}}$, where $m_{\pi,k}$ is the multiplicity of the irreducible representation π on $\mathcal{H}_k$ and $\mathcal{I}_k$ is some finite subset of $Rep(\mathcal{Q})$. By Theorem 2.3.12, R has the form

$$R = \oplus_{\pi\in\mathcal{I}_k, k\geq 1} F_\pi \otimes T_{\pi,k},$$

for some $T_{\pi,k} \in \mathcal{B}(\mathbb{C}^{m_{\pi,k}})$ so that $\Pi_V(v)^* R \Pi_V(v)$ is of the form $\oplus_{\pi,k} A_\pi^* F_\pi A_\pi \otimes T_{\pi,k}$, for some $T_{\pi,k}$ where A_π, F_π's are as in (ii) of Proposition 6.3.7. Now Recalling (ii) of Proposition 6.3.7, we see that $F_\pi^\sigma = c_\pi A_\pi^* F_\pi A_\pi$ for some positive constant c_π. Then R^σ is of the form $\oplus_{\pi\in\mathcal{I}_k, k\geq 1} F_\pi^\sigma \otimes c_\pi^{-1} T_{\pi,k}$, for some $T_{\pi,k}$. Hence $\mathrm{ad}_{\widetilde{V_\sigma}}$ preserves the R^σ-twisted volume by Theorem 2.3.12. □

Remark 6.5.2. *By looking at the proof we can easily conclude that if $(\mathcal{Q}, V)$ is an object in the category $\mathbf{Q}'(\mathcal{D})$, then $(\mathcal{Q}^\sigma, V_\sigma)$ is an object in the category $\mathbf{Q}'(\mathcal{D}_\sigma)$.*

Now replacing $\mathcal{Q}$ by $\widetilde{QISO_R^+}(\mathcal{D})$ and using the dual unitary 2-cocycle on $\widetilde{QISO_R^+}(\mathcal{D})$ induced from σ on its quantum subgroup, we get

Corollary 6.5.3. *$\widetilde{QISO_R^+}(\mathcal{D})^\sigma \leq \widetilde{QISO_{R^\sigma}^+}(\mathcal{D}_\sigma)$.*

Thus in particular, we have the dual unitary 2-cocycle σ^{-1} on $\mathcal{Q}^\sigma \leq \widetilde{QISO_{R^\sigma}^+}(\mathcal{D}_\sigma)$ and can deform $\mathcal{D}_\sigma$ by it. Consider $\mathcal{M}_1 = Sp\{P_{\pi_\sigma}(({\mathcal{M}_0})'') : \pi \in Rep(\mathcal{Q})\}$, where $P_{\pi_\sigma} = (id \otimes \rho^{\pi_\sigma})ad_{\tilde{V}_\sigma}$ (recall ρ_{π_σ} from Subsection 2.1). Then again as before $\rho_{\sigma^{-1}}$ is defined on $\mathcal{M}_1$ and it is the maximal subspace on which $ad_{\tilde{V}_\sigma}$ is algebraic. As $\mathrm{ad}_{\widetilde{U_\sigma}}$ is again a von Neumann algebraic action of $\mathcal{Q}^\sigma$ on $(\mathcal{A}_0)'' = \rho_\sigma(\mathcal{M}_0)''$, we have SOT dense subalgebra C_0 of $\rho_\sigma(\mathcal{M}_0)''$ over which $\mathrm{ad}_{\widetilde{U_\sigma}}$ is algebraic. Then $\mathrm{ad}_{\widetilde{V_\sigma}}$ is also algebraic over C_0. Hence by maximality $C_0 \subset \mathcal{M}_1$. Again by SOT continuity of $\rho_{\sigma^{-1}}$ on the image of P_{π_σ}, we have $\rho_{\sigma^{-1}}(C_0)'' = \rho_{\sigma^{-1}}(\mathcal{M}_1)''$. On the other hand as $\mathrm{ad}_{\widetilde{V_\sigma}}$ is algebraic over $\rho_\sigma(\mathcal{M}_0)$, we have

$$\begin{aligned} & \rho_\sigma(\mathcal{M}_0) \subset \mathcal{M}_1 \\ \Rightarrow \ & \rho_{\sigma^{-1}}(\rho_\sigma(\mathcal{M}_0)) \subset \rho_{\sigma^{-1}}(\mathcal{M}_1) \\ \Rightarrow \ & \mathcal{M}_0 \subset \rho_{\sigma^{-1}}(\mathcal{M}_1) \end{aligned}$$

By maximality of $\mathcal{M}_0$, we conclude that

$$\rho_{\sigma^{-1}}(C_0)'' = \mathcal{M}_0'' = (\mathcal{A}^\infty)'',$$

which implies that $\mathbf{Q}'_{(\mathbf{R}^\sigma)_{\sigma^{-1}}}((\mathcal{D}_\sigma)_{\sigma^{-1}}) = \mathbf{Q}'_{\mathbf{R}}(\mathcal{D})$

Lemma 6.5.4. $\widetilde{QISO^+_{R^\sigma}}(\mathcal{D}_\sigma)^{\sigma^{-1}} \leq \widetilde{QISO^+_R}(\mathcal{D})$

Proof:
Observe that $\widetilde{QISO^+_{R^\sigma}}(\mathcal{D}_\sigma)^{\sigma^{-1}}$ preserves volume τ_R and

$$\mathrm{ad}_{\widetilde{U}}(a) = \mathrm{ad}_{\widetilde{U_\sigma}}(\rho_{\sigma^{-1}}(a)),$$

for all $a \in C_0$.

But by definition $(\mathrm{id} \otimes \phi)\mathrm{ad}_{\widetilde{U_\sigma}}(\rho_{\sigma^{-1}}(a)) \subset \rho_{\sigma^{-1}}(C_0)'' = (\mathcal{A}^\infty)''$. □

Combining all these and using the fact that $\widetilde{QISO^+_R}(\mathcal{D})$'s are universal CQG's, we get

Theorem 6.5.5. $\widetilde{QISO^+_{R^\sigma}}(\mathcal{D}_\sigma) \cong (\widetilde{QISO^+_R}(\mathcal{D}))^\sigma$ *and hence* $QISO^+_{R^\sigma}(\mathcal{D}_\sigma) \cong (QISO^+_R(\mathcal{D}))^\sigma$.

Recall the category $\mathbf{Q}'(\mathcal{D})$ from Section 3. We know that in general we can not say anything about the existence of the universal object in this category. However by looking at the proofs in this section, with the notations used in this section we have the following

Corollary 6.5.6. *If* $QISO^+(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ *and* $QISO^+(\mathcal{A}_0^\sigma, \mathcal{H}, \mathcal{D})$ *both exist, then*

$$QISO^+(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})^\sigma \cong QISO^+(\mathcal{A}_0^\sigma, \mathcal{H}, \mathcal{D}).$$

Remark 6.5.7. *Viewing the Rieffel deformation as a special case of cocycle twist, the above theorem in fact improves the result (Theorem 5.13 of [10]) obtained by Bhowmick and Goswami by removing the assumption about a nice dense subalgebra on which the adjoint action of the quantum isometry group is algebraic. In fact, techniques of this paper have enabled us to prove existence of such nice algebra in general.*

Chapter 7

The averaging trick

7.1 Introduction

In case of a smooth group (compact) action on a classical, compact, Riemannian manifold we can average out the Riemannian inner product of the manifold with respect to the action using the the Haar state of the group. In this chapter we shall extend this classical averaging technique in the context of compact quantum group action on a classical compact Riemannian manifold. Similar averaging technique has been used in [18] to prove the nonexistence of genuine quantum isometry group for an arbitrary compact, connected, Riemannain manifold.

7.2 The averaging technique

Before we state and prove the main result in the next section, let us collect a few facts about a smooth faithful action of compact quantum groups on compact manifolds, for the details of which the reader may be referred to [18] and references therein.

Proposition 7.2.1. *If a CQG $\mathcal{Q}$ acts faithfully and smoothly on a smooth compact manifold M then we have:*
(i) $\mathcal{Q}$ has a tracial Haar state, i.e. it is Kac type CQG.
(ii) The action is injective.
(iii) The antipode κ satisfies $\kappa(a^*) = \kappa(a)^*$.

We usually denote by $\otimes$ algebraic tensor product of vector spaces or algebras. We also use Sweedler convention for Hopf algebra coproduct as well as its analogue for (co)-actions of Hopf algebras. That is, we simply write $\Delta(q) = q_{(1)} \otimes q_{(2)}$ suppressing finite summation, where Δ denote the co-product map of a Hopf algebra and q is an element of the Hopf algebra as in the previous chapters. Similarly, for an algebraic (co)action α of a Hopf algebra on some algebra $\mathcal{C}$, we write $\alpha(a) = a_{(0)} \otimes a_{(1)}$.

7.2.1 The main result

Fix a compact Riemannian manifold M (not necessarily orientable) and a smooth action α of a CQG $\mathcal{Q}$. We make the following assumptions for the rest of the paper.
Assumption I: There is a Fréchet dense unital $*$-subalgebra $\mathcal{A}$ of $C^\infty(M)$ such that $<< d\alpha(df), d\alpha(dg) >> \in \mathcal{A}$ for all $f, g \in \mathcal{A}$.
Assumption II: There is a well-defined representation Γ on $\Omega^1(M)$ in the sense discussed earlier, such that $\Gamma(df) = (d \otimes \mathrm{id})(\alpha(f))$ for all $f \in C^\infty(M)$. We'll denote this Γ by $d\alpha$.

We now state and prove the main result that we can equip M with a new Riemannian structure with respect to which the action becomes inner product preserving using an analogue of the averaging technique of classical differential geometry.

Theorem 7.2.2. *M has a Riemannian structure such that α is inner product preserving.*

Note that the first assumption holds for a large class of examples, such as algebraic actions of CQG's compact, smooth, real varieties where the complexified coordinate algebra of the variety can be chosen as $\mathcal{A}$. On the other hand, the second assumption means that the action on M in some sense lifts to the space of one-forms. This is always automatic for a smooth action by (not necessarily compact) groups, and in fact is nothing but the differential of the map giving the action. Moreover, it is easy to see that any CQG action which preserves the Riemannian inner product does admit such a lift on the bimodule of one-forms, i.e. satisfies the assumption II. Therefore, it is a reasonable assumption too.

Remark 7.2.3. *We have already mentioned in the introduction that the existence of an invariant Riemannian inner product, i.e. the conclusion of Theorem 7.2.2 has been proved in [18] as a part of much more general scheme; in fact, without even Assumption I, i.e. only under the assumption II. However, our aim in this chapter is to give a direct and easier construction of the invariant Riemannian inner product. But in doing this, we had to pay a price: the scope of our methods are slightly restrictive as we had to impose Assumption I.*

Nevertheless, we believe that the alternative construction of an ivariant Riemannian structure presented here should be valuable beyond classical manifolds, i.e. in the wider context of noncommutative geometry.

Proof of Theorem 7.2.2:

We break the proof of into a number of lemmas.

Lemma 7.2.4. *Define the following map* Ψ *from* $\mathcal{A} \otimes \mathcal{Q}_0$ *to* $\mathcal{A}$ *:*

$$\Psi(F) := (\mathrm{id} \otimes h)(\mathrm{id} \otimes m)(\mathrm{id} \otimes \kappa \otimes \mathrm{id})(\alpha \otimes \mathrm{id})(F).$$

Here $m : \mathcal{Q}_0 \otimes \mathcal{Q}_0 \to \mathcal{Q}_0$ *is the multiplication map. Then* Ψ *is a completely positive map.*

Proof:
As the range is a subalgebra of a unital commutative C^* algebra, it is enough to prove positivity. Let $F = G^*G$ in $\mathcal{A} \otimes \mathcal{Q}_0$ where $G = \sum_i f_i \otimes q_i$, (finite sum) for some $f_i \in \mathcal{A}, q_i \in \mathcal{Q}_0$. We write $\alpha(f) = f_{(0)} \otimes f_{(1)}$ in Sweedler notation as usual, and observe that

$$\begin{aligned}
&\Psi(F)\\
&= \sum_{ij} f^*_{i(0)} f_{j(0)} h(\kappa(f^*_{i(1)} f_{j(1)}) q^*_i q_j)\\
&= \sum_{ij} f_{j(0)} f^*_{i(0)} h(q_j (\kappa(f_{j(1)}))^* \kappa(f_{i(1)}) q^*_i)\\
&= (\mathrm{id} \otimes h)(\xi^* \xi) \geq 0,
\end{aligned}$$

where $\xi = \sum_i f^*_{i(0)} \otimes \kappa(f_{i(1)}) q^*_i$, and note also that we have used above the facts that h is tracial and κ is $*$-preserving. □

For $\omega, \eta \in \Omega^1(\mathcal{A})$ We define

$$<< \omega, \eta >>^{'} := \Psi(<< d\alpha(\omega), d\alpha(\eta) >>),$$

which is well defined as we have assumed that $<< d\alpha(ds_1), d\alpha(ds_2)) >> \in \mathcal{A} \otimes \mathcal{Q}_0$ for $s_1, s_2 \in \mathcal{A}$. Moreover, by complete positivity of Ψ this gives a non-negative definite sesquilinear form on $\Omega^1(\mathcal{A})$. As the action is algebraic over $\mathcal{A}$, we shall use Sweedler's notation to prove the following

Lemma 7.2.5. *For* $\omega, \eta \in \Omega^1(\mathcal{A})$*,* $f \in \mathcal{A}$*,* $<< \omega, \eta >>' = (<< \eta, \omega >>')^*$ *and* $<< \omega, \eta f >>' = << \omega, \eta >>' f$

Proof:
It is enough to prove the lemma for $\omega = d\phi$ and $\eta = d\psi$ for $\phi, \psi \in \mathcal{A}$. First observe that as we have $\kappa = \kappa^{-1}$, for $z \in \mathcal{Q}_0$ applying κ on $z_{(1)} \kappa(z_{(2)}) = \epsilon(z).1$, we get

$$z_{(2)} \kappa(z_{(1)}) = \epsilon(z).1. \tag{7.2.1}$$

We denote $<< d\phi_{(0)}, d\psi_{(0)} >>$ by x and $\phi^*_{(1)}\psi_{(1)}$ by y. Then

$$\begin{aligned}
& << d\phi, d\psi f >>' \\
= & (\mathrm{id} \otimes h)(\mathrm{id} \otimes m)(\mathrm{id} \otimes \kappa \otimes \mathrm{id})(\alpha \otimes \mathrm{id}) << d\alpha(d\phi), d\alpha(d\psi f) >> \\
= & (\mathrm{id} \otimes h)(\mathrm{id} \otimes m)(\mathrm{id} \otimes \kappa \otimes \mathrm{id})(\alpha \otimes \mathrm{id})(x f_{(0)} \otimes y f_{(1)}) \\
= & (\mathrm{id} \otimes h)(\mathrm{id} \otimes m)(\mathrm{id} \otimes \kappa \otimes \mathrm{id})(x_{(0)} f_{(0)(0)} \otimes x_{(1)} f_{(0)(1)} \otimes y f_{(1)}) \\
= & (\mathrm{id} \otimes h)(x_{(0)} f_{(0)(0)} \otimes \kappa(x_{(1)} f_{(0)(1)}) y f_{(1)}) \\
= & x_{(0)} f_{(0)(0)} h(f_{(1)} \kappa(f_{(0)(1)}) \kappa(x_{(1)}) y) (\textit{by tracial property of } h) \\
= & x_{(0)} f_{(0)} h(f_{(1)(2)} \kappa(f_{(1)(1)}) \kappa(x_{(1)}) y) \\
= & x_{(0)} f_{(0)} h(\epsilon(f_{(1)}).1.\kappa(x_{(1)}) y) \\
= & x_{(0)} (\mathrm{id} \otimes \epsilon)\alpha(f) h(\kappa(x_{(1)}) y) \\
= & x_{(0)} f h(\kappa(x_{(1)}) y).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
<< d\phi, d\psi >>' f &= [(\mathrm{id} \otimes h)(\mathrm{id} \otimes m)(\mathrm{id} \otimes \kappa \otimes \mathrm{id})(\alpha \otimes \mathrm{id}) << d\alpha(d\phi), d\alpha(d\psi) >>] f \\
&= [(\mathrm{id} \otimes h)(\mathrm{id} \otimes m)(\mathrm{id} \otimes \kappa \otimes \mathrm{id})(x_{(0)} \otimes x_{(1)} \otimes y)] f \\
&= x_{(0)} f h(\kappa(x_{(1)}) y).
\end{aligned}$$

Also we have

$$\begin{aligned}
& << d\phi, d\psi >>' \\
= & (\mathrm{id} \otimes h)(\mathrm{id} \otimes m)(\mathrm{id} \otimes \kappa \otimes \mathrm{id})(\alpha \otimes \mathrm{id})(<< d\phi_{(0)}, d\psi_{(0)} >> \otimes \phi^*_{(1)}\psi_{(1)}) \\
= & (\mathrm{id} \otimes h)(\mathrm{id} \otimes m)(\mathrm{id} \otimes \kappa \otimes \mathrm{id})(\alpha \otimes \mathrm{id})(<< d\psi_{(0)}, d\phi_{(0)} >>^* \otimes \phi^*_{(1)}\psi_{(1)}) \\
= & (\mathrm{id} \otimes h)(\mathrm{id} \otimes m)(\mathrm{id} \otimes \kappa \otimes \mathrm{id})(<< d\psi_{(0)}, d\phi_{(0)} >>^*_{(0)} \otimes << d\psi_{(0)}, d\phi_{(0)} >>^*_{(1)} \otimes \phi^*_{(1)}\psi_{(1)}) \\
= & << d\psi_{(0)}, d\phi_{(0)} >>^* h((\kappa(<< d\psi_{(0)}, d\phi_{(0)} >>))^* \phi^*_{(1)}\psi_{(1)}) (\textit{ since } \kappa \textit{ is } * \textit{ preserving})
\end{aligned}$$

Hence we have

$$<< d\phi, d\psi >>'^* = << d\psi_{(0)}, d\phi_{(0)} >> h((\kappa(<< d\psi_{(0)}, d\phi_{(0)} >>))\psi^*_{(1)}\phi_{(1)})$$

(*since* h *is tracial and* $h(a^*) = \overline{h(a)}$).

But we can readily see that

$$<< d\psi, d\phi >>' = << d\psi_{(0)}, d\phi_{(0)} >> h((\kappa(<< d\psi_{(0)}, d\phi_{(0)} >>))\psi^*_{(1)}\phi_{(1)}),$$

which completes the proof of the lemma. □

Actually we can extend $<<, >>'$ to a slightly bigger set than $\Omega^1(\mathcal{A})$ namely $\Omega^1(\mathcal{A})C^\infty(M) = Sp \{\omega f : \omega \in \Omega^1(\mathcal{A}), f \in C^\infty(M)\}$.

For $\omega, \eta \in \Omega^1(\mathcal{A})C^\infty(M)$, $\omega = \sum \omega_i f_i$ *and* $\eta = \sum \eta_i g_i($ *finite sums*$)$, $\omega_i, \eta_i \in \Omega^1(\mathcal{A})$ *and* $f_i, g_i \in C^\infty(M)$ (say) we can choose sequences $f_i^{(n)}, g_i^{(n)}$ from $\mathcal{A}$ such that $f_i^{(n)} \to f_i$ and $g_i^{(n)} \to g_i$ in the corresponding Fréchet topology and by Lemma 7.2.5 observe that

$$\begin{aligned} & << \sum_i \omega_i f_i^{(n)}, \sum_j \eta_j g_j^{(n)} >>' \\ = & \sum_{i,j} \overline{f_i^{(n)}} << \omega_i, \eta_j >>' g_j^{(n)} \\ \to & \sum_{i,j} \overline{f_i} << \omega_i, \eta_j >>' g_j := << \omega, \eta >>' \end{aligned} \tag{7.2.2}$$

Clearly this definition is independent of the choice of sequences $f_i^{(n)}$ and $g_i^{(n)}$. We next prove the following

Lemma 7.2.6. *For* $\phi, \psi \in \mathcal{A}$,

$$<< d\alpha(d\phi), d\alpha(d\psi) >>' = \alpha(<< d\phi, d\psi >>') \tag{7.2.3}$$

Proof:
With x, y as before we have

Claim 2: We can extend the definition of $<<, >>'$ for $\omega, \eta \in \Omega^1(\mathcal{A})C^\infty(M)$ such that

$$\forall \ f \in C^\infty(M), << (d\phi), (d\psi)f >>' = << d\phi, d\psi >>' f \tag{7.2.4}$$

Proof:
For $f \in C^\infty(M)$, define $<< (d\phi), (d\psi)f >>' := lim << d\phi, d\psi \ f_n >>'$, where $f_n \in \mathcal{A}$ with $lim \ f_n = f$, where the limits are taken in the Fréchet topology.
Observe that $<< d\phi, d\psi \ f_n >>'$ is Fréchet Cauchy as

$$\begin{aligned} & << d\phi, d\psi \ f_n >>' - << d\phi, d\psi \ f_m >>' \\ = & << d\phi, d\psi >>' (f_n - f_m) \end{aligned}$$

So $<< d\phi, d\psi f >>' = lim << d\phi, d\psi >>' f_n =<< d\phi, d\psi >>' f$, again the limit is taken in the corresponding Fréchet topology.
That proves the claim.

$$\begin{aligned} & << d\alpha(d\phi), d\alpha(d\psi) >>' \\ = & (\mathrm{id} \otimes h \otimes \mathrm{id})(\mathrm{id} \otimes m \otimes \mathrm{id})(\mathrm{id} \otimes \kappa \otimes \mathrm{id} \otimes \mathrm{id})(\alpha \otimes \mathrm{id} \otimes \mathrm{id})(x \otimes \Delta(y)) \\ = & (\mathrm{id} \otimes h \otimes \mathrm{id})(\mathrm{id} \otimes m \otimes \mathrm{id})(\mathrm{id} \otimes \kappa \otimes \mathrm{id} \otimes \mathrm{id})(x_{(0)} \otimes x_{(1)} \otimes y_{(1)} \otimes y_{(2)}) \\ = & (\mathrm{id} \otimes h \otimes \mathrm{id})(x_{(1)} \otimes \kappa(x_{(2)}) y_{(1)} \otimes y_{(2)}) \\ = & x_{(0)} \otimes h(\kappa(x_{(1)}) y_{(1)}) y_{(2)}. \end{aligned}$$

On the other hand

$$\begin{aligned} \alpha(<< d\phi, d\psi >>') &= x_{(0)(0)} h(\kappa(x_{(1)}) y) \otimes x_{(0)(1)} \\ &= x_{(0)} \otimes x_{(1)(1)} h(\kappa(x_{(1)(2)}) y) \\ &= x_{(0)} \otimes x_{(1)(1)} h(\kappa(y)(x_{(1)(2)}))(\text{ since } h(\kappa(a)) = h(a)) \end{aligned}$$

Hence it is enough to show that $h(\kappa(c) b_{(2)}) b_{(1)} = h(\kappa(b) c_{(1)}) c_{(2)}$ where $b, c \in \mathcal{Q}_0$, for then taking $x_{(1)} = b$ and $y = c$ we can complete the proof.

We make the transformation $T(a \otimes b) = \Delta(\kappa(a))(1 \otimes b)$.
Then

$$\begin{aligned} & (h \otimes \mathrm{id}) T(a \otimes b) \\ = & (h \otimes \mathrm{id}) \Delta(\kappa(a))(1 \otimes b) \\ = & ((h \otimes \mathrm{id}) \Delta(\kappa(a))) b \\ = & h(\kappa(a)) b \\ = & (h \otimes \mathrm{id})(a \otimes b) \end{aligned}$$

Hence $h(b_{(2)} \kappa(c)) b_{(1)} = (h \otimes id) T(b_{(2)} \kappa(c) \otimes b_{(1)})$.
So, by using traciality of h it is enough to show that $T(b_{(2)} \kappa(c) \otimes b_{(1)}) = c_{(1)} \kappa(b) \otimes c_{(2)}$.

$$
\begin{aligned}
& T(b_{(2)}\kappa(c)\otimes b_{(1)}) \\
= \;& \Delta(\kappa(b_{(2)}\kappa(c)))(1\otimes b_{(1)}) \\
= \;& \Delta(c\kappa(b_{(2)}))(1\otimes b_{(1)}) \\
= \;& (c_{(1)}\otimes c_{(2)})[\kappa(b_{(2)(2)})\otimes \kappa(b_{(2)(1)})](1\otimes b_{(1)}) \\
= \;& (c_{(1)}\otimes c_{(2)})m_{23}(\kappa(b_{(2)(2)})\otimes \kappa(b_{(2)(1)})\otimes b_{(1)}) \\
= \;& (c_{(1)}\otimes c_{(2)})m_{23}(\kappa\otimes\kappa\otimes \mathrm{id})\sigma_{13}(b_{(1)}\otimes b_{(2)(1)}\otimes b_{(2)(2)}) \\
= \;& (c_{(1)}\otimes c_{(2)})m_{23}(\kappa\otimes\kappa\otimes \mathrm{id})\sigma_{13}(b_{(1)(1)}\otimes b_{(1)(2)}\otimes b_{(2)}) \\
= \;& (c_{(1)}\otimes c_{(2)})m_{23}(\kappa(b_{(2)}\otimes \kappa(b_{(1)(2)})\otimes b_{(1)(1)}) \\
= \;& (c_{(1)}\otimes c_{(2)})(\kappa(b_{(2)})\otimes \epsilon(b_{(1)}).1_{\mathcal{Q}})(by\ (10)) \\
= \;& (c_{(1)}\otimes c_{(2)})(\kappa\otimes\kappa)((b_{(2)})\otimes \epsilon(b_{(1)}).1_{\mathcal{Q}}) \\
= \;& (c_{(1)}\otimes c_{(2)})(\kappa\otimes\kappa)(\epsilon(b_{(1)})b_{(2)}\otimes 1_{\mathcal{Q}}) \\
= \;& c_{(1)}\kappa(b)\otimes c_{(2)}
\end{aligned}
$$

Which proves the claim.

Now we proceed to define a new Riemannian structure on the manifold so that the action α will be inner product preserving. For that we are going to need the following

Lemma 7.2.7. *(i) For $m \in M$, Sp $\{ds(m) : s \in \mathcal{A}\}$ coincides with $T^*_m(M)$.*
(ii) If $\{s_1, ..., s_n\}$ and $\{s'_1, ..., s'_n\}$ are two sets of functions in $\mathcal{A}$ such that each of $\{ds_i(m) : i = 1, ..., n\}$ and $\{ds'_i(m) : i = 1, ..., n\}$ are bases for $T^*_m(M)$ and for $v, w \in T^*_m(M)$ with $v = \sum_i c_i ds_i(m) = \sum_i c'_i ds'_i(m)$ and $w = \sum_i d_i ds_i(m) = \sum_i d'_i ds'_i(m)$, then

$$\sum_{i,j} \bar{c_i} d_j << ds_i, ds_j >>' (m) = \sum_{i,j} \bar{c'_i} d'_j << ds'_i, ds'_j >>' (m),$$

where $<<, >>'$ is the new $C^\infty(M)$ valued inner product introduced earlier.

Proof:
Choosing a coordinate neighborhood U around m and a set of coordinates $x_1, ..., x_n$ we have $ds(m) = \sum_{i=1}^n \frac{\partial s}{\partial x_i}(m) dx_i(m)$.
Pick any $\eta \in T^*_m(M)$ i.e. we have $\eta = \sum_{i=1}^n c_i dx_i(m)$ for some c_i's in $\mathbb{R}$. Choose any $f \in C^\infty(M)$ with $\frac{\partial f}{\partial x_i}(m) = c_i$. For $f \in C^\infty(M)$, by Fréchet density of $\mathcal{A}$ we have a

sequence $s_n \in \mathcal{A}$ and an $n_0 \in \mathbb{N}$ such that

$$|\frac{\partial s}{\partial x_i}(m) - \frac{\partial f}{\partial x_i}(m)| < \epsilon \ \forall \ n \geq n_0.$$

So Sp $\{ds(m); s \in \mathcal{A}\}$ is dense in $T_m^*(M)$. $T_m^*(M)$ being finite dimensional Sp $\{ds(m) : s \in \mathcal{A}\}$ coincides with $T_m^*(M)$. Which proves (i).

For proving (ii) first we prove the following fact:

Let $m \in M$ and $\omega \in \Omega^1(\mathcal{A})$ such that $\omega = 0$ in a neighborhood U of m. Then $<< \omega, \eta >>' = 0$ for all $\eta \in \Omega^1(\mathcal{A})$

For the proof of the above fact Let $V \subset U$ such that $V \subset \bar{V} \subset U$. Choose $f \in C^\infty(M)_\mathbb{R}$ such that $supp(f) \subset \bar{V}$, $f \equiv 1$ on V and $f \equiv 0$ outside U. So we can write $\omega = (1-f)\omega$. Then

$$\begin{aligned} & << \omega, \eta >>' (m) \\ = \ & << (1-f)\omega, \eta >>' (m) \\ = \ & << \omega, \eta >>' (m)(1-f)(m) \ (by \ (7.2.4)) \\ = \ & 0. \end{aligned}$$

Applying the above fact we can show:

Let $m \in M$ and $\omega = \omega'$, $\eta = \eta'$ in a neighbourhood U of m. Then $<< \omega, \eta >>' = << \omega', \eta' >>'$, $\forall \omega, \omega', \eta, \eta' \in \Omega^1(\mathcal{A})$.

For the proof it is enough to observe that $<< \omega, \eta >>' (m) - << \omega', \eta' >>' (m) = << \omega - \omega', \eta >>' (m) + << \omega', \eta - \eta' >> (m)$.

As $\{ds_1(m), ..., ds_n(m)\}$ and $\{ds'_1(m), ..., ds'_n(m)\}$ are two bases for $T_m^*(M)$. Then they are actually bases for $T_x^*(M)$ for x in a neighborhood U of m. So there are $\{f_{ij} : i, j = 1(1)n\}$ in $C^\infty(M)$ such that

$$ds_i = \sum_{j=1}^n f_{ij} ds'_j$$

on U for all $i = 1, \ldots, n$. Hence by the previous discussion

$$<< ds_i, ds_j >>' (m) = << \sum_k f_{ik} ds'_k, \sum_l f_{jl} ds'_l >>' (m) \tag{7.2.5}$$

Let $v = \sum_{i=1}^n c_i ds_i(m) = \sum_{i=1}^n c'_i ds'_i(m)$ and $w = \sum_{i=1}^n d_i ds_i(m) = \sum_{i=1}^n d'_i ds'_i(m)$. So by definition

$$\begin{aligned} < v, w >' &= \sum_{ij} \bar{c_i} d_j << ds_i, ds_j >>' (m) \\ &= \sum_{ijkl} \bar{c_i} d_j \bar{f_{ik}}(m) f_{jl}(m) << ds'_k, ds'_l >>' (m) \text{ (by (7.2.4)} \\ &= \sum_{kl} \bar{c_k}' d'_l << ds'_k, ds'_l >>' (m) \end{aligned}$$

□

Proof of Theorem 7.2.2:
Now we can define a new inner product on the manifold M. For that let $v, w \in T^*_m(M)$ by (i) of Lemma 7.2.7 we choose $s_1, ..., s_n \in \mathcal{A}$ such that $ds_1(m), ..., ds_n(m)$ is a basis for $T^*_m(M)$. Let $\{c_i, d_i : i = 1, ..., n\}$ be such that $v = \sum_i c_i ds_i(m)$ and $w = \sum_i d_i ds_i$. Then we define

$$< v, w >' := \sum_{i,j} \bar{c_i} d_j << ds_i, ds_j >>' (m).$$

It is evident that this is a semi definite inner product. We have to show that this is a positive definite inner product. To that end let $< v, v >' = 0$ i.e.

$$\sum_{i,j} \bar{c_i} c_j << ds_i, ds_j >>' (x) = 0,$$

where $v = \sum_i c_i ds_i(x) \in T^*_x(M)$. Since the Haar state h is faithful on $\mathcal{Q}_0$ and by assumption $<< d\alpha(ds_i), d\alpha(ds_j) >> \in \Omega^1(\mathcal{A}) \otimes \mathcal{Q}_0$, we can deduce that

$$\sum_{i,j} \bar{c_i} c_j ((\mathrm{id} \otimes m)(\mathrm{id} \otimes \kappa \otimes \mathrm{id})(\alpha \otimes \mathrm{id}) << d\alpha(ds_i), d\alpha(ds_j) >>)(x) = 0.$$

Since $\epsilon \circ \kappa = \epsilon$ on $\mathcal{Q}_0$, applying $(\epsilon \otimes \epsilon)$ to the above equation, we get

$$\sum_{i,j} \bar{c_i} c_j ((\mathrm{id} \otimes m)(\mathrm{id} \otimes \epsilon \otimes \epsilon)(\alpha \otimes \mathrm{id}) << d\alpha(ds_i), d\alpha(ds_j) >>)(x) = 0.$$

Using the fact that ϵ is $*$-homomorphism we get

$$\sum_{i,j} \bar{c_i} c_j < \epsilon(d\alpha(ds_i)(x)), \epsilon(d\alpha(ds_j)(x)) >= 0.$$

It is easy to see that $\epsilon(d\alpha(ds_i)(x)) = ds_i(x)$ for all i. Hence we conclude that

$$< \sum_i c_i ds_i(x), \sum_i c_i ds_i(x) >= 0,$$

i.e $< v, v >= 0$ and hence $v = 0$ (as $< \cdot, \cdot >$ is strictly positive definite, being an inner product on $T_x^* M$) so that $< \cdot, \cdot >'$ is indeed strictly positive definite, i.e. inner product.

We have already noted ((ii) of Lemma 7.2.7) that our definition is independent of choice of s_i's, and also that with respect to this new Riemannian structure on the manifold, α is inner product preserving. This completes the proof of the Theorem 7.2.2 on $\Omega^1(\mathcal{A})$ and hence on $\Omega^1(C^\infty(M))$. □

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