Semimartingales and Stochastic Partial Differential Equations in the space of Tempered Distributions

Suprio Bhar

Indian Statistical Institute

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Supervisor: Prof. B. Rajeev

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Dedicated

to

My Parents and Teachers

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Contents

CHAPTER 1

Introduction

The study of stochastic differential equations (SDEs) and the semimartingales that arise as the solutions of these equations is central in the subject of stochastic analysis. The last 80 years have seen the birth and the subsequent growth of this subject. Topics such as stochastic flows ([70]), evolution equations ([20, 65]), stochastic filtering theory ([24, 37, 58, 68, 83]), stochastic control theory ([4, 17]) have given tremendous impetus to understand SDEs.

The development of SDEs in Euclidean spaces primarily centered on the properties of the diffusion coefficients and the drift terms. Classical results on the existence and uniqueness of the solutions of these equations are based on coefficients which are Lipschitz continuous. It is well-known that locally Lipschitz coefficients lead to solutions with possible explosions. Notions of weak and strong solutions, related semigroups and corresponding infinitesimal generators have yielded rich results. Most of these results are well-understood and the following texts give an idea of these basic results ([21, 46, 50, 54–56, 60, 74, 82, 87, 93, 107]). Extensions of these results dealt with processes which have jumps, like Lévy processes ([3, 71, 100]) and with processes which have more general state spaces. Such extensions include the notion of semimartingales ([27, 56, 74]) and general Markov processes ([14, 94, 95, 102, 105, 106]) and these have also been topics of research in their own right.

These developments in the theory of SDEs have taken place with a finite dimensional (Euclidean) state space. But the development of stochastic partial differential equations (SPDEs) has required an extension of this theory to infinite dimensions and in particular to suitable Hilbert spaces. The following books and monographs give some idea of the different directions that have been studied ([20, 22, 23, 26, 40, 59, 63, 64, 74, 117, 120]). This thesis is concerned with some mathematical problems that arise when an Itô type SDE is formulated as an SPDE driven by the same Brownian motion. In the rest of the introduction, we give an overview of results leading to such a formulation.

1.1 Stochastic partial differential equations

The subject of SPDEs is a relatively recent development. This subject was already present in the theory of stochastic flows in an 'embryo' form (see [70, Chapter 6]). While the classical theory of SDEs dealt with the time evolution of a single particle in a diffusive medium and its variants thereof in filtering and control theory, several applications involving deterministic systems perturbed by noise, as described for example in [117], required the incorporation of a spatial parameter into the SDE model to describe the effects of the spatial dependence of the noise as well as to model the evolution of a system of particles. SPDEs have emerged as a variant of the classical SDE model, incorporating features like the spatial dependence of the noise mentioned above. One of the distinctive features is an extension of the classical PDE results and techniques to situations where the physical system described by a PDE is now subject to random disturbances, modelled by the addition of a noise term analogous to the manner in which an Itô SDE is the perturbation of an ODE by a diffusion term involving Brownian motion or other types of noise like Lévy processes. A typical example is the stochastic heat equation [23, pp. 27-40]. Another recent application to Navier-Stokes equations was considered in [103]. While the SPDE model has drawn attention to the possibilities of a rigorous mathematical formulation of hitherto intractable physical models, like the KPZ equation ([43]), the connections of these models with the 'classical' diffusion models of Itô ([52]) or Stroock and Varadhan ([107]) have been less well researched. On the other hand a class of stochastic processes called Super processes ([15, 25, 29, 35, 72, 118]) that describe the evolution of a system of (interacting) particles are more explicitly modelled on the classical diffusion model (or more generally motion in a Markovian set up) and are at the same time less well described by SPDE models (see however [62, 120]).

One way of bridging the gap between the SPDE models and classical diffusion theory is to recast the equations of classical diffusive motion in the framework of SPDEs. This was done using the Itô formula as the principle tool, first in a series of papers [111–115] and later from a somewhat different perspective in another set of papers [88–92]. Both approaches used the framework of distributions to formulate the problem. The differences in the two approaches arose in the techniques used. To proceed further we have to consider the framework of distribution theory in which many of the results of SPDEs are formulated.

1.2 Random processes taking values in the space of Distributions

The development of the stochastic Calculus of variations (Malliavin Calculus, [81]) and White noise theory ([47]) had already made the theory of distributions due to L. Schwartz ([101]) an important tool in the study of stochastic processes. Further, given the fact that SPDEs involved both techniques from PDE and those dealing with spatially dependent noise, it is perhaps natural that the theory of distributions is of import in this subject. Two strands of the theory of distribution valued processes directly feed into the topic of this thesis, viz. the theory of $\mathcal{S}'(\mathbb{R}^d)$ - the space of tempered distributions (or more generally countably Hilbertian) - valued processes as developed in ([53, 59]) and certain analytic techniques like the 'Monotonicity inequality' ([65]) whose antecedents lie in the study of a class of SPDEs with solutions in certain Hilbert spaces that are Sobolev spaces ([65, 83]). In [53], Itô developed a theory of random processes taking values in $\mathcal{S}'(\mathbb{R}^d)$ or $\mathcal{D}'$ ([41]). This was further developed in [59]. The main advantage in this framework is that we are able to use the well developed theory of stochastic integration in Hilbert spaces ([74]) and at the same time deal with general $\mathcal{S}'(\mathbb{R}^d)$ or $\mathcal{D}'$ valued processes. Yet the techniques developed in [22] or [74] for solving SDEs or SPDEs in a single Hilbert space are insufficient for dealing with equations where the solutions take values in a single Hilbert space whereas the equations hold in a different space. As mentioned above, one needs here certain analytic techniques like the Monotonicity inequality to prove existence and uniqueness results. The Monotonicity inequality is a close relative of the so called coercivity inequality developed in [83] to prove existence and uniqueness results for stochastic evolution equations in the framework of a triple of Hilbert spaces (see [96]). It is used in [65] to prove uniqueness results for SPDEs. If the operators (A, L) respectively corresponding to the diffusion and drift terms in an SPDE viz.

$$dY_t = A(Y_t).\, dB_t + L(Y_t)\, dt \tag{1.1}$$

satisfy this inequality in a suitable Hilbert space, then pathwise uniqueness holds for this equation. It is to be noted that such techniques for proving existence and uniqueness are not available if one is dealing with equations as above in $\mathcal{S}'(\mathbb{R}^d)$ directly as in [112] without using its countable Hilbertian structure.

The problem of developing an SPDE framework for the classical diffusion models of Itô-Stroock-Varadhan can now be reformulated in the countable Hilbertian framework of $\mathcal{S}'(\mathbb{R}^d)$. Given the fact that for a finite dimensional diffusion $\{X_t\}$ its law is in some sense determined by Itô's formula via a martingale formulation, an important first step is to identify $\{X_t\}$ with the $\mathcal{S}'(\mathbb{R}^d)$ valued process $\{\delta_{X_t}\}$ via Itô's formula. This was done in

[89, 112]. It was shown in [89] that this can actually be done in the countable Hilbertian framework of $\mathcal{S}'(\mathbb{R}^d)$, arriving at an equation for $Y_t = \delta_{X_t}$ in the above form. As noted earlier the special feature of such equations is that the process $\{Y_t\}$ takes values in one of the Hermite-Sobolev spaces that constitute $\mathcal{S}'(\mathbb{R}^d)$, viz. $\mathcal{S}_p(\mathbb{R}^d)$ for some $p \in \mathbb{R}$ whereas the equation holds in a different (larger) space $\mathcal{S}_{p-1}(\mathbb{R}^d) \supset \mathcal{S}_p(\mathbb{R}^d)$. These are real separable Hilbert spaces (see [53]). It was also noted in [89] that $\delta_{X_t} = \tau_{X_t}(\delta_0)$ (where $\tau_x, x \in \mathbb{R}^d$ denotes the translation operators, see Example 2.11.6) and in this form the results could be stated for a general tempered distribution $\phi \in \mathcal{S}'(\mathbb{R}^d)$ in the form $\tau_{X_t}(\phi)$ (see also [112] for an expression for $\tau_{X_t}(\phi)$ in the equivalent form given by the convolution $\phi * \delta_{X_t}$). The SPDE (1.1) was solved in the special (linear) case when A and L were constant coefficient differential operators in [38] using the fact that the pair (A, L) satisfied the Monotonicity inequality, which was shown separately in [39]. The formulation of the problem in terms of the translation operators $\tau_x, x \in \mathbb{R}^d$ opened the way for applying analytic techniques based on the boundedness of these operators on the Hilbert spaces $\mathcal{S}_p(\mathbb{R}^d)$ ([91]) and for interpreting the expected value $\mathbb{E}(\tau_{X_t}\phi)$ as the convolution with the heat kernel, viz. $\phi * p_t$ when $\{X_t\}$ is a d-dimensional Brownian motion. In particular, these provide a stochastic representation of the well-known solutions of the heat equation (also see [5, Chapter II, (4.14) Theorem]).

These results were extended in [92] to the case of variable coefficients with heat equation for the Laplacian being replaced with the forward equation for the diffusion $\{X_t\}$. The results of [92] also provided an SPDE for stochastic flows generated by an Itô type SDE, where a solution of the SPDE was built up using the 'fundamental solutions' $\{\delta_{X_t^x}\}$, $\{X_t^x\}$ being the solution of the SDE generating the flow. More recently, it was shown in [90] that solutions of SPDE (1.1), with L - a non-linear second order elliptic operator arising in the diffusion theory and A - a suitable 'square root of $-L$', arise as the translations of the initial value $Y_0 (= y$, say $\in \mathcal{S}'(\mathbb{R}^d))$ by a process $\{Z_t(y)\}$ satisfying a finite dimensional SDE, i.e. $Y_t = \tau_{Z_t(y)}(y)$. Here the Monotonicity inequality plays an important role. The results of [90] also provide a notion of non-linear convolution needed to make sense of the non-linear evolution equation that arises on taking expectations in (1.1) analogous to the manner in which the usual notion of convolution appears in the solution of the heat equation for the Laplacian.

1.3 Some salient features of our methods

In this section, we describe certain technical aspects of the ideas mentioned in the previous sections. This thesis focuses on processes which take values in the countably Hilbertian Nuclear space $\mathcal{S}(\mathbb{R}^d)$ (the space of real valued rapidly decreasing smooth functions on $\mathbb{R}^d$)

or its dual $\mathcal{S}'(\mathbb{R}^d)$ and we use a 'Hilbert space approximation' to $\mathcal{S}'(\mathbb{R}^d)$, viz. we work with processes taking values in the Hermite Sobolev spaces $\mathcal{S}_p(\mathbb{R}^d)$, which are completions of $\mathcal{S}(\mathbb{R}^d)$ in the Hilbertian norms $\|\cdot\|_p$. We can describe the countably Hilbertian topology on $\mathcal{S}(\mathbb{R}^d)$ via these norms, which allows us to use the machinery from the theory of stochastic integration on Hilbert spaces to SPDEs in $\mathcal{S}'(\mathbb{R}^d)$.

Note that differentiation and multiplication by polynomials (more generally, multiplication by smooth functions) are standard operations on $\mathcal{S}'(\mathbb{R}^d)$ and these are basic constituents in the differential operators that one uses. A technical difficulty then arises due to the fact that these differential operators are unbounded operators on a Hermite Sobolev space $\mathcal{S}_p(\mathbb{R}^d)$. One usually has to take larger spaces as the range of these operators, which will typically be another Hermite Sobolev space. As a consequence and as observed earlier, the following situation repeats in multiple scenario: an $\mathcal{S}_p(\mathbb{R}^d)$ valued process satisfying a SPDE in $\mathcal{S}_{p-1}(\mathbb{R}^d)$ - which is a larger space (e.g. see [90, 92]).

One approach in constructing $\mathcal{S}'(\mathbb{R}^d)$ valued processes as well as studying SPDEs in $\mathcal{S}'(\mathbb{R}^d)$ is via a 'lifting' of finite dimensional processes to processes taking values in some $\mathcal{S}_p(\mathbb{R}^d)$. This 'lifting' procedure is used in this thesis and we describe two methods below.

(I) The first method uses the duality of function spaces with its dual (e.g. $\mathcal{S}(\mathbb{R}^d)$ with $\mathcal{S}'(\mathbb{R}^d)$, $C^\infty(\mathbb{R}^d)$ - the space of real valued smooth functions on $\mathbb{R}^d$ - with $\mathcal{E}'(\mathbb{R}^d)$ - the space of compactly supported distributions on $\mathbb{R}^d$) and can be thought of as a 'linear' method. If the flow $\{X_t^x\}$ generated by Itô's SDE

$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt, \tag{1.2}$$

is smooth enough in the initial condition x, then we can evaluate smooth functions on this flow. For 'nice' functions ϕ, observe that the evaluation can be written in terms of a duality $\phi(X_t^x) = \left\langle \delta_{X_t^x}, \phi \right\rangle$ and this is where the identification of $\{X_t^x\}$ with $\{\delta_{X_t^x}\}$ becomes paramount. In [92], the composition yielded a continuous linear map $X_t(\omega) : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$. Then using the dual map $X_t^*(\omega) : \mathcal{E}'(\mathbb{R}^d) \to \mathcal{E}'(\mathbb{R}^d)$, one generates distribution valued processes from the range of X_t^*. Since $\mathcal{E}'(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$, the processes generated via this method are also $\mathcal{S}'(\mathbb{R}^d)$ valued. We use this method in Chapter 4 to obtain results similar to [92].

(II) The second method involves translation operators $\tau_x, x \in \mathbb{R}^d$ on $\mathcal{S}'(\mathbb{R}^d)$ (Example 2.11.6) and can be thought of as a 'non-linear' method. The process $\{\tau_{X_t}\phi\}$ is an $\mathcal{S}_p(\mathbb{R}^d)$ valued process, where $\{X_t\}$ is an $\mathbb{R}^d$ valued process and $\phi \in \mathcal{S}_p(\mathbb{R}^d)$. In an Itô formula [89, Theorem 2.3], it was shown that $\{\tau_{X_t}\phi\}$ is a continuous semimartingale, if $\{X_t\}$ is so. As noted in the previous section, this method led to a correspondence ([90]) between a class of finite dimensional SDEs and a class of SPDEs in $\mathcal{S}'(\mathbb{R}^d)$ with deterministic initial condition in some $\mathcal{S}_p(\mathbb{R}^d)$. We use this method in Chapter 5 to extend the results in [90] to random initial conditions. In Chapter 6, we extend the Itô formula [89, Theorem 2.3] where $\{X_t\}$ is a semimartingale with jumps.

Another approach in the construction (and also to understand the properties) of $\mathcal{S}'(\mathbb{R}^d)$ valued random variables and processes - more generally processes taking values in the dual of Nuclear spaces - uses the technique of 'regularization' of random linear functionals on Nuclear spaces ([53, 57, 59, 77–80, 85]). We do not use this technique in this thesis; however some comments regarding this technique and our work have been made in Remark 6.3.4.

1.4 A chapter-wise summary

Unless stated otherwise, $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ will be a filtered complete probability space satisfying the usual conditions and $\{B_t\}$ a d dimensional $(\mathcal{F}_t)$ standard Brownian motion.

In **Chapter 2**, we recall basic results from analysis and the theory of stochastic processes. First we cover functions of bounded variation and Bochner integration in Sections 2 and 3 and then go on to list definitions and basic results related to real and Hilbert valued processes in Sections 4,5,6 and 7. In Section 9, the Schwartz topology on $\mathcal{S}(\mathbb{R}^d)$ ([110, Chapter 25], [98, Chapter 7, Section 3], [36, Chapter 8]) is described. In Section 10 we describe a countably Hilbertian Nuclear topology on $\mathcal{S}(\mathbb{R}^d)$ ([41, Chapter 1 Appendix], [53, Chapter 1.3]), which coincides with the Schwartz topology ([89, Proposition 1.1]. We also define the Hermite Sobolev spaces, denoted by $\mathcal{S}_p(\mathbb{R}^d)$, indexed by real numbers p ([53, Chapter 1.3]). Using the properties of the Hermite functions described in Section 8, we list examples of tempered distributions and operators on $\mathcal{S}'(\mathbb{R}^d)$ (and in particular, on $\mathcal{S}_p(\mathbb{R}^d)$) in Section 11. Section 12 covers results on stochastic integration tailored to $\mathcal{S}_p(\mathbb{R}^d)$ valued predictable integrands. Sections 13 and 14 contain some inequalities and results from semigroup theory, respectively.

In **Chapter 3**, we prove the Monotonicity inequality for $(A = (A_1, \cdots, A_r), L)$ in $\|\cdot\|_p$ in two different settings.

(i) In **Section 3**, we prove the inequality for constant coefficient differential operators (Theorem 3.3.1) given by

$$A_i = -\sum_{j=1}^{d} \sigma_{ji}\partial_j, \quad L = \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^t)_{ij}\partial_{ij}^2 - \sum_{i=1}^{d} b_i\partial_i.$$

This result was already proved in [39, Theorem 2.1]. We give a new proof, which involves a simplified computation via an identification of the adjoint of the operators $\partial_i, i = 1, \cdots, d$ on $\mathcal{S}_p(\mathbb{R}^d)$ as a sum $-\partial_i + T_i$ where T_i is a bounded linear operator on $\mathcal{S}_p(\mathbb{R}^d)$ (see Theorem 3.2.2).

(ii) In **Section 4**, we consider the inequality when the operators A, L contain variable coefficients, i.e. for

$$A_i\psi := -\sum_{k=1}^{d} \partial_k\left(\sigma_{ki}\psi\right), \forall \psi \in \mathcal{S}'(\mathbb{R}^d)$$

and

$$L\psi := \frac{1}{2}\sum_{i,j=1}^{d} \partial_{ij}^2\left((\sigma\sigma^t)_{ij}\psi\right) - \sum_{i=1}^{d}\partial_i\left(b_i\psi\right), \forall \psi \in \mathcal{S}'(\mathbb{R}^d)$$

where $\sigma_{ij}, b_i, 1 \leq i, j \leq d$ are smooth functions with bounded derivatives. This inequality was used in [92] to prove the uniqueness of the solution of the Cauchy problem for L as above. We prove the inequality when σ is a real $d \times d$ matrix and $b(x) := \alpha + Cx, \forall x \in \mathbb{R}^d$ with $\alpha \in \mathbb{R}^d$ and $C = (c_{ij})$ is a real $d \times d$ matrix (see Theorem 3.4.2). The proof is similar to that of Theorem 3.3.1 and uses the identification of the adjoint of a multiplication operator on the Hermite Sobolev spaces (see Theorem 3.4.1).

An important step in the proof shows the existence of some bilinear forms on $\mathcal{S}_p(\mathbb{R}^d)$. For example, we prove that the map $(\phi, \psi) \mapsto \langle \partial_i\phi\, , T_j\psi\rangle_p$ is a bounded bilinear form on $(\mathcal{S}(\mathbb{R}^d), \|\cdot\|_p) \times (\mathcal{S}(\mathbb{R}^d), \|\cdot\|_p)$ and hence extends to a bounded bilinear form on $\mathcal{S}_p(\mathbb{R}^d) \times \mathcal{S}_p(\mathbb{R}^d)$ (see Lemma 3.2.5, Theorem 3.4.1).

In **Chapter 4 Section 2**, we introduce and characterize a class of diffusions - that depend deterministicically on the initial condition - given by Itô's SDE (1.2) with Lipschitz coefficients, such that the general solution is the sum of the solution starting at 0 and the value of a deterministic function at the initial condition (see Definition 4.2.1). We show, under 'nice' conditions (Proposition 4.2.5, Theorem 4.2.4) that these diffusions correspond to the coefficients given as follows.

(i) σ is a real $d \times d$ matrix.
(ii) $b(x) := \alpha + Cx, \forall x \in \mathbb{R}^d$ where $\alpha \in \mathbb{R}^d$ and $C = (c_{ij})$ is a real $d \times d$ matrix.

These coefficients generate Gaussian flows and hence the above correspondence can be taken as characterization results on Gaussian flows in the class of flows that arise as the strong solutions of an Itô stochastic differential equation with smooth or Lipschitz coefficients and driven by a Brownian motion $\{B_t\}$.

In **Section 3**, continuing with these coefficients σ and b, we define continuous linear maps $X_t(\omega) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ (Lemma 4.3.4) and the corresponding adjoints $X_t^*(\omega) : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$. For any $\psi \in \mathcal{L}^1(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$, we define an $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued (for an appropriate p) continuous adapted process $\{Y_t(\psi)\}$ with two properties, viz

(i) $Y_t(\psi) = X_t^*(\psi)$ (see equation (4.17)).

(ii) $\{Y_t(\psi)\}$ solves the following equation in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$, a.s. (see Theorem 4.3.8)

$$Y_t(\psi) = \psi + \int_0^t A^*(Y_s(\psi))\,.dB_s + \int_0^t L^*(Y_s(\psi))\,ds,\ \forall t \geq 0.$$

Taking expectation on both sides of the previous equation, we show that $\overline{\psi}(t) := \mathbb{E}Y_t(\psi)$ solves the Cauchy problem for L^* with the initial condition $\psi \in \mathcal{L}^1(\mathbb{R}^d)$. Using Monotonicity inequality for (A^*, L^*) (Theorem 3.4.2), we show that both these solutions are unique. These results are motivated by the results in [92].

In **Chapter 5 Section 2**, we extend the correspondence obtained in [90] to allow random initial conditions for Y in SPDE (1.1). Let ξ be an $\mathcal{S}_p(\mathbb{R}^d)$ valued, $\mathcal{F}_0$ measurable, square integrable (independent of $\{B_t\}$) random variable. Let $(\mathcal{F}_t^\xi)$ denote the right continuous, complete filtration generated by ξ and $\{B_t\}$. Then under 'nice' conditions, the SPDE

$$dY_t = A(Y_t).\,dB_t + L(Y_t)\,dt; \quad Y_0 = \xi \tag{1.3}$$

has a unique $\mathcal{S}_p(\mathbb{R}^d)$ valued $(\mathcal{F}_t^\xi)$ adapted strong solution given by $Y_t = \tau_{Z_t}(\xi)$, $t \geq 0$ (see Theorem 5.2.15) where $\{Z_t\}$ solves the SDE

$$dZ_t = \bar{\sigma}(Z_t;\xi).\,dB_t + \bar{b}(Z_t;\xi)\,dt; \quad Z_0 = 0.$$

Note that $A, L, \bar{\sigma}, \bar{b}$ are defined in terms of $\sigma, b \in \mathcal{S}_{-p}(\mathbb{R}^d)$. The hypothesis requires a certain 'globally Lipschitz' nature of the coefficients, which depends on ξ. This 'globally Lipschitz' condition can be further relaxed to a 'locally Lipschitz' condition (Theorem 5.2.20).

In **Chapter 5 Section 3**, we construct stationary solutions of the infinite dimensional SPDE (1.3). Given a stationary solution, say $\{Z_t\}$, of some finite dimensional SDE, we identify a subset $\mathcal{C}$ (see equation (5.33)) of $\mathcal{S}_p(\mathbb{R}^d)$, which allows the 'lifting' of $\{Z_t\}$ (Theorem 5.3.4). This technique has been applied to Example 5.3.5 and Example 5.3.8.

In **Chapter 6**, we prove the following Itô formula: Let $p \in \mathbb{R}$. Given $\phi \in \mathcal{S}_{-p}(\mathbb{R}^d)$ and an $\mathbb{R}^d$ valued semimartingale $X_t = (X_t^1, \cdots, X_t^d)$, we have the equality in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$, a.s.

$$\begin{aligned}\tau_{X_t}\phi = \tau_{X_0}\phi - \sum_{i=1}^d \int_0^t \partial_i \tau_{X_{s-}}\phi\, dX_s^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial_{ij}^2 \tau_{X_{s-}}\phi\, d[X^i, X^j]_s^c \\ + \sum_{s\leq t}\left[\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^d (\triangle X_s^i\, \partial_i \tau_{X_{s-}}\phi)\right],\ t \geq 0,\end{aligned}$$

where $\triangle X$ denotes the jump of X (Theorem 6.2.3). If X is continuous, then the result follows from [89, Theorem 2.3]. We apply the Itô formula to a one-dimensional process X, which solves an SDE driven by a Lévy process and show the existence of a solution of a stochastic 'partial' integro-differential equation in the Hermite-Sobolev spaces (Theorem 6.3.1). This is similar to the solution obtained in [90] for continuous processes X. An identification of the local time process of a real valued semimartingale as an $\mathcal{S}'$ valued process is presented in Proposition 6.3.3.

We provide a list of publications (including preprints) which constitute the material of this thesis and a bibliography of books, monographs and research articles which have been referenced. A list of commonly used symbols, an index of terms and topics have been added at the end. We refer to a result due to Burkholder, Davis, Gundy (Theorem 2.5.28) as 'BDG inequalities'.

CHAPTER 2

Preliminaries

2.1 Introduction

In this chapter, we recall basic results from analysis and the theory of stochastic processes - which we use in this thesis. Our requirement in the context of stochastic integration with Hilbert valued (specifically those taking values in a Hermite-Sobolev space) processes amounts to integrating Hilbert valued predictable processes with respect to real semimartingales. While the stochastic integration of Hilbert valued predictable processes with respect to Hilbert valued Wiener processes and cylindrical Wiener processes ([22, 40]) or the stochastic integration of Hilbert valued predictable processes with respect to Hilbert valued semimartingales ([74]) are well-known, we have been unable to locate any reference in the literature that precisely deals with our requirement. We do not require the full generality (as in [74]) in which the results in the theory of Hilbert valued stochastic integration are proved. We prove well-known results of stochastic integration in this context, starting from the basic principles and this topic covers a major portion of this chapter.
Definitions and necessary results on real valued functions of bounded variation and Bochner integration are covered in Sections 2 (we refer to [2]) and 3 (we refer to [105, pp. 267-271]) respectively. In Section 4, we recall of filtrations and stochastic processes. Section 5 and Section 6 contain results on real valued stochastic processes and Section 7 is about Hilbert valued processes. For these sections, we refer to [27, 55, 56, 60, 74, 82, 87, 93].
In this thesis, we deal with processes taking values in the space of tempered distributions (denoted by $\mathcal{S}'(\mathbb{R}^d)$), in particular in an Hermite Sobolev space. In Section 8 we recall properties of the Hermite functions ([47, 51, 108, 109]). Two sections, viz. Section 9 and Section 10 are devoted to study the Schwartz topology ([110, Chapter 25], [98, Chapter 7, Section 3], [36, Chapter 8]) and a countably Hilbertian Nuclear topology ([53, Chapter 1.3], [41, Chapter 1 Appendix]) on the space of rapidly decreasing smooth functions on $\mathbb{R}^d$, denoted by $\mathcal{S}(\mathbb{R}^d)$. The fact that these two topologies coincide is well-known ([89, Proposition 1.1]).

Examples of tempered distributions and some operators on the Hermite Sobolev spaces are covered in Section 11. Some of the computations viz. Lemma 2.11.16, Lemma 2.11.20, Example 2.11.22 and Example 2.11.25 might be new. In Section 12, we restate some results from Section 7 for integrands taking values in the Hermite Sobolev spaces. Section 13 contains two basic inequalities including the Gronwall's inequality (Lemma 2.13.1). In Section 14 we cover two examples of semigroups of bounded linear operators, using the terminology and notations from [84, Chapter 1].

2.2 Functions of bounded variation

Let $a, b \in \mathbb{R}$ with $a < b$.

Definition 2.2.1 ([2, Definition 6.4])**.** A set of points $\mathbb{P} = \{x_0, x_1, \cdots, x_n\}$ satisfying the inequalities

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b,$$

is called a partition of $[a, b]$. The collection of all possible partitions of $[a, b]$ will be denoted by $\mathcal{P}[a, b]$.

We may write $\mathbb{P} = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$ to denote a partition of $[a, b]$.

Definition 2.2.2. (i) ([2, Definition 6.4]) Let f be a real valued function on $[a, b]$. If $\mathbb{P} = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$ is a partition of $[a, b]$, write $\triangle f_k = f(x_k) - f(x_{k-1})$ for $k = 1, 2, \cdots, n$. If there exists a positive real number M such that $\sum_{k=1}^{n} | \triangle f_k| \leq M$ for all partitions of $[a, b]$, then f is said to be of bounded variation on $[a, b]$. We denote the sum $\sum_{k=1}^{n} | \triangle f_k|$ by $Var(\mathbb{P}, f)$.

(ii) ([2, Definition 6.8]) Let f be of bounded variation on $[a, b]$. The real number $\sup\{Var(\mathbb{P}, f) : \mathbb{P} \in \mathcal{P}[a, b]\}$ is called the total variation of f on the interval $[a, b]$. We denote this supremum by $Var_{[a,b]}(f)$.

Theorem 2.2.3. *Let f be of bounded variation on $[a, b]$.*

(i) ([2, Theorem 6.11]) Let $c \in [a, b]$. Then f is of bounded variation on $[a, c]$ and on $[c, b]$ and we have

$$Var_{[a,b]}(f) = Var_{[a,c]}(f) + Var_{[c,b]}(f).$$

(ii) ([2, Theorem 6.12]) Let V be defined on $[a, b]$ as follows:

$$V(x) := \begin{cases} Var_{[a,x]}(f) \; if a < x \leq b \\ 0 \; if x = a. \end{cases}$$

Then V and $V - f$ are non-decreasing functions on $[a, b]$. Of course f is the difference of V and $V - f$.

(iii) ([2, Theorem 6.14]) Let V be as in (ii). Then

a) If f is right continuous on $[a,b)$, then so is V. The converse is also true.

b) If f is left continuous on $(a,b]$, then so is V. The converse is also true.

c) Every point of continuity of f is also a point of continuity of V. The converse is also true.

Theorem 2.2.4 (([2, Theorem 6.13]))**.** *Let $f:[a,b]\to\mathbb{R}$ be a function. Then f is bounded variation on $[a,b]$ if and only if f can be expressed as the difference of two increasing functions.*

The next result is well-known and we state it without proof.

Proposition 2.2.5. *Let $f:[0,\infty)\to\mathbb{R}$ be a function such that for any $t>0$, f is of bounded variation on $[0,t]$. Assume that f is right continuous. Then*

$$Var_{[0,t]}(f)=\sup_{m\geq 1}\sum_{k=1}^{2^m}\left|f\left(\frac{tk}{2^m}\right)-f\left(\frac{t(k-1)}{2^m}\right)\right|,\ \forall t\in[0,\infty).$$

2.3 Bochner integration

In this subsection, we recall basic results on Bochner integration. Our main reference for this subsection is [105, pp. 267-271].
Let μ be an arbitrary non-negative measure on a measurable space $(\Omega,\mathcal{F})$. Let $(\mathbb{B},\|\cdot\|)$ be a real separable Banach space with dual $\mathbb{B}'$.

Definition 2.3.1. (i) A function $X:\Omega\to\mathbb{E}$ is said to be μ-simple if X is $\mathcal{F}$ measurable, $\mu(X\neq 0)<\infty$ and X takes on only a finite number of distinct values.

(ii) Given a μ-simple function f, its integral with respect to μ is the element of $\mathbb{B}$ given by

$$\mathbb{E}^\mu[X]=\int_\Omega X(\omega)\,\mu(d\omega):=\sum_{x\in\mathbb{B}\setminus\{0\}}\mu(X=x)x.$$

Another description of $\mathbb{E}^\mu[X]$ is as the unique element of $\mathbb{B}$ with the property that

$$\langle\mathbb{E}^\mu[X]\,,\,\lambda\rangle=\mathbb{E}^\mu[\langle X\,,\,\lambda\rangle],\ \forall\lambda\in\mathbb{B}'.$$

(iii) If $X:\Omega\to\mathbb{B}$ is $\mathcal{F}$ measurable, then so is $\omega\in\Omega\mapsto\|X(\omega)\|\in\mathbb{R}$. We say X is μ-integrable if $\mathbb{E}^\mu[\|X\|]<\infty$ and we say X is μ-locally integrable if $\mathbb{1}_AX$ is μ-integrable for every $A\in\mathcal{F}$ with $\mu(A)<\infty$. The space of $\mathbb{B}$ valued μ-integrable functions will be denoted by $\mathcal{L}^1(\mu;\mathbb{B})$.

Theorem 2.3.2 ([105, Lemma 5.1.20])**.** *For each μ-integrable $X : \Omega \to \mathbb{B}$ there is a unique element $\mathbb{E}^\mu[X] \in \mathbb{B}$ such that*

$$\langle \mathbb{E}^\mu[X] \, , \, \lambda \rangle = \mathbb{E}^\mu[\langle X \, , \, \lambda \rangle], \, \forall \lambda \in \mathbb{B}'.$$

The mapping $X \in \mathcal{L}^1(\mu; \mathbb{B}) \mapsto \mathbb{E}^\mu[X] \in \mathbb{B}$ is linear and satisfies

$$\|\mathbb{E}^\mu[X]\| \leq \mathbb{E}^\mu[\|X\|].$$

Finally there exists a sequence $\{X_n\}$ of $\mathbb{B}$ valued μ-simple functions with the property that $\mathbb{E}^\mu[\|X_n - X\|] \xrightarrow{n \to \infty} 0$.

Theorem 2.3.3 ([105, Theorem 5.1.22])**.** *Let $(\Omega, \mathcal{F}, \mu)$ be a σ finite measure space and $X : \Omega \to \mathbb{B}$ a μ-locally integrable function. Then*

$$\mu(X \neq 0) = 0 \iff \mathbb{E}^\mu[\mathbb{1}_A X] = 0, \, \forall A \in \mathcal{F}, \mu(A) < \infty.$$

Assume that $\mathcal{B}$ is a sub σ field such that μ restricted to $\mathcal{B}$ is σ finite. Then for each μ-locally integrable $X : \Omega \to \mathbb{B}$ there is a μ almost everywhere unique μ-locally integrable, $\mathcal{B}$ measurable function $X_\mathcal{B} : \Omega \to \mathbb{B}$ such that

$$\mathbb{E}^\mu[\mathbb{1}_A X_\mathcal{B}] = \mathbb{E}^\mu[\mathbb{1}_A X], \, \forall A \in \mathcal{B}, \mu(A) < \infty.$$

In particular, if $Y : \Omega \to \mathbb{B}$ is another μ-locally integrable function, then for all $\alpha, \beta \in \mathbb{R}$,

$$(\alpha X + \beta Y)_\mathcal{B} = \alpha X_\mathcal{B} + \beta Y_\mathcal{B}, \, (\mu - a.e.)$$

Finally, $\|X_\mathcal{B}\| \leq (\|X\|)_\mathcal{B}$ μ-a.e. and hence the mapping $X \in \mathcal{L}^1(\mu; \mathbb{B}) \mapsto X_\mathcal{B} \in \mathcal{L}^1(\mu; \mathbb{B})$ is a linear contraction.

We call the μ equivalence class of $X_\mathcal{B}$'s (obtained in the previous theorem) the μ conditional expectation of X given $\mathcal{B}$. In general, we ignore the distinction between the equivalence class and a representative of the class. The μ equivalence class may also be denoted by $\mathbb{E}^\mu[X|\mathcal{B}]$. If $X : \Omega \to \mathbb{B}$ is μ-locally integrable and $\mathcal{C}$ is a sub σ-field of $\mathcal{B}$, then we have

$$\mathbb{E}^\mu[X|\mathcal{C}] = \mathbb{E}^\mu[\mathbb{E}^\mu[X|\mathcal{B}]|\mathcal{C}], \, (\mu - \text{a.e.})$$

Also given any bounded real valued $\mathcal{B}$ measurable function Y on $(\Omega, \mathcal{F}, \mu)$ we have

$$\mathbb{E}^\mu[YX|\mathcal{B}] = Y\mathbb{E}^\mu[X|\mathcal{B}], \, (\mu - \text{a.e.})$$

2.4 Filtrations, Stopping times and Stochastic processes

We recall some definitions from basic probability theory.

Definition 2.4.1. (i) Let $(\Omega, \mathcal{F}, P)$ be a probability space. The P completion of $\mathcal{F}$ is defined to be the σ-field generated by $\mathcal{F}$ and $\mathcal{N}$, where $\mathcal{N}$ denotes the class of all subsets of P-null sets in $\mathcal{F}$.

(ii) The probability space $(\Omega, \mathcal{F}, P)$ is said to be complete if the P completion of $\mathcal{F}$ is $\mathcal{F}$ itself.

We take $[0, \infty)$ to be our time set. Note that we write $\forall t \geq 0$ to mean $\forall t \in [0, \infty)$.

Definition 2.4.2. (i) Given a probability space $(\Omega, \mathcal{F}, P)$, a filtration on $[0, \infty)$ is defined as a non-decreasing family of σ-fields $\mathcal{F}_t \subset \mathcal{F}$, $t \geq 0$. We denote the family by $(\mathcal{F}_t)$.

(ii) We say $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ is a filtered probability space if $(\mathcal{F}_t)$ is a filtration of $(\Omega, \mathcal{F}, P)$.

(iii) We say a filtration $(\mathcal{F}_t)$ is right continuous if

$$\mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t, \, \forall t \geq 0.$$

(iv) For any $t \in (0, \infty)$, $\mathcal{F}_{t-}$ will denote the σ field generated by $\bigcup_{s<t} \mathcal{F}_s$. We also take $\mathcal{F}_{0-} := \mathcal{F}_0$. $\mathcal{F}_\infty$ will denote the σ-field generated by the collection $\bigcup_{t\geq 0} \mathcal{F}_t$.

Definition 2.4.3. A filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ is said to satisfy the usual conditions if

(i) $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$.

(ii) The filtration $(\mathcal{F}_t)$ is right continuous.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space. Let $\bar{\mathcal{F}}$ denote the P completion of $\mathcal{F}$ and put $\mathcal{N} := \{A \in \bar{\mathcal{F}} : P(A) = 0\}$. Define $\bar{\mathcal{F}}_t := \sigma\{\mathcal{F}_t, \mathcal{N}\}, t \geq 0$, i.e. the σ field generated by $\mathcal{F}_t$ and $\mathcal{N}$.

Lemma 2.4.4 ([56, Lemma 6.8])**.** *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space.*

(i) $\bar{\mathcal{F}}_{t+} = \overline{\mathcal{F}_{t+}}$ for all $t \geq 0$.

(ii) The filtration $(\bar{\mathcal{F}}_{t+})$ is the smallest right continuous and complete extension of $(\mathcal{F}_t)$.

Let $\mathbb{B}$ be a real separable Banach space with norm $\|\cdot\|$. Let $\mathcal{B}(\mathbb{B})$ denote the Borel σ field on $\mathbb{B}$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered complete probability space satisfying the usual conditions.

Definition 2.4.5. (i) We say $X = \{X_t : t \in [0, \infty)\}$ is a $\mathbb{B}$ valued stochastic process if X_t is a $\mathbb{B}$ valued $\mathcal{F}/\mathcal{B}(\mathbb{B})$ measurable random variable for each $t \in [0, \infty)$.

(ii) We say a stochastic process $\{X_t\}$ is $(\mathcal{F}_t)$ adapted if X_t is $\mathcal{F}_t/\mathcal{B}(\mathbb{B})$ measurable for all $t \in [0, \infty)$.

(iii) The stochastic process $\{X_t\}$ is called measurable, if the mapping

$$(t, \omega) \mapsto X_t(\omega) : ([0, \infty) \times \Omega, \mathcal{B}([0, \infty)) \otimes \mathcal{F}) \to (\mathbb{B}, \mathcal{B}(\mathbb{B}))$$

is measurable, where $\mathcal{B}([0, \infty))$ denotes the Borel σ-field on $[0, \infty)$.

(iv) Let $\{X_t\}$ be a stochastic process. A stochastic process $\{Y_t\}$ is said to be a modification of $\{X_t\}$ if

$$P(X_t = Y_t) = 1, \, \forall t \in [0, \infty).$$

(v) We say $\{X_t\}$ has continuous (respectively rcll) paths if a.s. the paths $t \mapsto X_t(\omega)$ are continuous functions (respectively right continuous function with left limits). We say $\{X_t\}$ is a continuous (respectively rcll) process if it has continuous (respectively rcll) paths.

(vi) A stochastic process $\{X_t\}$ is said to have a continuous (respectively rcll) modification if there exists a stochastic process $\{Y_t\}$ with continuous (respectively rcll) paths and

$$P(X_t = Y_t) = 1, \, \forall t \in [0, \infty).$$

(vii) Two stochastic processes $\{X_t\}$ and $\{Y_t\}$ are said to be indistinguishable if

$$P(X_t = Y_t, \, t \in [0, \infty)) = 1.$$

(viii) A stochastic process $\{X_t\}$ is progressively measurable if its restriction to $\Omega \times [0, t]$ is $\mathcal{F}_t \otimes \mathcal{B}([0, t])$ measurable for every $t \geq 0$, where $\mathcal{B}([0, t])$ denotes the Borel σ field on $[0, t]$. Such a process is $(\mathcal{F}_t)$ adapted.

Convention: Unless otherwise specified, we will assume the following:

(i) $\mathcal{F} = \mathcal{F}_\infty$.

(ii) The filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ is complete and satisfies the usual conditions.

(iii) Adapted processes will be with respect to the underlying filtration $(\mathcal{F}_t)$.

Definition 2.4.6. A $[0, \infty]$ valued random variable τ is said to be an $(\mathcal{F}_t)$ stopping time (or simply a stopping time if the filtration is understood from the context) if

$$(\tau \leq t) \in \mathcal{F}_t, \, \forall t \in [0, \infty).$$

Proposition 2.4.7. *(i) Let τ and σ be two $(\mathcal{F}_t)$ stopping times. Then* $\max\{\tau, \sigma\} = \tau \vee \sigma$ *and* $\min\{\tau, \sigma\} = \tau \wedge \sigma$ *are also $(\mathcal{F}_t)$ stopping times.*

(ii) Let $\{\tau_n\}$ be a sequence of $(\mathcal{F}_t)$ stopping times. Then $\sup_n \tau_n = \vee_n \tau_n$ *and* $\inf_n \tau_n = \wedge_n \tau_n$ *are also $(\mathcal{F}_t)$ stopping times.*

Definition 2.4.8. Let $\{X_t\}$ be an $(\mathcal{F}_t)$ adapted process and let τ be an $(\mathcal{F}_t)$ stopping time. Define the stopped process $\{X_t^\tau\}$ by

$$X_t^\tau(\omega) := X_{t\wedge\tau(\omega)}(\omega), \, \forall t \geq 0, \omega \in \Omega.$$

Definition 2.4.9. Let $\{X_t\}$ be an $(\mathcal{F}_t)$ adapted process.

(i) We say $\{X_t\}$ has the property Π locally if there exists a sequence of stopping times $\{\tau_n\}$ with $\tau_n \uparrow \infty$ and $\{X_t^{\tau_n}\}$ has the property Π for each n.

(ii) If $\{X_t\}$ has property Π locally corresponding to a sequence of stopping times $\{\tau_n\}$ with $\tau_n \uparrow \infty$, then we say $\{\tau_n\}$ is a localizing sequence of stopping times or simply a localizing sequence.

2.5 Real valued stochastic processes

In this section, we recall some basic results involving real valued stochastic processes.

2.5.1 Predictable processes

Definition 2.5.1. In the product space $\Omega \times [0, \infty)$, we define the predictable σ-field to be the σ-field generated by all real valued continuous $(\mathcal{F}_t)$ adapted processes. Elements of this σ-field are called predictable sets and any real valued measurable function on $\Omega \times [0, \infty)$ (with respect to this σ-field) is called a predictable process.

Lemma 2.5.2 ([56, Lemma 22.1])**.** *The predictable σ-field is generated by each of the following classes of sets or processes:*

(i) $\mathcal{F}_0 \times [0, \infty)$ and the sets $A \times (t, \infty)$ with $A \in \mathcal{F}_t, t \geq 0$.

(ii) the real valued left-continuous $(\mathcal{F}_t)$ adapted processes.

Proposition 2.5.3 ([55, Chapter I, 2.4 Proposition])**.** *If $\{X_t\}$ is a real valued predictable process and if τ is a stopping time, then $\{X_t^\tau\}$ is also a predictable process.*

Let $\{X_t\}$ be an $(\mathcal{F}_t)$ adapted process such that its paths have left limits. Then define an $(\mathcal{F}_t)$ adapted process $\{X_{t-}\}$ as follows:

$$X_{t-} := \begin{cases} X_0, \text{ if } t = 0. \\ \lim_{s\uparrow t} X_s, \text{ if } t > 0. \end{cases}.$$

Proposition 2.5.4 ([55, Chapter I, 2.6 Proposition])**.** *If $\{X_t\}$ is a real valued $(\mathcal{F}_t)$ adapted process with rcll paths then $\{X_{t-}\}$ is a predictable process.*

2.5.2 Processes of finite variation

Definition 2.5.5. Let $\{A_t\}$ be a real valued $(\mathcal{F}_t)$ adapted process with rcll paths.

(i) $\{A_t\}$ is an increasing process if the paths of the process, viz. $t \mapsto A_t(\omega)$ are non-decreasing for almost all ω and $A_0 = 0$.

(ii) $\{A_t\}$ is called a finite variation process (or a process of finite variation or simply an FV process) if almost all paths of the process are of bounded variation on each compact interval of $[0, \infty)$. For any $t > 0$ the total variation of $\{A_t\}$ will be denoted by $Var_{[0,t]}(A.)$.

Remark 2.5.6 (Regularity of paths of the total variation process)**.** Let $\{A_t\}$ be an FV process. Then the paths of the total variation process $\{Var_{[0,t]}(A.)\}$ are a.s. non-decreasing and in particular has left limits a.s. The paths are also a.s. right continuous (see Theorem 2.2.3(iii)). By Proposition 2.2.5, a.s.

$$Var_{[0,t]}(A.) = \sup_{m \geq 1} \sum_{k=1}^{2^m} \left| A_{\frac{tk}{2^m}} - A_{\frac{t(k-1)}{2^m}} \right|, \ \forall t \in [0, \infty).$$

Note that the random variables $A_{\frac{tk}{2^m}}, 1 \leq k \leq 2^m$ are $\mathcal{F}_t$ measurable and hence so is $\{Var_{[0,t]}(A.)\}$. Therefore $\{Var_{[0,t]}(A.)\}$ is an $(\mathcal{F}_t)$ adapted increasing process.

Theorem 2.5.7. *([27, Chapter VI, 52 Theorem]) Let A be an increasing process. There exist a continuous increasing process A^c, a sequence $\{T_n\}$ of stopping times (with graphs in general not disjoint) and a sequence $\{\lambda_n\}$ of constants > 0, such that*

$$A_t = A_t^c + \sum_n \lambda_n \mathbb{1}_{(T_n \leq t)}.$$

If A is predictable, the T_n can be chosen predictable.

2.5.3 Martingales

Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^d$. The dimension will be understood from the context.

Definition 2.5.8 ([74, 8.1 Definition])**.** An $\mathbb{R}^d$ valued $(\mathcal{F}_t)$ adapted stochastic process is called an $(\mathcal{F}_t)$ martingale (or simply a martingale, if the filtration is clear) if

(i) $\mathbb{E}|X_t| < \infty$ for all $t \geq 0$.

(ii) For every $s, t \geq 0$ with $s < t$ and every $A \in \mathcal{F}_s$,

$$\mathbb{E}(\mathbb{1}_A X_s) = \mathbb{E}(\mathbb{1}_A X_t).$$

Proposition 2.5.9 ([74, 10.9 Theorem]). *Let $\{X_t\}$ be a real valued $(\mathcal{F}_t)$ martingale. Then it has an rcll modification.*

Remark 2.5.10. In the definition of a martingale the regularity of paths, viz. rcll paths is not a requirement. But for theoretical development regularity of paths plays an important role. Unless otherwise specified we work with continuous or rcll processes.

Definition 2.5.11. Let $\{X_t\}$ be an $(\mathcal{F}_t)$ martingale.

(i) We say $\{X_t\}$ is an $\mathcal{L}^2$ martingale (or a square integrable martingale), if $\mathbb{E}|X_t|^2 < \infty$ for all $t \geq 0$.
(ii) We say $\{X_t\}$ is an $\mathcal{L}^2$-bounded martingale, if $\sup_{t\geq 0} \mathbb{E}|X_t|^2 < \infty$.

Definition 2.5.12. Let $\{X_t\}$ be a real valued $(\mathcal{F}_t)$ adapted process. Then $\{X_t\}$ is called a submartingale (respectively a supermartingale) if

(i) $\mathbb{E}|X_t| < \infty$ for all $t \geq 0$.
(ii) For every $s, t \geq 0$ with $s < t$ and every $A \in \mathcal{F}_s$,

$$\mathbb{E}(\mathbb{1}_A X_s) \leq \mathbb{E}(\mathbb{1}_A X_t)$$

(respectively $\mathbb{E}(\mathbb{1}_A X_s) \geq \mathbb{E}(\mathbb{1}_A X_t)$).

Remark 2.5.13. Condition (ii) in Definition 2.5.8 is often stated in terms of the conditional expectation as $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$ almost surely.

Definition 2.5.14. Let $\{X_t\}$ be a real valued $(\mathcal{F}_t)$ adapted process. It is called a local martingale (respectively local $\mathcal{L}^2$ martingale, locally square integrable martingale, local submartingale) if there exists a localizing sequence $\{\tau_n\}$ such that for each n, the stopped process $\{X_t^{\tau_n}\}$ is a martingale (respectively $\mathcal{L}^2$ martingale, square integrable martingale, submartingale).

Proposition 2.5.15 ([56, Lemma 6.11]). *Let $\{M_t\}$ be an $\mathbb{R}^d$ valued martingale and consider a convex function $f : \mathbb{R}^d \to \mathbb{R}$ such that $\{X_t\}$ defined by $X_t = f(M_t)$ is integrable for all t. Then $\{X_t\}$ is a real valued submartingale. The statement remains true for real submartingales $\{M_t\}$, provided that f is also non-decreasing.*

Note that $x \mapsto x^2$ and $x \mapsto |x|$ are convex functions on $\mathbb{R}$ and hence we get the next result.

Corollary 2.5.16. *Let $\{M_t\}$ be a real valued square integrable martingale. Then $\{M_t^2\}$ and $\{|M_t|\}$ are submartingales.*

The following result is a version of the Doob-Meyer decomposition theorem due to Meyer and Doléans.

Theorem 2.5.17 (Doob-Meyer decomposition, [56, Theorem 22.5])**.** *A process X is a local submartingale if and only if it has a decomposition $X = M + A$ where M is a local martingale and A is a locally integrable, increasing, predictable process. In that case M and A are a.s. unique.*

Lemma 2.5.18 (([74, 13.7 Corollary 1]))**.** *Every predictable right continuous martingale is continuous.*

Proposition 2.5.19. *([87, Chapter III, Theorem 12]) A predictable local martingale M of finite variation is a.s. constant i.e. a.s. $M_t = M_0, t \geq 0$.*

Let $\mathscr{M}^2$ (respectively $\mathscr{M}^{2,c}$) denote the vector space of real valued rcll $\mathcal{L}^2$ martingales (respectively continuous $\mathcal{L}^2$ martingales) with the locally convex structure defined by the seminorms $M \mapsto \mathbb{E}|M_t|^2$, $t \in [0, \infty)$. Let $\mathscr{M}^2_\infty$ (respectively $\mathscr{M}^{2,c}_\infty$) denote the vector space of real valued rcll $\mathcal{L}^2$-bounded martingales (respectively continuous $\mathcal{L}^2$-bounded martingales). Note that an $\mathcal{L}^2$-bounded martingale $\{M_t\}$ is closable, i.e. there exists an $\mathcal{F}_\infty$ measurable random variable M_∞ such that

$$M_t = \mathbb{E}[M_\infty | \mathcal{F}_t], \text{ a.s.}$$

The spaces $\mathscr{M}^2_\infty$ can be endowed with a Hilbert space structure by considering the scalar product $\langle M \, , \, N \rangle_{\mathscr{M}^2_\infty} := \mathbb{E}[M_\infty N_\infty]$. Then $\mathscr{M}^{2,c}_\infty$ is a Hilbert subspace of $\mathscr{M}^2_\infty$ (see [74, 16.4 Proposition]).

Proposition 2.5.20 (([74, 17.2 Proposition]))**.** *Let $\{M_t\}$ and $\{N_t\}$ be two real valued $\mathcal{L}^2$ martingales. There exists (up to indistinguishability) a unique predictable FV process $\{V_t\}$ with the property that $\{M_t N_t - V_t\}$ is a martingale and $V_0 = 0$.*

Definition 2.5.21 (([74, 17.3 Definition]))**.** If $\{M_t\}$ and $\{N_t\}$ are two real valued $\mathcal{L}^2$ martingales, the process $\{V_t\}$ (obtained in the previous proposition) will denoted by $\{\langle M, N \rangle_t\}$. We write $\{\langle M \rangle_t\}$ instead of $\{\langle M, M \rangle_t\}$ and this process is called the Meyer process of $\{M_t\}$.

Given a martingale or a local martingale $\{M_t\}$ we now assume $M_0 = 0$.

Theorem 2.5.22 (Decomposition of local martingales). *([56, Lemma 23.5] or [87, Chapter III, Theorem 25]) Given any real valued local martingale $\{M_t\}$, there exist two real valued local martingales $\{M'_t\}$, $\{M''_t\}$ one of which has bounded jumps and the other is of locally integrable variation and a.s.*

$$M_t = M'_t + M''_t,\ t \geq 0.$$

Lemma 2.5.23. *Any real valued local martingale with bounded jumps is locally $\mathcal{L}^2$-bounded.*

Proof. The arguments are taken from the discussion following [87, Chapter III, Theorem 25]. Let $\{M_t\}$ be a local martingale with bounded jumps. Let $T_n := \inf\{t \geq 0 : |M_t| \geq n\}$. Suppose the jumps of $\{M_t\}$ are bounded by a constant $\beta > 0$. Then $|M_{t\wedge T_n}| \leq n + \beta$. Hence $\{M_t\}$ is locally bounded and in particular, $\sup_{t\geq 0} \mathbb{E}(M_{t\wedge T_n})^2 \leq (n+\beta)^2$. Hence $\{M_t\}$ is locally $\mathcal{L}^2$-bounded. □

By the structure theorem for $\mathcal{L}^2$ martingales ([74, 17.7 Theorem]), for any $M \in \mathscr{M}^2$ (respectively $\mathscr{M}^2_\infty$) there exist a continuous martingale $M^c \in \mathscr{M}^2$ (respectively $\mathscr{M}^2_\infty$) such that $M^c, M - M^c$ are orthogonal in the following sense: $M^c(M - M^c)$ is a martingale, or equivalently $\langle M^c, M - M^c\rangle = 0$ ([74, 17.4 Proposition]). The process M^c will be called the continuous part of M and $M - M^c$ will be called the purely discontinuous part of M ([74, 17.8 Definition], also see [27, Chapter VIII, Section 2, 43 Theorem]).

Theorem 2.5.24 (Quadratic variation of a martingale). *([74, 18.6 Theorem and 18.9 Corollary 2]) Let $M, N \in \mathscr{M}^2$. Then there exists (up to indistinguishability) a unique FV process with rcll paths, denoted by $\{[M,N]_t\}$ with the following properties.*

(i) For every increasing sequence $\{\Pi_n\}$ of increasing subsequences of $[0,\infty)$ viz. $\Pi_n := \{0 < t_0 < t_1 < \cdots\}$ such that

$$\lim_{k\uparrow\infty} t_k = \infty, \quad \lim_{n\to\infty} \delta(\Pi_n) = 0,$$

where $\delta(\Pi_n) := \sup_{t_i\in\Pi_n}(t_{i+1} - t_i)$, one has

$$[M,N]_t \stackrel{\mathcal{L}^1}{=} \lim_{n\to\infty} \sum_{t_i\in\Pi_n} (M_{t_{i+1}\wedge t} - M_{t_i\wedge t})(N_{t_{i+1}\wedge t} - N_{t_i\wedge t}).$$

(ii) $MN - [M,N]$ is a martingale.

(iii) Let M^c, N^c denote the continuous part of M, N respectively. Then for every t

$$[M,N]_t = \langle M^c, N^c\rangle_t + \sum_{s\leq t} \triangle M_s \triangle N_s \ a.s.$$

with the series on the the right hand side being a.s. summable.

(iv) If M is continuous, then $[M, M] = \langle M, M \rangle$.

Definition 2.5.25. Let $M \in \mathscr{M}^2$. Then $[M, M]$ is called the quadratic variation process of the martingale M. For brevity we write $[M]$ instead of $[M, M]$.

2.5.4 Martingale inequalities

Theorem 2.5.26 ([87, Chapter I, Theorem 20])**.** *Let $\{X_t\}$ be a non-negative submartingale. For all $p > 1$, we have*

$$\mathbb{E}\left(\sup_{t\geq 0}|X_t|\right)^p \leq \left(\frac{p}{p-1}\right)^p \sup_{t\geq 0}\mathbb{E}|X_t|^p$$

Recall that if $\{M_t\}$ is a real valued square integrable martingale with rcll paths, then $\{|M_t|\}$ is a non-negative submartingale (see Corollary 2.5.16). Then taking $p = 2$ in the previous theorem we get the next result.

Proposition 2.5.27 (Doob's maximal quadratic inequality)**.** *([87, p.11]) Let $\{M_t\}$ be a real valued $\mathcal{L}^2$-bounded martingale with rcll paths. Then*

$$\mathbb{E}\left(\sup_{t\geq 0}|M_t|\right)^2 \leq 4\sup_{t\geq 0}\mathbb{E}|M_t|^2.$$

Next norm inequalities involving quadratic variation of a martingale are known as BDG inequalities.

Theorem 2.5.28 (Burkholder, Davis, Gundy)**.** *(i) ([56, Proposition 15.7]) There exist some constants $c_p \in (0, \infty)$, $p > 0$, such that for any continuous local martingale M with $M_0 = 0$*

$$c_p^{-1}\mathbb{E}\,[M]_\infty^{\frac{p}{2}} \leq \mathbb{E}\left(\sup_{t\geq 0}|M_t|\right)^p \leq c_p\mathbb{E}\,[M]_\infty^{\frac{p}{2}},\ p > 0.$$

(ii) ([56, Theorem 23.12]) There exist some constants $c_p \in (0, \infty)$, $p \geq 1$, such that for any local martingale M with $M_0 = 0$

$$c_p^{-1}\mathbb{E}\,[M]_\infty^{\frac{p}{2}} \leq \mathbb{E}\left(\sup_{t\geq 0}|M_t|\right)^p \leq c_p\mathbb{E}\,[M]_\infty^{\frac{p}{2}},\ p \geq 1.$$

Remark 2.5.29. In the BDG inequalities, the constant c_p can be chosen independent of the martingale M (see [87, Chapter IV, Theorem 48]).

2.5.5 Semimartingales

Definition 2.5.30. A real valued $(\mathcal{F}_t)$ adapted process $\{X_t\}$ with rcll paths is called a semimartingale if there exist a local martingale $\{M_t\}$ with $M_0 = 0$ and an FV process $\{A_t\}$ with $A_0 = 0$ such that a.s.

$$X_t = X_0 + M_t + A_t,\ t \geq 0. \tag{2.1}$$

Proposition 2.5.31. *The decomposition* (2.1) *is unique if the FV process* $\{A_t\}$ *is predictable, i.e. if there exist a local martingale* $\{N_t\}$ *and a predictable FV process* $\{V_t\}$ *such that a.s.*

$$X_t = X_0 + N_t + V_t,\ t \geq 0,$$

then a.s. $M_t = N_t, A_t = V_t,\ t \geq 0$.

Definition 2.5.32. A semimartingale $\{X_t\}$ with a decomposition a.s.

$$X_t = X_0 + M_t + A_t,\ t \geq 0$$

where $\{M_t\}$ is a local martingale with $M_0 = 0$ and $\{A_t\}$ a predictable FV process with $A_0 = 0$, is called a special semimartingale. We shall refer to the decomposition above as the canonical decomposition of X.

Example 2.5.33 (Examples of special semimartingales)**.** Any of the following two conditions imply the existence of a canonical decomposition of $\{X_t\}$.

(i) $\{X_t\}$ has bounded jumps ([87, Chapter III, Theorem 31]).
(ii) $\{X_t\}$ is a continuous semimartingale ([87, Chapter III, Corollary to Theorem 31]),

Lemma 2.5.34. *Let* $\{X_t\}$ *be a real semimartingale. Then* $\{X_t\}$ *has a decomposition, a.s.*

$$X_t = X_0 + M_t + A_t,\ t \geq 0$$

where $\{M_t\}$ *is a local* $\mathcal{L}^2$*-bounded martingale with* $M_0 = 0$ *and* $\{A_t\}$ *is a process of finite variation with* $A_0 = 0$.

Proof. This result is an observation pointed out during the course of the proof of [56, Theorem 23.4].
By definition there exists a local martingale $\{M_t\}$ with $M_0 = 0$ and a FV process $\{A_t\}$ with $A_0 = 0$ such that a.s. $X_t = X_0 + M_t + A_t$, $t \geq 0$. By Theorem 2.5.22 there exist local martingales $\{M'_t\}$, $\{M''_t\}$ such that a.s. $M_t = M'_t + M''_t$, $t \geq 0$, $\{M'_t\}$ has bounded jumps with $M'_0 = 0$ and $\{M''_t\}$ is of locally integrable variation with $M''_0 = 0$. Then

(i) $\{M'_t\}$ is a local $\mathcal{L}^2$-bounded martingale (see Lemma 2.5.23).
(ii) $\{M''_t\}$ is an FV process.

Hence a.s. $X_t = X_0 + M'_t + (M''_t + A_t)$ gives the required decomposition. □

Remark 2.5.35. The decomposition of a real semimartingale observed in the previous lemma is not necessarily unique.

Definition 2.5.36 ([56, pp.437-8])**.** Given two real valued semimartingales $\{X_t\}$ and $\{Y_t\}$, $\{X_{t-}\}$ and $\{Y_{t-}\}$ denote the left-continuous versions of the processes, respectively. Define the quadratic variation $[X]$ and the covariation $[X,Y]$ be the 'integration-by-parts' formulas

$$[X]_t := X_t^2 - X_0^2 - 2X_{t-}.X_t,$$
$$[X,Y]_t := X_tY_t - X_0Y_0 - X_{t-}Y_t - X_tY_{t-} = \frac{1}{4}([X+Y]_t - [X-Y]_t)$$

We list some properties of the covariation process $\{[X,Y]_t\}$ when

Theorem 2.5.37 ([56, Theorem 23.6])**.** *Let X, Y be two real valued semimartingales. Then*

(i) $[X,Y] = [X - X_0, Y - Y_0]$ *a.s.;*
(ii) $\{[X]\}$ *is a.s. non-decreasing, and* $\{[X,Y]\}$ *is a.s. symmetric and bilinear;*
(iii) $|[X,Y]| \leq \int |d[X,Y]| \leq [X]^{\frac{1}{2}} [Y]^{\frac{1}{2}}$ *a.s.;*
(iv) $\triangle[X] = (\triangle X)^2$ *and* $\triangle[X,Y] = \triangle X \, \triangle Y$ *a.s.;*
(v) $[\int_0^\cdot V_s \, dX_s, Y] = \int_0^\cdot V_s \, d[X,Y]_s$ *a.s. for any locally bounded, predictable process* $\{V_t\}$*;*
(vi) $[X^\tau, Y] = [X^\tau, Y^\tau] = [X,Y]^\tau$ *a.s. for any stopping time* τ*;*
(vii) if M, N *are locally* $\mathcal{L}^2$*-bounded martingales, then* $[M,N]$ *has a compensator* $\langle M,N\rangle$*, i.e.* $[M,N] - \langle M,N\rangle$ *is a local martingale;*
(viii) if A *has locally finite variation, then* $[X,A]_t = \sum_{s\leq t} \triangle X_s \, \triangle A_s$ *a.s.*

Definition 2.5.38 ([56, p. 445])**.** A semimartingale $X = M + A$ is said to be purely discontinuous if there exist some local martingales $M^1, M^2, \cdots$ of locally finite variation such that $\mathbb{E}(\sup_{s\leq t} |M - M^n|_s)^2 \to 0$ for every $t > 0$. Note that this property is independent of the choice of the decomposition $X = M + A$.

Theorem 2.5.39 (Decomposition of semimartingales, Yoeurp, Meyer)**.** *([56, p. 445]) Any semimartingale X has an a.s. unique decomposition $X = X_0 + X^c + X^d$, where X^c is a continuous local martingale with $X_0^c = 0$ and X^d is a purely discontinuous semimartingale. Furthermore, $[X^c] = [X]^c$ and $\left[X^d\right] = [X]^d$ a.s.*

Proposition 2.5.40 ([74, 25.5 Corollary 3])**.** *Let $\{X_t\}$ be a real semimartingale. Then for any $t \geq 0$, we have*

$$P(\sum_{s\leq t} (\triangle X_s)^2 < \infty) = 1$$

where $\triangle X_s$ denotes the jump of $\{X_t\}$ at time s.

Let $X = (X^1, \cdots, X^d)$ be an $\mathbb{R}^d$ valued semimartingale, i.e. each of $\{X^i\}, i = 1, \cdots, d$ is a real semimartingale. Let $\triangle X_s$ denote the jump of $\{X_t\}$ at time s. Then $| \triangle X_s|^2 = \sum_{i=1}^d (\triangle X_s^i)^2$ and we have a corollary to the previous result.

Corollary 2.5.41. *Let $X = (X^1, \cdots, X^d)$ be an $\mathbb{R}^d$ valued semimartingale. Then for each $t > 0$ a.s.*

$$P\left(\sum_{s \le t} | \triangle X_s|^2 < \infty\right) = 1.$$

Let $X = (X^1, \cdots, X^d)$ be an $\mathbb{R}^d$ valued semimartingale and consider the set $\Omega_t = \{\omega : \sum_{s\le t} | \triangle X_s(\omega)|^2 < \infty\}$, $t \ge 0$. Define $\widetilde{\Omega} := \bigcap_{n=1}^\infty \Omega_n$. Then $P(\widetilde{\Omega}) = 1$ and on the set $\widetilde{\Omega}$ we have

$$\sum_{s \le t} | \triangle X_s|^2 < \infty, \forall t > 0.$$

Lemma 2.5.42. *Fix $\omega \in \widetilde{\Omega}$.*

(i) Fix $t > 0$. Let $\{t_n\}$ be a strictly increasing sequence converging to t. Then

$$\lim_{n\to\infty} \sum_{s \le t_n} | \triangle X_s(\omega)|^2 = \sum_{s<t} | \triangle X_s(\omega)|^2.$$

(ii) Fix $t \ge 0$. Let $\{t_n\}$ be a strictly decreasing sequence converging to t. Then

$$\lim_{n\to\infty} \sum_{t_m < s \le t_1} | \triangle X_s(\omega)|^2 = \sum_{t<s\le t_1} | \triangle X_s(\omega)|^2.$$

Proof. We only prove part (i). Proof of part (ii) is similar.
Note that $\mathbb{1}_{(t_n<s<t)}| \triangle X_s(\omega)|^2 \le | \triangle X_s(\omega)|^2$ and $(t_n, t) \downarrow \emptyset$. Since $\sum_{s<t} | \triangle X_s(\omega)|^2 < \infty$, by the Dominated Convergence theorem, we get

$$\sum_{t_n<s<t} | \triangle X_s(\omega)|^2 \downarrow 0.$$

Since $\sum_{s<t} | \triangle X_s(\omega)|^2 - \sum_{s\le t_n} | \triangle X_s(\omega)|^2 = \sum_{t_n<s<t} | \triangle X_s(\omega)|^2$, part (i) follows. □

Alternative proof of Lemma 2.5.42(i). On $\widetilde{\Omega}$, $\sum_{s<t} |\triangle X_s|^2 < \infty$ and hence for any positive integer n the set

$$\{s : | \triangle X_s(\omega)| \ge \frac{1}{n}, s < t\}$$

is finite. Then for each n, there exists a positive integer $m = m(n)$ such that

$$\sup\{s : | \triangle X_s(\omega)| \ge \frac{1}{n}, s < t\} \le t_{m(n)}.$$

Hence

$$\sum_{\substack{s<t,\\ |\triangle X_s(\omega)|\geq \frac{1}{n}}} |\triangle X_s(\omega)|^2 \leq \sum_{s\leq t_{m(n)}} |\triangle X_s(\omega)|^2 \leq \sum_{s<t} |\triangle X_s(\omega)|^2,\ \forall n \in \mathbb{N}.$$

Since

$$\sum_{s<t} |\triangle X_s(\omega)|^2 = \sup_n \{ \sum_{\substack{s<t,\\ |\triangle X_s(\omega)|\geq \frac{1}{n}}} |\triangle X_s(\omega)|^2 \}$$

and $\{\sum_{s\leq t_n} |\triangle X_s(\omega)|^2\}$ is non-decreasing, we have part (i). □

2.6 Stochastic integration

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered complete probability space satisfying the usual conditions. Unless stated otherwise stopping times or adapted processes will be with respect to the filtration $(\mathcal{F}_t)$.

2.6.1 Stieltjes integration

Our main reference for this subsection is [87, Chapter I, Section 7]. Let $\{A_t\}$ be a real valued $(\mathcal{F}_t)$ adapted increasing process. Now fix an ω such that $t \mapsto A_t(\omega)$ is right continuous and non-decreasing. This function induces a measure $\mu_A(\omega, ds)$ on $[0, \infty)$ (with the Borel σ field).
If f is a real valued bounded Borel function on $[0, \infty)$, then $\int_0^t f(s)\mu_A(\omega, ds)$ is well-defined for each $t > 0$. We denote this integral by $\int_0^t f(s)\, dA_s(\omega)$.
If $F : [0, \infty) \times \Omega \to \mathbb{R}$ is bounded and jointly measurable, then we can define the integral $\int_0^t F(s, \omega)\, dA_s(\omega)$. The map $(t, \omega) \mapsto \int_0^t F(s, \omega)\, dA_s(\omega)$ is jointly measurable and a.s. for fixed ω, t, $(t, \omega) \mapsto \int_0^t F(s, \omega)\, dA_s(\omega)$ is right continuous.
If $\{A_t\}$ is a real valued $(\mathcal{F}_t)$ adapted process of finite variation, then it can be expressed as the difference of two increasing processes, viz. $\{Var_{[0,t]}(A.)\}$ and $\{Var_{[0,t]}(A.) - A_t\}$. Then for F as above we define

$$\int_0^t F(s)\, dA_s := \int_0^t F(s)\, dVar_{[0,s]}(A.) - \int_0^t F(s)\, d(Var_{[0,s]}(A.) - A_s),$$

which is a jointly measurable integral. We may use $F{\cdot}A$ to denote the process $\{\int_0^t F(s)\, dA_s\}$.

Definition 2.6.1 ([87, Chapter III, Section 3]). Let $\{A_t\}$ be a real valued $(\mathcal{F}_t)$ adapted process of finite variation with $A_0 = 0$.

(i) $\{A_t\}$ is of integrable variation if $\mathbb{E} \int_0^\infty |dA_s| < \infty$. We denote the random variable $\int_0^\infty |dA_s|$ by $Var_{[0,\infty)}(A.)$. Note that a.s. $Var_{[0,\infty)}(A.) = \lim_{t\to\infty} Var_{[0,t]}(A.)$.

(ii) $\{A_t\}$ is of locally integrable variation if there exists a localizing sequence $\{T_n\}$ such that $\mathbb{E} \int_0^{T_n} |dA_s| < \infty$, for each n.

We now recall a result from [27, Chapter VI]. Note that in this reference 'integrable' means 'of integrable variation'.

Proposition 2.6.2 ([27, Chapter VI, Theorem 80(a)])**.** *Any real valued predictable process of finite variation is of locally integrable variation.*

As an application of Theorem 2.5.7 we get the next result.

Proposition 2.6.3 ([27, Chapter VI, 53 Remarks (d)])**.** *Let $\{A_t\}$ be a predictable process of finite variation. Let $\{V_t\}$ be a bounded predictable process. Then $\{\int_0^t V_s \, dA_s\}$ is a predictable process.*

Remark 2.6.4. If the FV process $\{A_t\}$ is continuous, then one can define the integral $\int_0^t V_s \, dA_s$ for progressively measurable integrands $\{V_t\}$.

2.6.2 Stochastic integration with respect to a real $\mathcal{L}^2$-bounded martingale

Let $\mathscr{M}^2_{\infty,loc}$ denote the space of local $\mathcal{L}^2$-bounded martingales and let $\mathscr{M}^{2,c}_{\infty,loc}$ denote the subspace of $\mathscr{M}^2_{\infty,loc}$, consisting of continuous martingales. Let $\mathscr{M}^2_{\infty,0,loc}$ denote the subspace of $\mathscr{M}^2_{\infty,loc}$ with initial value 0. Let $M \in \mathscr{M}^2_{loc}$. Let $\{\langle M \rangle_t\}$ denote the predictable process such that $\{M_t^2 - \langle M \rangle_t\}$ is a local martingale.
Let $\mathcal{E}$ denote the class of bounded predictable step processes V with jumps at finitely many fixed times, viz

$$V_t = \sum_{k=1}^{n} \eta_k \mathbb{1}_{(\tau_k, \tau_{k+1}]}(t),$$

where τ_k are stopping times and η_k are $\mathcal{F}_{\tau_k}$ measurable random variables. For such processes define the elementary predictable integral as

$$\int_0^t V_s \, dM_s := \sum_{k=1}^{n} \eta_k (M_{t\wedge\tau_{k+1}} - M_{t\wedge\tau_k}).$$

We may use $V \cdot M$ to denote the process $\{\int_0^t V_s \, dM_s\}$. Let $\mathcal{L}^2(M)$ denote the class of real valued predictable processes $\{V_t\}$ such that a.s. $\int_0^t V_s^2 \, d\langle M \rangle_s < \infty$ for every $t > 0$. Given any $(\mathcal{F}_t)$ adapted rcll process $\{X_t\}$, we define the $(\mathcal{F}_t)$ adapted process $\{X_t^*\}$ by $X_t^* := \sup_{s \le t} |X_s|$.

Theorem 2.6.5 ([56, Theorem 23.2])**.** *The elementary predictable integral extends a.s. uniquely to a bilinear map of any $M \in \mathscr{M}^2_{\infty,loc}$ and $V \in \mathcal{L}^2(M)$ into $V \cdot M \in \mathscr{M}^2_{\infty,0,loc}$, such* *that if* $(V_n^2 \cdot \langle M_n \rangle)_t \xrightarrow{P} 0$ *for some* $V_n \in \mathcal{L}^2(M_n)$ *and* $t > 0$*, then* $(V_n \cdot M_n)_t^* \xrightarrow{P} 0$*. It has the following additional properties, the first of which characterizes the integral:*

(i) $\langle V \cdot M, N \rangle = V \cdot \langle M, N \rangle$ *a.s. for all* $N \in \mathscr{M}^2_{\infty,loc}$.
(ii) $U \cdot (V \cdot M) = (UV) \cdot M$ *a.s.*
(iii) $\triangle(V \cdot M) = V \triangle M$ *a.s.*
(iv) $(V \cdot M)^\tau = V \cdot M^\tau = (V \mathbb{1}_{[0,\tau]}) \cdot M$ *a.s. for any stopping time* τ.

Remark 2.6.6. If the martingale $\{M_t\}$ is continuous, then one can define the integral $\int_0^t V_s \, dM_s$ for progressively measurable integrands $\{V_t\}$.

Remark 2.6.7 ([60, Chapter 3, 2.11 Remark])**.** Let $\{M_t\}$ be a real valued continuous square integrable martingale such that the sample paths $t \mapsto \langle M \rangle_t (\omega)$ of the quadratic variation process $\{\langle M \rangle_t\}$ are absolutely continuous functions of t for P a.e. ω. Let $\mathscr{L}(M)$ denote the set of equivalence classes of all real valued measurable $(\mathcal{F}_t)$ adapted processes $\{X_t\}$ such that

$$\mathbb{E} \int_0^T X_t^2 d \langle M \rangle_t < \infty, \, \forall T > 0.$$

In what follows, we refer to the processes $\{X_t\}$ themselves as elements of $\mathscr{L}(M)$. Then we can define the process $\{\int_0^t X_s \, dM_s\}$ for all $X \in \mathscr{L}(M)$. Note that $\{\int_0^t X_s \, dM_s\}$ is in $\mathscr{M}^{2,c}_{loc}$ and for a standard Brownian Motion $\{B_t\}$ we have $\langle B \rangle_t = t$. Hence $\{\int_0^t X_s \, dB_s\}$ can be defined for all $X \in \mathscr{L}(B)$.

Proposition 2.6.8. *Let* $\mathscr{L}(B)$ *be as in previous remark. Let* $X \in \mathscr{L}(B)$*.*

(i) ([82, Theorem 3.2.5]) Let $T > 0$*. Then there exists a continuous* $(\mathcal{F}_t)$ *adapted stochastic process* $\{I_t\}$ *such that*

$$P\left(I_t = \int_0^t X_s \, dB_s\right) = 1, \, \forall t \in [0, T].$$

(ii) There exists a continuous $(\mathcal{F}_t)$ *adapted stochastic process* $\{I_t\}$ *such that*

$$P\left(I_t = \int_0^t X_s \, dB_s\right) = 1, \, \forall t \in [0, \infty).$$

Proof. Part (ii) follows from part (i) by consistency of the continuous modifications. Let $\{I_t\}$ and $\{J_t\}$ be some modifications on $[0, n]$ and $[0, m]$ respectively, when n, m are positive integers with $n < m$. Then a.s. $I_t = J_t$ for each $t \in [0, n]$. By continuity of these processes we conclude a.s. $I_t = J_t$, $\forall t \in [0, n]$. Using this consistency we can define a continuous modification on $[0, \infty)$. □

2.6.3 Stochastic integration with respect to a real semimartingale

Let $\{A_t\}$ be an $(\mathcal{F}_t)$ adapted FV process with rcll paths and $A_0 = 0$. Let $\mathcal{L}(A)$ denote the space of predictable processes $\{V_t\}$ such that a.s. the integral $\int_0^t V_s\, dA_s$ exists in the Stieltjes sense, for all $t \geq 0$.

Lemma 2.6.9 ([56, Lemma 23.3])**.** *Let V be a predictable process with $|V|^p \in \mathcal{L}(A)$, where A is increasing and $p \geq 1$. Then there exist some $V_1, V_2, \dots \in \mathcal{E}$ with $(|V_n - V|^p \cdot A)^* \to 0$ a.s. for all $t > 0$.*

Lemma 2.6.10. *Let $\{V_t\}$ be a real valued bounded predictable process. Let $\{M_t\}$ be an $\mathcal{L}^2$-bounded martingale and $\{A_t\}$ is a process of finite variation such that a.s.*

$$M_t = A_t, \forall t \in [0, T].$$

Then a.s.

$$\int_0^t V_s\, dM_s = \int_0^t V_s\, dA_s, \forall t \in [0, T].$$

Proof. This result is included in the proof of Theorem 23.4 in [56]. We present the argument for completeness sake.

The two integrals agree when $V \in \mathcal{E}$. For bounded and predictable V, there exists a sequence $\{V^{(n)}\}$ in $\mathcal{E}$ such that $((V^{(n)} - V)^2 \cdot \langle M \rangle)^* \to 0$ and $(|V^{(n)} - V| \cdot A)^* \to 0$ a.s. Then $(V^{(n)} \cdot M)_t \xrightarrow{P} (V \cdot M)_t$ and $(V^{(n)} \cdot A)_t \to (V \cdot A)_t$ for every $t > 0$. This proves the required equality. □

Theorem 2.6.11 ([56, Theorem 23.4])**.** *The $\mathcal{L}^2$ integral $V \cdot M$ and the ordinary Lebesgue-Stieltjes integral extend a.s. uniquely to a bilinear mapping of any semimartingale X and locally bounded predictable process V into a semimartingale $V \cdot X$. This mapping has the following properties.*

- *(i)* $U \cdot (V \cdot X) = (UV) \cdot X$ *a.s.*
- *(ii)* $\triangle(V \cdot X) = V \triangle X$ *a.s.*
- *(iii)* $(V \cdot X)^\tau = V \cdot X^\tau = (V \mathbb{1}_{[0,\tau]}) \cdot X$ *a.s. for any stopping time* τ*.*
- *(iv) For any locally bounded predictable processes $V, V_1, V_2, \cdots$ with $V \geq |V_n| \to 0$ pointwise, we have $(V_n \cdot X)_t^* \xrightarrow{P} 0$ for all $t > 0$.*
- *(v) If X is a local martingale, then so is $V \cdot X$.*

Remark 2.6.12. If the $(\mathcal{F}_t)$ semimartingale $\{X_t\}$ is continuous, then one can define the integral $\int_0^t V_s\, dX_s$ for progressively measurable integrands $\{V_t\}$.

2.7 Hilbert valued processes

We have already mentioned in the introduction that we do not require results on Hilbert valued stochastic integration in their full generality. Our requirement in this context amounts to integrating Hilbert valued predictable processes with respect to real semimartingales. Unless stated otherwise $\mathbb{H}$ will be a real separable Hilbert space. Let $\|\cdot\|, \langle\cdot\,,\,\cdot\rangle$ denote the norm and inner product respectively and let $\{e_n : n = 1, 2, \cdots\}$ denote an orthonormal basis for $\mathbb{H}$. We also assume that our filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ satisfies the usual conditions. Unless stated otherwise, adapted processes will be with respect to this filtration.
In [74], spaces of martingales were defined with only right continuous paths and processes with rcll paths are called R. R. C (regular right continuous) processes. Some of the results there (e.g. [74, 20.5 Theorem]) are stated for martingales with right continuous paths; however we will only need these results for martingales with rcll paths.

2.7.1 Basic definitions

Definition 2.7.1. (i) An $\mathbb{H}$ valued $(\mathcal{F}_t)$ adapted stochastic process $\{X_t\}$ is called an $(\mathcal{F}_t)$ martingale (or simply a martingale, if the filtration is clear) if

a) $\mathbb{E}\|X_t\| < \infty$ for all $t \geq 0$.

b) For every $s, t \geq 0$ with $s < t$ and every $A \in \mathcal{F}_s$,

$$\mathbb{E}(\mathbb{1}_A X_s) = \mathbb{E}(\mathbb{1}_A X_t).$$

(ii) Let $\{X_t\}$ be an $(\mathcal{F}_t)$ martingale.

a) We say $\{X_t\}$ is an $\mathcal{L}^2$ martingale (or a square integrable martingale), if $\mathbb{E}\|X_t\|^2 < \infty$ for all $t \geq 0$.

b) We say $\{X_t\}$ is an $\mathcal{L}^2$-bounded martingale, if $\sup_{t\geq 0} \mathbb{E}\|X_t\|^2 < \infty$.

(iii) Let $\{X_t\}$ be an $\mathbb{H}$ valued $(\mathcal{F}_t)$ adapted process. It is called a local martingale (respectively local $\mathcal{L}^2$ martingale, locally $\mathcal{L}^2$-bounded martingale) if there exists a localizing sequence $\{\tau_n\}$ such that for each n, the stopped process $\{X_t^{\tau_n}\}$ is a martingale (respectively $\mathcal{L}^2$ martingale, $\mathcal{L}^2$-bounded martingale).

Remark 2.7.2. In Section 2.3, we have pointed out the existence of conditional expectation for integrable $\mathbb{B}$ valued random variables (where $\mathbb{B}$ is a real separable Banach space). Therefore condition *b)* in the definition of an $\mathbb{H}$ valued martingale, can be stated in terms of conditional expectation (see [74, 8.3 Remarks]) as follows: for all $0 \leq s < t$,

$$X_s = \mathbb{E}[X_t|\mathcal{F}_s] \text{ a.s.}$$

Remark 2.7.3. In [74], most of the results have been stated for martingales which have right continuous paths. Unless stated otherwise, in this thesis we work with martingales which have rcll paths.

Lemma 2.7.4. *Let $\{M_t\}$ be an $\mathcal{L}^2$ martingale. Then $\{\|M_t\|^2\}$ is a submartingale.*

Proof. Let $\{e_n : n = 1, 2, \cdots\}$ denote an orthonormal normal basis for $\mathbb{H}$. Then $\{\langle M_t\,,\, e_n\rangle\}$ is a real valued $\mathcal{L}^2$ martingale for each n and hence the process $\{\langle M_t\,,\, e_n\rangle^2\}$ is a real valued submartingale.
Writing $\|M_t(\omega)\|^2 = \sum_{n=1}^{\infty} \langle M_t(\omega)\,,\, e_n\rangle^2$ we get the result. □

Definition 2.7.5. An $(\mathcal{F}_t)$ adapted $\mathbb{H}$ valued process $\{A_t\}$ with rcll paths is said to be a process of finite variation (simply an FV process) if a.s. for all $t > 0$

$$\sup_{\Pi} \sum_{i=1}^{n} \|A_{t_i} - A_{t_{i-1}}\| < \infty$$

where the supremum is taken over all partitions $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ of $[0, t]$.

Definition 2.7.6 ([74, 23.7 Definition])**.** An $\mathbb{H}$ valued process X with rcll paths is called an $(\mathcal{F}_t)$ semimartingale if X can be decomposed a.s. $X_t = X_0 + M_t + A_t$, $t \geq 0$ where M is a locally $\mathcal{L}^2$-bounded martingale with $M_0 = 0$ and A is an FV process with $A_0 = 0$.

Definition 2.7.7. A function $X : \Omega \times [0, \infty) \to \mathbb{H}$ is said to be a predictable process, if it is measurable with respect to the predictable σ field.

2.7.2 Stieltjes integration

We can proceed as in Subsection 2.6.1. Let $\{A_t\}$ be a real valued FV process with $A_0 = 0$. We denote the total variation process of $\{A_t\}$ by $\{V_{[0,t]}(A_\cdot)\}$ (see Definition 2.5.5). Let $\{G_t\}$ be an $\mathbb{H}$ valued predictable process such that a.s. for all $t > 0$,

$$\int_0^t \|G_s\|\, |dA_s| < \infty. \tag{2.2}$$

We denote the space of such predictable processes by $\mathcal{L}(A)$. For any $G \in \mathcal{L}(A)$, a.s. for all $t > 0$, the Stieltjes integral $\int_0^t G_s\, dA_s$ is defined as a Bochner Integral on $[0, t]$ with respect to the measure $|dA|$ (see Subsection 2.3).
For predictable step processes $\{G_t\}$, we can write down the explicit form of the integral $\int_0^t G_s\, dA_s$ as follows. Let $\{G_t\}$ be a predictable step process of the form $G_t =$

$\sum_{k=1}^{n} \mathbb{1}_{(t_k, t_{k+1}]}(t)\, a_k$ where $0 \leq t_1 < t_2 < \cdots t_n$ and a_k's are $\mathbb{H}$ valued $\mathcal{F}_{t_k}$ measurable random variables. Then

$$\int_0^t G_s\, dA_s = \sum_{k=1}^{n} (A_{t\wedge t_{k+1}} - A_{t\wedge t_k}) a_k.$$

Note that for any $G \in \mathcal{L}(A)$,

$$\left\| \int_0^t G_s\, dA_s \right\| \leq \int_0^t \|G_s\|\, |dA_s|. \tag{2.3}$$

Proposition 2.7.8. *(i) Let $G \in \mathcal{L}(A)$. Let $\mathbb{K}$ be a real separable Hilbert space and $T : \mathbb{H} \to \mathbb{K}$ be a bounded linear operator. Then a.s. $t \geq 0$, $T \int_0^t G_s\, dA_s = \int_0^t T G_s\, dA_s$. In particular, for any $h \in \mathbb{H}$, a.s. for all $t \geq 0$*

$$\left\langle \int_0^t G_s\, dA_s\, ,\, h \right\rangle = \int_0^t \langle G_s\, ,\, h \rangle\; dA_s.$$

(ii) Let $\{G_t\}$ be a locally norm-bounded $\mathbb{H}$ valued predictable process. Let $\mathbb{K}, T$ be as in part (i). Then the same conclusions are true.

Proof. Part (i) follows from the theory of Bochner integration. Fix an ω such that $\int_0^t \|G_s\|\, |dA_s| < \infty$. The result is easily verified when $s \mapsto G_s(\omega)$ is simple and then extended by continuity of T when $G_s^{(n)}(\omega)$ converges to $G_s(\omega)$ pointwise, where $s \mapsto G_s^{(n)}(\omega)$ are simple functions.
To prove part (ii), observe that there exists a localizing sequence $\{\tau_n\}$ such that the process $\{\int_0^{t\wedge\tau_n} T(G_s)\, dA_s\}$ is an $\mathbb{H}$ valued FV process, for each n. By part (i), we have a.s. for all $t \geq 0$

$$T\left(\int_0^{t\wedge\tau_n} G_s\, dA_s \right) = \int_0^{t\wedge\tau_n} T(G_s)\, dA_s.$$

But a.s. $\tau_k \uparrow \infty$ as $k \to \infty$. We get the result by letting n go to infinity in the previous equality. For the last part of the result, we need to use the bounded linear functional $h' \mapsto \langle h'\, ,\, h \rangle$ and the argument is similar to that in part (i). □

We now prove a technical lemma.

Lemma 2.7.9. *Let $y_1, y_2, \cdots$ be $\mathbb{H}$ valued rcll functions on $[0, \infty)$. Suppose that the sequence $\{y_n\}$ is Cauchy in the following sense: for any fixed $\epsilon > 0$ and for any $T > 0$ there exists a positive integer N such that*

$$\sup_{t\leq T} \|y_n(t) - y_m(t)\| \leq \epsilon,\ \forall n, m \geq N. \tag{2.4}$$

Then the following statements hold true.

(i) The function $y(t) := \lim_{n\to\infty} y_n(t)$, $t \geq 0$ *is well-defined and hence the sequence* $\{y_n\}$ *has a pointwise limit* y.

(ii) For any $T > 0$

$$\sup_{t\leq T} \|y(t) - y_n(t)\| \xrightarrow{n\to\infty} 0.$$

(iii) y *is rcll.*

Proof. For any fixed $t > 0$, the Cauchy condition (2.4) implies that $\{y_n(t)\}$ is an $\mathbb{H}$ valued Cauchy sequence and hence the existence of the point-wise limit y follows from the completeness of $\mathbb{H}$.
By our hypothesis for any fixed $\epsilon > 0$ and for any $T > 0$ there exists a positive integer N such that

$$\|y_n(t) - y_m(t)\| \leq \epsilon, \, \forall n, m \geq N, t \in [0, T].$$

Letting m go to infinity in the previous relation, we get

$$\|y_n(t) - y(t)\| \leq \epsilon, \, \forall n \geq N, t \in [0, T],$$

which proves (ii).
Now we prove the right continuity of y. Let $s, t \geq 0$ with $t < s \leq t + 1$ and let $\epsilon > 0$ be arbitrarily chosen.
By our hypothesis, there exists a positive integer N such that

$$\sup_{s\leq t+1} \|y(s) - y_n(s)\| \leq \epsilon, \, \forall n \geq N.$$

Fix an $n \geq N$. Observe that

$$\begin{aligned}\|y(t) - y(s)\| &\leq \|y(t) - y_n(t)\| + \|y_n(t) - y_n(s)\| + \|y_n(s) - y(s)\| \\ &\leq 2\epsilon + \|y_n(t) - y_n(s)\|.\end{aligned}$$

Since y_n is right continuous, right continuity of y follows.
An argument similar to above shows the existence of left limits of the function $t \mapsto y(t)$: fix any $t > 0$ and let $\{t_n\}$ be a monotonically increasing sequence converging to t. Then for positive integers k, l, n,

$$\|y(t_k) - y(t_l)\| \leq \|y(t_k) - y_n(t_k)\| + \|y_n(t_k) - y_n(t_l)\| + \|y_n(t_l) - y(t_l)\|.$$

For any fixed n, the function $t \mapsto y_n(t)$ has left limits and hence existence of left limits of the function $t \mapsto y(t)$ follows. □

Let $G \in \mathcal{L}(A)$. Properties of the process $\{\int_0^t G_s \, dA_s\}$ are pointed out in the next result.

Proposition 2.7.10. *Let $G \in \mathcal{L}(A)$.*

(i) $\{\int_0^t G_s \, dA_s\}$ *is* $(\mathcal{F}_t)$ *adapted.*
(ii) $\{\int_0^t G_s \, dA_s\}$ *has rcll paths.*
(iii) If $\{A_t\}$ *is predictable, then so is* $\{\int_0^t G_s \, dA_s\}$.
(iv) $\{\int_0^t G_s \, dA_s\}$ *is an FV process.*

Proof. Let $\{e_n : n = 1, 2, \cdots\}$ be an orthonormal basis. By Proposition 2.7.8 a.s.

$$\int_0^t G_s \, dA_s = \sum_{n=1}^{\infty} \left\langle \int_0^t G_s \, dA_s \, , \, e_n \right\rangle e_n = \sum_{n=1}^{\infty} \left(\int_0^t \langle G_s \, , \, e_n \rangle \; dA_s \right) e_n.$$

Note that $| \langle G_s \, , \, e_n \rangle | \leq \|G_s\|$ and hence a.s. for all $t \geq 0$ we have

$$\int_0^t |\langle G_s \, , \, e_n \rangle| \, |dA_s| < \infty.$$

Then for each n we have the real valued process $\{\int_0^t \langle G_s \, , \, e_n \rangle \; dA_s\}$ is $(\mathcal{F}_t)$ adapted. Hence $\{\int_0^t G_s \, dA_s\}$ is also $(\mathcal{F}_t)$ adapted.
For part (ii), let $\omega \in \Omega$ be such that $\int_0^t \|G_s(\omega)\| \, |dA_s(\omega)| < \infty$ for all $t > 0$. We show that $t \mapsto \int_0^t G_s(\omega) \, dA_s(\omega)$ is rcll.
Fix a positive real number $T > 0$. By Theorem 2.3.2, there exists a sequence of simple functions $\{t \mapsto g^n(t) : n = 1, 2, \cdots\}$ such that

$$\int_0^T \|G_s(\omega) - g^n(s)\| \, |dA_s(\omega)| \xrightarrow{n \to \infty} 0.$$

Fix $\epsilon > 0$ and choose a positive integer N, sufficiently large, such that $\int_0^T \|G_s(\omega) - g^n(s)\| \, |dA_s(\omega)| \leq \frac{\epsilon}{2}$ for all $n \geq N$.
Let $g^N(t) = \sum_{k=1}^{l} \mathbb{1}_{E_k} h_k$ where $E_1, \cdots, E_l$ are disjoint Borel subsets of $[0, T]$ and $h_1, \cdots, h_l$ are elements of $\mathbb{H}$. Let $R = 1 + \max\{\|h_k\| : 1 \leq k \leq l\}$. For each $k \in \{1, \cdots, l\}$, we can find sets U_k, which are finite unions of intervals of the form $(\alpha, \beta], 0 \leq \alpha < \beta \leq T$ such that

$$\int_0^T |\mathbb{1}_{E_k} - \mathbb{1}_{U_k}| \, |dA_s(\omega)| \leq \frac{\epsilon}{2Rl}.$$

Then

$$\begin{aligned}
\int_0^T \|G_s(\omega) - \sum_{k=1}^{l} \mathbb{1}_{U_k} h_k\| \, |dA_s(\omega)| &\leq \int_0^T \|G_s(\omega) - g^N(s)\| \, |dA_s(\omega)| \\
&\quad + \int_0^T \|g^N(s) - \sum_{k=1}^{l} \mathbb{1}_{U_k} h_k\| \, |dA_s(\omega)| \\
&\leq \frac{\epsilon}{2} + \int_0^T \sum_{k=1}^{l} \|h_k\| |\mathbb{1}_{E_k} - \mathbb{1}_{U_k}| \, |dA_s(\omega)|
\end{aligned}$$

$$\leq \frac{\epsilon}{2} + R\int_0^T \sum_{k=1}^{l} \|\mathbb{1}_{E_k} - \mathbb{1}_{U_k}|\, |dA_s(\omega)|$$
$$\leq \epsilon.$$

Continuing from the above estimate, using (2.3) we have

$$\sup_{t\leq T}\left\|\int_0^t \left(G_s(\omega) - \sum_{k=1}^{l} \mathbb{1}_{U_k} h_k\right)\, dA_s(\omega)\right\| \leq \int_0^T \|G_s(\omega) - \sum_{k=1}^{l} \mathbb{1}_{U_k} h_k\|\, |dA_s(\omega)| \leq \epsilon.$$

Since $t \in [0,T] \mapsto \int_0^t \sum_{k=1}^l \mathbb{1}_{U_k} h_k\, dA_s(\omega)$ is rcll, above estimate implies gives an uniform approximation of $t \in [0,T] \mapsto \int_0^t G_s(\omega)\, dA_s(\omega)$ in terms of rcll functions. Then by Lemma 2.7.9 $\{\int_0^t G_s\, dA_s\}$ has rcll paths.

Proof of part (iii) is similar to [27, Chapter VI, 53 Remarks (d)]. First assume $\{A_t\}$ is an increasing predictable process. By Theorem 2.5.7

$$A_t = A_t^c + \sum_n \lambda_n \mathbb{1}_{(T_n\leq t)},$$

where $\{A_t^c\}$ is a continuous increasing process, λ_n are constants and T_n are predictable times. Then

$$\int_0^t G_s\, dA_s = \int_0^t G_s\, dA_s^c + \sum_n \lambda_n G_{T_n} \mathbb{1}_{(T_n\leq t)}.$$

The sum $\sum_n \lambda_n G_{T_n} \mathbb{1}_{(T_n\leq t)}$ is predictable since G_{T_n} is $\mathcal{F}_{T_n} = \mathcal{F}_{T_n-}$ measurable. We now show $\{\int_0^t G_s\, dA_s^c\}$ is predictable. The proof is similar to part (i). We have a.s.

$$\int_0^t G_s\, dA_s^c = \sum_{n=1}^{\infty} \left\langle \int_0^t G_s\, dA_s^c\,,\, e_n\right\rangle e_n = \sum_{n=1}^{\infty} \left(\int_0^t \langle G_s\,,\, e_n\rangle\ dA_s^c\right) e_n.$$

By Proposition 2.6.3 $\{\int_0^t \langle G_s\,,\, e_n\rangle\ dA_s^c\}$ is predictable for each n and hence so is $\{\int_0^t G_s\, dA_s\}$. If $\{A_t\}$ is a predictable FV process, then we can express it as a difference of two predictable increasing processes and hence $\{\int_0^t G_s\, dA_s\}$ is also predictable.

We now prove part (iv). Let t be a positive real number and let $\{0 = t_0 < t_1 < \cdots < t_n = t\}$ be a partition of $[0,t]$. Observe that

$$\int_0^{t_{i+1}} G_s\, dA_s - \int_0^{t_i} G_s\, dA_s = \int_0^t \mathbb{1}_{(t_i,t_{i+1}]} G_s\, dA_s,\ i = 0, 1, \cdots, n-1.$$

Then using (2.3), we have

$$\sum_{i=0}^{n-1} \left\|\int_0^{t_{i+1}} G_s\, dA_s - \int_0^{t_i} G_s\, dA_s\right\| = \sum_{i=0}^{n-1} \left\|\int_0^t \mathbb{1}_{(t_i,t_{i+1}]} G_s\, dA_s\right\|$$
$$\leq \sum_{i=0}^{n-1} \int_0^t \mathbb{1}_{(t_i,t_{i+1}]} \|G_s\|\, |dA_s|$$

$$= \int_0^t \|G_s\| \, |dA_s|.$$

Since the upper bound $\int_0^t \|G_s\| \, |dA_s|$ is independent of the partition $\{0 = t_0 < t_1 < \cdots < t_n = t\}$, the previous inequality proves that $\{\int_0^t G_s \, dA_s\}$ is an FV process. □

2.7.3 Stochastic integration with respect to a real $\mathcal{L}^2$-bounded martingale

Let $\{M_t\}$ be a real valued $(\mathcal{F}_t)$ adapted $\mathcal{L}^2$-bounded martingale with rcll paths and $M_0 = 0$. Let $\{\langle M \rangle_t\}$ denote the predictable increasing process such that $\{M_t^2 - \langle M \rangle_t\}$ is an $(\mathcal{F}_t)$ martingale.

Let $\{G_t\}$ be an $\mathbb{H}$ valued predictable process such that for all $t > 0$,

$$\mathbb{E} \int_0^t \|G_s\|^2 \, d \langle M \rangle_s < \infty. \tag{2.5}$$

We denote the space of such predictable processes by $\mathcal{L}^2(M; \mathbb{H})$.

Proposition 2.7.11. *Let $G \in \mathcal{L}^2(M; \mathbb{H})$. Then there exists a sequence of predictable step processes $G^{(n)}$ such that*

$$\mathbb{E} \int_0^t \|G_s - G_s^{(n)}\|^2 \, d \langle M \rangle_s \xrightarrow{n \to \infty} 0, \, \forall t \geq 0.$$

Proof. First we write $G_t(\omega) = \sum_{k=1}^\infty g_t^k(\omega) e_k$ in the orthonormal basis. The convergence is pointwise. Since $\{G_t\}$ is predictable, so are $\{g_t^k\}$ for all k, since $g_t^k = \langle G_t \, , \, e_k \rangle$. Now Define

$$G_n(t, \omega) := \sum_{k=1}^n g_t^k(\omega) e_k$$

and

$$G^n(t, \omega) := \sum_{k=n+1}^\infty g_t^k(\omega) e_k.$$

For each fixed k, we have $g_t^k(\omega)^2 \leq \|G^n(t, \omega)\|^2$ and hence

$$\mathbb{E} \int_0^t \left(g_t^k \right)^2 \, d \langle M \rangle_s \leq \mathbb{E} \int_0^t \|G_s\|^2 \, d \langle M \rangle_s < \infty.$$

Then there exists a sequence of predictable step processes $\{g_t^{k,l}(\omega) : l = 1, 2, \cdots\}$ such that

$$\mathbb{E} \int_0^t \left(g_t^{k,l}(\omega) - g_t^k(\omega) \right)^2 \, d \langle M \rangle_s \xrightarrow{l \to \infty} 0, \, \forall t \geq 0.$$

Define $G_{n,l}(t, \omega) := \sum_{k=1}^n g_t^{k,l}(\omega) e_k$. Note that $\{G_{n,l}(t)\}$ is also a predictable step process. Then for each fixed n

$$\mathbb{E} \int_0^t \|G_n(s, \omega) - G_{n,l}(s, \omega)\|^2 \, d \langle M \rangle_s \xrightarrow{l \to \infty} 0, \, \forall t \geq 0. \tag{2.6}$$

Now $\|G^n(s,\omega)\| \leq \|G_s(\omega)\|$, $\|G^n(s,\omega)\| \xrightarrow{n\to\infty} 0$ for fixed s,ω and $G \in \mathcal{L}^2(M;\mathbb{H})$. Hence

$$\mathbb{E}\int_0^t \|G_s(\omega) - G_n(s,\omega)\|^2 \, d\langle M\rangle_s = \mathbb{E}\int_0^t \|G^n(s,\omega)\|^2 \, d\langle M\rangle_s \xrightarrow{n\to\infty} 0, \, \forall t \geq 0.$$

For any $\epsilon > 0$, for each $t > 0$ there exists a positive integer $n = n(t)$ such that

$$\mathbb{E}\int_0^t \|G_s(\omega) - G_n(s,\omega)\|^2 \, d\langle M\rangle_s \leq \frac{\epsilon}{4}.$$

By (2.6), for any t and $n = n(t)$, there exists a positive integer $l = l(n)$ such that

$$\mathbb{E}\int_0^t \|G_n(s,\omega) - G_{n,l}(s,\omega)\|^2 \, d\langle M\rangle_s \leq \frac{\epsilon}{4}.$$

Hence

$$\begin{aligned}\mathbb{E}\int_0^t \|G_s(\omega) - G_{n,l}(s,\omega)\|^2 \, d\langle M\rangle_s &\leq 2\mathbb{E}\int_0^t \|G_s(\omega) - G_n(s,\omega)\|^2 \, d\langle M\rangle_s \\ &\quad + 2\mathbb{E}\int_0^t \|G_n(s,\omega) - G_{n,l}(s,\omega)\|^2 \, d\langle M\rangle_s \\ &\leq \epsilon,\end{aligned}$$

which proves the result. □

Definition 2.7.12. We define the stochastic integral for predictable simple processes by

$$\int_0^t G_s \, dM_s := \sum_{i=1}^n (M_{t\wedge t_i} - M_{t\wedge t_{i-1}}) g_i$$

where n is a positive integer, $t_0, t_1, \cdots, t_n$ are real numbers satisfying $0 \leq t_0 < t_1 < \cdots t_n$, $G := \sum_{i=1}^n \mathbb{1}_{(t_{i-1},t_i]} \, g_i$, g_i is an $\mathbb{H}$ valued, $\mathcal{F}_{t_{i-1}}$ measurable random variable.

Proposition 2.7.13. *Let $\{G_t\}, \{M_t\}$ be as in the previous definition. The following are properties of the stochastic integral defined above.*

(i) $\{\int_0^t G_s \, dM_s\}$ is an $(\mathcal{F}_t)$ adapted $\mathcal{L}^2$ martingale with the isometry

$$\mathbb{E}\left\|\int_0^t G_s \, dM_s\right\|^2 = \mathbb{E}\int_0^t \|G_s\|^2 \, d\langle M\rangle_s \,. \tag{2.7}$$

(ii) $\{\int_0^t G_s \, dM_s\}$ has rcll paths.

Proof. Since each g_i is $\mathcal{F}_{t_{i-1}}$ measurable and $\{M_t\}$ is an $(\mathcal{F}_t)$ martingale, by definition $\{\int_0^t G_s \, dM_s\}$ is $(\mathcal{F}_t)$ adapted. We prove that the stochastic integral is a martingale. Let $s, t \geq 0$ with $s < t$. Fix any $i = 1, \cdots, n$. We claim that, a.s.

$$\mathbb{E}\left[(M_{t\wedge t_i} - M_{t\wedge t_{i-1}}) g_i | \mathcal{F}_s\right] = (M_{s\wedge t_i} - M_{s\wedge t_{i-1}}) g_i.$$

We prove the result for the case $t_i < s$. Proof of the result for the cases $s \leq t_{i-1}$ and $t_{i-1} < s \leq t_i$ are similar.
If $t_i < s$, then $\mathcal{F}_{t_{i-1}} \subseteq \mathcal{F}_{t_i} \subseteq \mathcal{F}_s$. Since g_i is $\mathcal{F}_{t_{i-1}}$ measurable and $\{M_t\}$ is a martingale, we have (see the notion of conditional expectation on separable Banach spaces in Subsection 2.3)

$$\begin{aligned}\mathbb{E}\left[(M_{t\wedge t_i} - M_{t\wedge t_{i-1}})g_i|\mathcal{F}_s\right] &= \mathbb{E}\left[(M_{t\wedge t_i} - M_{t\wedge t_{i-1}})|\mathcal{F}_s\right] g_i \\ &= (M_{s\wedge t_i} - M_{s\wedge t_{i-1}})g_i.\end{aligned}$$

Since a.s. $\mathbb{E}\left[\int_0^t G_u\, dM_u \Big| \mathcal{F}_s\right] = \sum_{i=1}^n \mathbb{E}\left[(M_{t_i\wedge t} - M_{s_i\wedge t})g_i|\mathcal{F}_s\right]$, above relation implies that the process $\{\int_0^t G_u\, dM_u\}$ is a martingale. Now we show that it is square integrable. Let $t > 0$. Then

$$\begin{aligned}\mathbb{E}\left\|\int_0^t G_s\, dM_s\right\|^2 &= \mathbb{E}\left\langle \sum_{i=1}^n (M_{t\wedge t_i} - M_{t\wedge t_{i-1}})g_i\,,\, \sum_{j=1}^n (M_{t\wedge t_j} - M_{t\wedge t_{j-1}})g_j \right\rangle \\ &= \mathbb{E}\sum_{i,j=1}^n \langle g_i\,,\, g_j\rangle\, (M_{t\wedge t_i} - M_{t\wedge t_{i-1}})(M_{t\wedge t_j} - M_{t\wedge t_{j-1}}).\end{aligned}$$

We show that the terms in the above sum are 0, if $i \neq j$. We show this for the case $i < j$ and the proof for $i > j$ is similar.
If $i < j$, then $t_i \leq t_{j-1}$. Then

$$\begin{aligned}&\mathbb{E}\,\langle g_i\,,\, g_j\rangle\, (M_{t\wedge t_i} - M_{t\wedge t_{i-1}})(M_{t\wedge t_j} - M_{t\wedge t_{j-1}}) \\ &= \mathbb{E}\left(\mathbb{E}[\langle g_i\,,\, g_j\rangle\, (M_{t\wedge t_i} - M_{t\wedge t_{i-1}})(M_{t\wedge t_j} - M_{t\wedge t_{j-1}})|\mathcal{F}_{t_i}]\right) \\ &= \mathbb{E}\left(\langle g_i\,,\, g_j\rangle\, (M_{t\wedge t_i} - M_{t\wedge t_{i-1}})\mathbb{E}[(M_{t\wedge t_j} - M_{t\wedge t_{j-1}})|\mathcal{F}_{t_i}]\right) \\ &= 0,\end{aligned}$$

since a.s. $\mathbb{E}[(M_{t\wedge t_j} - M_{t\wedge t_{j-1}})|\mathcal{F}_{t_i}] = M_{t\wedge t_i} - M_{t\wedge t_i} = 0$.
From the above computation, we have

$$\begin{aligned}\mathbb{E}\left\|\int_0^t G_s\, dM_s\right\|^2 &= \mathbb{E}\sum_{i=1}^n \|g_i\|^2 (M_{t\wedge t_i} - M_{t\wedge t_{i-1}})^2 \\ &= \mathbb{E}\sum_{i=1}^n \|g_i\|^2 (\langle M\rangle_{t\wedge t_i} - \langle M\rangle_{t\wedge t_{i-1}}) \\ &= \mathbb{E}\int_0^t \|G_s\|^2\, d\,\langle M\rangle_s \\ &< \infty.\end{aligned}$$

This completes the proof of part (i). Part (ii) follows from the rcll paths of $\{M_t\}$. □

Let $\{M_t\}, \{G_t\}, \{G_t^{(n)}\}$ be as in Proposition 2.7.11. Consider the measure μ_M on the product σ-field of $\Omega \times [0,t]$ (for any fixed $t \geq 0$) defined by

$$\mu_M(A \times (u_1, u_2]) := \mathbb{E} \int_0^t \mathbb{1}_{A \times (u_1,u_2]} \, d\langle M \rangle_s = \mathbb{E} \int_{u_1}^{u_2} \mathbb{1}_A \, d\langle M \rangle_s \,,$$

for any $A \in \mathcal{F}_t, 0 \leq u_1 < u_2 \leq t$. From now onwards, we denote $d\mu_M$ by $dP \times d\langle M \rangle$. By the isometry (2.7), the sequence $\{\int_0^t G_s^{(n)} \, dM_s : n = 1, 2, \cdots\}$ is Cauchy in $\mathcal{L}^2(\Omega \times [0,t], dP \times d\langle M \rangle ; \mathbb{H})$. By [28, Theorem III.6.6] this space is complete. Now define

$$\int_0^t G_s \, dM_s := \lim_n \int_0^t G_s^{(n)} \, dM_s.$$

Proposition 2.7.14. *Let $G \in \mathcal{L}^2(M; \mathbb{H})$. The following are the properties of the process $\{\int_0^t G_s \, dM_s\}$.*

(i) The definition of $\{\int_0^t G_s \, dM_s\}$ does not depend on the sequence $\{G^{(n)}\}$.
(ii) $\{\int_0^t G_s \, dM_s\}$ is an $(\mathcal{F}_t)$ adapted $\mathcal{L}^2$ martingale.
(iii) We have the isometry: $\mathbb{E} \left\| \int_0^t G_s \, dM_s \right\|^2 = \mathbb{E} \int_0^t \|G_s\|^2 \, d\langle M \rangle_s$.
(iv) $\{\int_0^t G_s \, dM_s\}$ has an rcll modification.

Proof. To prove (i), let $\{G^{(n)} : n = 1, 2, \cdots\}$ and $\{\bar{G}^{(n)} : n = 1, 2, \cdots\}$ be two sequences such that both $\{\int_0^t G_s^{(n)} \, dM_s : n = 1, 2, \cdots\}$ and $\{\int_0^t \bar{G}_s^{(n)} \, dM_s : n = 1, 2, \cdots\}$ converge to $\int_0^t G_s \, dM_s$. Define a new sequence of random variables where the odd-numbered and the even-numbered terms are from $\{\int_0^t G_s^{(n)} \, dM_s : n = 1, 2, \cdots\}$ and $\{\int_0^t \bar{G}_s^{(n)} \, dM_s : n = 1, 2, \cdots\}$ respectively. This new sequence is again Cauchy in $\mathcal{L}^2(\Omega \times [0,t], dP \times d\langle M \rangle)$. Hence it has a limit and which in turn shows that

$$\lim_n \int_0^t G_s^{(n)} \, dM_s = \lim_n \int_0^t \bar{G}_s^{(n)} \, dM_s.$$

This shows the uniqueness of the limits of $\{\int_0^t G_s^{(n)} \, dM_s : n = 1, 2, \cdots\}$ and $\{\int_0^t \bar{G}_s^{(n)} \, dM_s : n = 1, 2, \cdots\}$.

By the construction of $G^{(n)}$ in Proposition 2.7.11, all the terms $\int_0^t G_s^{(n)} \, dM_s$ are $(\mathcal{F}_t)$ measurable and hence so is $\int_0^t G_s \, dM_s$. To prove $\{\int_0^t G_u \, dM_u\}$ is a martingale, let $0 \leq s < t$ and $A \in \mathcal{F}_s$. Since convergence in $\mathcal{L}^2(\Omega \times [0,t], dP \times d\langle M \rangle)$ also imply the convergence in $\mathcal{L}^1(\Omega \times [0,t], dP \times d\langle M \rangle)$, we have

$$\begin{aligned} \mathbb{E} \left[\mathbb{1}_A \int_0^t G_u \, dM_u \right] &= \lim_n \mathbb{E} \left[\mathbb{1}_A \int_0^t G_u^{(n)} \, dM_u \right] \\ &= \lim_n \mathbb{E} \left[\mathbb{1}_A \int_0^s G_u^{(n)} \, dM_u \right] \\ &= \mathbb{E} \left[\mathbb{1}_A \int_0^s G_u \, dM_u \right]. \end{aligned}$$

Proof of the isometry in (iii) is similar to the proof of the martingale property. We use the same approximation along with the joint continuity of the inner product.
We now prove $\{\int_0^t G_s\, dM_s\}$ has an rcll modification. Note that $\int_0^t G_s\, dM_s$ is defined as a limit of a Cauchy sequence $\{\int_0^t G_s^{(n)}\, dM_s : n = 1, 2\cdots\}$ in $\mathcal{L}^2(\Omega \times [0,t], dP \times d\langle M\rangle; \mathbb{H})$. For each n, the process $\{\int_0^t G_s^{(n)}\, dM_s\}$ is an $\mathcal{L}^2$ martingale with rcll paths. By Doob's maximal quadratic inequality ([74, 20.6 Theorem]), we have

$$\begin{aligned}
&P\left(\sup_{t\leq T}\|\int_0^t G_s^{(n)}\, dM_s - \int_0^t G_s^{(m)}\, dM_s\| > \epsilon\right) \\
&\leq \frac{1}{\epsilon^2}\mathbb{E}\sup_{t\leq T}\|\int_0^t G_s^{(n)}\, dM_s - \int_0^t G_s^{(m)}\, dM_s\|^2 \\
&\leq \frac{4}{\epsilon^2}\mathbb{E}\int_0^T \|G_s^{(n)} - G_s^{(m)}\|^2\, d\langle M\rangle_s
\end{aligned}$$

for any $T, \epsilon > 0$. Hence $P\left(\sup_{t\leq T}\|\int_0^t G_s^{(n)}\, dM_s - \int_0^t G_s^{(m)}\, dM_s\| > \epsilon\right)$ is small for sufficiently large n, m. This implies $\{\int_0^t G_s^{(n)}\, dM_s : n = 1, 2, \cdots\}$ converges in probability uniformly in $[0,T]$. In particular, we have a.s. convergence along some subsequence. By Lemma 2.7.9, this limit, say $\{I_t\}_{t\in[0,T]}$ has rcll paths. But for each $t \in [0,T]$, $I_t = \int_0^t G_s\, dM_s$ a.s., since $\int_0^t G_s\, dM_s$ is the limit in $\mathcal{L}^2(\Omega \times [0,t], dP \times d\langle M\rangle; \mathbb{H})$ of $\{\int_0^t G_s^{(n)}\, dM_s : n = 1, 2\cdots\}$. Hence $\{\int_0^t G_s\, dM_s\}$ has an rcll modification. □

Let $\{M_t\}$ be a real valued local $\mathcal{L}^2$-bounded martingale with rcll paths and $M_0 = 0$. Let $\{G_t\}$ be an $\mathbb{H}$ valued predictable process such that there exists a localizing sequence $\{\tau_n\}$ with the following property: for all $t > 0$ and all positive integers n,

$$\mathbb{E}\int_0^{t\wedge\tau_n} \|G_s\|^2\, d\langle M\rangle_s < \infty.$$

Without loss of generality, we assume that for the same localizing sequence $\{\tau_n\}$, $\{M_t^{\tau_n}\}$ are $\mathcal{L}^2$-bounded martingales. Then for each n, we can define the $(\mathcal{F}_t)$ adapted process $\{\int_0^t \mathbb{1}_{(0,\tau_n]}(s) G_s\, dM_s\}$, which is an $\mathcal{L}^2$ martingale. Note that

$$\int_0^t \mathbb{1}_{(0,\tau_n]}(s) G_s\, dM_s = \int_0^t G_s\, dM_s^{\tau_n}. \tag{2.8}$$

Using this observation we now prove a property of the processes $\{\int_0^t G_s\, dM_s^{\tau_n}\}$.

Proposition 2.7.15. *Let $\{G_t\}, \{M_t\}, \{\tau_n : n = 1, 2, \cdots\}$ be as above. Fix a positive integer m. Let $\mathbb{K}$ be a real separable Hilbert space and $T : \mathbb{H} \to \mathbb{K}$ be a bounded linear operator. Then a.s. $t \geq 0$,*

$$T\int_0^t G_s\, dM_s^{\tau_m} = \int_0^t TG_s\, dM_s^{\tau_m}.$$

In particular, for any $h \in \mathbb{H}$, a.s. for all $t \geq 0$

$$\left\langle \int_0^t G_s \, dM_s^{\tau_m} \, , \, h \right\rangle = \int_0^t \langle G_s \, , \, h \rangle \; dM_s^{\tau_m}.$$

Proof. Note that $\mathbb{E} \int_0^t \|G_s\|^2 \, d\langle M^{\tau_m} \rangle_s < \infty$ for all $t \geq 0$. By Proposition 2.7.11, there exists a sequence of predictable step processes $\{G_t^{(l)} : l = 1, 2, \cdots\}$ such that

$$\mathbb{E} \int_0^t \|G_s - G_s^{(l)}\|^2 \, d\langle M^{\tau_m} \rangle_s \xrightarrow{l \to \infty} 0, \, \forall t \geq 0$$

and $\int_0^t G_s \, dM_s^{\tau_m}$ is the limit of the sequence $\int_0^t G_s^{(l)} \, dM_s^{\tau_m} : l = 1, 2, \cdots$ in $\mathcal{L}^2(\Omega \times [0,t], dP \times d\langle M^{\tau_m} \rangle ; \mathbb{H})$. Then we can show a.s. for all $t \geq 0$,

$$T \int_0^t G_s^{(l)} \, dM_s^{\tau_m} = \int_0^t T G_s^{(l)} \, dM_s^{\tau_m}, \forall l = 1, 2, \cdots .$$

Since $T : \mathbb{H} \to \mathbb{K}$ is a bounded linear operator, letting l go to infinity in the previous relation, we have a.s. $t \geq 0$,

$$T \int_0^t G_s \, dM_s^{\tau_m} = \int_0^t T G_s \, dM_s^{\tau_m}.$$

For any fixed $h \in \mathbb{H}$, consider the bounded linear functional $Th' := \langle h' \, , \, h \rangle$. Hence a.s. for all $t \geq 0$

$$\left\langle \int_0^t G_s \, dM_s^{\tau_m} \, , \, h \right\rangle = \int_0^t \langle G_s \, , \, h \rangle \; dM_s^{\tau_m}.$$

This completes the proof. □

Remark 2.7.16. Proposition 2.7.15 was also observed in [89, Proposition 1.3] when $\{M_t\}$ is continuous.

As a corollary of the Proposition 2.7.15, we get the next result.

Corollary 2.7.17. *Let $\{G_t\}, \{M_t\}, \{\tau_n : n = 1, 2, \cdots\}$ be as in Proposition 2.7.18. Then for any positive integer n, a.s. for all $t \geq 0$ we have*

$$\int_0^t \mathbb{1}_{(0,\tau_n]}(s) G_s \, dM_s^{\tau_{n+1}} = \int_0^t G_s \, dM_s^{\tau_n}.$$

Proof. For any $h \in \mathbb{H}$. we claim that a.s. for all $t \geq 0$

$$\left\langle \int_0^t \mathbb{1}_{(0,\tau_n]}(s) G_s \, dM_s^{\tau_{n+1}} \, , \, h \right\rangle = \left\langle \int_0^t G_s \, dM_s^{\tau_n} \, , \, h \right\rangle .$$

First we assume the claim and complete the proof. Let $\{e_m : m = 1, 2, \cdots\}$ be an orthonormal basis for $\mathbb{H}$. Since the processes $\{\int_0^t \mathbb{1}_{(0,\tau_n]}(s) G_s \, dM_s^{\tau_{n+1}}\}$, $\{\int_0^t G_s \, dM_s^{\tau_n}\}$ are determined

by the functionals $\langle \cdot\,, e_m \rangle\,, m = 1, 2, \cdots$ the proof follows from the claim.
We now prove the claim. By Proposition 2.7.15, we have a.s. for all $t \geq 0$,

$$\begin{aligned}\left\langle \int_0^t \mathbb{1}_{(0,\tau_n]}(s) G_s \, dM_s^{\tau_{n+1}}\,, h \right\rangle &= \int_0^t \mathbb{1}_{(0,\tau_n]}(s) \left\langle G_s\,, h \right\rangle \, dM_s^{\tau_{n+1}} \\ &= \int_0^{t\wedge\tau_{n+1}} \mathbb{1}_{(0,\tau_n]}(s) \left\langle G_s\,, h \right\rangle \, dM_s^{\tau_{n+1}} \\ &= \int_0^{t\wedge\tau_n} \left\langle G_s\,, h \right\rangle \, dM_s^{\tau_{n+1}} \\ &= \int_0^{t\wedge\tau_n} \left\langle G_s\,, h \right\rangle \, dM_s^{\tau_n} \\ &= \left\langle \int_0^t G_s \, dM_s^{\tau_n}\,, h \right\rangle .\end{aligned}$$

This completes the proof of the claim. □

Using the consistency relations obtained in the previous result, we define

$$\int_0^t G_s \, dM_s := \int_0^t G_s \, dM_s^{\tau_n}, t \leq \tau_n.$$

Since $\tau_n \uparrow \infty$, $\{\int_0^t G_s \, dM_s\}$ is a local $\mathcal{L}^2$ martingale with rcll paths. The next relation follows readily from the definition.

$$\int_0^{t\wedge\tau_n} G_s \, dM_s := \int_0^t G_s \, dM_s^{\tau_n}, t \geq 0.$$

Proposition 2.7.15 now can be extended to the next result.

Proposition 2.7.18. *Let $\{M_t\}$ be a real valued $(\mathcal{F}_t)$ adapted local $\mathcal{L}^2$ martingale with rcll paths and $M_0 = 0$. Let $\{G_t\}$ be an $\mathbb{H}$ valued predictable process such that there exists a localizing sequence $\{\tau_n\}$ with the following property: for all $t > 0$ and all positive integers n,*

$$\mathbb{E} \int_0^{t\wedge\tau_n} \|G_s\|^2 \, d \left\langle M \right\rangle_s < \infty.$$

Let $\mathbb{K}$ be a real separable Hilbert space and $T : \mathbb{H} \to \mathbb{K}$ be a bounded linear operator. Then a.s. $t \geq 0$,

$$T \int_0^t G_s \, dM_s = \int_0^t T G_s \, dM_s.$$

In particular, for any $h \in \mathbb{H}$, a.s. for all $t \geq 0$

$$\left\langle \int_0^t G_s \, dM_s\,, h \right\rangle = \int_0^t \left\langle G_s\,, h \right\rangle \, dM_s.$$

Proof. The proof uses Proposition 2.7.15. For any positive integer n, we have a.s. for all $t \geq 0$

$$\begin{aligned} T\int_0^{t\wedge\tau_n} G_s\, dM_s &= T\int_0^t G_s\, dM_s^{\tau_n} \\ &= \int_0^t TG_s\, dM_s^{\tau_n} \\ &= \int_0^{t\wedge\tau_n} TG_s\, dM_s \end{aligned}$$

Since $\tau_n \uparrow \infty$, we have a.s. for all $t \geq 0$

$$T\int_0^t G_s\, dM_s = \int_0^t TG_s\, dM_s.$$

For any fixed $h \in \mathbb{H}$, consider the bounded linear functional $Th' := \langle h' , h\rangle$. Hence a.s. for all $t \geq 0$

$$\left\langle \int_0^t G_s\, dM_s , h \right\rangle = \int_0^t \langle G_s , h\rangle \; dM_s.$$

This completes the proof. □

2.7.4 Stochastic integration with respect to a real semimartingale

Let $\{X_t\}$ be a real valued $(\mathcal{F}_t)$ semimartingale. Without loss of generality we assume $X_0 = 0$. By Lemma 2.5.34, there exists a local $\mathcal{L}^2$-bounded martingale $\{M_t\}$ with $M_0 = 0$ and a process of finite variation $\{A_t\}$ with $A_0 = 0$ such that a.s.

$$X_t = M_t + A_t,\ t \geq 0.$$

Let $\{G_t\}$ be an $\mathbb{H}$ valued norm-bounded (i.e. there exists a constant $R > 0$ such that a.s. $\|G_t\|_{-p} \leq R$ for all t) predictable process.
Define the stochastic integral of $\{G_t\}$ with respect to $\{X_t\}$ as

$$\int_0^t G_s\, dX_s := \int_0^t G_s\, dM_s + \int_0^t G_s\, dA_s,\ t \geq 0.$$

Proposition 2.7.19. *Let X, M, A, G be as above. Then $\{\int_0^t G_s\, dX_s\}$ is well-defined, i.e. it does not depend on the decomposition $X = M + A$.*

Proof. For any $h \in \mathbb{H}$, we have $|\langle G_t , h\rangle| \leq \|G_t\|\|h\| \leq R\|h\|$. Hence $\{\langle G_t , h\rangle\}$ is a bounded predictable process. Using Lemma 2.6.10, we conclude that for each $h \in \mathbb{H}$, the process

$$\left\{\int_0^t \langle G_s , h\rangle \; dM_s + \int_0^t \langle G_s , h\rangle \; dA_s\right\}$$

does not depend on the decomposition $X = M + A$. Now varying h in the orthonormal basis $\{e_n : n = 1, 2, \cdots\}$, we get a common null set $\widetilde{\Omega}$ such that for all $\omega \in \Omega \setminus \widetilde{\Omega}$, for all $n = 1, 2, \cdots$ and for all $t \geq 0$, we have by Proposition 2.7.8 and Proposition 2.7.18,

$$\left\langle \int_0^t G_s\, dM_s + \int_0^t G_s\, dA_s\, ,\, e_n \right\rangle = \int_0^t \langle G_s\, ,\, e_n\rangle\; dM_s + \int_0^t \langle G_s\, ,\, e_n\rangle\; dA_s.$$

If any element $h \in \mathbb{H}$ satisfies $\langle h\, ,\, e_n\rangle = 0, \forall n$, then $h = 0$. Hence the above relation identifies the $\mathbb{H}$ valued process $\{\int_0^t G_s\, dM_s + \int_0^t G_s\, dA_s\}$ independent of the decomposition $X = M + A$. □

The next result is an application of Itô formula ([56, Theorem 15.19]) to Hilbert valued continuous semimartingales.

Proposition 2.7.20. *Let $\{X_t\}$ be an $\mathbb{H}$ valued continuous semimartingale with a decomposition: a.s.*

$$X_t = X_0 + \int_0^t G_s\, dM_s + \int_0^t V_s\, dA_s,\; t \geq 0,$$

where $\{G_t\}$ and $\{V_t\}$ are locally norm-bounded $\mathbb{H}$ valued predictable processes, $\{M_t\}$ a real valued continuous $\mathcal{L}^2$ martingale with $M_0 = 0$ and $\{A_t\}$ a real valued continuous FV process with $A_0 = 0$. Then a.s. $t \geq 0$

$$\|X_t\|^2 = \|X_0\|^2 + 2\int_0^t \langle X_s\, ,\, G_s\rangle\; dM_s + 2\int_0^t \langle X_s\, ,\, V_s\rangle\; dA_s + \int_0^t \|G_s\|^2 d\,[M]_s\, .$$

Proof. First we assume $\{G_t\}$ and $\{V_t\}$ are norm-bounded and $\{M_t\}$ is an $\mathcal{L}^2$-bounded martingale. Let $\{e_n : n = 1, 2, \cdots\}$ denote an orthonormal basis for $\mathbb{H}$. Then the real valued processes $\{\langle G_t\, ,\, e_n\rangle\}, \{\langle V_t\, ,\, e_n\rangle\}$ are bounded and predictable for all n. Now a.s. for all $n = 1, 2, \cdots$ and for all $t \geq 0$ we have

$$\langle X_t\, ,\, e_n\rangle = \langle X_0\, ,\, e_n\rangle + \int_0^t \langle G_s\, ,\, e_n\rangle\; dM_s + \int_0^t \langle V_s\, ,\, e_n\rangle\; dA_s.$$

Since $\{X_t\}$ is continuous, $\{\langle X_t\, ,\, e_n\rangle\}$ is locally bounded and predictable for each n. Using Itô formula ([56, Theorem 15.19]) to the map $x \in \mathbb{R} \mapsto x^2$ we have a.s. for all $n = 1, 2, \cdots$ and for all $t \geq 0$

$$\begin{aligned}\langle X_t\, ,\, e_n\rangle^2 &= \langle X_0\, ,\, e_n\rangle^2 + 2\int_0^t \langle X_s\, ,\, e_n\rangle\; d\,\langle X_t\, ,\, e_n\rangle + \frac{1}{2}\int_0^t 2d\,[\langle X_t\, ,\, e_n\rangle]_s \\ &= \langle X_0\, ,\, e_n\rangle^2 + 2\int_0^t \langle X_s\, ,\, e_n\rangle\, \langle G_s\, ,\, e_n\rangle\; dM_s \\ &\quad + 2\int_0^t \langle X_s\, ,\, e_n\rangle\, \langle V_s\, ,\, e_n\rangle\; dA_s + \int_0^t \langle G_s\, ,\, e_n\rangle^2\, d\,[M]_s\, .\end{aligned}$$

Since $\{G_t\}, \{V_t\}$ are bounded predictable processes and $\{X_t\}$ is continuous, the processes $\{\langle X_t\,,\,G_t\rangle\}$, $\{\langle X_t\,,\,V_t\rangle\}$ are locally bounded and predictable. Then

$$\begin{aligned}
\|X_t\|^2 &= \lim_{m\to\infty}\sum_{n=1}^{m}\langle X_t\,,\,e_n\rangle^2 \\
&= \lim_{m\to\infty}\sum_{n=1}^{m}\langle X_0\,,\,e_n\rangle^2 + \lim_{m\to\infty} 2\int_0^t \sum_{n=1}^{m}\langle X_s\,,\,e_n\rangle\,\langle G_s\,,\,e_n\rangle\; dM_s \\
&+ \lim_{m\to\infty} 2\int_0^t \sum_{n=1}^{m}\langle X_s\,,\,e_n\rangle\,\langle V_s\,,\,e_n\rangle\; dA_s + \lim_{m\to\infty}\int_0^t \sum_{n=1}^{m}\langle G_s\,,\,e_n\rangle^2\, d\,[M]_s \\
&= \sum_{n=1}^{\infty}\langle X_0\,,\,e_n\rangle^2 + 2\int_0^t \sum_{n=1}^{\infty}\langle X_s\,,\,e_n\rangle\,\langle G_s\,,\,e_n\rangle\; dM_s \\
&+ 2\int_0^t \sum_{n=1}^{\infty}\langle X_s\,,\,e_n\rangle\,\langle V_s\,,\,e_n\rangle\; dA_s + \int_0^t \sum_{n=1}^{\infty}\langle G_s\,,\,e_n\rangle^2\, d\,[M]_s \\
&= \|X_0\|^2 + 2\int_0^t \langle X_s\,,\,G_s\rangle\; dM_s + 2\int_0^t \langle X_s\,,\,V_s\rangle\; dA_s \\
&+ \int_0^t \|G_s\|^2 d\,[M]_s\,.
\end{aligned}$$

Now we work with G, V, M as in the statement. Let $\{\tau_n\}$ be a localizing sequence such that for each n, $\{G_t^{\tau_n}\}$ and $\{V_t^{\tau_n}\}$ are norm-bounded and $\{M_t^{\tau_n}\}$ is an $\mathcal{L}^2$-bounded martingale. Then $\{\int_0^{t\wedge\tau_n} G_s\, dM_s\}$ is an $\mathbb{H}$ valued $\mathcal{L}^2$ martingale and $\{\int_0^{t\wedge\tau_n} V_s\, dA_s\}$ is an $\mathbb{H}$ valued FV process. Computation similar to above now yields a.s. $t \geq 0$

$$\begin{aligned}
\|X_t^{\tau_n}\|^2 &= \|X_0\|^2 + 2\int_0^{t\wedge\tau_n} \langle X_s\,,\,G_s\rangle\; dM_s + 2\int_0^{t\wedge\tau_n} \langle X_s\,,\,V_s\rangle\; dA_s \\
&+ \int_0^{t\wedge\tau_n} \|G_s\|^2 d\,[M]_s\,.
\end{aligned}$$

Since $\tau_n \uparrow \infty$, letting $n \to \infty$ we get the result. □

2.8 Hermite functions

Let $\{H_n(x) : n = 0, 1, \cdots\}$ be the Hermite polynomials on $\mathbb{R}$, which are the generating functions of $\exp(2xt - t^2)$, i.e.

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}.$$

Define the Hermite functions h_n on $\mathbb{R}$ as follows:

$$h_n(x) := (2^n n!\sqrt{\pi})^{-\frac{1}{2}} \exp\left(-\frac{x^2}{2}\right) H_n(x),\; x \in \mathbb{R}, n = 0, 1, \cdots .$$

Let $\mathbb{Z}_+^d := \{n = (n_1, \cdots, n_d) : n_i \text{ are non-negative integers}\}$. We refer to the elements of $\mathbb{Z}_+^d$ as multi-indices. If n is a multi-index, we define $|n| := n_1 + \cdots + n_d$ where $n = (n_1, \cdots, n_d)$. For multi-indices n, define the Hermite functions h_n on $\mathbb{R}^d$ as follows:

$$h_n(x_1, \cdots, x_d) := h_{n_1}(x_1) \times h_{n_2}(x_2) \times \cdots \times h_{n_d}(x_d),\ \forall (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$$

where the functions h_{n_i} on the right hand side are Hermite functions on $\mathbb{R}$.
Convention: This convention will be used throughout the thesis. We take $h_n \equiv 0$, if $n = (n_1, \cdots, n_d)$ with some $n_i < 0$.

Proposition 2.8.1. *We list some well-known properties of the Hermite functions.*

(i) *([51], [108]) Hermite functions on $\mathbb{R}$ are uniformly bounded, i.e. there exists a constant $C > 0$ such that*

$$|h_n(x)| \leq C,\ \forall x \in \mathbb{R}, n = 0, 1, \cdots .$$

(ii) *Hermite functions on $\mathbb{R}^d$ are uniformly bounded.*

(iii) $h_n(-x) = (-1)^n h_n(x), x \in \mathbb{R}$. *In particular, h_n is an odd function if n is odd and is an even function otherwise.*

(iv) $\{h_n : n \in \mathbb{Z}_+^d\}$ *is an orthonormal basis for $\mathcal{L}^2(\mathbb{R}^d)$ (see [109, Chapter 1]).*

(v) *([47, Appendix A.5, equation (A.26)]) Hermite functions on $\mathbb{R}$ satisfy the following recurrences.*

$$\partial h_n(x) = \sqrt{\frac{n}{2}} h_{n-1}(x) - \sqrt{\frac{n+1}{2}} h_{n+1}(x),\ x \in \mathbb{R}$$

and

$$x h_n(x) = \sqrt{\frac{n}{2}} h_{n-1}(x) + \sqrt{\frac{n+1}{2}} h_{n+1}(x),\ x \in \mathbb{R}.$$

The d dimensional version of the recurrences is stated as follows: Let $\{e_i : i = 1, \cdots, d\}$ be the standard basis vectors in $\mathbb{R}^d$. Let $n = (n_1, \cdots, n_d) \in \mathbb{Z}_+^d$ be a multi-index. Then

$$\partial_i h_n(x) = \sqrt{\frac{n_i}{2}} h_{n-e_i}(x) - \sqrt{\frac{n_i+1}{2}} h_{n+e_i}(x),\ x \in \mathbb{R}^d$$

and

$$x_i h_n(x) = \sqrt{\frac{n_i}{2}} h_{n-e_i}(x) + \sqrt{\frac{n_i+1}{2}} h_{n+e_i}(x),\ x \in \mathbb{R}^d.$$

(vi) *Consider the Hermite functions h_n on $\mathbb{R}$. Then*

$$h_n(0) = \begin{cases} 0, \text{ if } n \text{ is odd,} \\ (-1)^{\frac{n}{2}} \frac{1}{\sqrt[4]{\pi}} \frac{(n-1)!!}{\sqrt{n!}}, \text{ if } n \text{ is even,} \end{cases}$$

where $(2m-1)!!$ *is the double factorial given by*

$$(2m-1)!! = \begin{cases} 1 \times 3 \times \cdots \times (2m-1), \text{ if } m = 1, 2, \cdots \\ 1, \text{ if } m = 0. \end{cases}$$

Proof. Proof of part (ii) follows from part (i) as follows. By definition,

$$h_n(x_1, \cdots, x_d) := h_{n_1}(x_1) \times h_{n_2}(x_2) \times \cdots \times h_{n_d}(x_d), \, \forall (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$$

where the functions h_{n_i} on the right hand side are Hermite functions on $\mathbb{R}$. Then by part (i), we have

$$|h_n(x)| \le C^d, \, \forall x \in \mathbb{R}^d, n \in \mathbb{Z}_+^d.$$

This completes the proof of part (ii).
For part (iii), observe that for any $t, x \in \mathbb{R}$

$$\begin{aligned} \sum_{n=0}^{\infty} H_n(-x) \frac{t^n}{n!} &= \exp(2(-x)t - t^2) \\ &= \exp(2x(-t) - (-t)^2) \\ &= \sum_{n=0}^{\infty} H_n(x) \frac{(-t)^n}{n!} \\ &= \sum_{n=0}^{\infty} (-1)^n H_n(x) \frac{t^n}{n!} \end{aligned}$$

Comparing coefficients of t^n on both sides we get $H_n(-x) = (-1)^n H_n(x), \, \forall x \in \mathbb{R}$. Since $x \mapsto \exp(-\frac{x^2}{2})$ is an even function of x, the above relation implies part (iii).
We now prove part (vi). By [47, Appendix A.5, equation (A.20)] we have a recurrence relation of the Hermite polynomials

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \, n = 1, 2, \cdots, x \in \mathbb{R}.$$

Putting $x = 0$ in above relation we get

$$H_{n+1}(0) = -2nH_{n-1}(0), \, n = 1, 2, \cdots. \tag{2.9}$$

Note that $H_0(x) = 1$ and $H_1(x) = 2x$ for $x \in \mathbb{R}$. Then $H_0(0) = 1$ and $H_1(0) = 0$. Then using the principle of Mathematical induction we get

$$H_n(0) = \begin{cases} 0, \text{ if } n \text{ is odd} \\ (-1)^{\frac{n}{2}} 2^{\frac{n}{2}} (1 \times 3 \times \cdots \times (n-1)), \text{ if } n \text{ is even.} \end{cases}$$

Note that the empty product $1 \times 3 \times \cdots \times (n-1)$ for the case $n = 0$ is interpreted as 1. Now by definition of the Hermite functions we have $h_n(0) = (2^n n! \sqrt{\pi})^{-\frac{1}{2}} H_n(0)$ and hence

$$h_n(0) = \begin{cases} 0, \text{ if } n \text{ is odd,} \\ (-1)^{\frac{n}{2}} \frac{1}{\sqrt[4]{\pi}} \frac{(n-1)!!}{\sqrt{n!}}, \text{ if } n \text{ is even.} \end{cases}$$

□

Convention: Any $\phi \in \mathcal{L}^2(\mathbb{R}^d)$ can be written as $\phi = \sum_{k=0}^{\infty} \sum_{|n|=k} \phi_n h_n$ where $\phi_n \in \mathbb{R}$. We use the following convention throughout the thesis. We take $\phi_n = 0$, if $n = (n_1, \cdots, n_d)$ with some $n_i < 0$.

2.9 Schwartz topology on the space of rapidly decreasing smooth functions

Let $\mathcal{S}(\mathbb{R}^d)$ be the space of smooth rapidly decreasing real valued functions on $\mathbb{R}^d$ with the topology given by L. Schwartz (see [110, Chapter 25], [98, Chapter 7, Section 3], [36, Chapter 8]). The space is also called the Schwartz space. This is defined by

$$\mathcal{S}(\mathbb{R}^d) = \{\phi \in C^\infty(\mathbb{R}^d) : \forall k \geq 1, \max_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^d} (1 + |x|^2)^k |\partial^\alpha \phi(x)| < \infty\},$$

where

(i) $|x|$ stands for the Euclidean norm of $x \in \mathbb{R}^d$,

(ii) $\alpha = (\alpha_1, \cdots, \alpha_d)$ are elements of $\mathbb{Z}_+^d$ where

$$\mathbb{Z}_+^d = \{\alpha = (\alpha_1, \cdots, \alpha_d) : \alpha_i \text{ are non-negative integers}\}$$

with $|\alpha| = \alpha_1 + \cdots + \alpha_d$

(iii) $\partial^\alpha \phi = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} \phi$.

The space $\mathcal{S}(\mathbb{R}^d)$ is a real vector space. The family of seminorms

$$\max_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^d} (1 + |x|^2)^k |\partial^\alpha \phi(x)|, \; k \geq 1 \tag{2.10}$$

on $\mathcal{S}(\mathbb{R}^d)$ defines a locally convex topology and under this topology $\mathcal{S}(\mathbb{R}^d)$ is a Fréchet space. From now onwards, this topology on $\mathcal{S}(\mathbb{R}^d)$ will be called Schwartz topology. We state the following result without proof.

Lemma 2.9.1. *Let $\mathcal{L}^1(\mathbb{R}^d)$(respectively $\mathcal{L}^2(\mathbb{R}^d)$) denote the space of real valued functions on $\mathbb{R}^d$, which are integrable (respectively square integrable) with respect to the Lebesgue measure. Then $\mathcal{S}(\mathbb{R}^d) \subset \mathcal{L}^1(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d) \subset \mathcal{L}^2(\mathbb{R}^d)$.*

Usually we consider the following collection of complex valued functions

$$\mathcal{S}(\mathbb{R}^d;\mathbb{C}) = \{\phi \in C^\infty(\mathbb{R}^d;\mathbb{C}) : \forall k \geq 1, \max_{|\alpha|\leq k} \sup_{x\in\mathbb{R}^d} (1+|x|^2)^k |\partial^\alpha \phi(x)| < \infty\},$$

where $|\partial^\alpha \phi(x)|$ stands for the absolute value of a complex number.
Let $\mathcal{S}'(\mathbb{R}^d)$ denote the dual of $\mathcal{S}(\mathbb{R}^d)$ (as a real vector space). The space $\mathcal{S}'(\mathbb{R}^d)$ is also called the space of tempered distributions. In an analogous manner $\mathcal{S}'(\mathbb{R}^d;\mathbb{C})$ is defined as the dual of $\mathcal{S}(\mathbb{R}^d;\mathbb{C})$ (as a $\mathbb{C}$ vector space).

2.10 Hilbertian topology on $\mathcal{S}(\mathbb{R}^d)$

Let $\langle \cdot\,,\,\cdot\rangle$ represent the $\mathcal{L}^2(\mathbb{R}^d)$ inner product. For any fixed $p \in \mathbb{R}$, consider the following formal sums

$$\begin{cases} \langle \phi\,,\,\psi\rangle_p := \sum_{k=0}^\infty \sum_{|n|=k} (2k+d)^{2p} \langle \phi\,,\,h_n\rangle \langle \psi\,,\,h_n\rangle\,, \\ \|\phi\|_p^2 := \sum_{k=0}^\infty \sum_{|n|=k} (2k+d)^{2p} \langle \phi\,,\,h_n\rangle^2 \end{cases} \tag{2.11}$$

Then $(\mathcal{S}(\mathbb{R}^d), \|\cdot\|_p)$ are pre-Hilbert spaces and completing them one obtains the separable Hilbert spaces $(\mathcal{S}_p(\mathbb{R}^d), \|\cdot\|_p)$, known as the Hermite-Sobolev spaces (see [53, Chapter 1.3]). The collection $\{h_n^p : n \in \mathbb{Z}_+^d\}$ is an orthonormal basis for $(\mathcal{S}_p(\mathbb{R}^d), \|\cdot\|_p)$, where $h_n^p := (2k+d)^{-p} h_n$ with $k = |n|$.
In [53], it was shown that $(\mathcal{S}_{-p}(\mathbb{R}^d), \|\cdot\|_{-p})$ are dual to $(\mathcal{S}_p(\mathbb{R}^d), \|\cdot\|_p)$ for any $p \geq 0$. Furthermore, the following are also known:

$$\begin{cases} \mathcal{L}^2(\mathbb{R}^d) = (\mathcal{S}_0(\mathbb{R}^d), \|\cdot\|_0), \\ \text{for } p < q, (\mathcal{S}_q(\mathbb{R}^d), \|\cdot\|_q) \subset (\mathcal{S}_p(\mathbb{R}^d), \|\cdot\|_p), \\ \mathcal{S}(\mathbb{R}^d) = \bigcap_{p\in\mathbb{R}} (\mathcal{S}_p(\mathbb{R}^d), \|\cdot\|_p), \\ \mathcal{S}'(\mathbb{R}^d) = \bigcup_{p\in\mathbb{R}} (\mathcal{S}_p(\mathbb{R}^d), \|\cdot\|_p) \end{cases}$$

The following notations will be used throughout:

(i) $\mathcal{S}, \mathcal{S}', \mathcal{S}_p$ will stand for $\mathcal{S}(\mathbb{R}), \mathcal{S}'(\mathbb{R}), \mathcal{S}_p(\mathbb{R})$ respectively.
(ii) Given $\psi \in \mathcal{S}(\mathbb{R}^d)$ (or $\mathcal{S}_p(\mathbb{R}^d)$) and $\phi \in \mathcal{S}'(\mathbb{R}^d)$ (or $\mathcal{S}_{-p}(\mathbb{R}^d)$), the action of ϕ on ψ will be denoted by $\langle \phi\,,\,\psi\rangle$.

Since $\|\cdot\|_{-p}$ is the norm dual to $\|\cdot\|_p$ (for $p \geq 0$) we have

$$\|\phi\|_{-p} := \sup\{|\langle \phi\,,\,\psi\rangle| : \|\psi\|_p \leq 1, \psi \in \mathcal{S}_p(\mathbb{R}^d)\},\ \phi \in \mathcal{S}_{-p}(\mathbb{R}^d).$$

and

$$|\langle \phi\,,\,\psi\rangle| \leq \|\phi\|_{-p}\|\psi\|_p,\ \forall \phi \in \mathcal{S}_{-p}(\mathbb{R}^d), \psi \in \mathcal{S}_p(\mathbb{R}^d).$$

The next result provides an explicit expression for the norm $\|\cdot\|_p$.

Lemma 2.10.1. *Fix $p \in \mathbb{R}$ and $\psi, \phi \in \mathcal{S}_p(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$. Then*

(i) $\|\psi\|_p^2 = \sum_{k=0}^{\infty} \sum_{|n|=k} (2k+d)^{2p} \langle \psi , h_n \rangle^2$.

(ii) $\langle \psi , \phi \rangle_p = \sum_{k=0}^{\infty} \sum_{|n|=k} (2k+d)^{2p} \langle \psi , h_n \rangle \langle \phi , h_n \rangle$.

Proof. Suppose $\{\psi_m\}$ be a sequence in $\mathcal{S}$ converging to ψ in $\mathcal{S}_p$.
We first consider the case $p < 0$. Recall that $h_n^{-p} \in \mathcal{S}_{-p}(\mathbb{R}^d)$, $\forall n \in \mathbb{Z}_+^d$. Then

$$| \left\langle \psi_m , h_n^{-p} \right\rangle - \left\langle \psi , h_n^{-p} \right\rangle | \leq \|\psi_m - \psi\|_p \|h_n^{-p}\|_{-p} \xrightarrow{m\to\infty} 0 \ldots (*)$$

Again

$$| \langle \psi_m , h_n^p \rangle_p - \langle \psi , h_n^p \rangle_p | \leq \|\psi_m - \psi\|_p \|h_n^p\|_p \xrightarrow{m\to\infty} 0.$$

But

$$\langle \psi_m , h_n^p \rangle_p = (2k+d)^p \langle \psi_m , h_n \rangle_0 = \left\langle \psi_m , h_n^{-p} \right\rangle_0 = \left\langle \psi_m , h_n^{-p} \right\rangle.$$

Hence $\langle \psi , h_n^p \rangle_p = \langle \psi , h_n^{-p} \rangle = (2k+d)^p \langle \psi , h_n \rangle$ and

$$\|\psi\|_p^2 = \sum_{k=0}^{\infty} \sum_{|n|=k} \langle \psi , h_n^p \rangle_p^2 = \sum_{k=0}^{\infty} \sum_{|n|=k} (2k+d)^{2p} \langle \psi , h_n \rangle^2 .$$

If $p \geq 0$, observe that $\psi_m, \psi \in \mathcal{S}_0(\mathbb{R}^d) = \mathcal{L}^2(\mathbb{R}^d)$ and

$$\|\psi - \psi_m\|_0 \leq \|\psi - \psi_m\|_p \xrightarrow{m\to\infty} 0.$$

Now the statement $(*)$ can be proved as follows.

$$| \left\langle \psi_m , h_n^{-p} \right\rangle - \left\langle \psi , h_n^{-p} \right\rangle | \leq \|\psi_m - \psi\|_0 \|h_n^{-p}\|_0 \xrightarrow{m\to\infty} 0.$$

Hence $\langle \psi , h_n^p \rangle_p = \langle \psi , h_n^{-p} \rangle = (2k+d)^p \langle \psi , h_n \rangle$ and

$$\|\psi\|_p^2 = \sum_{k=0}^{\infty} \sum_{|n|=k} \langle \psi , h_n^p \rangle_p^2 = \sum_{k=0}^{\infty} \sum_{|n|=k} (2k+d)^{2p} \langle \psi , h_n \rangle^2 .$$

To prove (ii), let $\psi, \phi \in \mathcal{S}_p(\mathbb{R}^d)$. Then

$$\begin{aligned} \langle \psi , \phi \rangle_p &= \frac{1}{4}(\|\psi + \phi\|_p^2 - \|\psi - \phi\|_p^2) \\ &= \sum_{k=0}^{\infty} \sum_{|n|=k} (2k+d)^{2p} \frac{1}{4}(\langle \psi + \phi , h_n \rangle^2 - \langle \psi - \phi , h_n \rangle^2) \\ &= \sum_{k=0}^{\infty} \sum_{|n|=k} (2k+d)^{2p} \langle \psi , h_n \rangle \langle \phi , h_n \rangle \end{aligned}$$

This completes the proof. □

Proposition 2.10.2. *Let $\phi \in \mathcal{S}'(\mathbb{R}^d)$. Then ϕ is determined by the values $\{\langle \phi\,,\, h_n\rangle : n \in \mathbb{Z}_+^d\}$. In particular, any element $\phi \in \mathcal{S}_p(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$ (for some $p \in \mathbb{R}$) is determined by the same collection.*

Proof. Since $\mathcal{S}_{-q}(\mathbb{R}^d) \uparrow \mathcal{S}'(\mathbb{R}^d)$ as $q \to \infty$, there exists a positive real number p such that $\phi \in \mathcal{S}_{-p}(\mathbb{R}^d)$. Since $\{h_n^{-p} : n \in \mathbb{Z}_+^d\}$ forms an orthonormal basis for $\mathcal{S}_{-p}(\mathbb{R}^d)$, we have

$$\phi \overset{\mathcal{S}_{-p}(\mathbb{R}^d)}{=} \sum_{k=0}^{\infty} \sum_{|n|=k} \left\langle \phi\,,\, h_n^{-p} \right\rangle_{-p} h_n^{-p}.$$

Then for any $\psi \in \mathcal{S}(\mathbb{R}^d)$, $\langle \phi\,,\, \psi\rangle = \sum_{k=0}^{\infty}\sum_{|n|=k} \langle \phi\,,\, h_n^{-p}\rangle_{-p} \langle h_n^{-p}\,,\, \psi\rangle$. Therefore the tempered distribution $\phi : \psi \mapsto \langle \phi\,,\, \psi\rangle$ is determined by the values $\{\langle \phi\,,\, h_n^{-p}\rangle_{-p} : n \in \mathbb{Z}_+^d\}$. Now for any $m \in \mathbb{Z}_+^d$, using Lemma 2.10.1

$$\begin{aligned}
\left\langle \phi\,,\, h_m^{-p} \right\rangle_{-p} &= \sum_{k=0}^{\infty} \sum_{|n|=k} (2k+d)^{-2p} \left\langle \phi\,,\, h_n\right\rangle \left\langle h_m^{-p}\,,\, h_n \right\rangle \\
&= (2|m|+d)^{-2p} \left\langle \phi\,,\, h_m\right\rangle \left\langle h_m^{-p}\,,\, h_m \right\rangle \\
&= (2|m|+d)^{-p} \left\langle \phi\,,\, h_m\right\rangle .
\end{aligned}$$

Hence the values $\{\langle \phi\,,\, h_n^{-p}\rangle_{-p} : n \in \mathbb{Z}_+^d\}$ are determined by $\{\langle \phi\,,\, h_n\rangle : n \in \mathbb{Z}_+^d\}$ and so is ϕ. ∎

Proposition 2.10.3 ([89, Proposition 1.1])**.** *The topology on $\mathcal{S}(\mathbb{R}^d)$ induced by the collection $\{\|\cdot\|_p : p = 1, 2, \cdots\}$ is the Schwartz topology.*

Definition 2.10.4. [53, Definitions 1.1.1 and 1.1.2] Let p, q be two real numbers. Let $\{e_n : n = 1, 2, \cdots\}$ be an orthonormal basis for $(\mathcal{S}_q(\mathbb{R}^d), \|\cdot\|_q)$. Then define

$$(p : q)_{HS} := \left(\sum_{n=1}^{\infty} \|e_n\|_p^2 \right)^{\frac{1}{2}}.$$

Then $\|\cdot\|_p$ is said to be Hilbert-Schmidt bounded by $\|\cdot\|_q$ if $(p : q)_{HS} < \infty$. We denote this by $\|\cdot\|_p \prec_{HS} \|\cdot\|_q$.

Remark 2.10.5. [53, Remark 1.1.1] $(p : q)_{HS}$ is well defined independently of the choice of the orthonormal normal basis $\{e_n\}$.

Proposition 2.10.6. *Let $p \in \mathbb{R}$ and $q > p + \frac{d}{2}$. Then $\|\cdot\|_p \prec_{HS} \|\cdot\|_q$.*

Proof. The collection $\{h_n^q : n \in \mathbb{Z}_+^d\}$ forms an orthonormal basis for $\mathcal{S}_q(\mathbb{R}^d)$ and we have

$$\sum_{k=0}^{\infty} \sum_{|n|=k} \|h_n^q\|_p^2 \leq \sum_{k=0}^{\infty} (2k+d)^{2(p-q)} \#\{n \in \mathbb{Z}_+^d : |n| = k\}.$$

By [34, Chapter II, section 5], $\#\{n \in \mathbb{Z}_+^d : |n| = k\} = \binom{k+d-1}{d-1}$. Now there exists a constant $C > 0$ such that

$$\binom{k+d-1}{d-1} = \frac{(k+d-1)(k+d-2)\cdots(k+1)}{(d-1)!} \leq C.(2k+d)^{d-1}. \tag{2.12}$$

Hence

$$\sum_{k=0}^{\infty} \sum_{|n|=k} \|h_n^q\|_p^2 \leq C \sum_{k=0}^{\infty} (2k+d)^{2(p-q)+d-1}.$$

In particular for $q > p + \frac{d}{2}$ the series on the right hand side is finite, which proves the statement. □

2.11 Computations with Hilbertian Norms $\|\cdot\|_p$

2.11.1 Operators on Hermite Sobolev spaces

First we study some well-known operators on the space of tempered distributions.

Example 2.11.1 (Shift operators)**.** Let $\{e_i : 1 \leq i \leq d\}$ denote the standard basis for $\mathbb{R}^d$. Define linear operators U_{-e_i}, U_{+e_i} on $\mathcal{S}_p(\mathbb{R}^d)$ by the formal expressions: for $\phi \overset{\mathcal{S}_p(\mathbb{R}^d)}{=} \sum_{k=0}^{\infty} \sum_{|n|=k} \phi_n h_n \in \mathcal{S}_p(\mathbb{R}^d)$,

$$U_{+e_i}\phi := \sum_{k=0}^{\infty} \sum_{|n|=k} \phi_{n+e_i} h_n, \quad U_{-e_i}\phi := \sum_{k=0}^{\infty} \sum_{|n|=k} \phi_{n-e_i} h_n. \tag{2.13}$$

Lemma 2.11.2. *Fix any $1 \leq i \leq d$ and $p \in \mathbb{R}$. The linear operators U_{+e_i}, U_{-e_i} defined as above are bounded linear operators on $\mathcal{S}_p(\mathbb{R}^d)$.*

Proof. Given any $\phi \in \mathcal{S}_p(\mathbb{R}^d)$, we can write $\phi \overset{\mathcal{S}_p(\mathbb{R}^d)}{=} \sum_{k=0}^{\infty} \sum_{|n|=k} \phi_n h_n$. Then observe that

$$\begin{aligned}
\|U_{+e_i}\phi\|_p^2 &= \sum_{k=0}^{\infty} \sum_{|n|=k} (2k+d)^{2p} \phi_{n+e_i}^2 \\
&= \sum_{k=0}^{\infty} \frac{(2k+d)^{2p}}{(2k+d+2)^{2p}} \sum_{|m|=k+1, m=n+e_i} (2k+d+2)^{2p} |\phi_m|^2 \\
&\leq \left(\sup_{k \geq 0} \frac{(2k+d)^{2p}}{(2k+d+2)^{2p}} \right) \sum_{k=1}^{\infty} \sum_{|m|=k} (2k+d)^{2p} |\phi_m|^2 \\
&\leq \left(\sup_{k \geq 0} \left(\frac{2k+d}{2k+d+2} \right)^{2p} \right) \|\phi\|_p^2
\end{aligned}$$

which implies U_{+e_i} is a bounded operator on $\mathcal{S}_p(\mathbb{R}^d)$. Similarly U_{-e_i} is also a bounded operator on $\mathcal{S}_p(\mathbb{R}^d)$. □

Example 2.11.3 (Derivative operators)**.** Consider the derivative maps denoted by $\partial_i : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ for $i = 1, \cdots, d$. By duality we can extend these to $\partial_i : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ as follows: for $\psi \in \mathcal{S}'(\mathbb{R}^d)$,

$$\langle \partial_i \psi \,,\, \phi \rangle := -\langle \psi \,,\, \partial_i \phi \rangle \,,\ \forall \phi \in \mathcal{S}(\mathbb{R}^d).$$

Lemma 2.11.4. *Fix* $1 \leq i \leq d$ *and* $p \in \mathbb{R}$. *Then* $\partial_i : \mathcal{S}_{p+\frac{1}{2}}(\mathbb{R}^d) \to \mathcal{S}_p(\mathbb{R}^d)$ *is a bounded linear operator.*

Proof. Since $\mathcal{S}(\mathbb{R}^d) \subset \mathcal{L}^2(\mathbb{R}^d)$ (see Lemma 2.9.1), any element $\phi \in \mathcal{S}(\mathbb{R}^d)$ can be written as $\phi = \sum_{k=0}^{\infty} \sum_{|n|=k} \phi_n h_n$ where $\phi_n \in \mathbb{R}$. Then using Proposition 2.8.1(v) we get

$$\begin{aligned}
\partial_i \phi &= \sum_{k=0}^{\infty} \sum_{|n|=k} \phi_n (\partial_i h_n) \\
&= \sum_{k=0}^{\infty} \sum_{|n|=k} \phi_n \left[\sqrt{\frac{n_i}{2}} h_{n-e_i} - \sqrt{\frac{n_i+1}{2}} h_{n+e_i} \right] \\
&= \sum_{k=0}^{\infty} \sum_{\substack{|m|=k-1,\\ m=n-e_i}} \phi_{m+e_i} \sqrt{\frac{m_i+1}{2}} h_m - \sum_{k=0}^{\infty} \sum_{\substack{|m|=k+1,\\ m=n+e_i}} \phi_{m-e_i} \sqrt{\frac{m_i}{2}} h_m \\
&= \sum_{\substack{l=-1,\\ l=k-1}}^{\infty} \sum_{|m|=l} \phi_{m+e_i} \sqrt{\frac{m_i+1}{2}} h_m - \sum_{\substack{l=1,\\ l=k+1}}^{\infty} \sum_{|m|=l} \phi_{m-e_i} \sqrt{\frac{m_i}{2}} h_m \\
&= \sum_{l=0}^{\infty} \sum_{|m|=l} \phi_{m+e_i} \sqrt{\frac{m_i+1}{2}} h_m - \sum_{l=0}^{\infty} \sum_{|m|=l} \phi_{m-e_i} \sqrt{\frac{m_i}{2}} h_m .
\end{aligned}$$

In the last step we have made use of convention that $h_m = 0, \phi_m = 0$ if $|m| = l = -1$. From the above computation we get

$$\partial_i \phi = \sum_{k=0}^{\infty} \sum_{|n|=k} \left[\phi_{n+e_i} \sqrt{\frac{n_i+1}{2}} - \phi_{n-e_i} \sqrt{\frac{n_i}{2}} \right] h_n. \tag{2.14}$$

Then

$$\begin{aligned}
\|\partial_i \phi\|_p^2 &= \sum_{k=0}^{\infty} \sum_{|n|=k} (2k+d)^{2p} \left[\phi_{n+e_i} \sqrt{\frac{n_i+1}{2}} - \phi_{n-e_i} \sqrt{\frac{n_i}{2}} \right]^2 \\
&\leq \sum_{k=0}^{\infty} \sum_{|n|=k} (2k+d)^{2p}\, 2 \left[\left(\phi_{n+e_i} \sqrt{\frac{n_i+1}{2}} \right)^2 + \left(\phi_{n-e_i} \sqrt{\frac{n_i}{2}} \right)^2 \right] \\
&\leq \sum_{k=0}^{\infty} \sum_{|n|=k} (2k+d)^{2p} \left[\phi_{n+e_i}^2 (n_i+1) + \phi_{n-e_i}^2 n_i \right]
\end{aligned}$$

Since $n_i \leq |n| = k$ we have $n_i \leq k < 2k + d$. Also $n_i \leq k \leq 2k$ implies $n_i + 1 \leq 2k + 1 \leq 2k + d$. Hence using the shift operators (see Lemma 2.11.2)

$$\begin{aligned} \|\partial_i \phi\|_p^2 &\leq \sum_{k=0}^{\infty} \sum_{|n|=k} (2k+d)^{2p+1} \left[\phi_{n+e_i}^2 + \phi_{n-e_i}^2\right] \\ &\leq \left[\|U_{+e_i}\|^2_{\mathcal{S}_{p+\frac{1}{2}} \to \mathcal{S}_{p+\frac{1}{2}}} + \|U_{-e_i}\|^2_{\mathcal{S}_{p+\frac{1}{2}} \to \mathcal{S}_{p+\frac{1}{2}}}\right] \|\phi\|^2_{p+\frac{1}{2}}. \end{aligned}$$

Since $\mathcal{S}(\mathbb{R}^d)$ is dense in $\mathcal{S}_{p+\frac{1}{2}}(\mathbb{R}^d)$ in $\|\cdot\|_{p+\frac{1}{2}}$, above bound extends to all ϕ in $\mathcal{S}_{p+\frac{1}{2}}(\mathbb{R}^d)$. This completes the proof. □

For any $1 \leq i \leq d$ and $p \in \mathbb{R}$, observe that $\{\psi \in \mathcal{S}_p(\mathbb{R}^d) : \partial_i \psi \in \mathcal{S}_p(\mathbb{R}^d)\} \supseteq \mathcal{S}_{p+\frac{1}{2}}(\mathbb{R}^d)$. Let us denote the set $\{\psi \in \mathcal{S}_p(\mathbb{R}^d) : \partial_i \psi \in \mathcal{S}_p(\mathbb{R}^d)\}$ by $D(\partial_i, p)$. Then $\partial_i : D(\partial_i, p) \subseteq \mathcal{S}_p(\mathbb{R}^d) \to \mathcal{S}_p(\mathbb{R}^d)$ is a linear operator.

Lemma 2.11.5. *The linear operator $\partial_i : \mathcal{S}_p(\mathbb{R}^d) \to \mathcal{S}_p(\mathbb{R}^d)$ with domain $D(\partial_i, p)$ is a closed unbounded linear operator on $\mathcal{S}_p(\mathbb{R}^d)$.*

Proof. Let $\{\phi_m\}$ be a sequence in $D(\partial_i, p)$ converging to $\phi \in D(\partial_i, p)$ in the norm $\|\cdot\|_p$. Let the sequence $\{\partial_i \phi_m\}$ converge to ψ in $\|\cdot\|_p$. To prove ∂_i is closed, we need to show $\partial_i \phi = \psi$. Since

(i) $\partial_i \phi, \psi$ are elements of $\mathcal{S}_p(\mathbb{R}^d)$,
(ii) $\mathcal{S}_{-p}(\mathbb{R}^d)$ is dual to $\mathcal{S}_p(\mathbb{R}^d)$,
(iii) $\{h_n^{-p}\}$ is an orthonormal basis for the space $\mathcal{S}_{-p}(\mathbb{R}^d)$,

to complete the proof it is enough to show (see Proposition 2.10.2)

$$\left\langle \partial_i \phi \,,\, h_n^{-p} \right\rangle = \left\langle \psi \,,\, h_n^{-p} \right\rangle, \forall n \in \mathbb{Z}_+^d.$$

Fix $n \in \mathbb{Z}_+^d$ and observe that

$$|\left\langle \psi \,,\, h_n^{-p} \right\rangle - \left\langle \partial_i \phi_m \,,\, h_n^{-p} \right\rangle| \leq \|\psi - \partial_i \phi_m\|_p \|h_n^{-p}\|_{-p} \xrightarrow{m\to\infty} 0.$$

Therefore $\langle \psi \,,\, h_n^{-p} \rangle = \lim_m \langle \partial_i \phi_m \,,\, h_n^{-p} \rangle = -\lim_m \langle \phi_m \,,\, \partial_i h_n^{-p} \rangle$.
Now $\partial_i h_n^{-p} \in \mathcal{S}(\mathbb{R}^d) \subset \mathcal{S}_{-p}(\mathbb{R}^d)$. Hence

$$|\left\langle \phi_m \,,\, \partial_i h_n^{-p} \right\rangle - \left\langle \phi \,,\, \partial_i h_n^{-p} \right\rangle \leq \|\phi_m - \phi\|_p \|h_n^{-p}\|_{-p} \xrightarrow{m\to\infty} 0.$$

Continuing from above,

$$\left\langle \psi \,,\, h_n^{-p} \right\rangle = -\lim_m \left\langle \phi_m \,,\, \partial_i h_n^{-p} \right\rangle = -\left\langle \phi \,,\, \partial_i h_n^{-p} \right\rangle = \left\langle \partial_i \phi \,,\, h_n^{-p} \right\rangle.$$

This shows ∂_i is closed.
Using Proposition 2.8.1(v), we have for any $n \in \mathbb{Z}_+^d$ with $|n| = k$,

$$\partial_i h_n^p = (2k+d)^{-p} \left(\sqrt{\frac{n_i}{2}} h_{n-e_i} - \sqrt{\frac{n_i+1}{2}} h_{n+e_i} \right).$$

Suppose $p \geq 0$. Note that for $k \geq 0$

$$\begin{aligned}
\|\partial_i h_n^p\|_p^2 &= (2k+d)^{-2p} \left\| \sqrt{\frac{n_i}{2}} h_{n-e_i} - \sqrt{\frac{n_i+1}{2}} h_{n+e_i} \right\|_p^2 \\
&= (2k+d)^{-2p} \left\| \sqrt{\frac{n_i}{2}} h_{n-e_i} \right\|_p^2 + (2k+d)^{-2p} \left\| \sqrt{\frac{n_i+1}{2}} h_{n+e_i} \right\|_p^2 \\
&\geq (2k+d)^{-2p} \left\| \sqrt{\frac{n_i+1}{2}} h_{n+e_i} \right\|_p^2 \\
&= \frac{n_i+1}{2} \left(\frac{2(k+1)+d}{2k+d} \right)^{2p} \\
&\geq \frac{n_i+1}{2}.
\end{aligned}$$

In particular, $\|\partial_i h_{ke_i}^p\|_p^2 \geq \frac{k+1}{2}$ with $h_{ke_i}^p \in D(\partial_i, p)$. Hence $\partial_i : \mathcal{S}_p(\mathbb{R}^d) \to \mathcal{S}_p(\mathbb{R}^d)$ is an unbounded operator if $p \geq 0$.
If $p < 0$ then observe that for $k \geq 1$

$$\begin{aligned}
\|\partial_i h_n^p\|_p^2 &= (2k+d)^{-2p} \left\| \sqrt{\frac{n_i}{2}} h_{n-e_i} \right\|_p^2 + (2k+d)^{-2p} \left\| \sqrt{\frac{n_i+1}{2}} h_{n+e_i} \right\|_p^2 \\
&\geq (2k+d)^{-2p} \left\| \sqrt{\frac{n_i}{2}} h_{n-e_i} \right\|_p^2 \\
&= \frac{n_i}{2} \left(\frac{2(k-1)+d}{2k+d} \right)^{2p} \geq \frac{n_i}{2}.
\end{aligned}$$

Using arguments as in the case $p \geq 0$, we can prove $\partial_i : \mathcal{S}_p(\mathbb{R}^d) \to \mathcal{S}_p(\mathbb{R}^d)$ is an unbounded operator if $p < 0$. □

Example 2.11.6 (Translation operators)**.** For $x \in \mathbb{R}^d$, define translation operators on Schwartz class functions by

$$(\tau_x f)(y) := f(y-x), \forall y \in \mathbb{R}^d.$$

We can extend this operator to $\tau_x : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ by

$$\langle \tau_x \phi \,,\, \psi \rangle := \langle \phi \,,\, \tau_{-x} \psi \rangle \,,\, \forall \phi \in \mathcal{S}'(\mathbb{R}^d), \psi \in \mathcal{S}(\mathbb{R}^d).$$

Lemma 2.11.7. *The translation operators have the following properties:*

(i) ([91, Theorem 2.1]) For $x \in \mathbb{R}^d$ and any $p \in \mathbb{R}$, $\tau_x : \mathcal{S}_p(\mathbb{R}^d) \to \mathcal{S}_p(\mathbb{R}^d)$ is a bounded linear map. In particular, there exists a real polynomial P_k of degree $k = 2([|p|]+1)$ such that

$$\|\tau_x \phi\|_p \leq P_k(|x|)\|\phi\|_p, \; \forall \phi \in \mathcal{S}_p(\mathbb{R}^d),$$

where $|x|$ denotes the standard Euclidean norm of x.

(ii) Fix $\phi \in \mathcal{S}_p(\mathbb{R}^d)$. Then $x \mapsto \tau_x \phi$ is continuous.

(iii) For $x \in \mathbb{R}^d$ and any $i = 1, \cdots, d$ we have $\tau_x \partial_i = \partial_i \tau_x$.

Proof. Proof of part (ii) is contained in the proof of [92, Proposition 3.1].
We only prove part (iii). Fix an element $\psi \in \mathcal{S}(\mathbb{R}^d)$. Then for $y \in \mathbb{R}^d$,

$$(\tau_{-x}\partial_i \psi)(y) = (\partial_i \psi)(y+x) = \partial_i(\psi(y+x)) = (\partial_i \tau_{-x}\psi)(y)$$

i.e. $\tau_{-x}\partial_i\psi = \partial_i \tau_{-x}\psi$. Now for $\phi \in \mathcal{S}'(\mathbb{R}^d)$ and any $\psi \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned} \langle \partial_i \tau_x \phi \,,\, \psi \rangle &= - \langle \tau_x \phi \,,\, \partial_i \psi \rangle = - \langle \phi \,,\, \tau_{-x}\partial_i \psi \rangle \\ &= - \langle \phi \,,\, \partial_i \tau_{-x}\psi \rangle = \langle \partial_i \phi \,,\, \tau_{-x}\psi \rangle = \langle \tau_x \partial_i \phi \,,\, \psi \rangle \,. \end{aligned}$$

This completes the proof. □

The Hermite-Sobolev spaces $\mathcal{S}_p(\mathbb{R}^d;\mathbb{C})$ can also be defined for Schwartz space $\mathcal{S}(\mathbb{R}^d;\mathbb{C})$ where the functions are complex valued. For any fixed $p \in \mathbb{R}$, the inner product and the norm become

$$\begin{cases} \langle f \,,\, g \rangle_{\mathcal{S}_p(\mathbb{R}^d;\mathbb{C})} := \sum_{k=0}^{\infty} \sum_{|n|=k} (2k+d)^{2p} \langle f \,,\, h_n \rangle \, \langle g \,,\, h_n \rangle \,, \\ \|f\|^2_{\mathcal{S}_p(\mathbb{R}^d;\mathbb{C})} := \sum_{k=0}^{\infty} \sum_{|n|=k} (2k+d)^{2p} \, |\langle f \,,\, h_n \rangle|^2 \end{cases} \tag{2.15}$$

where $|\cdot|$ is the absolute value in the complex plane. If $T \in \mathcal{S}'(\mathbb{R}^d;\mathbb{C})$ is such that $\langle T \,,\, \phi \rangle \in \mathbb{R}, \forall \phi \in \mathcal{S}(\mathbb{R}^d)$ then we have

$$\|T\|_{\mathcal{S}_p(\mathbb{R}^d;\mathbb{C})} = \|T\big|_{\mathcal{S}_p(\mathbb{R}^d)}\|_p, \tag{2.16}$$

since $\langle T \,,\, h_n \rangle$, $n \in \mathbb{Z}^+_d$ are real.

Example 2.11.8 (Fourier transform)**.** Consider the Fourier transform of $\phi \in \mathcal{S}(\mathbb{R}^d;\mathbb{C})$ defined by

$$\widehat{\phi}(x) := \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix.y}\phi(y)\,dy.$$

We list some well-known properties of the Fourier transform.

(i) $\widehat{\cdot} : \mathcal{S}(\mathbb{R}^d;\mathbb{C}) \to \mathcal{S}(\mathbb{R}^d;\mathbb{C})$ is a continuous linear onto map. By duality, we can define the Fourier transform of tempered distributions: let $\psi \in \mathcal{S}'(\mathbb{R}^d)$, then define $\widehat{\psi} \in \mathcal{S}'(\mathbb{R}^d)$ by

$$\left\langle \widehat{\psi}\,,\,\phi \right\rangle := \left\langle \psi\,,\,\widehat{\phi} \right\rangle, \forall \phi \in \mathcal{S}(\mathbb{R}^d).$$

(ii) For $\phi, \psi \in \mathcal{S}(\mathbb{R}^d;\mathbb{C})$, $\widehat{\phi \star \psi} = \widehat{\phi}\widehat{\psi}$ where $\phi \star \psi$ denotes the convolution given by $\phi \star \psi(x) = \int_{\mathbb{R}^d} \phi(y)\psi(x-y)\,dy$.

(iii) Note that (see [47, Appendix A.5, equation (A.27)])

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix.y} h_n(y)\,dy = (-i)^n h_n(x),\ \forall x \in \mathbb{R}.$$

Then for any $n = (n_1, \cdots, n_d) \in \mathbb{Z}_+^d$ and $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$

$$\begin{aligned}
&\widehat{h}_n(x) \\
&= \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix.y} h_n(y)\,dy \\
&= \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix_1.y_1} h_{n_1}(y_1)\,dy_1\right) \times \cdots \times \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix_d.y_d} h_{n_d}(y_d)\,dy_d\right) \\
&= (-i^{n_1}) h_{n_1}(x_1) \cdots (-i^{n_d}) h_{n_d}(x_d) \\
&= (-i)^{|n|} h_n(x).
\end{aligned}$$

This implies $\widehat{\cdot} : \mathcal{S}_p(\mathbb{R}^d;\mathbb{C}) \to \mathcal{S}_p(\mathbb{R}^d;\mathbb{C})$ is an onto isometry, i.e.

$$\|\widehat{T}\|_{\mathcal{S}_p(\mathbb{R}^d;\mathbb{C})} = \|T\|_{\mathcal{S}_p(\mathbb{R}^d;\mathbb{C})}. \tag{2.17}$$

Example 2.11.9 (Multiplication operators)**.** Consider the multiplication operators $\mathscr{M}_i, i = 1, \cdots, d$ defined by

$$(\mathscr{M}_i\phi)(x) := x_i\phi(x),\ \phi \in \mathcal{S}(\mathbb{R}^d), x = (x_1, \cdots, x_d) \in \mathbb{R}^d.$$

By duality these operators can be extended to $\mathscr{M}_i : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$. Using arguments as in Lemma 2.11.4 and the recurrence formula in Proposition 2.8.1(v) we can show $\mathscr{M}_i : \mathcal{S}_{p+\frac{1}{2}}(\mathbb{R}^d) \to \mathcal{S}_p(\mathbb{R}^d)$ are bounded linear operators, for any $p \in \mathbb{R}$. Using arguments as in Lemma 2.11.5, we can show $\mathscr{M}_i : \mathcal{S}_p(\mathbb{R}^d) \to \mathcal{S}_p(\mathbb{R}^d)$ are closed unbounded linear operators on $\mathcal{S}_p(\mathbb{R}^d)$.

Example 2.11.10 (Hermite operator)**.** For $\phi \in \mathcal{S}(\mathbb{R}^d)$ define $\mathbf{H} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ by (see [109, Chapter 1, page 2] and [91, section 3])

$$\mathbf{H}\phi := \sum_{i=1}^{d} (\mathscr{M}_i^2 - \partial_i^2)\phi,$$

which can also be written as $(|x|^2 - \triangle)\phi$. It is well-known that

$$\mathbf{H}h_n = (2k+d)h_n \tag{2.18}$$

where $|n| = k$ for any multi-index $n = (n_1, \cdots, n_d)$. In particular, $\mathbf{H}$ is a positive operator on $\mathcal{L}^2(\mathbb{R}^d)$.
For $p \in \mathbb{R}$, define

$$\mathbf{H}^p\phi := \sum_{k=0}^{\infty} \sum_{|n|=k} (2k+d)^p \langle \phi\,,\, h_n\rangle\, h_n.$$

Example 2.11.11 (Creation and Annihilation operators)**.** For $1 \leq i \leq d$, define the Creation operators by

$$A_i^+\phi := (\mathscr{M}_i - \partial_i)\phi,\ \phi \in \mathcal{S}(\mathbb{R}^d)$$

and the Annihilation operators by

$$A_i^-\phi := (\mathscr{M}_i + \partial_i)\phi,\ \phi \in \mathcal{S}(\mathbb{R}^d).$$

Proposition 2.11.12 ([91, Proposition 3.1])**.** *Some properties of the operators* $\mathbf{H}, A_i^+, A_i^-$ *are listed below.*

(i) For any $p, q \in \mathbb{R}$, $\|\mathbf{H}^p\phi\|_{q-p} = \|\phi\|_q, \forall \phi \in \mathcal{S}(\mathbb{R}^d)$. *Consequently,* $\mathbf{H}^p : \mathcal{S}_q(\mathbb{R}^d) \to \mathcal{S}_{q-p}(\mathbb{R}^d)$ *extends as a linear isometry. Moreover this linear map is onto.*

(ii) $\mathbf{H} = \sum_{i=1}^d (A_i^- A_i^+ + A_i^+ A_i^-)$.

2.11.2 Some tempered distributions

Example 2.11.13 (Dirac distributions)**.** Fix $x \in \mathbb{R}^d$ and define the Dirac distribution δ_x by

$$\langle \delta_x\,,\, \phi\rangle := \phi(x),\ \forall \phi \in \mathcal{S}(\mathbb{R}^d).$$

$\delta_x \in \mathcal{S}'(\mathbb{R}^d)$ since $|\langle \delta_x\,,\, \phi\rangle| \leq \sup_{y\in\mathbb{R}^d} |\phi(y)|$.

The following result is an important property of the Dirac distributions.

Proposition 2.11.14 ([92, Theorem 4.1])**.** *(i) Let* $x \in \mathbb{R}^d$. *Then* $\delta_x \in \mathcal{S}_{-p}(\mathbb{R}^d)$ *for any* $p > \frac{d}{4}$. *Further if* $p > \frac{d}{4}$, *then* $\lim_{|x|\to\infty} \|\delta_x\|_{-p} = 0$.

(ii) Let $\gamma \in \mathbb{Z}_+^d$. *Let* $p > \frac{d}{4} + \frac{|\gamma|}{2}$. *Then*

$$\sup_{x\in\mathbb{R}^d} \|\partial^\gamma \delta_x\|_{-p} < \infty.$$

In particular, for any $p > \frac{d}{4}$, *there exists a constant* $C = C(p) > 0$ *such that* $\|\delta_x\|_{-p} \leq C, \forall x \in \mathbb{R}^d$.

We point out a well-known property of Dirac distributions. This property will be used in Chapter 6.

Lemma 2.11.15. *For any $x \in \mathbb{R}^d$, $\tau_x \delta_0 = \delta_x$.*

Proof. For any $\phi \in \mathcal{S}(\mathbb{R}^d)$,

$$\langle \tau_x \delta_0 \, , \, \phi \rangle = \langle \delta_0 \, , \, \tau_{-x} \phi \rangle = \langle \delta_0 \, , \, \phi(\cdot + x) \rangle = \phi(x) = \langle \delta_x \, , \, \phi \rangle \, .$$

Hence the required equality follows. □

We explicitly compute the norm of a Dirac distribution.

Lemma 2.11.16. $\|\delta_0\|^2_{-\frac{1}{2}} = \frac{1}{4\sqrt{2\pi}} \Gamma\left(\frac{1}{4}\right)^2$.

Proof. Using Lemma 2.10.1, we have

$$\begin{aligned} \|\delta_0\|^2_{-\frac{1}{2}} &= \sum_{n=0}^{\infty} (2n+1)^{-1} \langle \delta_0 \, , \, h_n \rangle^2 \\ &= \sum_{n=0}^{\infty} (2n+1)^{-1} h_n(0)^2 \\ &= \frac{1}{\sqrt{\pi}} \sum_{\substack{m=0, \\ n=2m}}^{\infty} \frac{1}{4m+1} \frac{((2m-1)!!)^2}{2m!}, \text{ (see Proposition 2.8.1(vi))} \end{aligned} \tag{2.19}$$

Call $a_m := \frac{1}{4m+1} \frac{((2m-1)!!)^2}{2m!}$, $m = 0, 1, \cdots$. Then $a_0 = 1$ and

$$\begin{aligned} \frac{a_{m+1}}{a_m} &= \frac{4m+1}{4m+5} \left(\frac{(2m+1)!!}{(2m-1)!!} \right)^2 \frac{2m!}{(2m+2)!} \\ &= \frac{4m+1}{4m+5} \frac{2m+1}{2m+2} = \frac{(m+\frac{1}{2})(m+\frac{1}{4})}{(m+\frac{5}{4})(m+1)} \end{aligned}$$

and hence using Pochhammer's symbol (see [1, 6.1.22, p. 256])

$$a_m = \frac{a_m}{a_{m-1}} \times \cdots \times \frac{a_1}{a_0} \times a_0 = \prod_{k=0}^{m-1} \frac{(k+\frac{1}{2})(k+\frac{1}{4})}{(k+\frac{5}{4})(k+1)} = \frac{(\frac{1}{2})_m (\frac{1}{4})_m}{(\frac{5}{4})_m} \frac{1}{m!}.$$

Then the sum of the series $\sum_{m=0}^{\infty} a_m$ in (2.19) is the evaluation of the Gauss Hypergeometric series ${}_2F_1(\frac{1}{2}, \frac{1}{4}; \frac{5}{4}; z)$ (see [1, 15.1.1, p. 558]) at $z = 1$. Note that $\frac{5}{4} - \frac{1}{2} - \frac{1}{4} = \frac{1}{2} > 0$ and hence using [1, 15.1.20, p. 558] we have

$$\|\delta_0\|^2_{-\frac{1}{2}} = \frac{1}{\sqrt{\pi}} \, {}_2F_1\left(\frac{1}{2}, \frac{1}{4}; \frac{5}{4}; 1 \right) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{5}{4}) \Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4}) \Gamma(1)}.$$

We recall some properties of the Gamma function (see [97, pp. 192-194]).

(i) $\Gamma(1) = 1$.
(ii) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ (see [97, equation (99), p. 194]).
(iii) By [97, Theorem 8.18] $\Gamma(\frac{5}{4}) = \frac{1}{4}\Gamma(\frac{1}{4})$.
(iv) For $0 < x < \infty$ we have the identity (see [97, equation (102), p. 194])

$$\Gamma(x) = \frac{2^{x-1}}{\sqrt{\pi}}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right).$$

Putting $x = \frac{1}{2}$ in the above identity, we have

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2\pi}\Gamma\left(\frac{1}{2}\right) = \pi\sqrt{2}.$$

Then

$$\|\delta_0\|^2_{-\frac{1}{2}} = \frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac{5}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})\Gamma(1)} = \frac{\frac{1}{4}\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} = \frac{1}{4}\frac{\left(\Gamma(\frac{1}{4})\right)^2}{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})} = \frac{1}{4\sqrt{2}\pi}\Gamma\left(\frac{1}{4}\right)^2.$$

□

Example 2.11.17 (Distributions given by constant functions)**.** For any $\phi \in \mathcal{S}(\mathbb{R}^d;\mathbb{C})$ we have (see Example 2.11.8)

$$\left\langle \widehat{\delta}_0\,,\,\phi\right\rangle = \left\langle \delta_0\,,\,\widehat{\phi}\right\rangle = \widehat{\phi}(0) = \left(\frac{1}{2\pi}\right)^{\frac{d}{2}}\int_{\mathbb{R}^d}\phi(y)\,dy = \left(\frac{1}{2\pi}\right)^{\frac{d}{2}}\langle 1\,,\,\phi\rangle\,,$$

where 1 represents the tempered distribution given by the constant function 1. Now for $p > \frac{d}{4}$,

$$\begin{aligned}\|1\|_{-p} = \|1\big|_{\mathcal{S}_{-p}(\mathbb{R}^d)}\|_{-p} &= \|1\|_{\mathcal{S}_{-p}(\mathbb{R}^d;\mathbb{C})},\ (\text{by } (2.16))\\ &= (2\pi)^{\frac{d}{2}}\|\widehat{\delta}_0\|_{\mathcal{S}_{-p}(\mathbb{R}^d;\mathbb{C})},\ (\text{by } (2.17))\\ &= (2\pi)^{\frac{d}{2}}\|\delta_0\|_{\mathcal{S}_{-p}(\mathbb{R}^d;\mathbb{C})}\\ &= (2\pi)^{\frac{d}{2}}\|\delta_0\|_{-p},\ (\text{by } (2.16)).\end{aligned}$$

Hence $1 \in \mathcal{S}_{-p}(\mathbb{R}^d)$ for $p > \frac{d}{4}$.

Example 2.11.18 (Distributions given by multiplication)**.** On $\mathbb{R}^d$ look at the mapping $x \mapsto x_i$. Since this map has linear growth, we get a tempered distribution, which we denote by x_i. Observe that for any $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $p > \frac{d}{4}$

$$\begin{aligned}|\langle x_i\,,\,\phi\rangle| = |\langle 1\,,\,\mathscr{M}_i\phi\rangle| &\leq \|1\|_{-p}\|\mathscr{M}_i\phi\|_p\\ &\leq \|1\|_{-p}\|\mathscr{M}_i\|_{\mathcal{S}_{p+\frac{1}{2}}(\mathbb{R}^d)\to\mathcal{S}_p(\mathbb{R}^d)}\cdot\|\phi\|_{p+\frac{1}{2}}.\end{aligned}$$

Since $1 \in \mathcal{S}_{-p}(\mathbb{R}^d)$ for $p > \frac{d}{4}$ (see Example 2.11.17), we have $x_i \in \mathcal{S}_{-p}(\mathbb{R}^d)$ for $p > \frac{d}{4} + \frac{1}{2} = \frac{d+2}{4}$.

Example 2.11.19 (Distributions given by integrable functions)**.** Given $f \in \mathcal{L}^1(\mathbb{R}^d)$, i.e. an integrable function, observe that

$$\left|\int_{\mathbb{R}^d} f(x)\phi(x)\,dx\right| \leq \sup_{x\in\mathbb{R}^d} |\phi(x)| \int_{\mathbb{R}^d} |f(x)|\,dx,\ \forall \phi \in \mathcal{S}(\mathbb{R}^d).$$

Therefore any integrable function acts as a tempered distribution. Abusing standard notations, we denote by $\mathcal{L}^1(\mathbb{R}^d)$ the space of tempered distributions which are given by integrable functions. If ψ is such a distribution, then we denote the corresponding integrable function again by ψ. Note that $\mathcal{S}(\mathbb{R}^d) \subset \mathcal{L}^1(\mathbb{R}^d)$ (see Lemma 2.9.1).

Next result will be used in Chapter 4.

Lemma 2.11.20. *Let* $p > \frac{d}{4}$. *Then* $\mathcal{L}^1(\mathbb{R}^d) \subset \mathcal{S}_{-p}(\mathbb{R}^d)$.

Proof. By Proposition 2.11.14(ii), there exists a constant $C = C(p) > 0$ such that $\|\delta_x\|_{-p} \leq C,\ \forall x \in \mathbb{R}^d$. Observe that for $\psi \in \mathcal{L}^1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} |\psi(x)|.\|\delta_x\|_{-p}\,dx \leq C \int_{\mathbb{R}^d} |\psi(x)|\,dx < \infty.$$

Hence $\int_{\mathbb{R}^d} \psi(x)\delta_x\,dx$ exists as a Bochner integral and is a well-defined element of $\mathcal{S}_{-p}(\mathbb{R}^d)$. But for any $\phi \in \mathcal{S}(\mathbb{R}^d)$,

$$\langle \psi\,,\,\phi\rangle = \int_{\mathbb{R}^d} \psi(x)\phi(x)\,dx = \int_{\mathbb{R}^d} \psi(x)\,\langle \delta_x\,,\,\phi\rangle\ dx = \left\langle \int_{\mathbb{R}^d} \psi(x)\delta_x\,dx\,,\,\phi\right\rangle.$$

Therefore as a tempered distribution $\psi = \int_{\mathbb{R}^d} \psi(x)\delta_x\,dx$ and hence $\psi \in \mathcal{S}_{-p}(\mathbb{R}^d)$, which proves (ii). □

The next result is an well-known application of Stirling's approximation and we use it in Example 2.11.22.

Lemma 2.11.21. *We have*

$$\lim_{n\to\infty} \frac{\sqrt{\pi n}\binom{2n}{n}}{4^n} = 1.$$

Proof. By Stirling's approximation (see [97, p. 194, equation (103)]),

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\,e^{-n}n^n} = 1.$$

Writing $\binom{2n}{n} = \dfrac{2n!}{(n!)^2}$ and using the above limit, we get the result. □

Example 2.11.22 (Distribution given by the Heaviside function)**.** Let $H = \mathbb{1}_{(0,\infty)}$. Now the distributional derivative of H is given by δ_0, since for any $\phi \in \mathcal{S}$,

$$\langle \partial H\, ,\, \phi \rangle = -\langle H\, ,\, \partial\phi \rangle = -\int_0^\infty \partial\phi(y)\, dy = \phi(0) = \langle \delta_0\, ,\, \phi \rangle\, .$$

Note that $\partial h_n = \sqrt{\frac{n}{2}}h_{n-1} - \sqrt{\frac{n+1}{2}}h_{n+1}$, $n \geq 0$ (see Proposition 2.8.1(v)). Set $a_n = \langle H\, ,\, h_n \rangle$, $n \geq 0$. Then

$$\|H\|_p^2 = \sum_{n=0}^{\infty}(2n+1)^{2p}a_n^2.$$

We want to identify a $p \in \mathbb{R}$ such that this series is finite (i.e. $H \in \mathcal{S}_p$) and with this goal in mind, we first obtain a growth estimate of $|a_n|$.
We also take $a_{-1} = 0$. Then from the previous relations we get

$$-\sqrt{\frac{n}{2}}a_{n-1} + \sqrt{\frac{n+1}{2}}a_{n+1} = h_n(0),\ n \geq 0.$$

A direct computation gives $a_0 = \int_0^\infty h_0(y)\, dy = \frac{\sqrt[4]{\pi}}{\sqrt{2}}$. From this recurrence relations, for $n = 0$ we get $a_1 = \sqrt{2}h_0(0) = \frac{1}{\sqrt{2}\sqrt[4]{\pi}}$. Recall that (see Proposition 2.8.1(vi))

$$h_n(0) = \begin{cases} 0, \text{ if } n \text{ is odd}, \\ (-1)^{\frac{n}{2}} \frac{1}{\sqrt[4]{\pi}} \frac{(n-1)!!}{\sqrt{n!}}, \text{ if } n \text{ is even} \end{cases}.$$

Simplifying the recurrence relation, we get for any integer $n \geq 1$,

$$a_{2n} = a_0 \frac{\sqrt{(2n)!}}{2^n.n!}, \quad \sqrt{\frac{2n+1}{2n}}a_{2n+1} = a_{2n-1} + \frac{1}{\sqrt{n}}h_{2n}(0). \tag{2.20}$$

Now multiplying the recurrence of the odd-numbered terms by $\sqrt{\frac{(2n-1)!!}{2^{n-1}(n-1)!}}$ we get

$$\begin{aligned}\sqrt{\frac{(2n+1)!!}{2^n n!}}a_{2n+1} &= \sqrt{\frac{(2n-1)!!}{2^{n-1}(n-1)!}}a_{2n-1} + \sqrt{\frac{(2n-1)!!}{2^{n-1}n!}}h_{2n}(0) \\ &= \sqrt{\frac{(2n-1)!!}{2^{n-1}(n-1)!}}a_{2n-1} + (-1)^n \frac{\sqrt{2}}{\sqrt[4]{\pi}} \frac{(2n)!}{4^n(n!)^2}\end{aligned}$$

A telescopic sum gives,

$$\sqrt{\frac{(2n+1)!!}{2^n n!}}a_{2n+1} = a_1 + \sum_{n=1}^{k}(-1)^n \frac{\sqrt{2}}{\sqrt[4]{\pi}} \frac{(2n)!}{4^n(n!)^2} = a_1 + \sum_{n=1}^{k}(-1)^n b_n \tag{2.21}$$

where $b_n = \dfrac{\sqrt{2}}{\sqrt[4]{\pi}} \dfrac{(2n)!}{4^n (n!)^2}$, $n \geq 1$. By Lemma 2.11.21, $\dfrac{\pi^{\frac{3}{4}}}{\sqrt{2}} \sqrt{n} b_n \to 1$ as $n \to \infty$. Hence $b_n \to 0$ as $n \to \infty$.
But $\frac{b_{n+1}}{b_n} = \frac{2n+1}{2(n+1)} < 1$, i.e. $\{b_n\}$ is a monotonically decreasing sequence. Hence $\sum_{n=1}^{\infty} (-1)^n b_n$ converges and hence the partial sum sequence is bounded. From the telescopic sum relation (2.21), we now conclude the existence of a constant $C > 0$ such that $|a_{2n+1}| \leq \frac{C}{\sqrt[4]{n}}$, $n \geq 1$.
Using the recurrence of the even-numbered terms (see equation (2.20)), by Lemma 2.11.21 we have $\sqrt[4]{\pi n} a_{2n} \to 1$ as $n \to \infty$.
Then, we can choose the constant C large enough so that $a_{2n} \leq \frac{C}{\sqrt[4]{n}}$, $n \geq 1$.
For H to belong to some $\mathcal{S}_{-p}$ we need the convergence of $\sum_{n=1}^{\infty} (2n+1)^{-2p} \frac{1}{\sqrt{n}}$, which happens if $p > \frac{1}{4}$. So $H \in \mathcal{S}_{-p}$ for $p > \frac{1}{4}$.

Remark 2.11.23. Computation of the coefficients (with respect to the basis of $\{h_n : n = 0, 1, \cdots\}$) of some tempered distributions are available in some texts. For Heaviside function see [19, Section 2, equation (8)], [99, p. 162].

Example 2.11.24 (Distribution given by the sign function)**.** Consider the sign function f on $\mathbb{R}$ given by

$$f(x) = \begin{cases} 1, \text{ if } x > 0 \\ -1, \text{ if } x \leq 0 \end{cases} .$$

Observe that $f = 2H - 1$ and hence the distribution given by f is in $\mathcal{S}_{-p}$ for $p > \frac{1}{4}$.

Example 2.11.25 (Distributions given by the Sine and Cosine functions)**.** Using the Fourier transform on the Hermite functions we have

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixy} h_n(y)\, dy = (-i)^n h_n(x). \tag{2.22}$$

Evaluating the previous relation at $x = \pm 1$ and adding we have

$$\int_{-\infty}^{\infty} \cos(y) h_n(y)\, dy = (-i)^n \sqrt{\frac{\pi}{2}} (h_n(1) + h_n(-1)) = (-i)^n \sqrt{\frac{\pi}{2}} \langle \delta_1 + \delta_{-1} \,,\, h_n \rangle .$$

Observe that the leftmost term in the above equality is real. There is no inconsistency in the previous relation since $h_n(1) + h_n(-1) = 0$ for odd values of n (if n is odd then so is h_n, see Proposition 2.8.1(iii)).
Evaluation of (2.22) at $x = \pm 1$ and subtraction gives

$$\int_{-\infty}^{\infty} \sin(y) h_n(y)\, dy = (i)^{n-1} \sqrt{\frac{\pi}{2}} \langle \delta_1 - \delta_{-1} \,,\, h_n \rangle .$$

Since $\delta_1, \delta_{-1} \in \mathcal{S}_{-p}$ for $p > \frac{1}{4}$ (see Proposition 2.11.14), the tempered distributions given by the Sine and Cosine functions are also in the same space.

Fix $p > d + \frac{1}{2}$ and $y \in \mathcal{S}_p(\mathbb{R}^d)$. Note that $\delta_x \in \mathcal{S}_{-p}(\mathbb{R}^d), \forall x \in \mathbb{R}^d$ (see Proposition 2.11.14). Hence $x \mapsto \langle \delta_x \,,\, y\rangle : \mathbb{R}^d \to \mathbb{R}$ is well-defined. Abusing notation, we denote this function by y. Next result is about the continuity and differentiability of the function y.

Proposition 2.11.26. *Let p, y be as above. Then the first order partial derivatives of function y exist and the distribution y is given by the differentiable function y. Furthermore, the first order distributional derivatives of y are given by the first order partial derivatives of y, which are continuous functions.*

Proof. We can write y in terms of the orthonormal basis $\{h_n^p : n \in \mathbb{Z}_+^d\}$, where $h_n^p = (2k+d)^{-p} h_n$ with $|n| = k$. Then $y \overset{\mathcal{S}_p(\mathbb{R}^d)}{=} \sum_{k=0}^\infty \sum_{|n|=k} y_n h_n$ for some $y_n \in \mathbb{R}$. Note that

(i) The Hermite functions h_n are uniformly bounded. Let $C > 0$ be a bound. For any $1 \leq i \leq d$, $\partial_i h_n = \sqrt{\frac{n_i}{2}} h_{n-e_i} - \sqrt{\frac{n_i+1}{2}} h_{n+e_i}$ where $\{e_1, \cdots, e_d\}$ is the standard orthonormal basis for $\mathbb{R}^d$ (see Proposition 2.8.1).

(ii) From Lemma 2.10.1, $\|y\|_p^2 = \sum_{k=0}^\infty \sum_{|n|=k} (2k+d)^{2p} y_n^2$. In particular, $(2k+d)^{2p} y_n^2 \leq \|y\|_p^2$ and hence $|y_n| \leq \|y\|_p (2k+d)^{-p}$ for any multi-index n with $|n| = k$.

(iii) There exists a constant $C' > 0$ such that the cardinality $\#\{n \in \mathbb{Z}_+^d : |n| = k\} \leq C'.(2k+d)^{d-1}$ (see equation (2.12) in Proposition 2.10.6).

Then

$$\begin{aligned}
\left|\sum_{k=0}^\infty \sum_{|n|=k} y_n h_n(x)\right| &\leq \sum_{k=0}^\infty \sum_{|n|=k} |y_n||h_n(x)| \\
&\leq C\|y\|_p \sum_{k=0}^\infty \sum_{|n|=k} (2k+d)^{-p} \\
&\leq CC'\|y\|_p \sum_{k=0}^\infty (2k+d)^{-p+d-1} \\
&< \infty, \; (\because p > d + \frac{1}{2}).
\end{aligned}$$

In particular the convergence of $\sum_{k=0}^\infty \sum_{|n|=k} y_n h_n(x)$ is uniform in x. Similarly we can show the convergence of $\sum_{k=0}^\infty \sum_{|n|=k} y_n \partial_i h_n(x)$ is uniform in x. Then partial derivatives of $y(x) = \sum_{k=0}^\infty \sum_{|n|=k} y_n h_n(x)$ exist, since term by term differentiability is allowed by the uniform convergence. The partial derivatives are given by $\sum_{k=0}^\infty \sum_{|n|=k} y_n \partial_i h_n(x), i = 1, \cdots, d$ and are continuous again due to the uniform convergence of the above series. $\square$

2.12 Stochastic integration with $\mathcal{S}_p(\mathbb{R}^d)$ valued integrands

In this subsection we consider $\mathcal{S}_p(\mathbb{R}^d)$ valued integrands and state results of stochastic integration with Hilbert valued integrands as considered in Section 2.7. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered complete probability space satisfying the usual conditions. Let $\{V_t\}$ be an $\mathcal{S}_p(\mathbb{R}^d)$ valued norm bounded predictable process. Let $\{M_t\}$ be a real valued $(\mathcal{F}_t)$ adapted $\mathcal{L}^2$-bounded martingale with $M_0 = 0$ and $\{A_t\}$ be a real valued FV process with $A_0 = 0$.

(i) We have a.s. $\int_0^t \|V_s\|_p^2 \, d\langle M\rangle_s < \infty$ for any $t > 0$, where $\langle M\rangle$ is the predictable process such that $M^2 - \langle M\rangle$ is a martingale. So we have the $\mathcal{S}_p(\mathbb{R}^d)$ valued process $\{\int_0^t V_s \, dM_s\}$.

(ii) Fix $1 \leq i \leq d$. Now $\partial_i : \mathcal{S}_p(\mathbb{R}^d) \to \mathcal{S}_{p-\frac{1}{2}}(\mathbb{R}^d)$ is a bounded operator (Lemma 2.11.4) and hence $\{\partial_i V_t\}$ is an $\mathcal{S}_{p-\frac{1}{2}}(\mathbb{R}^d)$ valued norm bounded predictable process. As in (i), we can define $\{\int_0^t \partial_i V_s \, dM_s\}$, which is an $\mathcal{S}_{p-\frac{1}{2}}(\mathbb{R}^d)$ valued process.

(iii) We have a.s. $\int_0^t \|V_s\|_p \, |dA_s| < \infty$ for any $t > 0$ and hence $\{\int_0^t V_s \, dA_s\}$ is an $\mathcal{S}_p(\mathbb{R}^d)$ valued $(\mathcal{F}_t)$ adapted process. If $\{A_t\}$ is predictable, then so is $\{\int_0^t V_s \, dA_s\}$.

(iv) Let $\phi \in \mathcal{S}_p(\mathbb{R}^d)$ and $\{X_t\}$ be an $\mathbb{R}^d$ valued $(\mathcal{F}_t)$ adapted continuous process. Then by Lemma 2.11.7(ii), $\{\tau_{X_t}\phi\}$ is an $\mathcal{S}_p(\mathbb{R}^d)$ valued $(\mathcal{F}_t)$ adapted continuous process and in particular it is locally bounded. Hence we can define the processes $\{\int_0^t \tau_{X_s}\phi \, dM_s\}$ and $\{\int_0^t \tau_{X_s}\phi \, dA_s\}$. If $\{X_t\}$ is a continuous semimartingale, then we can also define the process $\{\int_0^t \tau_{X_s}\phi \, dX_s\}$.

(v) As an application of [89, Theorem 2.3], we get the following Itô formula: Let $\phi \in \mathcal{S}_p(\mathbb{R}^d)$ and $X = (X^1, \cdots, X^d)$ be an $\mathbb{R}^d$ valued continuous $(\mathcal{F}_t)$ adapted semimartingale. Then we have the following equality in $\mathcal{S}_{p-1}(\mathbb{R}^d)$, a.s. for all $t \geq 0$

$$\tau_{X_t}\phi = \tau_{X_0}\phi - \sum_{i=1}^d \int_0^t \partial_i \tau_{X_s}\phi \, dX_s^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial_{ij}^2 \tau_{X_s}\phi \, d[X^i, X^j]_s.$$

(vi) We compute certain norms of Hermite Sobolev valued processes under 'nice' conditions. We show this as a proof of concept and to simplify the computations further, we assume $d = 1$. Similar expressions on the norm of Hermite Sobolev valued processes will be used at various points in this thesis (see Theorem 4.3.8, Lemma 5.2.16, Proposition 5.2.18).

Suppose that $\{X_t\}$ is given by the stochastic differential equation

$$dX_t = \sigma(X_t) \, dB_t + b(X_t) \, dt, \; t \geq 0$$

where $\sigma : \mathbb{R} \to \mathbb{R}, b : \mathbb{R} \to \mathbb{R}$ are bounded smooth functions and $\{B_t\}$ is a standard $(\mathcal{F}_t)$ Brownian motion. Then the following equality holds in $\mathcal{S}_{p-1}$ a.s. for all $t \geq 0$

$$\tau_{X_t}\phi = \tau_{X_0}\phi - \int_0^t \sigma(X_s)\partial\tau_{X_s}\phi \, dB_s - \int_0^t b(X_s)\partial\tau_{X_s}\phi \, ds$$

$$+\frac{1}{2}\int_0^t \sigma(X_s)^2\,\partial^2\tau_{X_s}\phi\,ds.$$

Using Proposition 2.7.20, we have

$$\begin{aligned}\|\tau_{X_t}\phi\|_{p-1}^2 &= \|\tau_{X_0}\phi\|_{p-1}^2 - 2\int_0^t \sigma(X_s)\,\langle\tau_{X_s}\phi\,,\,\partial\tau_{X_s}\phi\rangle_{p-1}\,dB_s\\ &\quad - 2\int_0^t b(X_s)\,\langle\tau_{X_s}\phi\,,\,\partial\tau_{X_s}\phi\rangle_{p-1}\,ds\\ &\quad + \int_0^t \sigma(X_s)^2\left\langle\tau_{X_s}\phi\,,\,\partial^2\tau_{X_s}\phi\right\rangle_{p-1}\,ds\\ &\quad + \int_0^t \sigma(X_s)^2\|\partial\tau_{X_s}\phi\|_{p-1}^2\,ds\end{aligned}$$

2.13 Some basic inequalities

The following result is usually called the Gronwall's inequality.

Lemma 2.13.1 ([56, Lemma 18.4], [87, Chapter V, Theorem 68])**.** *Let f be a continuous function on $[0,\infty)$ such that*

$$f(t) \le a + b\int_0^t f(s)\,ds,\; t \ge 0$$

for some $a, b \ge 0$. Then $f(t) \le ae^{bt}$ for all $t \ge 0$.
Moreover if f is non-negative and $a = 0$, then f vanishes identically.

The next result is a well-known inequality. For the sake of completeness, we include a proof.

Lemma 2.13.2. *Let k be a natural number. Then for positive real numbers $a_1, \cdots, a_n$ we have*

$$\frac{a_1^k + \cdots + a_n^k}{n} \ge \left(\frac{a_1 + \cdots + a_n}{n}\right)^k.$$

Proof. The inequality follows from the observation that the map $x \mapsto x^k : (0,\infty) \to (0,\infty)$ is convex. □

2.14 Semigroups of bounded linear operators

We recall some basic results for semigroups of bounded linear operators on a real Banach space. In what follows, X will be a real Banach space and $\|\cdot\|$ will stand for both the norm on X and also for the operator norm. The terminology used are standard and can be found in [84, Chapter 1]. For any bounded linear operator A, e^{tA} will denote the bounded linear operator defined by $\sum_{n=0}^{\infty}\frac{t^n}{n!}A^n$.
The following result is well-known and we state it without proof.

Theorem 2.14.1. *Fix $x \in X$ and let A be a bounded linear operator on X. Define $T_t := e^{tA}$, $t \geq 0$. Then*

(i) ([84, Chapter 1, Corollary 1.4 d) and Theorem 2.4 d)]) The map $t \mapsto T_t(x)$ is continuous on $[0, \infty)$ and we have

$$T_t(x) = x + \int_0^t A\, T_s(x)\, ds,\ t \geq 0.$$

(ii) $t \mapsto T_t(x)$ is the unique continuous map on $[0, \infty)$ satisfying above property.

Proof. Note that Corollary 1.4 d) and Theorem 2.4 d) in [84, Chapter 1] are proved for C_0 semigroups. In statement (i) since A is a bounded linear operator, the domain $D(A) = X$ and the semigroup $\{T_t : 0 \leq t < \infty\}$ is a uniformly continuous semigroup, which in particular is a C_0 semigroup.
For the sake of completeness, we give a proof of statement (ii). If $f, g : [0, \infty) \to X$ are continuous maps satisfying

$$f(t) = x + \int_0^t Af(s)\, ds, \quad g(t) = x + \int_0^t Ag(s)\, ds,\ \forall t \geq 0,$$

then for all $t \geq 0$

$$\|f(t) - g(t)\| \leq \left\| \int_0^t A[f(s) - g(s)]\, ds \right\| \leq \|A\| \int_0^t \|f(s) - g(s)\|\, ds.$$

By Lemma 2.13.1, $\|f(t) - g(t)\| = 0, t \geq 0$ which implies the required uniqueness. □

We mention two examples of semigroups of bounded linear operators which will be used in this thesis.

Example 2.14.2. Let C be a real square matrix of order d. Then C is a bounded linear operator on $\mathbb{R}^d$. Let $x \in \mathbb{R}^d$. By Theorem 2.14.1 $t \mapsto e^{tC}x$ is the unique continuous map on $[0, \infty)$ satisfying

$$e^{tC}x = x + \int_0^t e^{sC}x\, ds,\ t \geq 0.$$

Example 2.14.3. Fix $p \in \mathbb{R}$. Then τ_t is a bounded linear operator on $\mathcal{S}_p$ for any $t \in \mathbb{R}$ (see Lemma 2.11.7(i)). Again for $t, s \in \mathbb{R}$ and $\phi \in \mathcal{S}$, we have $\tau_s\phi(\cdot) = \phi(\cdot - s)$ and hence

$$(\tau_t(\tau_s\phi))(\cdot) = (\tau_s\phi)(\cdot - t) = \phi(\cdot - t - s) = \tau_{t+s}\phi(\cdot).$$

Since $\mathcal{S}$ is dense in $\mathcal{S}_p$, from the previous equality we have $\tau_t\tau_s = \tau_{t+s}$. Of course $\tau_0 = I$, the identity operator on $\mathcal{S}_p$. Therefore the family $\{\tau_t : -\infty < t < \infty\}$ is a group of bounded linear operators on $\mathcal{S}_p$. Using Lemma 2.11.7(ii), we conclude that the above family is a C_0

group. Since this C_0 group has the same infinitesimal generator as that of the C_0 semigroup $\{\tau_t : 0 \leq t < \infty\}$, by [84, Corollary 2.5], we conclude the infinitesimal generator is a closed linear operator on $\mathcal{S}_p$ with dense domain.

The following result is well-known in the case of $\mathcal{L}^2(\mathbb{R})$ (i.e. $\mathcal{S}_0$). We include a proof for completeness.

Lemma 2.14.4. *The infinitesimal generator of the C_0 group $\{\tau_t : -\infty < t < \infty\}$ is the operator $-\partial$ on $\mathcal{S}_p$ with domain*

$$D := \{\psi \in \mathcal{S} : \lim_{t\to 0} \frac{\tau_t - I}{t}\psi \text{ exists and is an element of } \mathcal{S}_p\}.$$

Proof. We claim

(i) $\mathcal{S} \subset D$.

(ii) For any $\psi \in \mathcal{S}$,

$$\lim_{t\to 0} \frac{\tau_t - I}{t}\psi \stackrel{\mathcal{S}}{=} -\partial\psi. \tag{2.23}$$

In particular, the equality above also holds in $\mathcal{S}_p$.

First we assume the claim and prove the statement of the result.
Let $\psi \in D$ and call $\widetilde{\psi} := \lim_{t\to 0} \frac{\tau_t - I}{t}\psi$. Then for any $\phi \in \mathcal{S}(\subset \mathcal{S}_{-p})$, we have

$$\left|\left\langle \widetilde{\psi} - \frac{\tau_t - I}{t}\psi \,,\, \phi \right\rangle\right| \leq \|\widetilde{\psi} - \frac{\tau_t - I}{t}\psi\|_p \|\phi\|_{-p} \xrightarrow{t\to 0} 0.$$

Then

$$\begin{aligned}
\left\langle \widetilde{\psi} \,,\, \phi \right\rangle &= \lim_{t\to 0} \left\langle \frac{\tau_t - I}{t}\psi \,,\, \phi \right\rangle \\
&= \lim_{t\to 0} \left\langle \psi \,,\, \frac{\tau_{-t} - I}{t}\phi \right\rangle \\
&= \left\langle \psi \,,\, \lim_{t\to 0} \frac{\tau_{-t} - I}{t}\phi \right\rangle \\
&= \langle \psi \,,\, \partial\phi \rangle \\
&= \langle -\partial\psi \,,\, \phi \rangle \,.
\end{aligned}$$

Hence $\widetilde{\psi} = -\partial\psi$ for any $\psi \in D$.
To complete the proof, we need to establish our claim. Let $\phi \in \mathcal{S}$ and fix $t \in \mathbb{R}\setminus\{0\}$. Since ϕ is a C^2 function, by Taylor's formula for any $x \in \mathbb{R}$ there exists $\theta_x \in (0,1)$ such that

$$\phi(x+t) = \phi(x) + t\partial\phi(x) + \frac{t^2}{2}\partial^2\phi(x + \theta_x t).$$

Then for any positive integer n, putting $y = x + \theta_x t$ we have

$$(1+x^2)^n \left| \frac{\phi(x+t)-\phi(x)}{t} - \partial\phi(x) \right| = (1+x^2)^n \frac{|t|}{2} \left| \partial^2\phi(x+\theta_x t) \right|$$
$$= (1+(y-\theta_x t)^2)^n \frac{|t|}{2} \left| \partial^2 \phi(y) \right| .$$

Since $\theta_x \in (0,1)$, for any t with $|t| \leq \frac{1}{\sqrt{2}}$ we have $2\theta_x^2 t^2 \leq 1$ and hence

$$1 + (y-\theta_x t)^2 \leq 1 + 2(y^2 + \theta_x^2 t^2) \leq 2(1+y^2).$$

Therefore for $|t| \leq \frac{1}{\sqrt{2}}$ with $y = x + \theta_x t$

$$(1+x^2)^n \left| \frac{\phi(x+t)-\phi(x)}{t} - \partial\phi(x) \right| \leq |t| 2^{n-1} (1+y^2)^n \left| \partial^2\phi(y) \right|$$

which implies

$$\sup_{x\in\mathbb{R}} (1+x^2)^n \left| \frac{\phi(x+t)-\phi(x)}{t} - \partial\phi(x) \right| \leq |t| 2^{n-1} \sup_{y\in\mathbb{R}} (1+y^2)^n \left| \partial^2\phi(y) \right| .$$

Since $\phi \in \mathcal{S}$, $\sup_{y\in\mathbb{R}}(1+y^2)^n \left|\partial^2\phi(y)\right| < \infty$ and hence

$$\sup_{x\in\mathbb{R}} (1+x^2)^n \left| \frac{\phi(x+t)-\phi(x)}{t} - \partial\phi(x) \right| \xrightarrow{t\to 0} 0,$$

i.e.

$$\sup_{x\in\mathbb{R}} (1+x^2)^n \left| \frac{\tau_{-t} - I}{t}\phi(x) - \partial\phi(x) \right| \xrightarrow{t\to 0} 0.$$

Any derivative of ϕ is again an element of $\mathcal{S}$ and hence above limit is true when ϕ is replaced by any derivative of ϕ. Since the seminorms given by the supremums (see equation (2.10)) defines the topology on $\mathcal{S}$ we have

$$\lim_{t\to 0} \frac{\tau_{-t} - I}{t}\phi \stackrel{\mathcal{S}}{=} \partial\phi.$$

Since the convergence in $\mathcal{S}$ is equivalent to the convergence in all $\|\cdot\|_p$ norms for $p = 1, 2, \cdots$ (see Proposition 2.10.3) we have

$$\lim_{t\to 0} \frac{\tau_{-t} - I}{t}\phi \stackrel{\mathcal{S}_p}{=} \partial\phi, \quad \forall p = 1, 2, \cdots .$$

Now for any real number p, we can choose a positive integer n such that $p \leq n$. Then $\|\frac{\tau_{-t}-I}{t}\phi - \partial\phi\|_p \leq \|\frac{\tau_{-t}-I}{t}\phi - \partial\phi\|_n$ and hence

$$\lim_{t\to 0} \frac{\tau_{-t} - I}{t}\phi \stackrel{\mathcal{S}_p}{=} \partial\phi, \quad \forall p \in \mathbb{R}.$$

This proves $\phi \in D$. This shows $\mathcal{S} \subset D$ and the proof of the claim is complete. $\square$

CHAPTER 3

Monotonicity inequality for stochastic partial differential equations in $\mathcal{S}'(\mathbb{R}^d)$

3.1 Introduction

Consider the existence and uniqueness problem for stochastic partial differential equations of the form

$$dY_t = L(Y_t)\ dt + A(Y_t).dB_t,$$

where (B_t) is a r-dimensional Brownian motion and (Y_t) an $\mathcal{S}'(\mathbb{R}^d)$ valued process, with Y_0 a given $\mathcal{S}'(\mathbb{R}^d)$-valued random variable and the operators L and $A = (A_1, \cdots, A_r)$ are certain differential operators. A sufficient condition for existence and uniqueness, the Monotonicity inequality for the pair of operators (A, L) has been studied by many authors (see [38,39,59,65,90,92]). Let $\|\cdot\|$ be a Hilbertian semi-norm on $\mathcal{S}'(\mathbb{R}^d)$ with corresponding inner product $\langle\cdot\,,\,\cdot\rangle$. Say that the pair of operators (A, L) satisfies the 'Monotonicity inequality' for the semi-norm $\|\cdot\|$ if

$$2\,\langle \phi\,,\, L\phi\rangle + \sum_{i=1}^{r}\|A_i\phi\|^2 \leq\ C\|\phi\|^2,\ \forall \phi \in \mathcal{S}(\mathbb{R}^d). \tag{3.1}$$

Here the semi-norm should be such that the space contains the range of A, L and the space $\mathcal{S}(\mathbb{R}^d)$. In practice, the norm $\|\cdot\|$ is taken as one of the Hermite-Sobolev norms $\|\cdot\|_p$, $p \in \mathbb{R}$. A related inequality, called the coercivity inequality is also considered in the context of stochastic partial differential equations, but in the setting of a Gelfand triple of Hilbert spaces (see [65, 83]). We prove the Monotonicity inequality in two different settings.

(i) In Section 3, we prove the inequality for constant coefficient differential operators given by

$$L = \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^t)_{ij}\partial_{ij}^2 - \sum_{i=1}^{d} b_i\partial_i$$

and

$$A_i = -\sum_{j=1}^{d} \sigma_{ji}\partial_j.$$

This result was already proved in [39, Theorem 2.1]. We give a new proof, the crux of which is outlined below.

(ii) In Section 4, we consider the inequality when the operators L, A contain variable coefficients, i.e. for

$$L\psi := \frac{1}{2}\sum_{i,j=1}^{d} \partial_{ij}^2 \left((\sigma\sigma^t)_{ij}\psi\right) - \sum_{i=1}^{d} \partial_i \left(b_i\psi\right), \forall \psi \in \mathcal{S}'(\mathbb{R}^d)$$

and

$$A_i\psi := -\sum_{k=1}^{d} \partial_k \left(\sigma_{ki}\psi\right), \forall \psi \in \mathcal{S}'(\mathbb{R}^d)$$

where $\sigma_{ij}, b_i, 1 \leq i, j \leq d$ are smooth functions with bounded derivatives. This inequality was used in [92] to prove the uniqueness of the solution of the Cauchy problem for L as above. We prove the inequality when σ is a real $d \times d$ matrix and $b(x) := \alpha + Cx, \forall x \in \mathbb{R}^d$ with $\alpha \in \mathbb{R}^d$ and $C = (c_{ij})$ is a real $d \times d$ matrix.

The problem of characterizing coefficients σ, b for which the Monotonicity inequality holds in the second case, remains unresolved, to our knowledge. For the first case, the proof given in [39] was essentially computational. It involved expanding ϕ along an orthonormal basis $\{h_n^p\}$ in $\mathcal{S}_p(\mathbb{R}^d)$, where $h_n^p := (2k+d)^{-p}h_n$ and $k = |n|$. The left hand side in the inequality above can then be computed using linearity, in terms of the action of L and A_i on the h_n^p, which in turn can be computed, using the recurrence relation for the action of the derivatives ∂_i on the Hermite functions, viz. $\partial_i h_n$ (Proposition 2.8.1). It was shown that the resulting series was essentially the same as that for $\|\phi\|^2$, by showing that certain sequences appearing in successive terms of the series were bounded ([39, Lemma 2.2]).

The method used in our proof can be described in the following steps.

(i) We identify the adjoints $\partial_i^*, \mathscr{M}_i^*, i = 1, \cdots, d$ of the operators $\partial_i, \mathscr{M}_i, i = 1, \cdots, d$ on $\mathcal{S}_p(\mathbb{R}^d)$. We show that $\partial_i^* = -\partial_i + T_i$ and $\mathscr{M}_i^* = \mathscr{M}_i + \widetilde{T}_i$ on $(\mathcal{S}(\mathbb{R}^d), \|\cdot\|_p)$ where $T_i, \widetilde{T}_i$ are bounded linear operators on $\mathcal{S}_p(\mathbb{R}^d)$, expressible in terms of certain shift and multiplication operators - both of which are bounded linear operators on $\mathcal{S}_p(\mathbb{R}^d)$ (see Theorem 3.2.2 and Theorem 3.4.1). We crucially use the recurrence relations for $\partial_i h_n, \mathscr{M}_i h_n$ in terms of other h_n's. The proof that $T_i, \widetilde{T}_i$ are bounded operators involves a 'first-order' version of the inequalities proved in [39, Lemma 2.2] (see Lemma 3.2.4).

(ii) This step can be broadly identified as an 'integration by parts' argument. We observe that the term in $2\langle \phi , L\phi\rangle_p$ corresponding to the second order term in L cancels with

$\sum_{i=1}^{r} \|A_i\phi\|_p^2$, leaving only terms of the form $\langle T_i\phi\,,\,\partial_j\phi\rangle_p$. We estimate these terms using certain bounded bilinear forms on $\mathcal{S}_p(\mathbb{R}^d)$ (see Lemma 3.2.5 and Theorem 3.4.1). Our proof is a generalization of the proof in the case $p = 0$ (i.e. $\mathcal{L}^2(\mathbb{R}^d)$), for which it follows trivially by 'integration by parts'.

(iii) The rest of the proof boils down to estimating the term coming from the first order term in L, which again follows from the identification of the adjoints.

In Remark 3.3.3, an interpretation of the Monotonicity inequality for the constant coefficient differential operators (A, L) is presented in terms of the C_0-group of translation operators.

In Chapter 4, we use the Monotonicity inequality for (A, L) involving variable coefficients (Theorem 3.4.2) to show the uniqueness of solutions of the Cauchy problem for L when the initial condition ψ is a tempered distribution given by an integrable function.

Most of the results in this chapter are from [10].

3.2 The Adjoint of the Derivative on the Hermite-Sobolev spaces

Since $\mathcal{S}(\mathbb{R}^d) \subset \mathcal{L}^2(\mathbb{R}^d)$ (see Lemma 2.9.1) and $\{h_n : n \in \mathbb{Z}_+^d\}$ is an orthonormal basis for $\mathcal{L}^2(\mathbb{R}^d)$ (see Proposition 2.8.1(iv)), any $\phi \in \mathcal{S}(\mathbb{R}^d)$ can be written as

$$\phi = \sum_{k=0}^{\infty} \sum_{|n|=k} \phi_n h_n.$$

Recall that we are using the convention: $\phi_n = 0, h_n = 0$ whenever $n_i < 0$, for some i.
For $i = 1, \cdots, d$ the derivative operators (see Example 2.11.3) $\partial_i : \mathcal{S}_{p+\frac{1}{2}}(\mathbb{R}^d) \to \mathcal{S}_p(\mathbb{R}^d)$ are bounded linear operators. Note that $\partial_i, 1 \leq i \leq d$ are densely defined unbounded closed linear operators on $\mathcal{S}_p(\mathbb{R}^d)$ (see Lemma 2.11.5).
Let ∂_i^* denote the Hilbert space adjoint of ∂_i on $\mathcal{S}_p(\mathbb{R}^d)$. For convenience of notation, we do not include p in ∂_i^*, though it should be understood that we are working for a fixed $p \in \mathbb{R}$. Now $\partial_i^* : Domain(\partial_i^*) \subset \mathcal{S}_p(\mathbb{R}^d) \to \mathcal{S}_p(\mathbb{R}^d)$ with

$$Domain(\partial_i^*) = \{\phi \in \mathcal{S}_p(\mathbb{R}^d) : Domain(\partial_i) \ni \psi \mapsto \langle \partial_i\psi\,,\,\phi\rangle_p\,, \text{ is a bounded linear functional}\}.$$

Note that ∂_i^* satisfies

$$\langle \partial_i\psi\,,\,\phi\rangle_p = \langle \psi\,,\,\partial_i^*\phi\rangle_p\,,\ \psi \in Domain(\partial_i), \phi \in Domain(\partial_i^*).$$

Lemma 3.2.1. *$Domain(\partial_i^*)$ contains $\mathcal{S}(\mathbb{R}^d)$.*

Proof. Let $\phi \in \mathcal{S}(\mathbb{R}^d)$. Then for any $\psi \in Domain(\partial_i)$, by Proposition 2.8.1(v) we have

$$\begin{aligned}\langle \partial_i \psi\, ,\, h_n\rangle &= -\langle \psi\, ,\, \partial_i h_n\rangle \\ &= -\sqrt{\frac{n_i}{2}}\, \langle \psi\, ,\, h_{n-e_i}\rangle + \sqrt{\frac{n_i+1}{2}}\, \langle \psi\, ,\, h_{n+e_i}\rangle\, .\end{aligned}$$

Then (see Lemma 2.10.1)

$$\begin{aligned}\langle \partial_i \psi\, ,\, \phi\rangle_p &= \sum_{k=0}^{\infty}\sum_{|n|=k} (2k+d)^{2p}\, \langle \partial_i\psi\, ,\, h_n\rangle\, \langle \phi\, ,\, h_n\rangle \\ &= -\sum_{k=0}^{\infty}\sum_{|n|=k} (2k+d)^{2p} \sqrt{\frac{n_i}{2}}\, \langle \psi\, ,\, h_{n-e_i}\rangle\, \langle \phi\, ,\, h_n\rangle \\ &\quad + \sum_{k=0}^{\infty}\sum_{|n|=k} (2k+d)^{2p} \sqrt{\frac{n_i+1}{2}}\, \langle \psi\, ,\, h_{n+e_i}\rangle\, \langle \phi\, ,\, h_n\rangle \\ &= \sum_{k=0}^{\infty}\sum_{\substack{|m|=k+1\\ m=n+e_i}} (2k+d)^{2p}\, \langle \psi\, ,\, h_m\rangle\, \langle \phi\, ,\, h_{m-e_i}\rangle \sqrt{\frac{m_i}{2}} \\ &\quad - \sum_{k=0}^{\infty}\sum_{\substack{|m|=k-1\\ m=n-e_i}} (2k+d)^{2p}\, \langle \psi\, ,\, h_m\rangle\, \langle \phi\, ,\, h_{m+e_i}\rangle \sqrt{\frac{m_i+1}{2}} \\ &= \sum_{k=1}^{\infty}\sum_{|m|=k} (2k+d-2)^{2p}\, \langle \psi\, ,\, h_m\rangle\, \langle \phi\, ,\, h_{m-e_i}\rangle \sqrt{\frac{m_i}{2}} \\ &\quad - \sum_{k=-1}^{\infty}\sum_{|m|=k} (2k+d+2)^{2p}\, \langle \psi\, ,\, h_m\rangle\, \langle \phi\, ,\, h_{m+e_i}\rangle \sqrt{\frac{m_i+1}{2}}.\end{aligned}$$

Observe that the term for $k = 0$ in the first sum evaluates to 0 because of $h_{m-e_i} = 0$. Again the term for $k = -1$ in the second sum is 0 because of $h_m = 0$. Hence

$$\begin{aligned}&\langle \partial_i \psi\, ,\, \phi\rangle_p \\ &= \sum_{k=1}^{\infty}\sum_{|m|=k} (2k+d)^{2p}\, \langle \psi\, ,\, h_m\rangle \left[\langle \phi\, ,\, h_{m-e_i}\rangle \sqrt{\frac{m_i}{2}} \left(\frac{2k+d-2}{2k+d}\right)^{2p}\right] \\ &- \sum_{k=0}^{\infty}\sum_{|m|=k} (2k+d)^{2p}\, \langle \psi\, ,\, h_m\rangle \left[\langle \phi\, ,\, h_{m+e_i}\rangle \sqrt{\frac{m_i+1}{2}} \left(\frac{2k+d+2}{2k+d}\right)^{2p}\right]\end{aligned} \tag{3.2}$$

We now prove an estimate of the first sum in terms of $\|\psi\|_p$. Note that $\lim_{k\to\infty} \left(\frac{2k+d-2}{2k+d}\right)^{2p} = 1$ and hence $\sup\{\left(\frac{2k+d-2}{2k+d}\right)^{2p} : k \geq 1\} < \infty$. Also for a multi-index $m = (m_1, \cdots, m_d)$ with $|m| = k$ we have $m_i \leq |m| = k \leq 2k < (2k+d)$. Then

$$\left|\sum_{k=1}^{\infty}\sum_{|m|=k} (2k+d)^{2p}\, \langle \psi\, ,\, h_m\rangle \left[\langle \phi\, ,\, h_{m-e_i}\rangle \sqrt{\frac{m_i}{2}} \left(\frac{2k+d-2}{2k+d}\right)^{2p}\right]\right|$$

$$\leq \sup_{k\geq 1}\left(\frac{2k+d-2}{2k+d}\right)^{2p}\sum_{k=1}^{\infty}\sum_{|m|=k}(2k+d)^{2p}|\langle\psi\,,\,h_m\rangle||\langle\phi\,,\,h_{m-e_i}\rangle|\sqrt{\frac{m_i}{2}}$$
$$\leq \frac{1}{\sqrt{2}}\sup_{k\geq 1}\left(\frac{2k+d-2}{2k+d}\right)^{2p}\sum_{k=1}^{\infty}\sum_{|m|=k}(2k+d)^{2p+\frac{1}{2}}|\langle\psi\,,\,h_m\rangle||\langle\phi\,,\,h_{m-e_i}\rangle|$$
$$\leq \frac{1}{\sqrt{2}}\sup_{k\geq 1}\left(\frac{2k+d-2}{2k+d}\right)^{2p}\left(\sum_{k=1}^{\infty}\sum_{|m|=k}(2k+d)^{2p}\langle\psi\,,\,h_m\rangle^2\right)^{\frac{1}{2}}$$
$$\times\left(\sum_{k=1}^{\infty}\sum_{|m|=k}(2k+d)^{2p+1}\langle\phi\,,\,h_{m-e_i}\rangle^2\right)^{\frac{1}{2}},\ \text{(by Cauchy-Schwarz inequality)}$$
$$= \frac{1}{\sqrt{2}}\sup_{k\geq 1}\left(\frac{2k+d-2}{2k+d}\right)^{2p}\|\psi\|_p\left(\sum_{k=1}^{\infty}\sum_{|m|=k}(2k+d)^{2p+1}\langle\phi\,,\,h_{m-e_i}\rangle^2\right)^{\frac{1}{2}}$$

Now

$$\sum_{k=1}^{\infty}\sum_{|m|=k}(2k+d)^{2p+1}\langle\phi\,,\,h_{m-e_i}\rangle^2$$
$$=\sum_{k=1}^{\infty}\sum_{\substack{n=m-e_i,\\|m|=k}}(2k+d)^{2p+1}\langle\phi\,,\,h_n\rangle^2$$
$$=\sum_{k=1}^{\infty}\sum_{|n|=k-1}(2k+d)^{2p+1}\langle\phi\,,\,h_n\rangle^2$$
$$=\sum_{k=0}^{\infty}\sum_{|n|=k}(2k+2+d)^{2p+1}\langle\phi\,,\,h_n\rangle^2$$
$$\leq\sup_{k\geq 0}\left(\frac{2k+d+2}{2k+d}\right)^{2p+1}\sum_{k=0}^{\infty}\sum_{|n|=k}(2k+2)^{2p+1}\langle\phi\,,\,h_n\rangle^2$$
$$=\sup_{k\geq 0}\left(\frac{2k+d+2}{2k+d}\right)^{2p+1}\|\phi\|^2_{p+\frac{1}{2}}.$$

Last two estimates gives us an estimate of the first sum on the right hand side of (3.2)

$$\left|\sum_{k=1}^{\infty}\sum_{|m|=k}(2k+d)^{2p}\langle\psi\,,\,h_m\rangle\left[\langle\phi\,,\,h_{m-e_i}\rangle\sqrt{\frac{m_i}{2}}\left(\frac{2k+d-2}{2k+d}\right)^{2p}\right]\right|$$
$$\leq\frac{1}{\sqrt{2}}\sup_{k\geq 1}\left(\frac{2k+d-2}{2k+d}\right)^{2p}\|\psi\|_p\ \sup_{k\geq 0}\left(\frac{2k+d+2}{2k+d}\right)^{p+\frac{1}{2}}\|\phi\|_{p+\frac{1}{2}}$$

Since $\phi\in\mathcal{S}(\mathbb{R}^d)\subset\mathcal{S}_{p+\frac{1}{2}}(\mathbb{R}^d)$, we have $\|\phi\|_{p+\frac{1}{2}}<\infty$. We can obtain a similar estimate of the second sum on the right hand side of (3.2) in terms of $\|\psi\|_p$. Then from (3.2), there exists a constant $C = C(\phi)$ such that

$$|\langle \partial_i \psi \,,\, \phi \rangle_p | \le C \|\psi\|_p, \, \forall \psi \in Domain(\partial_i).$$

This shows $\psi \mapsto \langle \partial_i \psi \,,\, \phi \rangle_p$ is a bounded linear functional on $Domain(\partial_i)$ when $\phi \in \mathcal{S}(\mathbb{R}^d)$. Hence $\mathcal{S}(\mathbb{R}^d) \subset Domain(\partial_i^*)$. □

Using integration by parts, we have $\partial_i^* = -\partial_i$ on $(\mathcal{S}(\mathbb{R}^d), \|\cdot\|_0)$. In the next Theorem, we compute ∂_i^* in $(\mathcal{S}(\mathbb{R}^d), \|\cdot\|_p)$ explicitly and the resulting formula generalizes the above relation to the case $p \neq 0$.

For each $i = 1, \cdots, d$ we define two sequences:

$$\begin{aligned} a_{n,i} &:= \sqrt{\frac{n_i}{2}} \left[\frac{(2k+d-2)^{2p} - (2k+d)^{2p}}{(2k+d)^{2p}} \right], \\ b_{n,i} &:= \sqrt{\frac{n_i+1}{2}} \left[\frac{(2k+d)^{2p} - (2k+d+2)^{2p}}{(2k+d)^{2p}} \right] \end{aligned} \tag{3.3}$$

where $n = (n_1, \cdots, n_d)$ is a multi-index with $|n| = k \ge 0$.
Let $\{e_i : 1 \le i \le d\}$ denote the standard basis for $\mathbb{R}^d$. Define linear operators $\tilde{A}_i, \tilde{B}_i$ on $\mathcal{S}(\mathbb{R}^d)$ by the formal expressions: for $\psi = \sum_{k=0}^{\infty} \sum_{|n|=k} \psi_n h_n \in \mathcal{S}(\mathbb{R}^d)$,

$$\tilde{A}_i \psi := \sum_{k=0}^{\infty} \sum_{|n|=k} a_{n,i} \psi_n h_n, \quad \tilde{B}_i \psi := \sum_{k=0}^{\infty} \sum_{|n|=k} b_{n,i} \psi_n h_n. \tag{3.4}$$

Theorem 3.2.2. *For any $1 \le i \le d$, each of $\tilde{A}_i, \tilde{B}_i$ is a bounded operator on $(\mathcal{S}(\mathbb{R}^d), \|\cdot\|_p)$ and hence extends to $(\mathcal{S}_p(\mathbb{R}^d), \|\cdot\|_p)$ as bounded linear operators. Furthermore, for any $1 \le i \le d$ and for any $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$,*

$$\langle \partial_i \phi \,,\, \psi \rangle_p + \langle \phi \,,\, \partial_i \psi \rangle_p = \left\langle \phi \,,\, (\tilde{A}_i U_{-e_i} + \tilde{B}_i U_{+e_i}) \psi \right\rangle_p \tag{3.5}$$

where U_{-e_i}, U_{+e_i} are the shift operators defined in Example 2.11.1. Hence we have

$$\partial_i^* = -\partial_i + T_i \text{ on } (\mathcal{S}(\mathbb{R}^d), \|\cdot\|_p)$$

where $T_i = \tilde{A}_i U_{-e_i} + \tilde{B}_i U_{+e_i}$ is a bounded linear operator on $\mathcal{S}_p(\mathbb{R}^d)$. By density arguments, (3.5) can be extended to any $\phi, \psi \in \mathcal{S}_{p+\frac{1}{2}}(\mathbb{R}^d)$.

Before proving this Theorem, we first prove some necessary results.

Lemma 3.2.3. *For any $1 \leq i \leq d$ and $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$,*

$$\begin{aligned} &\langle \partial_i \phi \,,\, \psi \rangle_p + \langle \phi \,,\, \partial_i \psi \rangle_p \\ &= \sum_{k=0}^{\infty} \sum_{|n|=k} (2k+d)^{2p} \phi_n a_{n,i} \psi_{n-e_i} + \sum_{k=0}^{\infty} \sum_{|n|=k} (2k+d)^{2p} \phi_n b_{n,i} \psi_{n+e_i} \end{aligned} \tag{3.6}$$

Lemma 3.2.4. *Fix $i = 1, \cdots, d$. Then there exists a constant $M_p > 0$, (independent of i) such that*

$$\sup_{\{n:|n|=k\}} |a_{n,i}| \leq \frac{M_p}{\sqrt{k}}, \qquad \sup_{\{n:|n|=k\}} |b_{n,i}| \leq \frac{M_p}{\sqrt{k}} \tag{3.7}$$

for any $k \geq 1$. In particular, the sequences $\{a_{n,i}\}$ and $\{b_{n,i}\}$ are bounded.

Proof of Lemma 3.2.3. Since $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$, we can write

$$\phi = \sum_{k=0}^{\infty} \sum_{|n|=k} \phi_n h_n, \quad \psi = \sum_{k=0}^{\infty} \sum_{|n|=k} \psi_n h_n.$$

By equation (2.14), we have

$$\partial_i \phi = \sum_{k=0}^{\infty} \sum_{|n|=k} \left[\phi_{n+e_i} \sqrt{\frac{n_i+1}{2}} - \phi_{n-e_i} \sqrt{\frac{n_i}{2}} \right] h_n.$$

Similar expression is true for $\partial_i \psi$.
Therefore, $\langle \phi \,,\, \partial_i \psi \rangle_p = \sum_{k=0}^{\infty} (2k+d)^{2p} \sum_{|n|=k} \phi_n \left[\sqrt{\frac{n_i+1}{2}} \psi_{n+e_i} - \sqrt{\frac{n_i}{2}} \psi_{n-e_i} \right]$
and

$$\begin{aligned} \langle \partial_i \phi \,,\, \psi \rangle_p &= \sum_{k=0}^{\infty} (2k+d)^{2p} \sum_{|n|=k} \psi_n \left[\sqrt{\frac{n_i+1}{2}} \phi_{n+e_i} - \sqrt{\frac{n_i}{2}} \phi_{n-e_i} \right] \\ &= \sum_{k=0}^{\infty} \sum_{\substack{|m|=k+1 \\ m=n+e_i}} (2k+d)^{2p} \phi_m \psi_{m-e_i} \sqrt{\frac{m_i}{2}} \\ &\quad - \sum_{k=0}^{\infty} \sum_{\substack{|m|=k-1 \\ m=n-e_i}} (2k+d)^{2p} \phi_m \psi_{m+e_i} \sqrt{\frac{m_i+1}{2}} \\ &= \sum_{k=1}^{\infty} \sum_{|m|=k} (2k+d-2)^{2p} \phi_m \psi_{m-e_i} \sqrt{\frac{m_i}{2}} \\ &\quad - \sum_{k=-1}^{\infty} \sum_{|m|=k} (2k+d+2)^{2p} \phi_m \psi_{m+e_i} \sqrt{\frac{m_i+1}{2}} \end{aligned}$$

the corresponding term for $k = 0$ in the first sum evaluates to 0 because of $\psi_{m-e_i}\sqrt{\frac{m_i}{2}}$, also the term for $k = -1$ in the second sum is 0 because of ϕ_m

$$= \sum_{k=0}^{\infty} \sum_{|m|=k} (2k+d)^{2p} \phi_m \left[\psi_{m-e_i} \sqrt{\frac{m_i}{2}} \left(\frac{2k+d-2}{2k+d} \right)^{2p} \right]$$
$$- \sum_{k=0}^{\infty} \sum_{|m|=k} (2k+d)^{2p} \phi_m \left[\psi_{m+e_i} \sqrt{\frac{m_i+1}{2}} \left(\frac{2k+d+2}{2k+d} \right)^{2p} \right]$$

Then

$$\begin{aligned}
&\langle \partial_i \phi \,,\, \psi \rangle_p + \langle \phi \,,\, \partial_i \psi \rangle_p \\
&= \sum_{k=0}^{\infty} \sum_{|m|=k} (2k+d)^{2p} \phi_m \left[\psi_{m-e_i} \sqrt{\frac{m_i}{2}} \left\{ \left(\frac{2k+d-2}{2k+d} \right)^{2p} - 1 \right\} \right] \\
&+ \sum_{k=0}^{\infty} \sum_{|m|=k} (2k+d)^{2p} \phi_m \left[\psi_{m+e_i} \sqrt{\frac{m_i+1}{2}} \left\{ 1 - \left(\frac{2k+d+2}{2k+d} \right)^{2p} \right\} \right] \\
&= \sum_{k=0}^{\infty} \sum_{|m|=k} (2k+d)^{2p} \phi_m a_{m,i} \psi_{m-e_i} + \sum_{k=0}^{\infty} \sum_{|m|=k} (2k+d)^{2p} \phi_m b_{m,i} \psi_{m+e_i}
\end{aligned}$$

This completes the proof. □

Proof of Lemma 3.2.4. We prove for $a_{n,i}$'s. Proof for $b_{n,i}$'s are similar.
We can safely ignore the term for $|n| = k = 0$.
Now for $|n| = k \in \mathbb{N}$,

$$\begin{aligned}
|a_{n,i}| &= \sqrt{\frac{n_i}{2|n|}} \sqrt{|n|} \left[\frac{(2k+d-2)^{2p} - (2k+d)^{2p}}{(2k+d)^{2p}} \right] \\
&\leq \sqrt{k} \left[\frac{(2k+d-2)^{2p} - (2k+d)^{2p}}{(2k+d)^{2p}} \right]
\end{aligned}$$

To find an upper bound of $a_{n,i}$'s, we follow the method in Lemma (2.2) of [39].
Choose an analytic branch of $z \mapsto z^{2p}$ in a domain containing the positive real axis and then we can define

$$f(z) := \left(1 - \frac{2z}{2+dz} \right)^{2p} - 1$$

in a sufficiently small neighbourhood of 0, say in a ball of radius $\delta > 0$, i.e. $B(0, \delta)$.

Since $f(0) = 0$, $\exists$ an analytic function g defined on $B(0, \delta)$ such that $f(z) = zg(z)$, $\forall z \in B(0, \delta)$. But on the compact set $\overline{B(0, \frac{\delta}{2})}$ the function g is bounded, say by some constant

$R > 0$.
Fix a positive integer N such that $\frac{1}{N} < \frac{\delta}{2}$. Then $\forall k \geq N$ and n with $|n| = k$,

$$|a_{n,i}| \leq \sqrt{k}\left|f\left(\frac{1}{k}\right)\right| \leq \frac{1}{\sqrt{k}}\left|g\left(\frac{1}{k}\right)\right| \leq \frac{R}{\sqrt{k}}.$$

Then taking $M := \max\{\max_{\{(n,i):1\leq|n|<N,1\leq i\leq d\}}\{\sqrt{|n|}\,|a_{n,i}|\}, R\}$, we have

$$|a_{n,i}| \leq \frac{M}{\sqrt{k}},\ \forall n, \text{ with } |n| = k \geq 1.$$

From this inequality required bound can be obtained.
Proof for $b_{n,i}$'s are similar. Finally we choose M_p as the larger of the two constants which were obtained for $a_{n,i}$'s and $b_{n,i}$'s separately and we have

$$|a_{n,i}| \leq \frac{M_p}{\sqrt{k}}, \quad |b_{n,i}| \leq \frac{M_p}{\sqrt{k}},\ \forall n, \text{ with } |n| = k \geq 1.$$

Taking $M_p' = \max\{M_p, \max\{a_{0,i}, b_{0,i} : 1 \leq i \leq d\}\}$, we get

$$|a_{n,i}| \leq M_p', \quad |b_{n,i}| \leq M_p',\ \forall n.$$

This completes the proof. □

Alternative proof of Lemma 3.2.4. This approach was suggested by an anonymous referee. We use mean value theorems to establish the bounds.
We present the proof for the case $d = 1$. The sequences become

$$a_n = \sqrt{\frac{n}{2}}\left[\frac{(2n-1)^{2p} - (2n+1)^{2p}}{(2n+1)^{2p}}\right], b_n = \sqrt{\frac{n+1}{2}}\left[\frac{(2n+1)^{2p} - (2n+3)^{2p}}{(2n+1)^{2p}}\right]$$

We show an upper bound for $\{a_n\}$. Proof for $\{b_n\}$ is similar.
Consider the following continuously differentiable function

$$f(x) := \left(\frac{2-x}{2+x}\right)^{2p},\ x \in [-1, 1].$$

Observe that

(a) $a_n = \sqrt{\frac{n}{2}}[f(\frac{1}{n}) - f(0)]$
(b) $f'(x) = \left(\frac{2-x}{2+x}\right)^{2p-1} \frac{-4}{(2+x)^2}$, $x \in (-1, 1)$.
(c) $x \mapsto \left(\frac{2-x}{2+x}\right)^{2p-1}$ is a real valued continuous function on $[0, 1]$ and hence there exists a constant $R > 0$ such that $0 < \left(\frac{2-x}{2+x}\right)^{2p-1} \leq R$, $x \in [0, 1]$.

Now using mean value theorem for any $n \in \mathbb{N}$

$$\begin{aligned}
|f(\tfrac{1}{n}) - f(0)| &\leq \int_0^{\frac{1}{n}} |f'(t)|\, dt \\
&= \int_0^{\frac{1}{n}} \left(\frac{2-t}{2+t}\right)^{2p-1} \frac{4}{(2+t)^2}\, dt \\
&\leq 4R \int_0^{\frac{1}{n}} \frac{dt}{(2+t)^2} = 4R \left[\frac{1}{2} - \frac{1}{2+\frac{1}{n}}\right] \\
&= \frac{2R}{2n+1} \leq \frac{R}{n}
\end{aligned}$$

and hence $|a_n| \leq \frac{R}{\sqrt{n}}$ for all $n \in \mathbb{N}$. □

Proof of Theorem 3.2.2. Linearity of $\tilde{A}_i, \tilde{B}_i, U_{+e_i}, U_{-e_i}$ is clear from definition. Given $\phi \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned}
\|\tilde{A}_i\phi\|_p^2 &= \sum_{k=0}^{\infty} \sum_{|n|=k} (2k+d)^{2p} |a_{n,i}|^2 \phi_n^2 \\
&\leq \left(\sup_n |a_{n,i}|^2\right) \sum_{k=0}^{\infty} \sum_{|n|=k} (2k+d)^{2p} \phi_n^2 \\
&\leq (M_p')^2 \|\phi\|_p^2 \text{ (by Lemma (3.2.4))}
\end{aligned}$$

Therefore $\|\tilde{A}_i\phi\|_p \leq M_p'\|\phi\|_p$ for all $\phi \in \mathcal{S}(\mathbb{R}^d)$ and hence $\|\tilde{A}_i\|_{\mathcal{S}_p(\mathbb{R}^d)\to\mathcal{S}_p(\mathbb{R}^d)} \leq M_p'$. Similarly, $\|\tilde{B}_i\|_{\mathcal{S}_p(\mathbb{R}^d)\to\mathcal{S}_p(\mathbb{R}^d)} \leq M_p'$. Hence $\tilde{A}_i, \tilde{B}_i$ are bounded linear operators on $\mathcal{S}_p(\mathbb{R}^d)$.
Using Lemma (3.2.3), we now have

$$\langle \partial_i\phi\,,\, \psi\rangle_p + \langle \phi\,,\, \partial_i\psi\rangle_p = \left\langle \phi\,,\, (\tilde{A}_i U_{-e_i} + \tilde{B}_i U_{+e_i})\psi \right\rangle_p.$$

By Lemma 2.11.2, U_{+e_i}, U_{-e_i} are bounded linear operators on $\mathcal{S}_p(\mathbb{R}^d)$. Hence $T_i = (\tilde{A}_i U_{-e_i} + \tilde{B}_i U_{+e_i})$ is also a bounded linear operator on $\mathcal{S}_p(\mathbb{R}^d)$. □

The operators T_i have the following important property that will be needed in the next section.

Lemma 3.2.5. *For any $1 \leq i,j \leq d$, the map $\langle \partial_i(\cdot)\,,\, T_j(\cdot)\rangle_p : \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ defined by*

$$(\phi,\psi) \mapsto \langle \partial_i\phi\,,\, T_j\psi\rangle_p\,,\ \forall \phi,\psi \in \mathcal{S}(\mathbb{R}^d)$$

is a bounded bilinear form in $\|\cdot\|_p$ and hence extends to a bounded bilinear form on $(\mathcal{S}_p(\mathbb{R}^d), \|\cdot\|_p) \times (\mathcal{S}_p(\mathbb{R}^d), \|\cdot\|_p)$.

Proof. For $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned}
&\langle \partial_i \phi \,,\, T_j \psi \rangle_p \\
&= \sum_{k=0}^{\infty} \sum_{|n|=k} (2k+d)^{2p} \langle \partial_i \phi \,,\, h_n \rangle \langle T_j \psi \,,\, h_n \rangle \\
&= -\sum_{k=0}^{\infty} \sum_{|n=k|} (2k+d)^{2p} \langle \phi \,,\, \partial_i h_n \rangle \left\langle (\tilde{A}_j U_{-e_j} + \tilde{B}_j U_{+e_j}) \psi \,,\, h_n \right\rangle \\
&= -\sum_{k=0}^{\infty} \sum_{|n|=k} (2k+d)^{2p} \left\langle \phi \,,\, \sqrt{\frac{n_i}{2}} h_{n-e_i} - \sqrt{\frac{n_i+1}{2}} h_{n+e_i} \right\rangle \\
&\qquad \times \left\langle (\tilde{A}_j U_{-e_j} + \tilde{B}_j U_{+e_j}) \psi \,,\, h_n \right\rangle \\
&= -\sum_{k=0}^{\infty} \sum_{|n|=k} (2k+d)^{2p} \left(\sqrt{\frac{n_i}{2}} \phi_{n-e_i} - \sqrt{\frac{n_i+1}{2}} \phi_{n+e_i} \right) \\
&\qquad \times \left(a_{n,j} \psi_{n-e_j} + b_{n,j} \psi_{n+e_j} \right)
\end{aligned}$$

From Lemma (3.2.4), we have $a_{n,j} \sim O(\frac{1}{\sqrt{|n|}}), b_{n,j} \sim O(\frac{1}{\sqrt{|n|}})$.
Now using the Cauchy-Schwarz inequality, we get a constant $C > 0$, such that

$$|\langle \partial_i \phi \,,\, T_j \psi \rangle_p| \leq C \|\phi\|_q \|\psi\|_q.$$

This completes the proof. □

3.3 The Monotonicity inequality

Let $\{f_i : 1 \leq i \leq r\}$ denote the standard orthonormal basis for $\mathbb{R}^r$. Let $\sigma = (\sigma_{ij})$ be a constant $d \times r$ matrix with $(a_{ij}) = (\sigma \sigma^t)_{ij}$ and $b = (b_1, ..., b_d) \in \mathbb{R}^d$. For $\phi \in \mathcal{S}$, we define

$$\left.\begin{array}{l}
L\phi := \frac{1}{2} \sum_{i,j=1}^d a_{ij} \partial_{ij}^2 \phi - \sum_{i=1}^d b_i \partial_i \phi, \\
A_i \phi := -\sum_{j=1}^d \sigma_{ji} (\partial_j \phi),\ i = 1, \cdots, r \\
A\phi = (A_1 \phi, \ldots, A_r \phi)
\end{array}\right\}$$

So that for $l \in \mathbb{R}^r$,

$$A\phi(l) := -\sum_{i=1}^r \sum_{j=1}^d \sigma_{ji} (\partial_j \phi) l_i = \sum_{i=1}^r A\phi(f_i) l_i.$$

The following result has already been established in [39] and we present another proof using the results obtained in the previous section.

Theorem 3.3.1. *For every $p \in \mathbb{R}, \exists$ a constant $C = C(p, d, (\sigma_{ij}), (b_j)) > 0$, such that*

$$2 \langle \phi \,,\, L\phi \rangle_p + \|A\phi\|_{HS(p)}^2 \leq C.\|\phi\|_p^2 \tag{3.8}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^d)$, where $\|A\phi\|_{HS(p)}^2 := \sum_{i=1}^r \|A_i\phi\|_p^2$. Furthermore, by density arguments the above inequality can be extended to all $\phi \in \mathcal{S}_{p+1}(\mathbb{R}^d)$.

Proof. Let $\phi \in \mathcal{S}(\mathbb{R}^d)$. For convenience, we introduce two notations

$$L_1\phi := -\sum_{i=1}^d b_i\partial_i\phi, \quad L_2\phi := \frac{1}{2}\sum_{i,j=1}^d a_{ij}\partial_{ij}^2\phi.$$

Observe that for any $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$, by Theorem 3.2.2 we have

$$\langle \phi\,,\, \partial_i\psi\rangle_p + \langle \partial_i\phi\,,\, \psi\rangle_p = \langle T_i\phi\,,\, \psi\rangle_p \le \|T_i\|_{\mathcal{S}_p(\mathbb{R}^d)\to\mathcal{S}_p(\mathbb{R}^d)}\|\phi\|_p\|\psi\|_p.$$

Therefore,

$$\begin{aligned}\langle \phi\,,\, L_1\psi\rangle_p + \langle L_1\phi\,,\, \psi\rangle_p &= -\sum_{i=1}^d b_i\left[\langle \phi\,,\, \partial_i\psi\rangle_p + \langle \partial_i\phi\,,\, \psi\rangle_p\right]\\ &\le \left(\sum_i |b_i|\|T_i\|_{\mathcal{S}_p(\mathbb{R}^d)\to\mathcal{S}_p(\mathbb{R}^d)}\right)\|\phi\|_q\|\psi\|_q.\end{aligned}$$

Taking $\phi = \psi$, we obtain

$$2\langle \phi\,,\, L_1\phi\rangle_p \le \left(\sum_i |b_i|\|T_i\|_{\mathcal{S}_p(\mathbb{R}^d)\to\mathcal{S}_p(\mathbb{R}^d)}\right)\|\phi\|_p^2 \tag{3.9}$$

Now using Theorem 3.2.2,

$$\begin{aligned}2\langle \phi\,,\, L_2\phi\rangle_p &= \sum_{i,j=1}^d (\sigma\sigma^t)_{ij}\langle \phi, \partial_{ij}^2\phi\rangle_p = \sum_{i,j=1}^d (\sigma\sigma^t)_{ij}\langle \partial_i^*\phi, \partial_j\phi\rangle_p\\ &= -\sum_{i,j=1}^d (\sigma\sigma^t)_{ij}\langle \partial_i\phi\,,\, \partial_j\phi\rangle_p + \sum_{i,j=1}^d (\sigma\sigma^t)_{ij}\langle T_i\phi\,,\, \partial_j\phi\rangle_p \end{aligned}\tag{3.10}$$

Note that $(\sigma\sigma^t)_{ij} = \sum_{k=1}^r \sigma_{ik}\sigma_{jk}$. Then

$$\begin{aligned}-\sum_{i,j=1}^d (\sigma\sigma^t)_{ij}\langle \partial_i\phi\,,\, \partial_j\phi\rangle_p &= -\sum_{k=1}^r \left\langle \sum_{i=1}^d \sigma_{ik}\partial_i\phi\,,\, \sum_{j=1}^d \sigma_{jk}\partial_j\phi\right\rangle_p\\ &= -\sum_{k=1}^r \langle A\phi(f_k)\,,\, A\phi(f_k)\rangle_p\\ &= -\|A\phi\|_{HS(q)}^2\end{aligned}$$

Hence from (3.10) we have

$$2\langle \phi\,,\, L_2\phi\rangle_p + \|A\phi\|_{HS(q)}^2 = \sum_{i,j=1}^d (\sigma\sigma^t)_{ij}\langle T_i\phi\,,\, \partial_j\phi\rangle_p\,. \tag{3.11}$$

Using Lemma 3.2.5 we get

$$2\langle \phi\,,\, L_2\phi\rangle_p + \|A\phi\|^2_{HS(q)} \le C'\|\phi\|^2_p$$

for some constant $C' > 0$. Combining with (3.9) we get the result. □

Remark 3.3.2. From the proof it is clear that the constant C in the Monotonicity inequality actually depends on the upper bound of $|\sigma_{ij}|$ and $|b_i|$. In [39, Remark 3.1]), it was observed that the Monotonicity inequality can be extended to the case where the coefficients are bounded random processes.

Remark 3.3.3 (An interpretation using the C_0 group of translation operators)**.** We consider the simple case when $d = 1$ and $A = -\partial$. Fix $p \in \mathbb{R}$. Note that the translation operators (see Example 2.14.3) $\{\tau_t : t \in \mathbb{R}\}$ forms a C_0-group of bounded linear operators on $\mathcal{S}_p$ with A as the infinitesimal generator (see Lemma 2.14.4) and A is a densely defined closed linear operator on $\mathcal{S}_p$ with the domain of A containing $\mathcal{S}$. Then for any $\psi \in \mathcal{S}$,

$$A\psi \overset{\mathcal{S}_p}{=} \lim_{t\to 0} \frac{\tau_t - I}{t}\psi.$$

Now given $\phi, \psi \in \mathcal{S}$ we have

$$\begin{aligned}
\langle \phi\,,\, A\psi\rangle_p &= \lim_{t\to 0}\left\langle \phi\,,\, \frac{\tau_t - I}{t}\psi\right\rangle_p, \text{ (I being the identity operator)}\\
&= \lim_{t\to 0}\frac{1}{t}\langle \phi\,,\, (\tau_t - I)\psi\rangle_p\\
&= \lim_{t\to 0}\frac{1}{t}\langle \mathbf{H}^p\phi\,,\, \mathbf{H}^p(\tau_t - I)\psi\rangle_0, \text{ ($\mathbf{H}$ as in Example 2.11.11)}\\
&= \lim_{t\to 0}\frac{1}{t}\left\langle \mathbf{H}^{2p}\phi\,,\, (\tau_t - I)\psi\right\rangle_0, \text{ ($\because \mathbf{H}$ is a positive operator on $\mathcal{L}^2$)}\\
&= \lim_{t\to 0}\frac{1}{t}\left\langle (\tau_{-t} - I)\mathbf{H}^{2p}\phi\,,\, \psi\right\rangle_0\\
&= \lim_{t\to 0}\frac{1}{t}\left\langle \mathbf{H}^{2p}\mathbf{H}^{-2p}(\tau_{-t} - I)\mathbf{H}^{2p}\phi\,,\, \psi\right\rangle_0\\
&= \lim_{t\to 0}\frac{1}{t}\left\langle \mathbf{H}^{p}\mathbf{H}^{-2p}(\tau_{-t} - I)\mathbf{H}^{2p}\phi\,,\, \mathbf{H}^p\psi\right\rangle_0\\
&= \lim_{t\to 0}\frac{1}{t}\left\langle \mathbf{H}^{-2p}(\tau_{-t} - I)\mathbf{H}^{2p}\phi\,,\, \psi\right\rangle_p\\
&= \left\langle \mathbf{H}^{-2p}(-A)\mathbf{H}^{2p}\phi\,,\, \psi\right\rangle_p
\end{aligned}$$

Hence $A^* = -\mathbf{H}^{-2p}A\mathbf{H}^{2p}$ on $(\mathcal{S}, \|\cdot\|_p)$. Now

$$\begin{aligned}
\left\langle \phi\,,\, A^2\phi\right\rangle_p + \|A\phi\|^2_p &= -\left\langle \mathbf{H}^{-2p}A\mathbf{H}^{2p}\phi\,,\, A\phi\right\rangle_p + \langle A\phi\,,\, A\phi\rangle_p\\
&= \left\langle (A - \mathbf{H}^{-2p}A\mathbf{H}^{2p})\phi\,,\, A\phi\right\rangle_p \qquad (3.12)\\
&= \left\langle \mathbf{H}^{-2p}(\mathbf{H}^{2p}A - A\mathbf{H}^{2p})\phi\,,\, A\phi\right\rangle_p
\end{aligned}$$

We show $\mathbf{H}^{-2p}(\mathbf{H}^{2p}A - A\mathbf{H}^{2p})$ is the bounded operator obtained in Theorem 3.2.2. By Proposition 2.8.1(v)

$$Ah_n = -\sqrt{\frac{n}{2}}h_{n-1} + \sqrt{\frac{n+1}{2}}h_{n+1}.$$

For $\phi \in \mathcal{S}$ with $\phi = \sum_{n=0}^{\infty} \phi_n h_n$, using (2.18) we have

$$\begin{aligned}
(\mathbf{H}^{2p}A - A\mathbf{H}^{2p})\phi &= -\sum_{n=0}^{\infty} \phi_n \mathbf{H}^{2p}\left[\sqrt{\frac{n}{2}}h_{n-1} - \sqrt{\frac{n+1}{2}}h_{n+1}\right] \\
&\quad - \sum_{n=0}^{\infty} \phi_n (2n+1)^{2p} A h_n \\
&= -\sum_{n=0}^{\infty} \phi_n \left[\sqrt{\frac{n}{2}}(2n-1)^{2p}h_{n-1} - \sqrt{\frac{n+1}{2}}(2n+3)^{2p}h_{n+1}\right] \\
&\quad + \sum_{n=0}^{\infty} \phi_n (2n+1)^{2p}\left[\sqrt{\frac{n}{2}}h_{n-1} - \sqrt{\frac{n+1}{2}}h_{n+1}\right] \\
&= -\sum_{n=0}^{\infty} \phi_n \left[\sqrt{\frac{n}{2}}\left\{(2n-1)^{2p} - (2n+1)^{2p}\right\} h_{n-1}\right. \\
&\quad \left. + \sqrt{\frac{n+1}{2}}\left\{(2n+1)^{2p} - (2n+3)^{2p}\right\} h_{n+1}\right]
\end{aligned}$$

Then

$$\begin{aligned}
\mathbf{H}^{-2p}(\mathbf{H}^{2p}A - A\mathbf{H}^{2p})\phi &= -\sum_{n=0}^{\infty} \phi_n \left[\sqrt{\frac{n}{2}}\,\frac{(2n-1)^{2p} - (2n+1)^{2p}}{(2n-1)^{2p}}h_{n-1}\right. \\
&\quad \left. + \sqrt{\frac{n+1}{2}}\,\frac{(2n+1)^{2p} - (2n+3)^{2p}}{(2n+3)^{2p}}h_{n+1}\right] \\
&= -\sum_{n=0}^{\infty}\left[\phi_{n+1}\sqrt{\frac{n+1}{2}}\,\frac{(2n+1)^{2p} - (2n+3)^{2p}}{(2n+1)^{2p}}\right. \\
&\quad \left. + \phi_{n-1}\sqrt{\frac{n}{2}}\,\frac{(2n-1)^{2p} - (2n+1)^{2p}}{(2n+1)^{2p}}\right] h_n \\
&= -\sum_{n=0}^{\infty}(b_n\phi_{n+1} + a_n\phi_{n-1})h_n
\end{aligned}$$

Here we have used the notations $\{a_n\}$ and $\{b_n\}$ instead of $\{a_{n,1}\}$ and $\{b_{n,1}\}$ (see equation (3.3)). Now write T instead of T_1 (see Theorem 3.2.2). Then we have

$$\mathbf{H}^{-2p}(\mathbf{H}^{2p}A - A\mathbf{H}^{2p}) = -T$$

and hence by (3.12) and Lemma 3.2.5 there exists a constant $C > 0$ such that

$$\left\langle \phi\,,\, A^2\phi\right\rangle_p + \|A\phi\|_p^2 = \langle T\phi\,,\, \partial\phi\rangle_p \leq C\,\|\phi\|_p^2$$

for any $\phi \in \mathcal{S}$.

3.4 The Monotonicity inequality for (A^*, L^*)

In the introduction to this chapter, we have used the notations A, L for the differential operators with variable coefficients. This was done for the sake of brevity. The inequality used in [92] is denoted in terms of (A^*, L^*), because there a duality formulation transformed certain flows and the stochastic partial differential equation solved by the dual flow involved the adjoint operators of the original pair (A, L). To bear this in mind, we continue to use the notation (A^*, L^*).

Suppose that $\sigma = (\sigma_{ij}), i = 1, \cdots, d; j = 1, \cdots, r$ and $b = (b_1, \cdots, b_d)$ where σ_{ij}, b_i are C^∞ functions on $\mathbb{R}^d$ with bounded derivatives. Consider the differential operators $A^* = (A_1^*, \cdots, A_r^*), L^*$ on $\mathcal{S}'(\mathbb{R}^d)$ given as follows: for $\psi \in \mathcal{S}'(\mathbb{R}^d)$

$$\begin{cases} A_i^* \psi := -\sum_{k=1}^d \partial_k \left(\sigma_{ki}\psi\right), \\ L^*\psi := \frac{1}{2}\sum_{i,j=1}^d \partial_{ij}^2 \left((\sigma\sigma^t)_{ij}\psi\right) - \sum_{i=1}^d \partial_i \left(b_i\psi\right) \end{cases} \tag{3.13}$$

From now onwards we consider the case $r = d$. If σ_{ij}, b_i, $1 \le i, j \le d$ are real constants, i.e. σ is a real square matrix of order d and $b \in \mathbb{R}^d$, then the Monotonicity inequality for (A^*, L^*) follows from [39, Theorem 2.1] (also see Theorem 3.3.1), since $A_i^*, i = 1, \cdots, d$ and L^* are now constant coefficient differential operators (see [92, Remark after Theorem 4.4]). In this section, we prove the inequality for a slightly more general class of examples, viz. σ is a real square matrix of order d and $b(x) := \alpha + Cx$, $\forall x \in \mathbb{R}^d$ where $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{R}^d$ and $C = (c_{ij})$ is a real square matrix of order d. Unless otherwise specified, p will be an arbitrary but fixed real number.

First we identify the adjoint of the multiplication operators $\mathscr{M}_i, i = 1, \cdots, d$ (see Example 2.11.9) on $(\mathcal{S}(\mathbb{R}^d), \|\cdot\|_p)$.

Theorem 3.4.1. *The following are some properties of the operators* $\mathscr{M}_i$.

(i) For any $1 \le i \le d$ *and* $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned} &\langle \mathscr{M}_i\phi\,,\,\psi\rangle_p - \langle \phi\,,\,\mathscr{M}_i\psi\rangle_p \\ &= \sum_{k=0}^\infty \sum_{|n|=k} (2k+d)^{2p}\phi_n a_{n,i}\psi_{n-e_i} - \sum_{k=0}^\infty \sum_{|n|=k} (2k+d)^{2p}\phi_n b_{n,i}\psi_{n+e_i} \end{aligned} \tag{3.14}$$

and hence

$$\mathscr{M}_i^* = \mathscr{M}_i + \widetilde{T}_i \text{ on } (\mathcal{S}(\mathbb{R}^d), \|\cdot\|_p)$$

with $\widetilde{T}_i$ *is a bounded linear operator on* $(\mathcal{S}_p(\mathbb{R}^d), \|\cdot\|_p)$ *given by*

$$\widetilde{T}_i = \tilde{A}_i U_{-e_i} - \tilde{B}_i U_{+e_i}$$

where $\tilde{A}_i, U_{-e_i}, \tilde{B}_i, U_{+e_i}, a_{n,i}, b_{n,i}$ *are as in Theorem 3.2.2.*

(ii) For any $1 \leq i, j \leq d$*, the map* $\left\langle \partial_i(\cdot)\, ,\, \widetilde{T}_j(\cdot)\right\rangle_p : \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ *defined by*

$$(\phi, \psi) \mapsto \left\langle \partial_i \phi\, ,\, \widetilde{T}_j \psi\right\rangle_p,\ \forall \phi, \psi \in \mathcal{S}(\mathbb{R}^d)$$

is a bounded bilinear form in $\|\cdot\|_p$ *and hence extends to a bounded bilinear form on* $(\mathcal{S}_p(\mathbb{R}^d), \|\cdot\|_p) \times (\mathcal{S}_p(\mathbb{R}^d), \|\cdot\|_p)$.

(iii) For any $1 \leq i \leq d$*, let* T_i *be as in Theorem 3.2.2. Then for any* $1 \leq i, j \leq d$*, the map* $\langle \mathscr{M}_i(\cdot)\, ,\, T_j(\cdot)\rangle_p : \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ *defined by*

$$(\phi, \psi) \mapsto \langle \mathscr{M}_i \phi\, ,\, T_j \psi\rangle_p,\ \forall \phi, \psi \in \mathcal{S}(\mathbb{R}^d)$$

is a bounded bilinear form in $\|\cdot\|_p$ *and hence extends to a bounded bilinear form on* $(\mathcal{S}_p(\mathbb{R}^d), \|\cdot\|_p) \times (\mathcal{S}_p(\mathbb{R}^d), \|\cdot\|_p)$.

Proof. Since $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$, we can write

$$\phi = \sum_{k=0}^{\infty} \sum_{|n|=k} \phi_n h_n, \psi = \sum_{k=0}^{\infty} \sum_{|n|=k} \psi_n h_n.$$

Now $\mathscr{M}_i \phi = \sum_{k=0}^{\infty} \sum_{|n|=k} \phi_n (\mathscr{M}_i h_n)$, where $\mathscr{M}_i h_n = \sqrt{\frac{n_i}{2}} h_{n-e_i} + \sqrt{\frac{n_i+1}{2}} h_{n+e_i}$ for all $n = (n_1, .., n_d)$ (see Proposition 2.8.1). Therefore,

$$\begin{aligned}
\mathscr{M}_i \phi &= \sum_{k=0}^{\infty} \sum_{|n|=k} \phi_n \left[\sqrt{\frac{n_i}{2}} h_{n-e_i} + \sqrt{\frac{n_i+1}{2}} h_{n+e_i}\right] \\
&= \sum_{k=0}^{\infty} \sum_{\substack{|m|=k-1,\\ m=n-e_i}} \phi_{m+e_i} \sqrt{\frac{m_i+1}{2}} h_m + \sum_{k=0}^{\infty} \sum_{\substack{|m|=k+1,\\ m=n+e_i}} \phi_{m-e_i} \sqrt{\frac{m_i}{2}} h_m \\
&= \sum_{\substack{l=-1,\\ l=k-1}}^{\infty} \sum_{|m|=l} \phi_{m+e_i} \sqrt{\frac{m_i+1}{2}} h_m + \sum_{\substack{l=1,\\ l=k+1}}^{\infty} \sum_{|m|=l} \phi_{m-e_i} \sqrt{\frac{m_i}{2}} h_m \\
&= \sum_{l=0}^{\infty} \sum_{|m|=l} \phi_{m+e_i} \sqrt{\frac{m_i+1}{2}} h_m + \sum_{l=0}^{\infty} \sum_{|m|=l} \phi_{m-e_i} \sqrt{\frac{m_i}{2}} h_m
\end{aligned}$$

Similar expression is true for $\mathscr{M}_i \psi$. Therefore,

$$\langle \phi\, ,\, \mathscr{M}_i \psi\rangle_p = \sum_{k=0}^{\infty} (2k+d)^{2p} \sum_{|n|=k} \phi_n \left[\sqrt{\frac{n_i+1}{2}} \psi_{n+e_i} + \sqrt{\frac{n_i}{2}} \psi_{n-e_i}\right]$$

and

$$\begin{aligned}\langle \mathscr{M}_i\phi\,,\,\psi\rangle_p &= \sum_{k=0}^{\infty}(2k+d)^{2p}\sum_{|n|=k}\psi_n\left[\sqrt{\frac{n_i+1}{2}}\phi_{n+e_i}+\sqrt{\frac{n_i}{2}}\phi_{n-e_i}\right]\\
&=\sum_{k=0}^{\infty}\sum_{\substack{|m|=k+1\\ m=n+e_i}}(2k+d)^{2p}\phi_m\psi_{m-e_i}\sqrt{\frac{m_i}{2}}\\
&+\sum_{k=0}^{\infty}\sum_{\substack{|m|=k-1\\ m=n-e_i}}(2k+d)^{2p}\phi_m\psi_{m+e_i}\sqrt{\frac{m_i+1}{2}}\\
&=\sum_{k=1}^{\infty}\sum_{|m|=k}(2k+d-2)^{2p}\phi_m\psi_{m-e_i}\sqrt{\frac{m_i}{2}}\\
&+\sum_{k=-1}^{\infty}\sum_{|m|=k}(2k+d+2)^{2p}\phi_m\psi_{m+e_i}\sqrt{\frac{m_i+1}{2}}\end{aligned}$$

the term for $k=0$ in the first sum evaluates to 0 because of $\psi_{m-e_i}\sqrt{\frac{m_i}{2}}$, also the term for $k=-1$ in the second sum is 0 because of ϕ_m

$$\begin{aligned}&=\sum_{k=0}^{\infty}\sum_{|m|=k}(2k+d)^{2p}\phi_m\left[\psi_{m-e_i}\sqrt{\frac{m_i}{2}}\left(\frac{2k+d-2}{2k+d}\right)^{2p}\right]\\
&+\sum_{k=0}^{\infty}\sum_{|m|=k}(2k+d)^{2p}\phi_m\left[\psi_{m+e_i}\sqrt{\frac{m_i+1}{2}}\left(\frac{2k+d+2}{2k+d}\right)^{2p}\right]\end{aligned}$$

Combining expressions for $\langle \mathscr{M}_i\phi\,,\,\psi\rangle_p$ and $\langle \phi\,,\,\mathscr{M}_i\psi\rangle_p$ we get

$$\begin{aligned}&\langle \mathscr{M}_i\phi\,,\,\psi\rangle_p-\langle \phi\,,\,\mathscr{M}_i\psi\rangle_p\\
&=\sum_{k=0}^{\infty}\sum_{|m|=k}(2k+d)^{2p}\phi_m\left[\psi_{m-e_i}\sqrt{\frac{m_i}{2}}\left\{\left(\frac{2k+d-2}{2k+d}\right)^{2p}-1\right\}\right]\\
&-\sum_{k=0}^{\infty}\sum_{|m|=k}(2k+d)^{2p}\phi_m\left[\psi_{m+e_i}\sqrt{\frac{m_i+1}{2}}\left\{1-\left(\frac{2k+d+2}{2k+d}\right)^{2p}\right\}\right]\\
&=\sum_{k=0}^{\infty}\sum_{|m|=k}(2k+d)^{2p}\phi_m a_{m,i}\psi_{m-e_i}-\sum_{k=0}^{\infty}\sum_{|m|=k}(2k+d)^{2p}\phi_m b_{m,i}\psi_{m+e_i}\end{aligned}$$

Proof of part (ii) and (iii) are similar to Lemma 3.2.5. We give the details for part (ii). For $\phi,\psi\in\mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned}&\left\langle \partial_i\phi\,,\,\widetilde{T}_j\psi\right\rangle_p\\
&=\sum_{k=0}^{\infty}\sum_{|n|=k}(2k+d)^{2p}\left\langle \partial_i\phi\,,\,h_n\right\rangle\left\langle \widetilde{T}_j\psi\,,\,h_n\right\rangle\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{k=0}^{\infty}\sum_{|n=k|}(2k+d)^{2p}\left\langle \phi\,,\,\partial_i h_n\right\rangle \left\langle (\tilde{A}_j U_{-e_j} - \tilde{B}_j U_{+e_j})\psi\,,\,h_n\right\rangle \\
&= -\sum_{k=0}^{\infty}\sum_{|n|=k}(2k+d)^{2p}\left\langle \phi\,,\,\sqrt{\frac{n_i}{2}}h_{n-e_i} - \sqrt{\frac{n_i+1}{2}}h_{n+e_i}\right\rangle \\
&\qquad\qquad\qquad \times \left\langle (\tilde{A}_j U_{-e_j} - \tilde{B}_j U_{+e_j})\psi\,,\,h_n\right\rangle \\
&= -\sum_{k=0}^{\infty}\sum_{|n|=k}(2k+d)^{2p}\left(\sqrt{\frac{n_i}{2}}\phi_{n-e_i} - \sqrt{\frac{n_i+1}{2}}\phi_{n+e_i}\right) \\
&\qquad\qquad\qquad \times \left(a_{n,j}\psi_{n-e_j} - b_{n,j}\psi_{n+e_j}\right)
\end{aligned}
$$

From Lemma (3.2.4), we have $a_{n,j} \sim O(\frac{1}{\sqrt{|n|}}), b_{n,j} \sim O(\frac{1}{\sqrt{|n|}})$.
Now using the Cauchy-Schwarz inequality, we get a constant $C > 0$, such that

$$
|\left\langle \partial_i\phi\,,\,\widetilde{T}_j\psi\right\rangle_p| \le C\|\phi\|_q\|\psi\|_q.
$$

This completes the proof. □

The following is the main result of the section.

Theorem 3.4.2. *Let $\alpha = (\alpha_1,\cdots,\alpha_d) \in \mathbb{R}^d$ and $C = (c_{ij})$ be a real square matrix of order d. Let σ be a constant function, i.e. $\sigma(x) \equiv (\sigma_{ij})$, $\forall x \in \mathbb{R}^d$ where $\sigma_{ij} \in \mathbb{R}$, $i,j = 1,\cdots,d$. Let $b = (b_1,\cdots,b_d)$ with $b(x) := \alpha + Cx$, $\forall x \in \mathbb{R}^d$. Fix $p \in \mathbb{R}$. Then*

- *(i) The maps A_i^* are bounded linear operators from $\mathcal{S}_{p+\frac{1}{2}}(\mathbb{R}^d)$ to $\mathcal{S}_p(\mathbb{R}^d)$ and L^* is a bounded linear operator from $\mathcal{S}_{p+1}(\mathbb{R}^d)$ to $\mathcal{S}_p(\mathbb{R}^d)$.*
- *(ii) Monotonicity inequality for A^*, L^* holds, i.e. there exists a positive constant $R = R(p,d,(\sigma_{ij}),(b_j))$, such that*

$$
2\left\langle \phi\,,\,L^*\phi\right\rangle_p + \|A^*\phi\|^2_{HS(p)} \le R\,\|\phi\|_p^2 \tag{3.15}
$$

for all $\phi \in \mathcal{S}_{p+1}(\mathbb{R}^d)$.

Proof. Let $\phi \in \mathcal{S}'(\mathbb{R}^d)$. Then, $A_i^*\phi = -\sum_{k=1}^d \partial_k(\sigma_{ki}\phi) = -\sum_{k=1}^d \sigma_{ki}\partial_k(\phi)$, for $1 \le i \le d$. Also

$$
\begin{aligned}
L^*\phi &= \frac{1}{2}\sum_{i,j=1}^d \partial_{ij}^2\left((\sigma\sigma^t)_{ij}\phi\right) - \sum_{i=1}^d \partial_i(b_i\phi) \\
&= \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^t)_{ij}\partial_{ij}^2(\phi) - \sum_{i=1}^d \alpha_i\partial_i(\phi) - \sum_{i,j=1}^d c_{ij}\partial_i(M_j\phi).
\end{aligned}
$$

For any $q \in \mathbb{R}$, $\partial_i : \mathcal{S}_{q+\frac{1}{2}}(\mathbb{R}^d) \to \mathcal{S}_q(\mathbb{R}^d)$ and $\mathscr{M}_i : \mathcal{S}_{q+\frac{1}{2}}(\mathbb{R}^d) \to \mathcal{S}_q(\mathbb{R}^d)$ are bounded linear operators (see Example 2.11.3 and Example 2.11.9). Hence we get the boundedness of A_i^* and L^* as mentioned in part (i).
To prove (ii), we first introduce the notations: for $\phi \in \mathcal{S}'(\mathbb{R}^d)$,

$$L_1^*(\phi) := \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^t)_{ij}\partial_{ij}^2(\phi) - \sum_{i=1}^d \alpha_i\partial_i(\phi), \quad L_2^*(\phi) := -\sum_{i,j=1}^d c_{ij}\partial_i\left(M_j\phi\right)$$

By Theorem 3.3.1, there exists a constant $\tilde{C} = \tilde{C}(p, d, (\sigma_{ij}), (b_j)) > 0$,

$$2\left\langle \phi\,,\, L_1^*\phi\right\rangle_p + \|A^*\phi\|_{HS(p)}^2 \leq \tilde{C}\,\|\phi\|_p^2,\ \forall \phi \in \mathcal{S}_{p+1}(\mathbb{R}^d).$$

To complete the proof, it is enough to show that

$$2\left\langle \phi\,,\, L_2^*\phi\right\rangle_p \leq C'\,\|\phi\|_p^2,\ \forall \phi \in \mathcal{S}(\mathbb{R}^d)$$

for some constant $C' = C'(p, d, (\sigma_{ij}), (b_j)) > 0$ and then the same inequality extends to $\phi \in \mathcal{S}_{p+1}(\mathbb{R}^d)$ via density arguments.
For $\phi \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{cases} \text{for } i = j,\ \partial_i\left(M_j\phi\right) = \phi + M_i(\partial_i\phi) \\ \text{for } i \neq j,\ \partial_i\left(M_j\phi\right) = M_j(\partial_i\phi) \end{cases}.$$

Then for any i, using Theorem 3.2.2 and Theorem 3.4.1

$$\begin{aligned} \left\langle \phi\,,\, \partial_i\left(M_i\phi\right)\right\rangle_p &= \|\phi\|_p^2 + \left\langle \phi\,,\, M_i(\partial_i\phi)\right\rangle_p \\ &= \|\phi\|_p^2 + \left\langle M_i\phi\,,\, \partial_i\phi\right\rangle_p + \left\langle \widetilde{T}_i\phi\,,\, \partial_i\phi\right\rangle_p \\ &= \|\phi\|_p^2 + \left\langle M_i\phi\,,\, (-\partial_i^* + T_i)\phi\right\rangle_p + \left\langle \widetilde{T}_i\phi\,,\, \partial_i\phi\right\rangle_p \\ &= \|\phi\|_p^2 - \left\langle \partial_i\left(M_i\phi\right)\,,\, \phi\right\rangle_p + \left\langle M_i\phi\,,\, T_i\phi\right\rangle_p + \left\langle \widetilde{T}_i\phi\,,\, \partial_i\phi\right\rangle_p \end{aligned}$$

and hence $2\left\langle \phi\,,\, \partial_i\left(M_i\phi\right)\right\rangle_p = \|\phi\|_p^2 + \left\langle M_i\phi\,,\, T_i\phi\right\rangle_p + \left\langle \widetilde{T}_i\phi\,,\, \partial_i\phi\right\rangle_p$.
For $i \neq j$, a similar computation yields $2\left\langle \phi\,,\, \partial_i\left(M_j\phi\right)\right\rangle_p = \left\langle M_j\phi\,,\, T_i\phi\right\rangle_p + \left\langle \widetilde{T}_j\phi\,,\, \partial_i\phi\right\rangle_p$.
Hence

$$\begin{aligned} 2\left\langle \phi\,,\, L_2^*\phi\right\rangle_p &= -2\sum_{i,j=1}^d c_{ij}\left\langle \phi\,,\, \partial_i\left(M_j\phi\right)\right\rangle_p \\ &= -2\sum_{i\neq j} c_{ij}\left\langle \phi\,,\, \partial_i\left(M_j\phi\right)\right\rangle_p - 2\sum_{i=1}^d c_{ii}\left\langle \phi\,,\, \partial_i\left(M_i\phi\right)\right\rangle_p \\ &= -\sum_{i,j=1}^d c_{ij}\left[\left\langle M_j\phi\,,\, T_i\phi\right\rangle_p + \left\langle \widetilde{T}_j\phi\,,\, \partial_i\phi\right\rangle_p\right] - \|\phi\|_p^2\sum_{i=1}^d c_{ii}. \end{aligned}$$

Bilinearity of $(\phi, \psi) \mapsto \langle M_j\phi\,,\, T_i\phi\rangle_p + \left\langle \widetilde{T}_j\phi\,,\, \partial_i\phi\right\rangle_p$ (see Theorem 3.4.1) gives the required estimate on $2\,\langle \phi\,,\, L_2^*\phi\rangle_p$ and this completes the proof. □

Remark 3.4.3. Theorem 3.4.2 covers a class of examples where the Monotonicity inequality for the pair (A^*, L^*) holds. The problem of characterizing all (σ, b) such that the inequality holds is a problem for the future. This question remains unresolved to date, to our knowledge.

CHAPTER 4

Gaussian flows and probabilistic representation of solutions of the Forward equations

4.1 Introduction

Itô's stochastic differential equations provide a concrete model for stochastic flows, on which topic there is a considerable literature (see [6, 13, 16, 18, 31, 32, 49, 66, 69, 70, 73, 76, 104, 119] and the references therein). In this chapter, we study three interrelated properties (which we call property I, II and III) of stochastic flows arising as solutions of finite dimensional stochastic differential equations, viz.

$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt,\ t \geq 0. \tag{4.1}$$

In Section 2 we explore property I and in Section 3 we consider properties II, III.
Property I: We want to identify the pairs (σ, b) (assumed to be Lipschitz continuous) such that the general solutions $\{X_t^x\}$ (x denotes the deterministic initial conditions) of the corresponding diffusions are the sum of the solution starting at 0, i.e. $\{X_t^0\}$ and the value of a deterministic function at the initial condition, viz. $f(t, x)$. We call this class of diffusions as diffusions depending deterministically on the initial condition (see Definition 4.2.1). A consequence of this notion is that the map $t \mapsto f(t, x)$ is C^1 for each fixed $x \in \mathbb{R}^d$ (see Lemma 4.2.3).
Property II: We want to identify the pairs (σ, b) (assumed to be sufficiently smooth) such that the map $x \mapsto \psi(X_t^x)$ is in $\mathcal{S}(\mathbb{R}^d)$ whenever $\psi \in \mathcal{S}(\mathbb{R}^d)$. In [92], a similar composition of maps led to the existence of a solution of (see [92, Theorem 3.3])

$$Y_t(\psi) = \psi + \int_0^t A^*(Y_s(\psi))\, .dB_s + \int_0^t L^*(Y_s(\psi))\, ds,\ \forall t \geq 0, \tag{4.2}$$

in some Hermite Sobolev space $\mathcal{S}_p(\mathbb{R}^d)$, where $\psi \in \mathcal{E}'(\mathbb{R}^d)$ - the space of compactly supported distributions on $\mathbb{R}^d$, $\{B_t\}$ - a r dimensional standard Brownian motion and the

operators $A^* = (A_1^*, \cdots, A_r^*), L$ are as in equation (3.13).
Property III: We want to identify the pairs (σ, b) such that the solution to the SPDE (4.2) is unique. This question remains unresolved to date, to our knowledge. The Monotonicity inequality for the pair (A^*, L^*) is a sufficient condition for the uniqueness.

We consider the case $r = d$ and show that all three properties hold if

(i) σ is a real $d \times d$ matrix.
(ii) $b(x) := \alpha + Cx, \forall x \in \mathbb{R}^d$ where $\alpha \in \mathbb{R}^d$ and $C = (c_{ij})$ is a real $d \times d$ matrix.

Under 'nice' conditions we show that property I holds if and only if the pair σ, b is given by (i) and (ii) (Proposition 4.2.5, Theorem 4.2.4). Since the flows generated by these coefficients are Gaussian, these results can be considered as characterization results on Gaussian flows. In Proposition 4.2.10 and Theorem 4.2.12, we discuss some generalizations of Definition 4.2.1.

For σ, b in our class, we observe that $\{X_t^0 + e^{tC}x\}$ solves equation (4.1) (Lemma 4.3.1). In particular this result implies property II. Using this result, we define continuous linear maps $X_t(\omega) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ (Lemma 4.3.4) and the corresponding adjoints $X_t^*(\omega)$: $\mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$. Recall that the space of tempered distributions given by integrable functions, viz. $\mathcal{L}^1(\mathbb{R}^d)$ is a subset of $\mathcal{S}_{-p}(\mathbb{R}^d)$ whenever $p > \frac{d}{4}$ (Lemma 2.11.20). For any $\psi \in \mathcal{L}^1(\mathbb{R}^d)$, we construct an $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued continuous adapted process $\{Y_t(\psi)\}$ with the property $X_t^*(\psi) = Y_t(\psi)$ (see equation (4.17)). We then show that the process $\{Y_t(\psi)\}$ satisfies the stochastic partial differential equation (4.2) in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$ with $Y_0 = \psi \in \mathcal{L}^1(\mathbb{R}^d)$ (Theorem 4.3.8).

Note that the Monotonicity inequality for (A^*, L^*), mentioned in property III, was proved in Chapter 3 for σ, b as given by (i) and (ii) (Theorem 3.4.2).

Taking expectation on both sides of equation (4.2), we show in Theorem 4.3.9 that $\overline{\psi}(t) := \mathbb{E}\, Y_t(\psi)$ solves the Cauchy problem for L^*, viz.

$$\frac{d\psi(t)}{dt} = L^*\psi(t); \quad \psi(0) = \psi, \tag{4.3}$$

where $\psi \in \mathcal{L}^1(\mathbb{R}^d)$. Furthermore, the uniqueness of solutions of (4.3) follows from the Monotonicity inequality for (A^*, L^*). This result was motivated by [92, Theorem 4.4], where the uniqueness of (4.3) was obtained for $\psi \in \mathcal{E}'(\mathbb{R}^d)$. Note that we have explicitly proved the Monotonicity inequality for the pair (A^*, L^*) (corresponding to σ, b in our class), whereas in [92] it was stated as an assumption.

It was shown in [90] that the solutions of certain stochastic partial differential equations can be represented as translates of the initial condition by the solution of a finite dimensional diffusion. In Proposition 4.3.10, we prove a similar result, viz. the tempered distribution $Y_t(\psi)$ is given by the integrable function $e^{-t\,tr(C)}\,\tau_{X(t,0)}\psi(e^{-tC}\cdot)$ where $tr(C)$ is the trace of the matrix C.

Most of the results in this chapter are from [8].

4.2 Characterizing diffusions with the general solution depending deterministically on the initial condition

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered complete probability space satisfying the usual conditions. Let $\{B_t\}$ be a standard d-dimensional $(\mathcal{F}_t)$ Brownian motion. Now consider the diffusion:

$$dX_t = \sigma(X_t)\,dB_t + b(X_t)\,dt,\ \forall t \geq 0, \tag{4.4}$$

where the coefficients $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}, b : \mathbb{R}^d \to \mathbb{R}^d$ are Lipschitz continuous. Note that σ, b satisfy a linear growth condition, i.e. there exists a constant $K > 0$ such that

$$|\sigma(x)| + |b(x)| \leq K(1 + |x|),\ \forall x \in \mathbb{R}^d,$$

where $|\cdot|$ denotes the Euclidean norm in the appropriate spaces. For any $x \in \mathbb{R}^d$, let $\{X_t^x\}$ denote the solution of (4.4) with $X_0 = x$.

Definition 4.2.1. We say the general solution to the diffusion (4.4) depends deterministically on the initial condition, if there exists a function $f : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ such that for any $x \in \mathbb{R}^d$, we have a.s.

$$X_t^x(\omega) = f(t,x) + X_t^0(\omega),\ t \geq 0. \tag{4.5}$$

Remark 4.2.2. A motivation to look for this type of diffusions is to have 'nice' solution $\{X_t^x\}$ so that the composition $x \mapsto \phi(X_t^x)$ has 'smoothness' for ϕ in a suitable function class. A special case of this type of diffusions and subsequent composition will be used in Section 3.

If equation (4.5) is satisfied, then the function f has 'nice' properties. This is our next result. The component functions of f are denoted by $f_1, \cdots, f_d$.

Lemma 4.2.3. *If the general solution of the diffusion* (4.4) *depends deterministically on the initial condition, then for all* $(t,x) \in [0,\infty) \times \mathbb{R}^d$*, the partial derivative* $\frac{\partial f}{\partial t}(t,x)$ *exists and for every fixed* $x \in \mathbb{R}^d$*, the map* $t \mapsto \frac{\partial f}{\partial t}(t,x)$ *is continuous, where* $\frac{\partial f}{\partial t}(t,x) = (\frac{\partial f_1}{\partial t}(t,x), \cdots, \frac{\partial f_d}{\partial t}(t,x))$.

Proof. Due to the linear growth of the coefficients σ, b, the first and second moments of X_t^x exist for all x, t. If equation (4.5) is satisfied, then for all x, t we have

$$f(t,x) = \mathbb{E}X_t^x - \mathbb{E}X_t^0 = x + \int_0^t \left[\mathbb{E}b(X_s^x) - \mathbb{E}b(X_s^0)\right] ds.$$

This implies the differentiability of f and the continuity of $t \mapsto \frac{\partial f}{\partial t}(t,x)$ for every $x \in \mathbb{R}^d$. □

In Theorem 4.2.4 and Proposition 4.2.5 we characterize diffusions depending deterministically on the initial condition. In the first result we obtain a characterization under a non-degeneracy condition on σ and smoothness assumptions on certain derivatives of b, f. In the second result we consider the case when f is in a product form.

For any $d \times d$ matrix C, the bounded linear operator on $\mathbb{R}^d$ given by the matrix $\sum_{n=0}^{\infty} \frac{t^n}{n!} C^n$ will be denoted by e^{tC}.

Theorem 4.2.4. *Let σ, b be Lipschitz continuous functions. Suppose the following happen:*

(i) there exists an $x \in \mathbb{R}^d$ such that the determinant of $(\sigma_{ij}(x))$ is not zero,
(ii) $b_i \in C^2(\mathbb{R}^d, \mathbb{R}), i = 1, \cdots, d$ where $b = (b_1, \cdots, b_d)$,
(iii) for every fixed $x \in \mathbb{R}^d$, the map $t \mapsto \frac{\partial f}{\partial t}(t,x)$ is of bounded variation.

Then the general solution of the diffusion (4.4) *depends deterministically on the initial condition through* (4.5) *if and only if σ is a real non-singular matrix of order d and b is of the form $b(x) = \alpha + Cx$ and $f(t,x) = e^{tC}x$ where $\alpha \in \mathbb{R}^d$ and C is a real square matrix of order d.*

Proof. Suppose that the solution of the diffusion depends deterministically on the initial condition through (4.5). Then for any $x \in \mathbb{R}^d$, a.s. $t \geq 0$

$$\begin{aligned} f(t,x) &= X_t^x - X_t^0 \\ &= f(0,x) + \int_0^t \left[\sigma(X_s^x) - \sigma(X_s^0)\right] . dB_s + \int_0^t \left[b(X_s^x) - b(X_s^0)\right] ds. \end{aligned}$$

Note that necessarily we must have $f(0,x) = x$. Now rewriting above relation

$$\int_0^t \left[\sigma(X_s^x) - \sigma(X_s^0)\right] . dB_s + \int_0^t \left[b(X_s^x) - b(X_s^0) - \frac{\partial f}{\partial t}(s,x)\right] ds = 0.$$

But the first integral is a continuous martingale and the second is a continuous process of finite variation. Hence the martingale is almost surely constant (see Proposition 2.5.19).

Since it starts at 0, the martingale term is 0 a.s. and hence so is the finite variation term, i.e. a.s. $t \geq 0$

$$\int_0^t \left[\sigma(X_s^x) - \sigma(X_s^0)\right] . dB_s = 0, \tag{4.6a}$$

$$\int_0^t \left[b(X_s^x) - b(X_s^0) - \frac{\partial f}{\partial t}(s,x)\right] ds = 0. \tag{4.6b}$$

The quadratic variation of the martingale in (4.6a) is also 0 and hence for any fixed $x \in \mathbb{R}^d$ and for any $i, j = 1, \cdots, d$ a.s.

$$\int_0^t \left[\sigma_{ij}(X_s^x) - \sigma_{ij}(X_s^0)\right]^2 ds = 0,\ t \geq 0.$$

But for fixed x and a.s. ω the map $t \mapsto [\sigma_{ij}(X_t^x) - \sigma_{ij}(X_t^0)]^2$ is continuous and hence for all $x \in \mathbb{R}^d, i, j = 1, \cdots, d$ a.s. $[\sigma_{ij}(X_t^x) - \sigma_{ij}(X_t^0)]^2 = 0$, $\forall t \geq 0$. Putting $t = 0$ we have $\sigma_{ij}(x) = \sigma_{ij}(0)$, $x \in \mathbb{R}^d$ i.e. σ is a constant $d \times d$ matrix. The fact that the determinant of σ is non-zero, follows from our hypothesis.

On the other hand, from equation (4.6b), for each $x \in \mathbb{R}^d$, a.s. $t \geq 0$,

$$b(X_t^x) - b(X_t^0) - \frac{\partial f}{\partial t}(t,x) = 0. \tag{4.7}$$

Evaluating at $t = 0$ yields

$$b_i(x) = b_i(0) + \frac{\partial f_i}{\partial t}(0,x),\ i = 1, \cdots, d.$$

Let $\{B_t^{(i)}\}$ denote the ith component of $\{B_t\}$. Since $b_i \in C^2(\mathbb{R}^d, \mathbb{R})$, by Itô formula we have for $x \in \mathbb{R}^d$, a.s. $t \geq 0$,

$$\begin{aligned} b_i(X_t^x) = b_i(x) &+ \sum_{j=1}^d \int_0^t \partial_j b_i(X_s^x)\, d(X^x)_s^{(j)} \\ &+ \frac{1}{2} \sum_{j,k=1}^d \int_0^t \partial_j \partial_k b_i(X_s^x)\, d\left[(X^x)^{(j)}, (X^x)^{(k)}\right]_s \end{aligned}$$

Using (4.7) and the Itô formula above, we have a.s. $t \geq 0$,

$$\begin{aligned} \frac{\partial f_i}{\partial t}(t,x) &= b_i(X_t^x) - b_i(X_t^0) \\ &= [b_i(x) - b_i(0)] + \sum_{j,k=1}^d \int_0^t \left(\partial_j b_i(X_s^x) - \partial_j b_i(X_s^0)\right) \sigma_{jk} dB_s^{(k)} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{d}\int_0^t \Big[\partial_j b_i(X_s^x)b_i(X_s^x)-\partial_j b_i(X_s^0)b_i(X_s^0)\Big]\,ds\\
&+\frac{1}{2}\sum_{j,k=1}^{d}\int_0^t(\sigma\sigma^t)_{jk}\Big[\partial_j\partial_k b_i(X_s^x)-\partial_j\partial_k b_i(X_s^0)\Big]\,ds\\
&=\frac{\partial f_i}{\partial t}(0,x)+\sum_{j,k=1}^{d}\int_0^t\Big(\partial_j b_i(X_s^x)-\partial_j b_i(X_s^0)\Big)\sigma_{jk}dB_s^{(k)}\\
&+\sum_{j=1}^{d}\int_0^t \Big[\partial_j b_i(X_s^x)b_i(X_s^x)-\partial_j b_i(X_s^0)b_i(X_s^0)\Big]\,ds\\
&+\frac{1}{2}\sum_{j,k=1}^{d}\int_0^t(\sigma\sigma^t)_{jk}\Big[\partial_j\partial_k b_i(X_s^x)-\partial_j\partial_k b_i(X_s^0)\Big]\,ds.
\end{aligned}
$$

Then a.s. $t \geq 0$,

$$
\begin{aligned}
\frac{\partial f_i}{\partial t}(t,x)-\frac{\partial f_i}{\partial t}(0,x)&=\sum_{j,k=1}^{d}\int_0^t\Big(\partial_j b_i(X_s^x)-\partial_j b_i(X_s^0)\Big)\sigma_{jk}dB_s^{(k)}\\
&+\sum_{j=1}^{d}\int_0^t \Big[\partial_j b_i(X_s^x)b_j(X_s^x)-\partial_j b_i(X_s^0)b_j(X_s^0)\Big]\,ds\\
&+\frac{1}{2}\sum_{j,k=1}^{d}\int_0^t(\sigma\sigma^t)_{jk}\Big[\partial_j\partial_k b_i(X_s^x)-\partial_j\partial_k b_i(X_s^0)\Big]\,ds.
\end{aligned}
$$

Again, the martingale term must be zero. Then for any $i,k = 1,\cdots,d$ we have a.s. $\sum_{j=1}^d \sigma_{jk}^2\Big(\partial_j b_i(X_t^x)-\partial_j b_i(X_t^0)\Big)^2 = 0$, $t \geq 0$. Evaluating at $t = 0$ and simplifying we have $\sum_{j=1}^d \sigma_{jk}(\partial_j b_i(x)-\partial_j b_i(0)) = 0$. The last equation we can write as

$$
\sigma^t\begin{pmatrix}\partial_1 b_i(x)-\partial_1 b_i(0)\\ \cdots \\ \partial_d b_i(x)-\partial_d b_i(0)\end{pmatrix}=0.
$$

Since σ is non-singular, we have for each $i,j = 1,\cdots,d$ the function $x \mapsto \partial_j b_i(x)$ is a constant function. Define $c_{ij} := \partial_j b_i(0)$ and write $C = (c_{ij})$. Then

$$
\begin{aligned}
b_i(x)-b_i(0)&=\Big(b_i(x_1,x_2,\cdots,x_d)-b_i(0,x_2,\cdots,x_d)\Big)\\
&+\Big(b_i(0,x_2,\cdots,x_d)-b_i(0,0,x_3,\cdots,x_d)\Big)\\
&+\cdots+\Big(b_i(0,\cdots,0,x_d)-b_i(0,\cdots,0)\Big)\\
&=\int_0^{x_1}\partial_1 b_i(y,x_2,\cdots,x_d)\,dy+\cdots+\int_0^{x_d}\partial_d b_i(0,\cdots,0,y)\,dy\\
&=\sum_{j=1}^{d}c_{ij}x_j
\end{aligned}
$$

Now for any fixed $x \in \mathbb{R}^d$, we have $f(0,x) = x$ and

$$f(t,x) - x = \int_0^t \left[b(X_s^x) - b(X_s^0)\right] ds = \int_0^t C\left[X_s^x - X_s^0\right] ds = \int_0^t Cf(s,x)\, ds.$$

Hence $f(t,x) = e^{tC}x$ (see Example 2.14.2). This completes the proof of necessity.
To prove the converse, observe that there exists a P null set $\mathcal{N}$ such that for all $\omega \in \Omega \setminus \mathcal{N}, t \geq 0$

$$X_t^0 = \int_0^t \sigma\, dB_s + \alpha t + \int_0^t CX_s^0\, ds.$$

Again for any $t \geq 0, x \in \mathbb{R}^d$ we have (see Example 2.14.2)

$$e^{tC}x = x + \int_0^t Ce^{sC}x\, ds.$$

Hence on $\Omega \setminus \mathcal{N}$ for all $t \geq 0, x \in \mathbb{R}^d$ we have

$$X_t^0 + e^{tC}x = x + \int_0^t \sigma\, dB_s + \alpha t + \int_0^t C(X_s^0 + e^{sC}x)\, ds \tag{4.8}$$

so that the sum $\{X_t^0 + e^{tC}x\}$ solves equation (4.4). □

In Definition 4.2.1, if the function f is in a product form , then a similar characterization can be obtained without additional smoothness assumptions on b, f.

Proposition 4.2.5. *Let σ, b be Lipschitz continuous functions.*

(i) Suppose the general solution of the diffusion (4.4) depends deterministically on the initial condition, where the function f has the decomposition $f(t,x) = g(t)h(x)$ with $g \in C^1([0,\infty),\mathbb{R}), h \in C(\mathbb{R}^d,\mathbb{R}^d)$. Then $f(t,x) = \tilde{g}(t)x$ for some $\tilde{g} \in \mathscr{D}$ where

$$\mathscr{D} := \{g \in C^1([0,\infty),\mathbb{R}) : g(0) = 1\}.$$

(ii) The solution to (4.4) is linear in the initial condition in the following sense

$$X_t = g(t)X_0 + X_t^0;\ t \geq 0 \tag{4.9}$$

for some $g \in \mathscr{D}$ if and only if σ is a constant $d \times d$ matrix and b is of the form $b(x) = \alpha + \beta x$ where $\alpha \in \mathbb{R}^d, \beta \in \mathbb{R}$. In this case, the solution has the form

$$X_t = \begin{cases} e^{\beta t}X_0 + \sigma \int_0^t e^{\beta(t-s)}\, dB_s + \frac{e^{\beta t}-1}{\beta}\alpha, \text{ if } \beta \neq 0 \\ X_0 + t\alpha + \sigma B_t, \text{ if } \beta = 0 \end{cases}.$$

and $g(t) = e^{\beta t}$, $t \geq 0$.

Proof. Since a.s. $X_0^x = f(0,x) + X_0^0$, we have $x = g(0)h(x), \forall x \in \mathbb{R}^d$. So $g(0) \neq 0$. Without loss of generality, we may assume $g(0) = 1$. Then $h(x) = x, f(t,x) = g(t)x$. This proves part (i).
If (4.9) holds for some $g \in \mathscr{D}$, then as in Theorem 4.2.4, we can show σ is a constant $d \times d$ matrix. Using equation (4.6b), we have for all $x \in \mathbb{R}^d$, a.s. $t \geq 0$

$$\int_0^t \Big[b(X_s^x) - b(X_s^0) - g'(s)x\Big]\, ds = 0. \tag{4.10}$$

For fixed x and a.s. ω the map $t \mapsto b(X_t^x) - b(X_t^0) - g'(t)x$ is continuous and hence from (4.10) we have, for all $x \in \mathbb{R}^d$

$$\text{a.s.} \quad b(X_t^x) - b(X_t^0) - g'(t)x = 0, \, \forall t \geq 0. \tag{4.11}$$

Putting $t = 0$ we have for all $x \in \mathbb{R}^d$, $b(x) = b(0) + g'(0)x$. Now for all $x \in \mathbb{R}^d$, a.s. for $t \geq 0$,

$$\begin{aligned} &b(X_t^x) - b(X_t^0) - g'(t)x = b(X_t^0 + g(t)x) - b(X_t^0) - g'(t)x \\ &\quad = \{b(0) + g'(0)X_t^0 + g'(0)g(t)x\} - \{b(0) + g'(0)X_t^0\} - g'(t)x = \{g'(0)g(t) - g'(t)\}x \end{aligned}$$

Then using (4.11), we have

$$g'(0)g(t) = g'(t);\, t \geq 0; \quad g(0) = 1.$$

Solution to the previous differential equation is given by $g(t) = e^{g'(0)t}, t \geq 0$. Then $b(x) = b(0) + g'(0)x$ and is determined by the values $b(0), g'(0)$.
The converse part can be verified through direct computation. □

Remark 4.2.6. If σ is a $d \times d$ real matrix and b is of the form $b(x) = \alpha + Cx, x \in \mathbb{R}^d$, then the flow generated by equation (4.4) is Gaussian. Consequently, Theorem 4.2.4 and Proposition 4.2.5 can be considered as characterization results on Gaussian flows in the class of flows that arise as the strong solutions of an Itô stochastic differential equation with smooth or Lipschitz coefficients and driven by a Brownian motion $\{B_t\}$.

In dimension $d = 1$, for convex functions we can apply the following generalization of Itô formula.

Theorem 4.2.7 ([93, Chapter VI, (1.1) Theorem])**.** *If $\{X_t\}$ is a continuous real valued semimartingale and $f : \mathbb{R} \to \mathbb{R}$ is a convex function, then there exists a continuous increasing process $\{A_t^f\}$ such that a.s. $t \geq 0$*

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s)\, dX_s + \frac{1}{2} A_t^f$$

where f'_- is the left-hand derivative of f.

Using the previous theorem, we get the following version of Theorem 4.2.4.

Proposition 4.2.8. *Let σ, b be Lipschitz continuous functions on $\mathbb{R}$. Suppose the following happen:*

(i) there exists an $x \in \mathbb{R}$ such that $\sigma(x)$ is not zero,
(ii) b is continuously differentiable and is a finite linear combination of convex functions,
(iii) for every fixed $x \in \mathbb{R}^d$, the map $t \mapsto \frac{\partial f}{\partial t}(t,x)$ is of bounded variation.

Then the general solution of the diffusion (4.4) *depends deterministically on the initial condition through* (4.5) *if and only if σ is a non-zero constant function and b is of the form $b(x) = \alpha + Cx$ and $f(t,x) = e^{tC}x$ where $\alpha, C \in \mathbb{R}$.*

Proof. The proof remain the same as in Theorem 4.2.4, except the following minor change in the proof of necessity.

First observe that if $h : \mathbb{R} \to \mathbb{R}$ is convex, then so is βh for any scalar $\beta > 0$. Again the sum of two convex functions is convex. If $b = \sum_{i=1}^{k} \beta_i h_i$ for scalars $\beta_i \in \mathbb{R}$ and convex functions h_i, then without loss of generality we may assume $|\beta_i| = 1$, i.e. b will be a difference of convex functions $b = \bar{h}_1 - \bar{h}_2$.

Note that $b'(\cdot) = b'_-(\cdot) = (\bar{h}_1)'_-(\cdot) - (\bar{h}_2)'_-(\cdot)$. Now use Theorem 4.2.7 instead of the Itô formula for C^2 functions for the computations involving $b(X_t^x)$ in the necessity part of Theorem 4.2.4. □

Remark 4.2.9. (i) One may formulate and prove similar results for the following type of condition

$$X_t^{s,x}(\omega) = f(t,s,x) + X_t^{s,0}(\omega),\ t \geq s; X_s^{s,x} = x$$

for $s \geq 0, x \in \mathbb{R}^d$.

(ii) If equation (4.5) is satisfied then we have $f(t,x) = \mathbb{E}[X_t^x - X_t^0]$. As such the conditions on f (in Theorem 4.2.4, Proposition 4.2.5) can be stated in terms of the means $\mathbb{E}X_t^x, x \in \mathbb{R}^d$.

(iii) Diffusions satisfying (4.5) also satisfy the following condition: for any $x, y \in \mathbb{R}^d$, a.s. $t \geq 0$

$$X_t^x - X_t^y = f(t,x) - f(t,y).$$

In certain situations such differences were shown to be diffusions (see [116, Proposition 2.2]).

(iv) Semimartingales with independent increments have been considered in [55, Chapter II]. In particular, it was shown that any rcll process with independent increments must be a sum of a semimartingale with independent increments and a deterministic part ([55, Chapter II, 5.1 Theorem]). This is similar to (4.5), but we are interested in the dependence of a possible deterministic part of the flows (generated by stochastic differential equations) on the initial condition.

In the next proposition, we present an example where Definition 4.2.1 appears. Given a random field $(X_t^x, x \in \mathbb{R}^d, t \geq 0)$ in many situations it is reasonable to assume that the field can be decomposed as $X_t^x = Y_t^x + Z_t$, where $\{Y_t^x\}$ is a 'local' component and $\{Z_t\}$ is a 'global' component. We show that under certain conditions the 'local' component has to be deterministic.

Proposition 4.2.10. *Suppose that* $Z_t = X_t^0$ *and that for all* $x \in \mathbb{R}^d$*, the field* $Y_t^x = X_t^x - X_t^0$ *is independent of* Z*. In addition assume that* $\{X_t^x\}$ *solves*

$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt, t \geq 0;\ X_0 = x$$

and the sigma-fields generated by the processes $\{X_t^0\}$ *and* $\{B_t\}$ *are the same. Then* $\{Y_t^x\}$ *is deterministic.*

Proof. Under our hypothesis, $\{Y_t^x\}$ is both adapted to the said sigma-field and is independent of it. Hence $\{Y_t^x\}$ is deterministic. □

Example 4.2.11. We note that not all Gaussian flows are of the form (4.5). Consider the stochastic differential equations in dimension one:

$$dX_t = x\, dB_t + (\alpha - X_t)\, dt;\ X_0 = \frac{x^2}{2},$$

where α is some fixed real number. The solution is given by

$$X_t^x = e^{-t}\frac{x^2}{2} + x \int_0^t e^{-(t-s)}\, dB_s - \alpha(e^{-t} - 1),$$

which is not of the form (4.5), but the flow is Gaussian.

In Proposition 4.2.10, we can allow σ, b to be random, but independent of $\{B_t\}$ and then the conclusion still holds, conditional on the σ-fields of σ, b. We take this to be in a product form in the next theorem.

Theorem 4.2.12. *Let* $(\Omega', \mathcal{F}', P')$ *be a complete probability space and* $(\Omega'', \mathcal{F}'', (\mathcal{F}_t''), P'')$ *a filtered complete probability space satisfying the usual conditions. Define* $\Omega := \Omega' \times \Omega''$*. Consider the filtered probability space* $(\Omega, \mathcal{F}' \otimes \mathcal{F}'', (\mathcal{F}' \otimes \mathcal{F}_t''), P' \times P'')$*. Let* $\{B_t\}$ *be an* $(\mathcal{F}_t'')$ *Brownian motion. Assume that* $\mathcal{F}_t'' = \sigma\{B_s : 0 \leq s \leq t\}$ *and* $\mathcal{F}'' = \sigma\{B_t : t \geq 0\}$*.*

Let $b : \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ *be* $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}' \otimes \mathcal{F}_0''/\mathcal{B}(\mathbb{R}^d)$ *measurable and* $\sigma : \mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times d}$ *be* $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}' \otimes \mathcal{F}_0''/\mathcal{B}(\mathbb{R}^{d\times d})$ *measurable, where* $\mathcal{B}(\mathbb{R}^d)$ *denotes the Borel sigma field on* $\mathbb{R}^d$*. Suppose that a unique strong solution to the following stochastic differential equation*

$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt, t \geq 0;\ X_0 = x$$

exists for each $x \in \mathbb{R}^d$*. Denote the solution by* $\{X_t^x\}$*. Suppose that*

(i) $\sigma\{B_t : t \geq 0\} = \sigma\{X_t^0 : t \geq 0\}$,

(ii) $\{X_t^x - X_t^0 : x \in \mathbb{R}^d, t \geq 0\}$ *and* $\{B_t : t \geq 0\}$ *are independent.*

Then a.s. $\omega'\,(P')$*, a.s.* $\omega''\,(P'')$ *the process* $\{X_t^x - X_t^0\}$ *depends on* ω' *alone.*

Proof. By condition (ii), a.s. $\omega'\,(P')$, $\{B_t : t \geq 0\}$ and $\{X_t^x(\omega', \cdot) - X_t^0(\omega', \cdot) : t \geq 0, x \in \mathbb{R}^d\}$ are independent.
Since $\{X_t^x\}$ is the strong solution of a stochastic differential equation, there exists a P'-null set $\mathcal{N}' \subset \Omega'$ such that for every $\omega \in \Omega' \setminus \mathcal{N}'$, a.s. $\omega''\,(P'')$,

$$X_t^x(\omega', \omega'') = x + \left(\int_0^t \sigma(X_s)\, dB_s\right)(\omega', \omega'') + \int_0^t b(X_s(\omega', \omega''), \omega', \omega'')\, ds,\ t \geq 0.$$

Hence a.s. $\omega'\,(P')$, the random variables $X_t^x(\omega', \cdot), t \geq 0, x \in \mathbb{R}^d$ are measurable with respect to $\sigma\{B_t : t \geq 0\}$ and by condition (i), so are $X_t^x(\omega', \cdot) - X_t^0(\omega', \cdot), t \geq 0, x \in \mathbb{R}^d$. Hence a.s. $\omega'\,(P')$, $X_t^x(\omega', \omega'') - X_t^0(\omega', \omega''), t \geq 0, x \in \mathbb{R}^d$ is deterministic in ω'', i.e. the random variables depend on ω' alone. □

4.3 A probabilistic representation of the solutions of the Forward equations

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered complete probability space satisfying the usual conditions and let $\{B_t\}$ denote a standard $(\mathcal{F}_t)$ r-dimensional Brownian motion. We obtain the existence and uniqueness of solutions of equations (4.2) and (4.3) where

(i) $r = d$ and the coefficients of the stochastic differential equation (4.1) are as follows: σ is a real square matrix of order d and $b(x) := \alpha + Cx, \forall x \in \mathbb{R}^d$ where $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{R}^d$ and $C = (c_{ij})$ is a real square matrix of order d.

(ii) ψ is a tempered distribution on $\mathbb{R}^d$ given by an integrable function (see Example 2.11.19). Recall that $\mathcal{L}^1(\mathbb{R}^d)$ denotes the space of all such distributions and $\mathcal{L}^1(\mathbb{R}^d) \subset \mathcal{S}_{-p}(\mathbb{R}^d)$ for any $p > \frac{d}{4}$ (see Lemma 2.11.20).

Since σ, b are C^∞ functions with bounded derivatives, a diffeomorphic modification of the solution of (4.1) exists (see [70], [92, Theorem 2.1]). We first observe that such a modification can be written in an explicit form. The proof actually follows from the converse part of Theorem 4.2.4. We observe that the proof there actually produces a null set independent of the initial conditions $x \in \mathbb{R}^d$, even though as per Definition 4.2.1 the null set may well vary over x.

Lemma 4.3.1. *Let σ, b be as above. Let $\{X(t,0)\}$ denote the solution of*

$$dX_t = \sigma(X_t).dB_t + b(X_t)dt; \quad X_0 = 0.$$

Then a.s. for all $t \geq 0, x \in \mathbb{R}^d$

$$X(t,0) + e^{tC}x = x + \int_0^t \sigma\, dB_s + \alpha t + \int_0^t C(X(s,0) + e^{sC}x)\, ds \tag{4.12}$$

so that the sum $\{X(t,0) + e^{tC}x\}$ solves the stochastic differential equation

$$dX_t = \sigma(X_t).dB_t + b(X_t)dt; \quad X_0 = x.$$

Example 4.3.2. For the case $\sigma = Id$ (Id denotes the $d \times d$ identity matrix), $b(x) = -x$, we get the well-known Ornstein-Uhlenbeck diffusion, whose solution is given by

$$X(t,x) = e^{-t}x + \int_0^t e^{-(t-s)}\, dB_s,\ 0 \leq t < \infty. \tag{4.13}$$

In what follows, $\{X(t,x)\}$ and $\mathcal{N}$ will denote the solution and the null set mentioned in Lemma 4.3.1 respectively.

As in [92, equation (3.3)]), for any $\psi \in \mathcal{L}^1(\mathbb{R}^d)$ we define

$$Y_t(\omega)(\psi) := \int_{\mathbb{R}^d} \psi(x)\delta_{X(t,x,\omega)}\, dx,\ \omega \in \Omega \setminus \mathcal{N} \tag{4.14}$$

and set $Y_t(\omega)(\psi) := 0$, if $\omega \in \mathcal{N}$.

Proposition 4.3.3. *Let $\psi, \{X_t\}, \{Y_t(\psi\})$ be as above. Let $p > \frac{d}{4}$. Then $\{Y_t(\psi)\}$ is an $(\mathcal{F}_t)$ adapted $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued continuous process. Furthermore, $\{Y_t(\psi)\}$ is norm-bounded.*

Proof. By Proposition 2.11.14(ii), for any $p > \frac{d}{4}$ there exists a positive constant $\gamma = \gamma(p)$ such that $\|\delta_x\|_{-p} \leq \gamma$, $\forall x \in \mathbb{R}^d$. Then

$$\int_{\mathbb{R}^d} |\psi(x)|.\|\delta_{X(t,x,\omega)}\|_{-p}\, dx \leq \gamma \int_{\mathbb{R}^d} |\psi(x)|\, dx < \infty.$$

Therefore the right hand side of equation (4.14) is Bochner integrable for any $\omega \in \Omega \setminus \mathcal{N}$ and $Y_t(\psi)$ is a well-defined element of $\mathcal{S}_{-p}(\mathbb{R}^d)$ for any $p > \frac{d}{4}$. Similar arguments were used to show $\mathcal{L}^1(\mathbb{R}^d) \subset \mathcal{S}_{-p}(\mathbb{R}^d)$ (see Lemma 2.11.20). The following equation gives an upper bound of the norm of $\{Y_t(\psi)\}$, viz.

$$\left\| \int_{\mathbb{R}^d} \psi(x) \delta_{X(t,x,\omega)} \, dx \right\|_{-p} \leq \gamma \int_{\mathbb{R}^d} |\psi(x)| \, dx. \tag{4.15}$$

Since $\{X(t,x)\}$ is $(\mathcal{F}_t)$ adapted for each x, so is $\{Y_t(\psi)\}$. We now prove $\{Y_t(\psi)\}$ has continuous paths.
Note that $\tau_z, z \in \mathbb{R}^d$ denote the translation operators on $\mathcal{S}_{-p}(\mathbb{R}^d)$ (see Example 2.11.6) and $\delta_0 \in \mathcal{S}_{-p}$ (Proposition 2.11.14(i)) and $\tau_z \delta_0 = \delta_z$. Since $\{X(t,x)\}$ has continuous paths for each $x \in \mathbb{R}^d$, by Lemma 2.11.7(ii), $\{\delta_{X(t,x)}\}$ also has continuous paths in $\mathcal{S}_{-p}(\mathbb{R}^d)$ for each $x \in \mathbb{R}^d$. The upper bound in (4.15) allows us to apply the Dominated Convergence theorem and continuity of $\{Y_t(\psi)\}$ follows. □

Since $\{Y_t(\psi)\}$ is norm-bounded (equation (4.15)), it is also square integrable, i.e.

$$\mathbb{E} \, \|Y_t(\psi)\|_{-p}^2 \leq \gamma^2 \left(\int_{\mathbb{R}^d} |\psi(x)| \, dx \right)^2 < \infty. \tag{4.16}$$

For any $t \geq 0, \omega \in \Omega \setminus \mathcal{N}$, the map $x \mapsto X(t,x,\omega)$ is an affine map and hence it is a C^∞ map with bounded derivatives. Then the map $x \mapsto \phi(X(t,x,\omega))$ is in $\mathcal{S}(\mathbb{R}^d)$ whenever $\phi \in \mathcal{S}(\mathbb{R}^d)$. This allows us to define a linear map, denoted by $X_t(\omega) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ and given by $(X_t(\omega)\phi)(x) := \phi(X(t,x,\omega))$, $x \in \mathbb{R}^d$.

Lemma 4.3.4. *Fix any $t \geq 0, \omega \in \Omega \setminus \mathcal{N}$. The linear map $X_t(\omega) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is continuous.*

Proof. Let $\phi \in \mathcal{S}(\mathbb{R}^d)$. To simplify the notations, we write $\beta = X(t,0,\omega)$ and $\Gamma = (\gamma_{ij})$ for the matrix e^{tC}. Then Γ is invertible and

$$(X_t(\omega)\phi)(x) = \phi(\beta + \Gamma x), x \in \mathbb{R}^d.$$

Let N be a non-negative integer. For any $d \times d$ matrix D, $\|D\|$ and $|D|$ will denote the operator norm and Euclidean norm respectively. Then

$$\begin{aligned}
&\sup_{x \in \mathbb{R}^d} (1 + |x|^2)^N |(X_t(\omega)\phi)(x)| \\
&= \sup_{x \in \mathbb{R}^d} (1 + |x|^2)^N |\phi(\beta + \Gamma x)| \\
&= \sup_{y \in \mathbb{R}^d} \left(1 + |\Gamma^{-1}(y - \beta)|^2\right)^N |\phi(y)|, \text{ (putting } y = \beta + \Gamma x) \\
&\leq \sup_{y \in \mathbb{R}^d} \left(1 + \|\Gamma^{-1}\|^2 |(y - \beta)|^2\right)^N |\phi(y)|
\end{aligned}$$

$$\leq \sup_{y\in\mathbb{R}^d} \left(1 + \|\Gamma^{-1}\|^2.2(|y|^2 + |\beta|^2)\right)^N |\phi(y)|$$
$$\leq M^N \sup_{y\in\mathbb{R}^d} \left(1 + |y|^2\right)^N |\phi(y)|$$

where $M = \max\{2\|\Gamma^{-1}\|^2, (1 + 2|\beta|^2\|\Gamma^{-1}\|^2)\}$. Now let $1 \leq i \leq d$. Then $\partial_i(X_t(\omega)\phi)(x) = \sum_{k=1}^d \gamma_{ki}\, \partial_k\phi(\beta + \Gamma x)$ and hence

$$|\partial_i(X_t(\omega)\phi)(x)| \leq \sum_{k=1}^d |\gamma_{ki}|\, |\partial_k\phi(\beta + \Gamma x)| \leq |\Gamma| \sum_{k=1}^d |\partial_k\phi(\beta + \Gamma x)|.$$

We now combine the two estimates above to obtain

$$\sup_{x\in\mathbb{R}^d} (1 + |x|^2)^N |\partial_i(X_t(\omega)\phi)(x)| \leq |\Gamma| M^N \sum_{k=1}^d \sup_{y\in\mathbb{R}^d} \left(1 + |y|^2\right)^N |\partial_k\phi(y)|.$$

Hence

$$\max_{1\leq i\leq d} \sup_{x\in\mathbb{R}^d} (1 + |x|^2)^N |\partial_i(X_t(\omega)\phi)(x)| \leq \alpha \max_{1\leq k\leq d} \sup_{y\in\mathbb{R}^d} \left(1 + |y|^2\right)^N |\partial_k\phi(y)|,$$

for some constant $\alpha > 0$. Similar estimates can be obtained for higher derivatives of $X_t(\omega)\phi$. Since the seminorms in equation (2.10) determine the topology on $\mathcal{S}(\mathbb{R}^d)$, the above estimate proves the continuity of the linear map $X_t(\omega)$. □

Remark 4.3.5. We point out an observation regarding the constants obtained in the previous proof. For fixed $\omega \in \Omega \setminus \mathcal{N}$, the map $s \mapsto X(s, 0, \omega)$ is continuous. So is the map $s \mapsto e^{sC}$. This implies that the terms $|e^{sC}|$ and $\max\{2\|e^{-sC}\|^2, (1+2|X(s, 0, \omega)|^2\|e^{-sC}\|^2)\}$ can be dominated uniformly in s when $s \in [0, t]$, for any fixed $t > 0$. This fact will be used in the proof of Theorem 4.3.8.

Let $X_t^*(\omega) : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ denote the transpose of the map $X_t(\omega)$. Then for any $\theta \in \mathcal{S}'(\mathbb{R}^d)$,

$$\langle X_t^*(\theta)\, ,\, \phi\rangle = \langle\theta\, ,\, X_t(\phi)\rangle\, ,\, \forall\phi \in \mathcal{S}(\mathbb{R}^d).$$

Using (4.14), for any $\phi \in \mathcal{S}(\mathbb{R}^d), \psi \in \mathcal{L}^1(\mathbb{R}^d)$ we have

$$\langle Y_t(\psi)\, ,\, \phi\rangle = \int_{\mathbb{R}^d} \psi(x)\, \phi(X(t, x))\, dx = \int_{\mathbb{R}^d} \psi(x)\, (X_t(\phi))(x)\, dx = \langle\psi\, ,\, X_t(\phi)\rangle\, .$$

This implies

$$Y_t(\psi) = X_t^*(\psi),\, \forall\psi \in \mathcal{L}^1(\mathbb{R}^d). \tag{4.17}$$

The operators A, L are given as follows: for $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$\begin{cases} A\phi := (A_1\phi, \cdots, A_d\phi), \\ A_i\phi(x) := \sum_{k=1}^d \sigma_{ki}(x)\partial_k\phi(x), \\ L\phi(x) := \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^t)_{ij}(x)\partial_{ij}^2\phi(x) + \sum_{i=1}^d b_i(x)\partial_i\phi(x), \end{cases} \tag{4.18}$$

where σ^t denotes the transpose of σ. For $\psi \in \mathcal{S}'(\mathbb{R}^d)$ consider the adjoint operators A^*, L^* as follows.

$$\begin{cases} A^*\psi := (A_1^*\psi, \cdots, A_d^*\psi), \\ A_i^*\psi := -\sum_{k=1}^d \partial_k\left(\sigma_{ki}\psi\right), \\ L^*\psi := \frac{1}{2}\sum_{i,j=1}^d \partial_{ij}^2\left((\sigma\sigma^t)_{ij}\psi\right) - \sum_{i=1}^d \partial_i\left(b_i\psi\right). \end{cases} \tag{4.19}$$

We now look at A^*, L^* as operators on $\mathcal{S}_p(\mathbb{R}^d)$.

Proposition 4.3.6. *Fix $p \in \mathbb{R}$. There exist constants $C_1 = C_1(p), C_2 = C_2(p) > 0$ such that*

$$\|A_i^*\theta\|_{p-\frac{1}{2}} \leq C_1\|\theta\|_p,\ \|L^*\theta\|_{p-1} \leq C_2\|\theta\|_p,\ \forall \theta \in \mathcal{S}_p.$$

Furthermore, we have the Monotonicity inequality for (A^*, L^*), i.e. there exist a constant $C_p > 0$ such that

$$2\left\langle \theta\,,\, L^*\theta\right\rangle_p + \|A^*\theta\|_{HS(p)}^2 \leq C_p\|\theta\|_p^2,\ \forall \theta \in \mathcal{S}_{p+1},$$

where $\|A^*\theta\|_{HS(p)}^2 := \sum_{i=1}^d \|A_i^*\theta\|_p^2$.

Proof. For any $q \in \mathbb{R}$, $\partial_i, \mathscr{M}_i : \mathcal{S}_q(\mathbb{R}^d) \to \mathcal{S}_{q-\frac{1}{2}}(\mathbb{R}^d)$ are bounded linear operators (see Example 2.11.3 and Example 2.11.9). Using the definitions of A^*, L^* estimates on the norms follows.
Proof of the Monotonicity inequality for (A^*, L^*) follows from Theorem 3.4.2. □

Proposition 4.3.7. *Let $p > \frac{d}{4}$. Then $\{\int_0^t A^*(Y_s(\psi))\,.dB_s\}$ is an $(\mathcal{F}_t)$ adapted $\mathcal{S}_{-p-\frac{1}{2}}(\mathbb{R}^d)$ valued continuous martingale.*

Proof. Since $\{Y_t(\psi)\}$ is an $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued continuous adapted process, to complete the proof it is enough to show that

$$\mathbb{E}\int_0^t \|A_i^*(Y_s(\psi))\|_{-p-\frac{1}{2}}^2 ds < \infty,\ \forall i = 1, \cdots, d, t > 0.$$

But A_i^* is a bounded linear operator from $\mathcal{S}_{-p}(\mathbb{R}^d)$ to $\mathcal{S}_{-p-\frac{1}{2}}(\mathbb{R}^d)$ for each $i = 1, \cdots, d$ and the process $\{Y_t(\psi)\}$ is norm-bounded (see (4.15)). Hence the required estimate follows. □

Theorem 4.3.8. *Let $p > \frac{d}{4}$ and $\psi \in \mathcal{L}^1(\mathbb{R}^d)$. Then the $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued continuous adapted process $\{Y_t(\psi)\}$ satisfies the following equation in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$, a.s.*

$$Y_t(\psi) = \psi + \int_0^t A^*(Y_s(\psi))\,.dB_s + \int_0^t L^*(Y_s(\psi))\,ds,\ \forall t \geq 0. \tag{4.20}$$

This solution is also unique.

Proof. By Itô's formula for any $\phi \in \mathcal{S}(\mathbb{R}^d)$, and any $x \in \mathbb{R}^d$

$$\begin{aligned}(X_t(\phi))(x) &= \phi(X(t,x)) \\ &= \phi(x) + \int_0^t A\phi(X(s,x)).\,dB_s + \int_0^t L\phi(X(s,x))\,ds \\ &= \phi(x) + \int_0^t (X_s(A\phi))(x).\,dB_s + \int_0^t (X_s(L\phi))(x)\,ds\end{aligned}$$

Note that $L\phi \in \mathcal{S}(\mathbb{R}^d)$ since $\phi \in \mathcal{S}(\mathbb{R}^d)$ and hence $\{x \mapsto (X_t(L\phi))(x)\}$ is an $\mathcal{S}(\mathbb{R}^d)$ valued process. Using differentiation under the sign of integration we can establish the existence of all derivatives of $x \mapsto \int_0^t (X_s(L\phi))(x)\,ds$ for any $t \geq 0$. Given non-negative integers $N, \alpha_1, \cdots, \alpha_d$, the terms

$$\sup_{x\in\mathbb{R}^d} (1+|x|^2)^N |\partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d}(X_s(L\phi))(x)|, s \in [0,t]$$

can be dominated uniformly in s (see Remark 4.3.5, the upper bound may depend on ω) and hence

$$\begin{aligned}&\sup_{x\in\mathbb{R}^d} (1+|x|^2)^N \left|\int_0^t \partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d}(X_s(L\phi))(x)\,ds\right| \\ &\leq \int_0^t \left|\sup_{x\in\mathbb{R}^d} (1+|x|^2)^N \partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d}(X_s(L\phi))(x)\right|\,ds \\ &< \infty.\end{aligned}$$

Hence $\{x \mapsto \int_0^t L\phi(X(s,x))\,ds\}$ is an $\mathcal{S}(\mathbb{R}^d)$ valued process. So are $\{x \mapsto \phi(X(t,x))\}, \{x \mapsto \phi(x)\}$. Hence from the equality obtained via the Itô formula, we conclude that the process $\{x \mapsto \int_0^t (X_s(A\phi))(x).\,dB_s\}$ is also an $\mathcal{S}(\mathbb{R}^d)$ valued process.
Then for $\phi \in \mathcal{S}(\mathbb{R}^d)$, by (4.17), a.s. $t \geq 0$

$$\begin{aligned}\langle Y_t(\psi)\,,\,\phi\rangle &= \langle \psi\,,\,X_t(\phi)\rangle \\ &= \left\langle \psi\,,\,\phi + \int_0^t X_s(A\phi).\,dB_s + \int_0^t X_s(L\phi)\,ds\right\rangle\end{aligned}$$

using Proposition 2.7.18,

$$= \langle \psi\,,\,\phi\rangle + \int_0^t \langle \psi\,,\,X_s(A\phi)\rangle\,.dB_s + \int_0^t \langle \psi\,,\,X_s(L\phi)\rangle\,ds$$

$$= \langle \psi \, , \, \phi \rangle + \int_0^t \langle A^* Y_s(\psi) \, , \, \phi \rangle \, . \, dB_s + \int_0^t \langle L^* Y_s(\psi) \, , \, \phi \rangle \; ds$$

again using Proposition 2.7.18,

$$= \left\langle \psi + \int_0^t A^* Y_s(\psi). \, dB_s + \int_0^t L^* Y_s(\psi) \, ds \, , \, \phi \right\rangle$$

Since $\{h_n : n \in \mathbb{Z}_+^d\}$ is countable, a common P-null set can be obtained outside with the previous relation holds for all $\phi \in \{h_n : n \in \mathbb{Z}_+^d\}$. But this set is total in $\mathcal{S}_{p+1}(\mathbb{R}^d)$ and $A_i^* : \mathcal{S}_{-p}(\mathbb{R}^d) \to \mathcal{S}_{-p-\frac{1}{2}}(\mathbb{R}^d), L^* : \mathcal{S}_{-p}(\mathbb{R}^d) \to \mathcal{S}_{-p-1}(\mathbb{R}^d)$ are bounded linear operators (by Proposition 4.3.6). This proves $Y_t(\psi)$ solves (4.20) in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$.
Now we are going to show that the solution of the equation

$$dY_t = A^* Y_t \, dB_t + L^* Y_t \, dt; \quad Y_0 = \psi$$

with $\psi \in \mathcal{L}^1(\mathbb{R}^d)$ must be unique.
Let $\{Y_t^1\}, \{Y_t^2\}$ be two continuous solutions of the previous equation. Define $Z_t := Y_t^1 - Y_t^2, t \geq 0$. Then in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$ a.s.

$$Z_t = \int_0^t A^* Z_s \, dB_s + \int_0^t L^* Z_s \, ds, \, \forall t \geq 0.$$

Note that $\{Z_t\}$ is $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued and we want the uniqueness in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$. Using Itô formula for $\|\cdot\|_{-p-1}^2$, (see Proposition 2.7.20, also see Section 2.12 Item (vi)) we obtain a.s. $t \geq 0$

$$\|Z_t\|_{-p-1}^2 = \int_0^t \left[2 \, \langle Z_s \, , \, L^* Z_s \rangle_{-p-1} + \sum_{i=1}^d \|A_i^* Z_s\|_{-p-1}^2 \right] ds + M_t,$$

where $\{M_t\}$ is some continuous local martingale with $M_0 = 0$. Let $\{\eta_n\}$ be a localizing sequence such that for each n, $\{M_t^{\eta_n}\}$ is a continuous martingale and $\|Z_t^{\eta_n}\|_{-p-1}$ is bounded. Then

$$\begin{aligned}
\|Z_t^{\eta_n}\|_{-p-1}^2 &= \int_0^{t \wedge \eta_n} \left[2 \, \langle Z_s \, , \, L^* Z_s \rangle_{-p-1} + \sum_{i=1}^d \|A_i^* Z_s\|_{-p-1}^2 \right] ds + M_t^{\eta_n} \\
&\leq R_{-p-1} \int_0^{t \wedge \eta_n} \|Z_s\|_{-p-1}^2 \, ds + M_t^{\eta_n} \\
&\leq R_{-p-1} \int_0^t \|Z_s^{\eta_n}\|_{-p-1}^2 \, ds + M_t^{\tau_n}
\end{aligned}$$

where $R_{-p-1} > 0$ is a constant obtained from the Monotonicity inequality (Proposition 4.3.6). Taking expectation in the above inequality, we have

$$\mathbb{E} \|Z_t^{\eta_n}\|_{-p-1}^2 \leq R_{-p-1} \int_0^t \mathbb{E} \|Z_s^{\eta_n}\|_{-p-1}^2 \, ds.$$

Then using Gronwall's inequality (Lemma 2.13.1), we have $\mathbb{E}\|Z_t^{\eta_n}\|_{-p-1}^2 = 0$, which shows a.s. $Y_t^1 = Y_t^2, t \leq \eta_n$. Since $\eta_n \uparrow \infty$, we have a.s. $Y_t^1 = Y_t^2, t \geq 0$. This completes the proof of uniqueness. □

The next result is about the existence and uniqueness of solution to equation (4.3) with initial condition $\psi \in \mathcal{L}^1(\mathbb{R}^d)$. By a solution to equation (4.3) we mean an $\mathcal{S}_p(\mathbb{R}^d)$ valued continuous function $\psi(\cdot) : [0, \infty) \to \mathcal{S}_p(\mathbb{R}^d)$ for some p such that

$$\psi(t) = \psi(0) + \int_0^t L^* \psi(s)\, ds$$

holds in $\mathcal{S}_{p-1}(\mathbb{R}^d)$.

Theorem 4.3.9. *Let $p > \frac{d}{4}$ and $\psi \in \mathcal{L}^1(\mathbb{R}^d)$. Then $\overline{\psi}(t) := \mathbb{E}\, Y_t(\psi)$ solves the initial value problem* (4.3)*, i.e.*

$$\mathbb{E}\, Y_t(\psi) = \psi + \int_0^t L^*\, \mathbb{E}\, Y_s(\psi)\, ds$$

holds in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$. Furthermore this is the unique solution.

Proof. We first observe some properties of the terms appearing on both sides of the stochastic partial differential equation (4.20).

(i) Since the random variables $Y_t(\psi)$ are bounded in the norm $\|\cdot\|_{-p}$, independent of $t \geq 0$ (equation (4.15)), $\overline{\psi}(t) := \mathbb{E}\, Y_t(\psi)$ are well-defined elements of $\mathcal{S}_{-p}(\mathbb{R}^d)$. Furthermore, the continuity of $t \mapsto \overline{\psi}(t)$ follows from the Dominated Convergence Theorem, using the continuity of the process $\{Y_t(\psi)\}$.

(ii) $\{\int_0^t A^* Y_s(\psi).\, dB_s\}$ is a continuous martingale and in particular, $\mathbb{E}\int_0^t A^* Y_s(\psi).\, dB_s = 0$ (see Proposition 4.3.7).

(iii) Another consequence of the existence of a bound of $\|Y_t(\psi)\|_{-p}, t \geq 0$ independent of t (equation (4.15)) is that the random variables $\int_0^t L^*\, Y_s(\psi)\, ds$ are bounded in $\|\cdot\|_{-p-1}$ for each t. Here we have used the fact that $L^* : \mathcal{S}_{-p}(\mathbb{R}^d) \to \mathcal{S}_{-p-1}(\mathbb{R}^d)$ is a bounded linear operator (Proposition 4.3.6). The same boundedness and linearity of L^* also imply $L^*\mathbb{E}\, Y_s(\psi) = \mathbb{E}\, L^*\, Y_s(\psi)$ and hence for each $t \geq 0$,

$$\mathbb{E}\int_0^t L^*\, Y_s(\psi)\, ds = \int_0^t \mathbb{E} L^*\, Y_s(\psi)\, ds = \int_0^t L^*\mathbb{E}\, Y_s(\psi)\, ds.$$

In view of the above observations, taking term by term expectation on both sides of (4.20) we obtain

$$\mathbb{E}\, Y_t(\psi) = \psi + \int_0^t L^*\mathbb{E}\, Y_s(\psi)\, ds$$

in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$.

The proof of uniqueness of the solution is same as in [92, Theorem 4.4]. We use the

Monotonicity inequality in Proposition 4.3.6 and the Gronwall's inequality (Lemma 2.13.1). Let $\widetilde{\psi}(t)$ be another $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued continuous solution. Define $\phi(t) := \overline{\psi}(t) - \widetilde{\psi}(t), t \geq 0$. Then $\phi(t)$ is continuous in t and it satisfies

$$\phi(t) = \int_0^t L^* \phi(s)\, ds,\; t \geq 0$$

in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$ and

$$\begin{aligned}
\|\phi(t)\|_{-p-1}^2 &= 2 \int_0^t \langle \phi(s)\,,\, L^*\phi(s)\rangle_{-p-1}\, ds \\
&\leq \int_0^t \left[2\langle \phi(s)\,,\, L^*\phi(s)\rangle_{-p-1} + \sum_{i=1}^d \|A_i^* \phi(s)\|_{-p-1}^2 \right] ds \\
&\leq R_{-p-1} \int_0^t \|\phi(s)\|_{-p-1}^2\, ds,
\end{aligned}$$

where $R_{-p-1} > 0$ is a constant obtained in the Monotonicity inequality. Then the Gronwall's inequality imply $\phi(t) \equiv 0, t \geq 0$, which proves the required uniqueness. □

The process $\{Y_t(\psi)\}$ can also be described in terms of $\{X(t,0)\}$ without using the integral representation in (4.14). We show that the tempered distribution $Y_t(\psi)(\omega)$ is given by an integrable function. This representation of $Y_t(\psi)$ is similar to the representation obtained in [90, Lemma 3.6], where the author looked at the solution of stochastic partial differential equations governed by certain non-linear operators. Given a $d \times d$ matrix D, $det(D), tr(D)$ will denote the determinant and trace of the matrix respectively.

Proposition 4.3.10. *Let $\psi \in \mathcal{L}^1(\mathbb{R}^d)$ and $\omega \in \mathcal{N}$. Then*

$$Y_t(\psi) = e^{-t\,tr(C)}\, \tau_{Z_t} \psi_t(\cdot),$$

where $Z_t := X(t,0), \psi_t(x) := \psi(e^{-tC}x)$ for $t \geq 0, x \in \mathbb{R}^d$.

Proof. For $\phi \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\begin{aligned}
\langle Y_t(\psi)\,,\, \phi\rangle &= \int_{\mathbb{R}^d} \psi(x)\phi(X(t,x))\, dx = \int_{\mathbb{R}^d} \psi(x)\phi(e^{tC}x + Z_t)\, dx \\
&= |det(e^{-tC})| \int_{\mathbb{R}^d} \psi(e^{-tC}(z - Z_t))\phi(z)\, dz, \text{ (putting } z = e^{tC}x + Z_t) \\
&= e^{-t\,tr(C)} \int_{\mathbb{R}^d} \psi_t(z - Z_t)\phi(z)\, dz, \\
&= e^{-t\,tr(C)} \int_{\mathbb{R}^d} (\tau_{Z_t}\psi_t)(z)\phi(z)\, dz \\
&= \left\langle e^{-t\,tr(C)}(\tau_{Z_t}\psi_t)\,,\, \phi \right\rangle.
\end{aligned}$$

Here we have used the equality $det(e^{-tC}) = e^{-t\,tr(C)}$ ([48, Problem 5.6.P43]). Using Proposition 2.10.2 we conclude $Y_t(\psi) = e^{-t\,tr(C)}\, \tau_{Z_t}\psi_t(\cdot)$. □

Remark 4.3.11. In this section, we obtained the probabilistic representation of solutions of (4.3), when the initial condition $\psi \in \mathcal{L}^1(\mathbb{R}^d)$ and the coefficients σ, b of the stochastic differential equation (4.1) are in a specific form. Possible extensions of these results to the case - when the initial condition ψ is an $\mathcal{L}^q(\mathbb{R}^d)$ function for some $q > 1$ or more generally a finite linear combination of the distributional derivatives of $\mathcal{L}^q(\mathbb{R}^d)$ functions ($q \geq 1$) - will be taken up in future.

Remark 4.3.12. It may be possible to obtain more examples of coefficients σ, b by relaxing the conditions of Theorem 4.2.4. These coefficients may be the 'right' candidates for which we can define the necessary compositions and continuous linear maps as in Lemma 4.3.4, leading to existence of solutions of the stochastic partial differential equation (4.20).

CHAPTER 5

Stationary solutions of stochastic partial differential equations in $\mathcal{S}'$

5.1 Introduction

In [90], a correspondence was shown between finite dimensional stochastic differential equations and stochastic partial differential equations in $\mathcal{S}'(\mathbb{R}^d)$ via an Itô formula. The results there involves deterministic initial conditions in some Hermite Sobolev space $\mathcal{S}_p(\mathbb{R}^d)$. In this chapter we extend this correspondence to random initial conditions. Assuming the existence of stationary solutions of finite dimensional stochastic differential equations, we then show the existence of stationary solutions of infinite dimensional stochastic partial differential equations, via an Itô formula which is used in proving the said correspondence.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered complete probability space satisfying the usual conditions. In Section 2, we consider the problem of existence and uniqueness of solutions of the stochastic partial differential equation

$$dY_t = A(Y_t).\, dB_t + L(Y_t)\, dt; \quad Y_0 = \xi, \tag{5.1}$$

where

(i) $\{B_t\}$ is a d dimensional standard $(\mathcal{F}_t)$ Brownian motion.

(ii) ξ is an $\mathcal{S}_p(\mathbb{R}^d)$ valued $\mathcal{F}_0$ measurable random variable, independent of $\{B_t\}$.

(iii) the operators $A := (A_1, \cdots, A_d), L$ on $\mathcal{S}_p(\mathbb{R}^d)$ are given as follows: for $\phi \in \mathcal{S}_p(\mathbb{R}^d)$

$$A_i\phi := -\sum_{j=1}^{d} \langle \sigma\,,\, \phi\rangle_{ji}\, \partial_j\phi,\ i = 1, \cdots, d \tag{5.2}$$

and

$$L\phi := \frac{1}{2}\sum_{i,j=1}^{d} (\langle \sigma\,,\, \phi\rangle\, \langle \sigma\,,\, \phi\rangle^t)_{ij}\, \partial^2_{ij}\phi - \sum_{i=1}^{d} \langle b\,,\, \phi\rangle_i\, \partial_i\phi, \tag{5.3}$$

where $\langle \sigma\,,\,\phi\rangle^t$ stands for the transpose of the matrix $\langle \sigma\,,\,\phi\rangle$,

(iv) $\sigma = (\sigma_{ij})_{d\times d}, b = (b_1, b_2, \cdots, b_d)$ with $\sigma_{ij}, b_i \in \mathcal{S}_{-p}(\mathbb{R}^d)$ for $i, j = 1, 2, \cdots, d$. For any $\phi \in \mathcal{S}_p(\mathbb{R}^d)$, by $\langle \sigma\,,\,\phi\rangle$ we denote the $d \times d$ matrix with entries $\langle \sigma\,,\,\phi\rangle_{ij} := \langle \sigma_{ij}\,,\,\phi\rangle$. Similarly $\langle b\,,\,\phi\rangle$ is a vector in $\mathbb{R}^d$ with $\langle b\,,\,\phi\rangle_i := \langle b_i\,,\,\phi\rangle$.

We show that the above problem is related to the problem of existence and uniqueness of solutions of the finite dimensional stochastic differential equation:

$$dZ_t = \bar{\sigma}(Z_t; \xi).\, dB_t + \bar{b}(Z_t; \xi)\, dt; \quad Z_0 = 0, \tag{5.4}$$

where the functions $\bar{\sigma}(\cdot\,;\psi) : \mathbb{R}^d \to \mathbb{R}^{d^2}$ and $\bar{b}(\cdot\,;\psi) : \mathbb{R}^d \to \mathbb{R}^d$ are given by $\bar{\sigma}(x;\psi) := (\langle \sigma_{ij}\,,\,\tau_x\psi\rangle)$ and $\bar{b}(x;\psi) := (\langle b_i\,,\,\tau_x\psi\rangle)$, with the parameter $\psi \in \mathcal{S}_p(\mathbb{R}^d)$ and $\tau_x, x \in \mathbb{R}^d$ denoting the translation operators (see Example 2.11.6). In particular if $\{Z_t\}$ solves equation (5.4), then $Y_t = \tau_{Z_t}(\xi)$ solves equation (5.1), .

We first prove an Itô formula (Theorem 5.2.2) which is an extension of Proposition 5.2.1 (an implication of [89, Theorem 2.3]). Next we prove Theorem 5.2.4 which gives an existence and uniqueness of the solutions of the finite dimensional stochastic differential equation

$$dZ_t = \bar{\sigma}(Z_t; \xi).\, dB_t + \bar{b}(Z_t; \xi)\, dt; \quad Z_0 = \zeta \tag{5.5}$$

where ξ is square integrable and $\zeta = c \in \mathbb{R}^d$. Note that the hypothesis requires a certain 'globally Lipschitz' nature of the coefficients, which depends on ξ - the initial condition for Y. We need control on the norm of ξ to make the usual proof via Picard iteration work. We also note that the same proof works if the random variable ζ is square integrable.

The 'globally Lipschitz' condition can be further relaxed to 'locally Lipschitz' conditions. We prove this in Theorem 5.2.9 and show that the solution involves a possible explosion. We also provide a criterion on ξ, which imply the 'locally Lipschitz' condition (Proposition 5.2.11). Using this result, we prove Theorem 5.2.12, which is a version of Theorem 5.2.9.

We continue with 'globally Lipschitz' coefficients and obtain a characterization result of solutions of equation (5.1) viz. Lemma 5.2.16 (an extension of [90, Lemma 3.6]) which allows us to prove the pathwise uniqueness of the solutions of equation (5.1) in Theorem 5.2.15. The existence of solutions of equation (5.1) follows from the Itô formula (Theorem 5.2.2) and the existence of solutions of the finite dimensional stochastic differential equation (5.4). A version of this result for the 'locally Lipschitz' coefficients is proved in Theorem 5.2.20. Note that the equation (5.4) for Z involves the initial condition for Y i.e. ξ, but with $Z_0 = 0$. These results extend results in [90, Section 3], where ξ was taken to be deterministic.

For ‘globally Lipschitz’ coefficients we prove $\mathcal{L}^2$ estimates on the supremum of the norms of the solutions of equation (5.1), in terms of the initial condition (see Proposition 5.2.17, Proposition 5.2.18).

A motivation for studying the existence and uniqueness problem for the stochastic partial differential equation (5.1) is to study stationary solutions of these equations. In Section 3, we construct stationary solutions of equation (5.1) by a ‘lifting’ of stationary solutions of the finite dimensional stochastic differential equation (5.5). We define a subset $\mathcal{C}$ of the Hermite Sobolev space with the following property: if the initial random variable ξ is deterministic and takes values in the set $\mathcal{C}$, then the associated finite dimensional stochastic differential equations (5.4) remain the same, i.e. the coefficients $\bar{\sigma}, \bar{b}$ are ‘constant’ on $\mathcal{C}$. This property is observed in Lemma 5.3.3 and using which we show the existence of stationary solutions of stochastic partial differential equations in our class (Theorem 5.3.4). To guarantee non-explosion for finite dimensional stochastic differential equations with locally Lipschitz coefficients, we use a ‘Liapunov’ type criteria ([105, 7.3.14 Corollary]). Two examples of stationary solutions are given in Example 5.3.5 and Example 5.3.8. In Proposition 5.3.9, we obtain $\mathcal{L}^1$ estimates on the supremum of the norms of the stationary solutions of equation (5.1), in terms of the initial condition.

Most of the results in this chapter are from [9].

5.2 Stochastic partial differential equations involving random initial conditions

An outline of the approach taken in this section was set out in Section 1. Let ξ be an $\mathcal{S}_p(\mathbb{R}^d)$ valued $\mathcal{F}_0$ measurable random variable. We need an Itô formula (Theorem 5.2.2), a ‘deterministic’ version (Proposition 5.2.1) of which follows from [89, Theorem 2.3].

Proposition 5.2.1. *Let $p \in \mathbb{R}$ and $\phi \in \mathcal{S}_{-p}(\mathbb{R}^d)$. Let $X = (X^1, \cdots, X^d)$ be an $\mathbb{R}^d$ valued continuous $(\mathcal{F}_t)$ adapted semimartingale. Then we have the following equality in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$, a.s.*

$$\tau_{X_t}\phi = \tau_{X_0}\phi - \sum_{i=1}^{d}\int_0^t \partial_i\tau_{X_s}\phi\, dX_s^i + \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \partial_{ij}^2\tau_{X_s}\phi\, d[X^i, X^j]_s,\ \forall t \geq 0. \tag{5.6}$$

We need to extend above result to allow random ϕ.

Theorem 5.2.2. *Let $p \in \mathbb{R}$. Let ξ be an $\mathcal{S}_p(\mathbb{R}^d)$ valued $\mathcal{F}_0$ measurable random variable with $\mathbb{E}\|\xi\|_p^2 < \infty$. Let $X = (X^1, \cdots, X^d)$ be an $\mathbb{R}^d$ valued continuous semimartingale.*

Then we have the following equality in $\mathcal{S}_{p-1}(\mathbb{R}^d)$, *a.s.*

$$\tau_{X_t}\xi = \tau_{X_0}\xi - \sum_{i=1}^{d}\int_0^t \partial_i\tau_{X_s}\xi\, dX_s^i + \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \partial_{ij}^2\tau_{X_s}\xi\, d[X^i,X^j]_s,\ \forall t\geq 0. \tag{5.7}$$

Proof. First we show the existence of $\sum_{i=1}^d \int_0^t \partial_i\tau_{X_s}\xi\, dX_s^i$ as an $\mathcal{S}_{p-\frac{1}{2}}(\mathbb{R}^d)$ valued integral. Let $X_t^i = X_0^i + M_t^i + V_t^i$ be the decomposition of X^i, where M^i, V^i are the continuous local martingale part and the continuous finite variation part of X^i. We use localization under stopping times and hence without loss of generality, assume that $M^i, i = 1,\cdots,d$ are continuous martingales. Let $Var_{[0,t]}(V_\cdot^i)$ denote the total variation process of V^i.
For $i,j = 1,\cdots,d$ define $\bar{\eta}_k^{i,j} := \inf\{t\geq 0 : |[M^i,M^j]_t| \geq k\}$ and $\eta_k' := \inf\{t\geq 0 : |X_t|\geq k\}$ and $\tilde{\eta}_k^i := \inf\{t\geq 0 : Var_{[0,t]}(V_\cdot^i)\geq k\}$. Set $\eta_k = (\bigwedge_{i,j}\bar{\eta}_k^{i,j})\wedge\eta_k'\wedge(\bigwedge_i\tilde{\eta}_k^i)$. Note that $\eta_k\uparrow\infty$. Consider the following two cases:

(i) If $|X_0(\omega)| > k$ for some w, then $\eta_k(\omega) = 0$. Such ω does not contribute to the integral $\sum_{i=1}^d\int_0^{t\wedge\eta_k}\|\partial_i\tau_{X_s}\xi\|_{p-\frac{1}{2}}^2\, d[M^i]_s$, where $[M^i] := [M^i,M^i]$.
(ii) If $|X_0(\omega)|\leq k$ for some w, then $|X_{t\wedge\eta_k(\omega)}(\omega)|\leq k$.

In view of these observations, to establish the existence of $\sum_{i=1}^d\int_0^t \partial_i\tau_{X_s}\xi\, dM_s^i$ we assume $\{X_t^{\eta_k}\}$ is bounded. Since $\partial_i : \mathcal{S}_p(\mathbb{R}^d)\to\mathcal{S}_{p-\frac{1}{2}}(\mathbb{R}^d)$ is a bounded linear operator, there exist constants C, C' such that

$$\|\partial_i\tau_{X_s^{\eta_k}}\xi\|_{p-\frac{1}{2}} \leq C.\|\tau_{X_s^{\eta_k}}\xi\|_p \leq C'\|\xi\|_p,\ (\text{using Lemma 2.11.7}(i)).$$

Since $\mathbb{E}\|\xi\|_p^2 < \infty$, we have the required integrability condition for the existence of the $\mathcal{S}_{p-\frac{1}{2}}(\mathbb{R}^d)$ valued integral $\sum_{i=1}^d\int_0^{t\wedge\eta_k}\partial_i\tau_{X_s}\xi\, dM_s^i$ and hence $\sum_{i=1}^d\int_0^t\partial_i\tau_{X_s}\xi\, dM_s^i$ also exists. Similarly, we can show the existence of $\sum_{i=1}^d\int_0^t\partial_i\tau_{X_s}\xi\, dV_s^i$ as an $\mathcal{S}_{p-\frac{1}{2}}(\mathbb{R}^d)$ valued integral and that of $\sum_{i,j=1}^d\int_0^t\partial_{ij}^2\tau_{X_s}\xi\, d[X^i,X^j]_s$ as an $\mathcal{S}_{p-1}(\mathbb{R}^d)$ valued integral.
Fix $\phi\in\mathcal{S}(\mathbb{R}^d)$. Then $\phi\in\mathcal{S}_{-p+1}(\mathbb{R}^d)$ and by Proposition 5.2.1, we have in $\mathcal{S}_{-p}(\mathbb{R}^d)$ a.s. for all $t\geq 0$

$$\tau_{-X_t}\phi = \tau_{-X_0}\phi + \sum_{i=1}^d\int_0^t\partial_i\tau_{-X_s}\phi\, dX_s^i + \frac{1}{2}\sum_{i,j=1}^d\int_0^t\partial_{ij}^2\tau_{-X_s}\phi\, d[X^i,X^j]_s.$$

Then a.s.

$$\begin{aligned}\langle\xi\,,\,\tau_{-X_t}\phi\rangle =& \langle\xi\,,\,\tau_{-X_0}\phi\rangle + \left\langle\xi\,,\,\sum_{i=1}^d\int_0^t\partial_i\tau_{-X_s}\phi\, dX_s^i\right\rangle\\ &+\left\langle\xi\,,\,\frac{1}{2}\sum_{i,j=1}^d\int_0^t\partial_{ij}^2\tau_{-X_s}\phi\, d[X^i,X^j]_s\right\rangle,\ \forall t\geq 0.\end{aligned}\tag{5.8}$$

Now using Proposition 2.7.8, Proposition 2.7.18 and Lemma 2.11.7(iii), we have

$$\begin{aligned}\left\langle \xi\,,\, \sum_{i=1}^{d}\int_0^t \partial_i \tau_{-X_s}\phi\, dX_s^i \right\rangle &= \sum_{i=1}^{d}\left\langle \xi\,,\, \int_0^t \partial_i \tau_{-X_s}\phi\, dX_s^i \right\rangle \\ &= \sum_{i=1}^{d}\int_0^t \langle \xi\,,\, \partial_i \tau_{-X_s}\phi\rangle\, dX_s^i \\ &= -\sum_{i=1}^{d}\int_0^t \langle \partial_i \tau_{X_s}\xi\,,\, \phi\rangle\ dX_s^i \\ &= \left\langle -\sum_{i=1}^{d}\int_0^t \partial_i \tau_{X_s}\xi\, dX_s^i\,,\, \phi \right\rangle\end{aligned}$$

Similarly,

$$\left\langle \xi\,,\, \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \partial_{ij}^2 \tau_{-X_s}\phi\, d[X^i,X^j]_s \right\rangle = \left\langle \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \partial_{ij}^2 \tau_{X_s}\xi\, d[X^i,X^j]_s\,,\, \phi \right\rangle.$$

For each $\phi \in \mathcal{S}(\mathbb{R}^d)$, using (5.8) we have a.s.

$$\begin{aligned}\langle \tau_{X_t}\xi\,,\, \phi\rangle = \langle \tau_{X_0}\xi\,,\, \phi\rangle &- \left\langle \sum_{i=1}^{d}\int_0^t \partial_i \tau_{X_s}\xi\, dX_s^i\,,\, \phi \right\rangle \\ &+ \left\langle \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \partial_{ij}^2 \tau_{X_s}\xi\, d[X^i,X^j]_s\,,\, \phi \right\rangle,\ \forall t \geq 0.\end{aligned}$$

In particular we get a P-null set $\mathcal{N}$ such that for $\omega \in \Omega \setminus \mathcal{N}$ and for any multi-index $n = (n_1, \cdots, n_d)$ we have

$$\begin{aligned}\langle \tau_{X_t}\xi\,,\, h_n\rangle = \langle \tau_{X_0}\xi\,,\, h_n\rangle &- \left\langle \sum_{i=1}^{d}\int_0^t \partial_i \tau_{X_s}\xi\, dX_s^i\,,\, h_n \right\rangle \\ &+ \left\langle \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \partial_{ij}^2 \tau_{X_s}\xi\, d[X^i,X^j]_s\,,\, h_n \right\rangle,\ \forall t \geq 0\end{aligned}$$

where h_n are the Hermite functions. Since the process $\{\tau_{X_t}\xi - \tau_{X_0}\xi + \sum_{i=1}^{d}\int_0^t \partial_i \tau_{X_s}\xi\, dX_s^i - \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \partial_{ij}^2 \tau_{X_s}\xi\, d[X^i,X^j]_s\}$ is $\mathcal{S}_{p-1}(\mathbb{R}^d)$ valued and $\{h_n : n \in \mathbb{Z}_+^d\}$ is a total set in $\mathcal{S}_{1-p}(\mathbb{R}^d)$, we have the equality in $\mathcal{S}_{p-1}(\mathbb{R}^d)$ a.s. (see Proposition 2.10.2)

$$\tau_{X_t}\xi - \tau_{X_0}\xi + \sum_{i=1}^{d}\int_0^t \partial_i \tau_{X_s}\xi\, dX_s^i - \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \partial_{ij}^2 \tau_{X_s}\xi\, d[X^i,X^j]_s = 0,\ t \geq 0.$$

This completes the proof. $\square$

Alternative proof of Theorem 5.2.2. In the previous proof we have shown the existence of the integrals $\sum_{i=1}^d \int_0^t \partial_i \tau_{X_s}\xi \, dX_s^i$ and $\sum_{i,j=1}^d \int_0^t \partial_{ij}^2 \tau_{X_s}\xi \, d[X^i, X^j]_s$. In this argument we make use of a property of stochastic integrals, viz. (5.9).

Let $\{\xi^{(n)}\}$ be a sequence of $\mathcal{S}_p(\mathbb{R}^d)$ valued simple $\mathcal{F}_0$ measurable functions such that $\xi^{(n)} \xrightarrow[\mathcal{L}^2]{n\to\infty} \xi$. Observe that

(a) Given any $\mathcal{F}_0$ measurable set F, an $\mathcal{S}_p(\mathbb{R}^d)$ valued predictable step process $\{G_t\}$ and a continuous $\mathbb{R}^d$ valued semimartingale $\{X_t\}$, we have a.s.

$$\mathbb{1}_F \int_0^t G_s \, dX_s = \int_0^t \mathbb{1}_F G_s \, dX_s, \; t \geq 0. \tag{5.9}$$

Above equality can be extended to the case involving norm-bounded $\mathcal{S}_p(\mathbb{R}^d)$ valued predictable process $\{G_t\}$.

(b) Given any $\mathcal{F}_0$ measurable set F, $\phi \in \mathcal{S}_p(\mathbb{R}^d)$, $\psi \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ we have

$$\begin{aligned} \langle \mathbb{1}_F \tau_x \phi \, , \, \psi \rangle &= \mathbb{1}_F \, \langle \tau_x \phi \, , \, \psi \rangle = \mathbb{1}_F \, \langle \phi \, , \, \tau_{-x}\psi \rangle \\ &= \langle \mathbb{1}_F \phi \, , \, \tau_{-x}\psi \rangle = \langle \tau_x(\mathbb{1}_F \phi) \, , \, \psi \rangle \end{aligned} \tag{5.10}$$

and hence $\mathbb{1}_F \tau_x \phi = \tau_x(\mathbb{1}_F \phi)$. Similarly $\mathbb{1}_F \tau_x \phi = \tau_{\mathbb{1}_F x}(\mathbb{1}_F \phi)$.

Using Proposition 5.2.1 and equations (5.9), (5.10) we can establish the required result when X is bounded and ξ is an $\mathcal{S}_p(\mathbb{R}^d)$ valued simple $\mathcal{F}_0$ measurable random variable. In particular, the following equality holds in $\mathcal{S}_{p-1}(\mathbb{R}^d)$ a.s. for all $t \geq 0$

$$\begin{aligned} \tau_{X_{t\wedge\eta_k}}\xi^{(n)} = \tau_{X_0}\xi^{(n)} &- \sum_{i=1}^d \int_0^{t\wedge\eta_k} \partial_i \tau_{X_s}\xi^{(n)} dX_s^i \\ &+ \frac{1}{2}\sum_{i,j=1}^d \int_0^{t\wedge\eta_k} \partial_{ij}^2 \tau_{X_s}\xi^{(n)} d[X^i, X^j]_s, \end{aligned}$$

where the localizing sequence $\{\eta_k\}$ is as in the previous proof. Now letting n go to infinity we get the equality in $\mathcal{S}_{p-1}(\mathbb{R}^d)$ a.s. for all $t \geq 0$

$$\tau_{X_{t\wedge\eta_k}}\xi = \tau_{X_0}\xi - \sum_{i=1}^d \int_0^{t\wedge\eta_k} \partial_i \tau_{X_s}\xi \, dX_s^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^{t\wedge\eta_k} \partial_{ij}^2 \tau_{X_s}\xi \, d[X^i, X^j]_s.$$

Letting k go to infinity, we get the result. □

We need an existence and uniqueness of solution to the following equation:

$$dZ_t = \bar{\sigma}(Z_t; \xi). \, dB_t + \bar{b}(Z_t; \xi) \, dt; \quad Z_0 = \zeta, \tag{5.11}$$

where ξ is an $\mathcal{S}_p(\mathbb{R}^d)$ valued $\mathcal{F}_0$ measurable random variable and ζ is an $\mathbb{R}^d$ valued $\mathcal{F}_0$ measurable random variable. Unless stated otherwise, we assume that both ξ, ζ are independent of the Brownian motion $\{B_t\}$.

We now introduce some notations and terminology. Let $(\mathcal{G}_t)$ denote the filtration generated by ξ, ζ and $\{B_t\}$. Let $\mathcal{G}_\infty$ denote the smallest sub σ-field of $\mathcal{F}$ containing $\mathcal{G}_t$ for all $t \geq 0$. Let $\mathcal{G}_\infty^P$ be the P-completion of $\mathcal{G}_\infty$ and let $\mathcal{N}^P$ be the collection of all P-null sets in $\mathcal{G}_\infty^P$. Define

$$\mathcal{F}_t^{\xi,\zeta} := \bigcap_{s>t} \sigma(\mathcal{G}_s \cup \mathcal{N}^P),\, t \geq 0$$

where $\sigma(\mathcal{G}_s \cup \mathcal{N}^P)$ denotes the smallest σ-field generated by the collection $\mathcal{G}_s \cup \mathcal{N}^P$. This filtration satisfies the usual conditions. $\mathcal{F}_\infty^{\xi,\zeta}$ will denote the σ field generated by the collection $\bigcup_{t\geq 0} \mathcal{F}_t^{\xi,\zeta}$. If ζ is a constant, then we write $(\mathcal{F}_t^{\xi})$ instead of $(\mathcal{F}_t^{\xi,\zeta})$.

Proposition 5.2.3. *Suppose the following conditions are satisfied.*

(i) *ξ is norm-bounded in $\mathcal{S}_p(\mathbb{R}^d)$, i.e. there exists a constant $K > 0$ such that $\|\xi\|_p \leq K$.*
(ii) $\mathbb{E}|\zeta|^2 < \infty$.
(iii) *(Globally Lipschitz in x, locally in y) For any fixed $y \in \mathcal{S}_p(\mathbb{R}^d)$, the functions $x \mapsto \bar{\sigma}(x;y)$ and $x \mapsto \bar{b}(x;y)$ are globally Lipschitz functions in x and the Lipschitz coefficient is independent of y when y varies over any bounded set G in $\mathcal{S}_p(\mathbb{R}^d)$; i.e. for any bounded set G in $\mathcal{S}_p(\mathbb{R}^d)$ there exists a constant $C(G) > 0$ such that for all $x_1, x_2 \in \mathbb{R}^d, y \in G$*

$$|\bar{\sigma}(x_1;y) - \bar{\sigma}(x_2;y)| + |\bar{b}(x_1;y) - \bar{b}(x_2;y)| \leq C(G)|x_1 - x_2|.$$

Then (5.11) *has a continuous $(\mathcal{F}_t^{\xi,\zeta})$ adapted strong solution $\{X_t\}$ with the property that $\mathbb{E}\int_0^T |X_t|^2\, dt < \infty$ for any $T > 0$. Pathwise uniqueness of solutions also holds, i.e. if $\{X_t^1\}$ is another solution, then $P(X_t = X_t^1,\, t \geq 0) = 1$.*

Proof. We follow the proof in [82, Theorem 5.2.1] with appropriate modifications. First we show the uniqueness of the strong solution.
Let $\{Z_t^1\}$ and $\{Z_t^2\}$ be two strong, continuous solutions of (5.11). Define $a(t,\omega) = \bar{\sigma}(Z_t^1(\omega);\xi(\omega)) - \bar{\sigma}(Z_t^2(\omega);\xi(\omega))$ and $\gamma(t,\omega) = \bar{b}(Z_t^1(\omega);\xi(\omega)) - \bar{b}(Z_t^2(\omega);\xi(\omega))$. Since ξ is norm-bounded, then by our hypothesis

$$|a(t,\omega)|^2 \leq C^2 \left|Z_t^1(\omega) - Z_t^2(\omega)\right|^2,\, |\gamma(t,\omega)|^2 \leq C^2 \left|Z_t^1(\omega) - Z_t^2(\omega)\right|^2$$

where $C = C(Range(\xi))$. Then

$$\left|Z_t^1 - Z_t^2\right|^2 = \left|\int_0^t a(s).\, dB_s + \int_0^t \gamma(s)\, ds\right|^2$$

$$\leq 2\left|\int_0^t a(s).\,dB_s\right|^2 + 2\left|\int_0^t \gamma(s)\,ds\right|^2$$

Consider the localizing sequence $\{\eta_k\}$ defined by $\eta_k = \inf\{t \geq 0 : |Z_t^1 - Z_t^2| \geq k\}$. Then using Itô isometry and Cauchy-Schwarz Inequality,

$$\begin{aligned}\mathbb{E}\left|Z^1_{t\wedge\eta_k} - Z^2_{t\wedge\eta_k}\right|^2 &\leq 2\,\mathbb{E}\int_0^{t\wedge\eta_k} |a(s)|^2\,ds + 2t\,\mathbb{E}\int_0^{t\wedge\eta_k} |\gamma(s)|^2\,ds \\ &\leq 2C^2(1+t)\int_0^{t\wedge\eta_k} \mathbb{E}\left|Z_s^1 - Z_s^2\right|^2\,ds \\ &\leq 2C^2(1+t)\int_0^t \mathbb{E}\left|Z^1_{s\wedge\eta_k} - Z^2_{s\wedge\eta_k}\right|^2\,ds\end{aligned} \tag{5.12}$$

For any positive integer k, consider the function $v(t) = \mathbb{E}\left|Z^1_{t\wedge\eta_k} - Z^2_{t\wedge\eta_k}\right|^2$ on any compact time interval $[0,T]$. Then using Gronwall's inequality (see Lemma 2.13.1) and the fact that $t \to v(t)$ is continuous, we get $v \equiv 0$. Now using Fatou's Lemma,

$$\mathbb{E}\left|Z_t^1 - Z_t^2\right|^2 \leq \liminf_{k\to\infty} \mathbb{E}\left|Z^1_{t\wedge\eta_k} - Z^2_{t\wedge\eta_k}\right|^2 = 0, \forall t \in [0,T].$$

This proves the uniqueness.
To show the existence of a strong solution, we use a Picard type iteration. Set $Z_t^{(0)} = \zeta$ and then successively define

$$Z_t^{(k+1)} := \zeta + \int_0^t \bar{\sigma}(Z_s^{(k)};\xi)\,dB_s + \int_0^t \bar{b}(Z_s^{(k)};\xi)\,ds,\ \forall k \geq 0. \tag{5.13}$$

Fix any compact time interval $[0,N]$. For $k \geq 1, t \in [0,N]$ we have

$$\mathbb{E}|Z_t^{(k+1)} - Z_t^{(k)}|^2 \leq 2C^2(1+N)\int_0^t \mathbb{E}|Z_s^{(k)} - Z_s^{(k-1)}|^2\,ds. \tag{5.14}$$

Proof of the above estimate is similar to (5.12).
Using the Lipschitz continuity for any $x \in \mathbb{R}^d, y \in Range(\xi)$ we have, $|\bar{\sigma}(x;y) - \bar{\sigma}(0;y)| + |\bar{b}(x;y) - \bar{b}(0;y)| \leq C|x|$. But $|\bar{\sigma}(0;y)| = |\langle \sigma\,,\,y\rangle| \leq \|\sigma_{ij}\|_{-p}\|y\|_p$ and $|\bar{b}(0;y)| = |\langle b\,,\,y\rangle| \leq \|b_i\|_{-p}\|y\|_p$. This shows $\bar{\sigma}, \bar{b}$ has linear growth in x, i.e. there exists a constant $D = D(Range(\xi)) > 0$ such that $|\bar{\sigma}(x;y)| \leq D(1+|x|), |\bar{b}(x;y)| \leq D(1+|x|)$ for $x \in \mathbb{R}^d, y \in Range(\xi)$. Since $Z_t^{(0)} = \zeta$ using (5.14) we get

$$\begin{aligned}\mathbb{E}|Z_t^{(1)} - Z_t^{(0)}|^2 &\leq 2\,\mathbb{E}\int_0^t |\bar{\sigma}(\zeta;\xi)|^2\,ds + 2t\,\mathbb{E}\int_0^t |\bar{b}(\zeta;\xi)|^2\,ds \\ &\leq 4D^2(1+N)(1+\mathbb{E}|\zeta|^2)t,\ \forall t \in [0,N].\end{aligned} \tag{5.15}$$

Now we use an induction on k with (5.14) as the recurrence relations and (5.15) as our base step. Then there exists a constant $R > 0$ such that

$$\mathbb{E}|Z_t^{(k+1)} - Z_t^{(k)}|^2 \leq \frac{(Rt)^{k+1}}{(k+1)!},\ \forall k \geq 0, t \in [0,N]. \tag{5.16}$$

Let λ denote the Lebesgue measure on $[0,N]$. We are going to show that the iteration converges in $\mathcal{L}^2(\lambda\times P)$ and the limit satisfy (5.11). For positive integers m,n with $m>n$ we have

$$\begin{aligned}\|Z^{(m)}-Z^{(n)}\|_{\mathcal{L}^2(\lambda\times P)} &= \|\sum_{k=n}^{m-1}\left(Z^{(k+1)}-Z^{(k)}\right)\|_{\mathcal{L}^2(\lambda\times P)}\\ &\le \sum_{k=n}^{m-1}\|Z^{(k+1)}-Z^{(k)}\|_{\mathcal{L}^2(\lambda\times P)}\\ &= \sum_{k=n}^{m-1}\left(\mathbb{E}\int_0^N |Z_t^{(k+1)}-Z_t^{(k)}|^2\,dt\right)^{\frac{1}{2}}\\ &\le \sum_{k=n}^{m-1}\left(\int_0^N \frac{(Rt)^{k+1}}{(k+1)!}\,dt\right)^{\frac{1}{2}} = \sum_{k=n}^{m-1}\left(\frac{(RN)^{k+2}}{(k+2)!}\right)^{\frac{1}{2}}\end{aligned}$$

As $m,n\to\infty$, $\|Z^{(m)}-Z^{(n)}\|_{\mathcal{L}^2(\lambda\times P)}\to 0$. Using completeness of $\mathcal{L}^2(\lambda\times P)$ we have a limit, which we denote by $\{X_t\}_{t\in[0,N]}$. Using (5.16), we also have $\lim_{n\to\infty} Z_t^{(n)} \overset{\mathcal{L}^2(P)}{=} X_t$ for each $t\in[0,N]$.

This $\{X_t\}$ is measurable and $(\mathcal{F}_t^{\xi,\zeta})$ adapted. Now using the linear growth of $x\mapsto\bar{\sigma}(x;y)$ (for every fixed $y\in\mathcal{S}_p(R^d)$) we have

$$\begin{aligned}\mathbb{E}\int_0^N \bar{\sigma}(X_s;\xi)^2\,ds &\le D^2\,\mathbb{E}\int_0^N (1+|X_s|)^2\,ds\\ &\le 2D^2\,\mathbb{E}\int_0^N (1+|X_s|^2)\,ds\\ &= 2D^2N+2D^2\|X\|^2_{\mathcal{L}^2(\lambda\times P)}<\infty.\end{aligned}$$

Hence $\{\int_0^t \bar{\sigma}(X_s;\xi)\,dB_s\}_{t\in[0,N]}$ exists (see Remark 2.6.7). Since $\mathbb{E}\int_0^N |X_s|^2\,ds<\infty$, we have $\int_0^N |X_s|^2\,ds<\infty$ almost surely. Now using the linear growth of $x\mapsto\bar{b}(x;y)$ (for every fixed $y\in\mathcal{S}_p(R^d)$) and Cauchy-Schwarz inequality, we can establish the existence of $\{\int_0^t \bar{b}(X_s;\xi)\,ds\}_{t\in[0,N]}$.

Now using Itô isometry and the Lipschitz property of $\bar{\sigma}$ we get

$$\begin{aligned}\mathbb{E}\left|\int_0^t \bar{\sigma}(Z_s^{(k)};\xi)\,dB_s-\int_0^t \bar{\sigma}(X_s;\xi)\,dB_s\right|^2 &= \mathbb{E}\int_0^t |\bar{\sigma}(Z_s^{(k)};\xi)-\bar{\sigma}(X_s;\xi)|^2 ds\\ &\le C^2\,\mathbb{E}\int_0^t |Z_s^{(k)}-X_s|^2\,ds\\ &\le C^2\,\mathbb{E}\int_0^N |Z_s^{(k)}-X_s|^2\,ds.\end{aligned}$$

Using Jensen's inequality and the Lipschitz property of $\bar{b}$ we get

$$\mathbb{E}\left|\int_0^t \bar{b}(Z_s^{(k)};\xi)\,ds-\int_0^t \bar{b}(X_s;\xi)\,ds\right|^2 \le t\mathbb{E}\int_0^t \left|\bar{b}(Z_s^{(k)};\xi)-\bar{b}(X_s;\xi)\right|^2\,ds$$

$$\leq C^2 t\,\mathbb{E}\int_0^t |Z_s^{(k)} - X_s|^2\,ds$$
$$\leq C^2 N\,\mathbb{E}\int_0^N |Z_s^{(k)} - X_s|^2\,ds$$

Using above estimates, for each $t \in [0, N]$ we have

$$\int_0^t \bar{\sigma}(Z_s^{(k)}; \xi)\,dB_s \xrightarrow[k\to\infty]{\mathcal{L}^2(P)} \int_0^t \bar{\sigma}(X_s; \xi)\,dB_s,$$

and

$$\int_0^t \bar{b}(Z_s^{(k)}; \xi)\,dB_s \xrightarrow[k\to\infty]{\mathcal{L}^2(P)} \int_0^t \bar{b}(X_s; \xi)\,dB_s.$$

From (5.13) we conclude that for each $t \in [0, N]$, a.s.

$$X_t = \zeta + \int_0^t \bar{\sigma}(X_s; \xi)\,dB_s + \int_0^t \bar{b}(X_s; \xi)\,ds.$$

The integral $\int_0^t \bar{\sigma}(X_s; \xi)\,dB_s$ has a continuous version (see Proposition 2.6.8). We denote the continuous version of $\{\zeta + \int_0^t \bar{\sigma}(X_s; \xi)\,dB_s + \int_0^t \bar{b}(X_s; \xi)\,ds\}_{t\in[0,N]}$ by $\{\widetilde{X}_t\}_{t\in[0,N]}$. Then for each $t \in [0, N]$, a.s.

$$\widetilde{X}_t = \zeta + \int_0^t \bar{\sigma}(X_s; \xi)\,dB_s + \int_0^t \bar{b}(X_s; \xi)\,ds = X_t, \text{ a.s.}$$

In particular, for all $t \in [0, N]$ we have $\mathbb{E}|X_t - \widetilde{X}_t|^2 = 0$. Then $\int_0^t \bar{\sigma}(X_s; \xi)\,dB_s = \int_0^t \bar{\sigma}(\widetilde{X}_s; \xi)\,dB_s$ a.s. We can also show $\int_0^t \bar{b}(X_s; \xi)\,dB_s = \int_0^t \bar{b}(\widetilde{X}_s; \xi)\,dB_s$ a.s. for each $t \in [0, N]$. Then for each $t \in [0, N]$, a.s.

$$\widetilde{X}_t = \zeta + \int_0^t \bar{\sigma}(\widetilde{X}_s; \xi)\,dB_s + \int_0^t \bar{b}(\widetilde{X}_s; \xi)\,ds, \text{ a.s.}$$

Since $\{\widetilde{X}_t\}$ is continuous, we have, a.s.

$$\widetilde{X}_t = \zeta + \int_0^t \bar{\sigma}(\widetilde{X}_s; \xi)\,dB_s + \int_0^t \bar{b}(\widetilde{X}_s; \xi)\,ds,\ t \in [0, N].$$

So we have obtained a continuous $(\mathcal{F}_t^{\xi,\zeta})$ adapted solution up to any positive integer N. The uniqueness of this continuous solution follows from the proof of uniqueness given at the beginning of this proof.
Let $\{X_t^{(N)}\}$ and $\{X_t^{(N+1)}\}$ be the solutions up to time N and $N+1$ respectively. Then $\{X_{t\in[0,N]}^{(N+1)}\}$ is also a continuous solution up to time N and hence by the uniqueness, is indistinguishable from $\{X_t^{(N)}\}$ on $[0, N]$. Using this consistency, we can patch up the solutions $\{X_t^{(N)}\}$ to obtain the solution of (5.11) on the time interval $[0, \infty)$. □

We now come to a main result regarding the existence and uniqueness of solutions of (5.11).

Theorem 5.2.4. *Suppose the following are satisfied.*

(i) $\mathbb{E}\|\xi\|_p^2 < \infty$.

(ii) $\zeta = c$, *where* c *is some element in* $\mathbb{R}^d$.

(iii) (Globally Lipschitz in x, locally in y) For any fixed $y \in \mathcal{S}_p(\mathbb{R}^d)$, *the functions* $x \mapsto \bar{\sigma}(x; y)$ *and* $x \mapsto \bar{b}(x; y)$ *are globally Lipschitz functions in* x *and the Lipschitz coefficient is independent of* y *when* y *varies over any bounded set* G *in* $\mathcal{S}_p(\mathbb{R}^d)$*; i.e. for any bounded set* G *in* $\mathcal{S}_p(\mathbb{R}^d)$ *there exists a constant* $C(G) > 0$ *such that for all* $x_1, x_2 \in \mathbb{R}^d, y \in G$

$$|\bar{\sigma}(x_1; y) - \bar{\sigma}(x_2; y)| + |\bar{b}(x_1; y) - \bar{b}(x_2; y)| \leq C(G)|x_1 - x_2|.$$

Then (5.11) *has a continuous* $(\mathcal{F}_t^\xi)$ *adapted strong solution* $\{X_t\}$ *such that there exists a localizing sequence of stopping times* $\{\eta_n\}$ *with* $\mathbb{E}\int_0^{T\wedge\eta_n} |X_t|^2\, dt < \infty$ *for any* $T > 0$. *Pathwise uniqueness of solutions also holds, i.e. if* $\{\tilde{X}_t\}$ *is another solution, then* $P(X_t = \tilde{X}_t,\, t \geq 0) = 1$.

Remark 5.2.5. Theorem 5.2.4 is also true if ζ is an $\mathbb{R}^d$ valued $\mathcal{F}_0$ measurable square integrable random variable, which is also independent of the Brownian motion $\{B_t\}$. However, we only need the version for $\zeta = 0$ (see the proof of Theorem 5.2.15).

Proof of Theorem 5.2.4. For any positive integer k, define $\xi^{(k)} := \xi \mathbb{1}_{(\|\xi\|_p \leq k)}$. Since ξ is $\mathcal{S}_p(\mathbb{R}^d)$ valued, we have $(\|\xi\|_p < \infty) = \Omega$. Since $\mathbb{E}\|\xi\|_p^2 < \infty$, we have $\xi^{(k)} \xrightarrow[\mathcal{L}^2]{k\to\infty} \xi$ and the convergence is also almost sure. Note that $\mathbb{1}_{(\|\xi\|_p \leq k)}\xi^{(k+1)} = \xi^{(k)}$.
By (5.10), we have for any $x \in \mathbb{R}^d, y \in \mathcal{S}_p(\mathbb{R}^d), F \in \mathcal{F}$

$$\begin{aligned} \mathbb{1}_F \bar{\sigma}(x; y) &= \bar{\sigma}(x; \mathbb{1}_F y) = \bar{\sigma}(\mathbb{1}_F x; \mathbb{1}_F y) \\ \mathbb{1}_F \bar{b}(x; y) &= \bar{b}(x; \mathbb{1}_F y) = \bar{b}(\mathbb{1}_F x; \mathbb{1}_F y) \end{aligned} \tag{5.17}$$

By Proposition 5.2.3 we have the $(\mathcal{F}_t^{\xi^{(k)}})$ adapted (and hence $(\mathcal{F}_t^\xi)$ adapted) strong solution denoted by $\{Z_t^{(k)}\}$, satisfying a.s.

$$Z_t^{(k)} = c + \int_0^t \bar{\sigma}(Z_s^{(k)}; \xi^{(k)}).\, dB_s + \int_0^t \bar{b}(Z_s^{(k)}; \xi^{(k)})\, ds,\ t \geq 0.$$

Using (5.9) and (5.17), we have a.s. for all $t \geq 0$

$$\mathbb{1}_{(\|\xi\|_p \leq k)} Z_t^{(k)} = \mathbb{1}_{(\|\xi\|_p \leq k)} c + \int_0^t \bar{\sigma}(\mathbb{1}_{(\|\xi\|_p \leq k)} Z_s^{(k)}; \xi^{(k)}).\, dB_s$$

$$+\int_0^t \bar{b}(\mathbb{1}_{(\|\xi\|_p\leq k)} Z_s^{(k)}; \xi^{(k)})\, ds.$$

and

$$\begin{aligned}\mathbb{1}_{(\|\xi\|_p\leq k)} Z_t^{(k+1)} &= \mathbb{1}_{(\|\xi\|_p\leq k)} c + \int_0^t \bar{\sigma}(\mathbb{1}_{(\|\xi\|_p\leq k)} Z_s^{(k+1)}; \mathbb{1}_{(\|\xi\|_p\leq k)} \xi^{(k+1)}).\, dB_s\\ &+ \int_0^t \bar{b}(\mathbb{1}_{(\|\xi\|_p\leq k)} Z_s^{(k+1)}; \mathbb{1}_{(\|\xi\|_p\leq k)} \xi^{(k+1)})\, ds\\ &= \mathbb{1}_{(\|\xi\|_p\leq k)} c + \int_0^t \bar{\sigma}(\mathbb{1}_{(\|\xi\|_p\leq k)} Z_s^{(k+1)}; \xi^{(k)}).\, dB_s\\ &+ \int_0^t \bar{b}(\mathbb{1}_{(\|\xi\|_p\leq k)} Z_s^{(k+1)}; \xi^{(k)})\, ds\end{aligned}$$

Using the uniqueness obtained in Proposition 5.2.3 (applied to $(\mathcal{F}_t^\xi)$ adapted processes), we have a.s.

$$\mathbb{1}_{(\|\xi\|_p\leq k)} Z_t^{(k+1)} = \mathbb{1}_{(\|\xi\|_p\leq k)} Z_t^{(k)},\ t \geq 0 \tag{5.18}$$

with the null set possibly depending on k. Let $\tilde{\Omega}_k$ be the set of probability 1 where the above relation holds. Then on $\Omega' := \bigcap_{k=1}^\infty \tilde{\Omega}_k$, which is a set of probability 1, (5.18) holds for all k.

Note that $(\|\xi\|_p < \infty) = \Omega$ and hence for any $\omega \in \Omega$, there exists a positive integer k such that $\|\xi(\omega)\|_p \leq k$. Then we can write

$$\Omega' = \bigcup_{k=1}^\infty \left(\Omega' \cap (\|\xi\|_p \leq k)\right).$$

Note that Ω' is an element of $\mathcal{F}$ with probability 1 and hence $(\Omega')^c$ is a null set in $\mathcal{F}$. Since $(\mathcal{F}_t)$ satisfies the usual conditions, we have $(\Omega')^c \in \mathcal{F}_0$ and hence $\Omega' \in \mathcal{F}_0$.

We define a process $\{X_t\}$ as follows: for any $t \geq 0$

$$X_t(\omega) := \begin{cases} Z_t^{(k)}(\omega),\ \text{if}\, \omega \in \Omega' \cap (\|\xi\|_p \leq k),\ k = 1, 2, \cdots \\ 0,\ \text{if}\, \omega \in (\Omega')^c \end{cases}$$

From equation (5.18), $Z_t^{(k+1)} = Z_t^{(k)}$, $\forall t \geq 0$ on $\Omega' \cap (\|\xi\|_p \leq k)$ and hence $\{X_t\}$ is well-defined.

Since each $\{Z_t^{(k)}\}$ is $(\mathcal{F}_t^\xi)$ adapted and $\Omega' \cap (\|\xi\|_p \leq k) \in \mathcal{F}_0$, adaptedness of $\{X_t\}$ follows.

Since each $\{Z_t^{(k)}\}$ has continuous paths and $\Omega' \cap (\|\xi\|_p \leq k) \uparrow \Omega'$, $\{X_t\}$ also has continuous paths on Ω'. On $(\Omega')^c$, $X \equiv 0$ and hence has continuous paths.

We now show that $\{X_t\}$ solves equation (5.11). On Ω' we have

$$\mathbb{1}_{(\|\xi\|_p\leq k)} X_t = \mathbb{1}_{(\|\xi\|_p\leq k)} Z_t^{(k)},\ \forall t \geq 0, k = 1, 2, \cdots \tag{5.19}$$

i.e. above relation holds almost surely.
Then for each $k = 1, 2, \cdots$, a.s. $t \geq 0$

$$\begin{aligned}
\mathbb{1}_{(\|\xi\|_p \leq k)} X_t &= \mathbb{1}_{(\|\xi\|_p \leq k)} Z_t^{(k)} \\
&= \mathbb{1}_{(\|\xi\|_p \leq k)} c + \int_0^t \bar{\sigma}(\mathbb{1}_{(\|\xi\|_p \leq k)} Z_s^{(k)}; \xi^{(k)}).\, dB_s \\
&+ \int_0^t \bar{b}(\mathbb{1}_{(\|\xi\|_p \leq k)} Z_s^{(k)}; \xi^{(k)})\, ds \\
&= \mathbb{1}_{(\|\xi\|_p \leq k)} c + \int_0^t \bar{\sigma}(\mathbb{1}_{(\|\xi\|_p \leq k)} X_s; \xi^{(k)}).\, dB_s \\
&+ \int_0^t \bar{b}(\mathbb{1}_{(\|\xi\|_p \leq k)} X_s; \xi^{(k)})\, ds, \text{ (using (5.19))} \\
&= \mathbb{1}_{(\|\xi\|_p \leq k)} c + \int_0^t \mathbb{1}_{(\|\xi\|_p \leq k)} \bar{\sigma}(X_s; \xi).\, dB_s \\
&+ \int_0^t \mathbb{1}_{(\|\xi\|_p \leq k)} \bar{b}(X_s; \xi)\, ds, \text{ (using (5.17))} \\
&= \mathbb{1}_{(\|\xi\|_p \leq k)} c + \mathbb{1}_{(\|\xi\|_p \leq k)} \int_0^t \bar{\sigma}(X_s; \xi).\, dB_s \\
&+ \mathbb{1}_{(\|\xi\|_p \leq k)} \int_0^t \bar{b}(X_s; \xi)\, ds, \text{ (using (5.9))}
\end{aligned}$$

Let $\bar{\Omega}_k$ denote the set of probability 1 where the above relation holds. Then $\bar{\Omega} := \bigcap_{k=1}^{\infty} \bar{\Omega}_k$ is also a set of probability 1 and on $\bar{\Omega}$, for all $k = 1, 2, \cdots$ and for all $t \geq 0$

$$\mathbb{1}_{(\|\xi\|_p \leq k)} X_t = \mathbb{1}_{(\|\xi\|_p \leq k)} \left(c + \int_0^t \bar{\sigma}(X_s; \xi).\, dB_s + \int_0^t \bar{b}(X_s; \xi)\, ds \right).$$

Then on $\bar{\Omega} \cap (\|\xi\|_p \leq k)$ we have for all $t \geq 0$

$$X_t = c + \int_0^t \bar{\sigma}(X_s; \xi).\, dB_s + \int_0^t \bar{b}(X_s; \xi)\, ds.$$

But $\bar{\Omega} \cap (\|\xi\|_p \leq k) \uparrow \bar{\Omega}$ and hence on $\bar{\Omega}$ above relation holds for all $t \geq 0$. So $\{X_t\}$ is a solution of (5.11).
Taking $\eta_n := \inf\{t \geq 0 : |X_t| \geq n\}$ it follows that $\mathbb{E} \int_0^{t \wedge \eta_n} |X_t|^2\, dt < \infty$ for any $t > 0$.
To prove the uniqueness, let $\{\tilde{X}_t\}$ be a continuous $(\mathcal{F}_t^{\xi})$ adapted strong solution of (5.11). Then a.s. for all $t \geq 0$

$$\begin{aligned}
&\mathbb{1}_{(\|\xi\|_p \leq k)} \tilde{X}_t \\
&= \mathbb{1}_{(\|\xi\|_p \leq k)} \left(c + \int_0^t \bar{\sigma}(\tilde{X}_s; \xi).\, dB_s + \int_0^t \bar{b}(\tilde{X}_s; \xi)\, ds \right) \\
&= \mathbb{1}_{(\|\xi\|_p \leq k)} c + \int_0^t \bar{\sigma}(\mathbb{1}_{(\|\xi\|_p \leq k)} \tilde{X}_s; \xi^{(k)}).\, dB_s + \int_0^t \bar{b}(\mathbb{1}_{(\|\xi\|_p \leq k)} \tilde{X}_s; \xi^{(k)})\, ds
\end{aligned}$$

From the uniqueness obtained in Proposition 5.2.3 and using equation (5.19), we now conclude a.s. for all $t \geq 0$

$$\mathbb{1}_{(\|\xi\|_p \leq k)} \tilde{X}_t = \mathbb{1}_{(\|\xi\|_p \leq k)} Z_t^{(k)} = \mathbb{1}_{(\|\xi\|_p \leq k)} X_t.$$

Since $(\|\xi\|_p \leq k) \uparrow \Omega$, this proves $P(X_t = \tilde{X}_t,\, t \geq 0) = 1$. $\square$

The next result Proposition 5.2.6 is a corollary of Theorem 5.2.4. In Theorem 5.2.4 we used $\xi^{(k)}$ to approximate ξ and then established the existence of the solution. However if we have more regularity of the coefficients $\bar{\sigma}, \bar{b}$ then Proposition 5.2.6 can be established independently by using the Picard iteration as in Proposition 5.2.3. We present the details about this technique.

Proposition 5.2.6. *Suppose the following happens.*

(i) $\mathbb{E}\|\xi\|_p^2 < \infty$.
(ii) $\zeta = c$, *where* c *is some element in* $\mathbb{R}^d$.
(iii) (Globally Lipschitz in x, globally in y) For any fixed $y \in \mathcal{S}_p(\mathbb{R}^d)$, *the functions* $x \mapsto \bar{\sigma}(x; y)$ *and* $x \mapsto \bar{b}(x; y)$ *are globally Lipschitz functions in* x *and the Lipschitz coefficient can be taken to be independent of* $y \in \mathcal{S}_p(\mathbb{R}^d)$*; i.e. there exists a constant* $C > 0$ *such that for all* $x_1, x_2 \in \mathbb{R}^d, y \in \mathcal{S}_p(\mathbb{R}^d)$

$$|\bar{\sigma}(x_1; y) - \bar{\sigma}(x_2; y)| + |\bar{b}(x_1; y) - \bar{b}(x_2; y)| \leq C|x_1 - x_2|.$$

Then (5.11) *has a continuous* $(\mathcal{F}_t^\xi)$ *adapted strong solution* $\{X_t\}$ *with the property that* $\mathbb{E}\int_0^T |X_t|^2\, dt < \infty$ *for any* $T > 0$*. Pathwise uniqueness of solutions also holds, i.e. if* $\{X_t^1\}$ *is another solution, then* $P(X_t = X_t^1,\, t \geq 0) = 1$.

Proof. The proof is similar to that of Proposition 5.2.3. We indicate the necessary changes and use the same notations.
In proving the uniqueness, we have estimates on $|a(t,\omega)|, |\gamma(t,\omega)|$ involving a constant $C > 0$ which is now independent of ξ, i.e.

$$|a(t,\omega)|^2 \leq C^2 \left|Z_t^1(\omega) - Z_t^2(\omega)\right|^2,\ |\gamma(t,\omega)|^2 \leq C^2 \left|Z_t^1(\omega) - Z_t^2(\omega)\right|^2.$$

In Proposition 5.2.3, ξ was norm bounded and the constant C depended on $Range(\xi)$. Now C is independent of ξ because the coefficients $\bar{\sigma}, \bar{b}$ are ‘globally Lipschitz’. The uniqueness follows using Gronwall’s inequality (see Lemma 2.13.1).
For the existence, we again define the iteration (5.13) with $\zeta = c$. Then we get (5.14) with the constant $C > 0$ independent of ξ. In Proposition 5.2.3, we had shown the linear

growth (in x) of $\bar{\sigma}$ and $\bar{b}$. Now we use the following estimate $|\bar{\sigma}(\zeta;\xi)| = |\langle \tau_{-c}\sigma\,,\,\xi\rangle| \leq \|\tau_{-c}\sigma_{ij}\|_{-p}\|\xi\|_p$ and $|\bar{b}(\zeta;\xi)| = |\langle \tau_{-c}b\,,\,\xi\rangle| \leq \|\tau_{-c}b_i\|_{-p}\|\xi\|_p$. Then

$$\begin{aligned}\mathbb{E}|Z_t^{(1)} - Z_t^{(0)}|^2 &\leq 2\,\mathbb{E}\int_0^t |\bar{\sigma}(\zeta;\xi)|^2\,ds + 2t\,\mathbb{E}\int_0^t |\bar{b}(\zeta;\xi)|^2\,ds\\ &\leq 2D(1+N)\mathbb{E}\|\xi\|_p^2\,t,\ \forall t \in [0,N],\end{aligned}$$

where $D > 0$ is some constant depending on $\|\tau_{-c}\sigma_{ij}\|_{-p}$ and $\|\tau_{-c}b_i\|_{-p}$.
Rest of the proof is same as that of Proposition 5.2.3. □

In Theorem 5.2.4 we can assume locally Lipschitz nature of the coefficients $\bar{\sigma}, \bar{b}$ instead of those being globally Lipschitz. This extension from globally Lipschitz to locally Lipschitz is a well-known technique (see [56, Theorem 18.3 and the discussion in page 340 about explosion], [93, Chapter IX, Exercise 2.10], [50, Theorem 2.3 and 3.1]). We denote the one point compactification of $\mathbb{R}^d$ by $\widehat{\mathbb{R}^d} := \mathbb{R}^d \cup \{\infty\}$.
We now recall a result about an extension of a Lipschitz function defined on a subset of a metric space to the whole space. M. D. Kirszbraun proved a version of this result for Euclidean spaces and the result is referred to as Kirszbraun Theorem.

Theorem 5.2.7 ([33, p. 202])**.** *Let (X,d) be a metric space and let U be a subset of X. Suppose $f : U \to \mathbb{R}$ be Lipschitz continuous, i.e.*

$$|f(x_1) - f(x_2)| \leq K\,d(x_1,x_2),\ \forall x_1, x_2 \in U$$

where $K > 0$ is independent of the choice of x_1, x_2. Then there is an extension of f to X, viz $\tilde{f}$ defined by

$$\tilde{f}(x) := \inf_{u\in U}\{f(u) + K\,d(x,u)\},\ x \in X$$

such that $\tilde{f}$ is globally Lipschitz on X with the Lipschitz constant K, i.e.

$$|\tilde{f}(x_1) - \tilde{f}(x_2)| \leq K\,d(x_1,x_2),\ \forall x_1, x_2 \in X.$$

As a consequence of the previous result, given a locally Lipschitz function on $\mathbb{R}^d$, we can define globally Lipschitz functions which agree with the locally Lipschitz functions on certain sets.

Corollary 5.2.8. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz function, i.e.*

$$|f(x_1) - f(x_2)| \leq C_n\,|x_1 - x_2|,\ \forall x_1, x_2 \in B(0,n)$$

where $B(0,n) = \{x \in \mathbb{R}^d : |x| \leq n\}$ (for any positive integer n) and $C_n > 0$ is a constant, depending only on n. Then there exist globally Lipschitz functions $f^{(n)}$ on $\mathbb{R}^d$ such that *(i)* $f(x) = f^{(n)}(x)$ *for all* $x \in B(0, n)$.

(ii) C_n *can be taken as the Lipschitz constant for* $f^{(n)}$, *i.e.*

$$|f^{(n)}(x_1) - f^{(n)}(x_2)| \leq C_n\, |x_1 - x_2|, \,\forall x_1, x_2 \in \mathbb{R}^d.$$

Proof. For any n, the function f is a Lipschitz continuous function on $B(0, n)$ and hence by the previous Theorem has a globally Lipschitz extension $f^{(n)}$. This function satisfies the required properties. □

We use this corollary to extend Theorem 5.2.4 to the case involving locally Lipschitz coefficients. If $x \mapsto \bar{\sigma}(x; y)$ is Lipschitz continuous in x for each fixed $y \in \mathcal{S}_p(\mathbb{R}^d)$, then we can obtain globally Lipschitz functions as given in the previous corollary. If the Lipschitz constant can be chosen independent of y, then the globally Lipschitz extensions will also have the same Lipschitz constant independent of y.
Note that $\widehat{\mathbb{R}^d} := \mathbb{R}^d \cup \{\infty\}$ is the one point compactification of $\mathbb{R}^d$.

Theorem 5.2.9. *Suppose the following are satisfied.*

(i) $\mathbb{E}\|\xi\|_p^2 < \infty$.

(ii) $\zeta = 0$.

(iii) (Locally Lipschitz in x, locally in y) for any fixed $y \in \mathcal{S}_p(\mathbb{R}^d)$ *the functions* $x \mapsto \bar{\sigma}(x; y)$ *and* $x \mapsto \bar{b}(x; y)$ *are locally Lipschitz functions in* x *and the Lipschitz coefficient is independent of* y *when* y *varies over any bounded set* G *in* $\mathcal{S}_p(\mathbb{R}^d)$*; i.e. for any bounded set* G *in* $\mathcal{S}_p(\mathbb{R}^d)$ *and any positive integer* n *there exists a constant* $C(G, n) > 0$ *such that for all* $x_1, x_2 \in B(0, n), y \in G$

$$|\bar{\sigma}(x_1; y) - \bar{\sigma}(x_2; y)| + |\bar{b}(x_1; y) - \bar{b}(x_2; y)| \leq C(G, n)|x_1 - x_2|,$$

where $B(0, n) = \{x \in \mathbb{R}^d : |x| \leq n\}$.

Then there exists an $(\mathcal{F}_t^\xi)$ *stopping time* η *and an* $(\mathcal{F}_t^\xi)$ *adapted* $\widehat{\mathbb{R}^d}$ *valued process* $\{X_t\}$ *such that*

(a) $\{X_t\}$ *solves equation* (5.11) *up to* η *i.e. a.s.*

$$X_t = \int_0^t \bar{\sigma}(X_s; \xi).\, dB_s + \int_0^t \bar{b}(X_s; \xi)\, ds, \; 0 \leq t < \eta$$

and $X_t = \infty$ *for* $t \geq \eta$.

(b) $\{X_t\}$ *has continuous paths on the interval* $[0, \eta)$.

(c) $\eta = \lim_m \theta_m$ *where* $\{\theta_m\}$ *are* $(\mathcal{F}_t^\xi)$ *stopping times defined by* $\theta_m := \inf\{t \geq 0 : |X_t| \geq m\}$.

This is also pathwise unique in this sense: if $(\{X'_t\}, \eta')$ is another solution satisfying $(a), (b), (c)$, then $P(X_t = X'_t, 0 \le t < \eta \wedge \eta') = 1$.

Proof. We first prove the existence in Steps 1 to 13. Unless otherwise specified the symbols k, m, n will stand for positive integers.

Step 1: Fix a positive integer n. Now for each $y \in \mathcal{S}_p(\mathbb{R}^d)$, using Corollary 5.2.8 we get a globally Lipschitz function viz. $\bar{\sigma}_n(x; y)$ such that

(i) $\bar{\sigma}(x; y) = \bar{\sigma}_n(x; y)$ for all $x \in B(0, n)$.

(ii) For any bounded set G in $\mathcal{S}_p(\mathbb{R}^d)$,

$$|\bar{\sigma}_n(x_1; y) - \bar{\sigma}_n(x_2; y)| \le C(G, n)\, |x_1 - x_2|,\ \forall x_1, x_2 \in \mathbb{R}^d,\ y \in G.$$

Similarly define $\bar{b}_n(x; y)$ from $\bar{b}(x; y)$. Note that $\bar{\sigma}_n(0; y) = \bar{\sigma}(0; y)$ and $\bar{b}_n(0; y) = \bar{b}(0; y)$ for any $y \in \mathcal{S}_p(\mathbb{R}^d)$. For $y \in G$, the linear growth of $x \mapsto \bar{\sigma}_n(x; y)$ and $x \mapsto \bar{b}_n(x; y)$ is established as done for $x \mapsto \bar{\sigma}(x; y)$ and $x \mapsto \bar{b}(x; y)$ in Proposition 5.2.3.

Step 2: Assume that ξ is norm-bounded. We want to establish the existence and uniqueness of strong solution of

$$dX_t = \bar{\sigma}_n(X_t; \xi).\, dB_t + \bar{b}_n(X_t; \xi)\, dt,\ t \ge 0; \quad X_0 = 0.$$

The arguments of Proposition 5.2.3 will produce the required result. For fixed x, $y \mapsto \bar{\sigma}(x; y)$ is linear, whereas $y \mapsto \bar{\sigma}_n(x; y)$ might not be linear. So the proof for $\bar{\sigma}_n(x; y), \bar{b}_n(x; y)$ is similar to Proposition 5.2.3, but is not an implication of it.

Step 3: Now we consider the case $\mathbb{E}\|\xi\|_p^2 < \infty$. For any positive integer k, define $\xi^{(k)} := \xi \mathbb{1}_{(\|\xi\|_p \le k)}$. By the previous step we have the existence and uniqueness of strong solution of

$$dX_t = \bar{\sigma}_n(X_t; \xi^{(k)}).\, dB_t + \bar{b}_n(X_t; \xi^{(k)})\, dt,\ t \ge 0; \quad X_0 = 0.$$

Let $\{X_t^{(n,k)}\}$ denote the solution. Define $\{Z_t^{(n)}\}$ as $Z_t^{(n)} := X_t^{(n,n)}$, $t \ge 0$ for all $n \in \mathbb{N}$.

Step 4: The integer n was arbitrary but fixed. To construct the solution as mentioned in the statement we use approximation by varying n.

First fix any positive integer $m < n$. Note that

$$\bar{\sigma}_m(x; y) = \bar{\sigma}(x; y) = \bar{\sigma}_n(x; y),\ \forall x \in B(0, m), y \in \mathcal{S}_p(\mathbb{R}^d). \tag{5.20}$$

Define the stopping time $T_m := \inf\{t \ge 0 : |Z_t^{(m)}| \ge m, \text{ or } |Z_t^{(n)}| \ge m\}$. Then a.s. for all $t \ge 0$

$$Z_{t\wedge T_m}^{(n)} = \int_0^{t\wedge T_m} \bar{\sigma}_n(Z_s^{(n)}; \xi^{(n)}).\, dB_s + \int_0^{t\wedge T_m} \bar{b}_n(Z_s^{(n)}; \xi^{(n)})\, ds$$

$$= \int_0^{t\wedge T_m} \bar{\sigma}(Z_s^{(n)};\xi^{(n)}).\, dB_s + \int_0^{t\wedge T_m} \bar{b}(Z_s^{(n)};\xi^{(n)})\, ds, \text{ (using (5.20))}.$$

Using (5.9) and (5.17), we have a.s. for all $t \geq 0$

$$\begin{aligned}
\mathbb{1}_{(\|\xi\|_p \leq m)} Z_{t\wedge T_m}^{(n)} &= \mathbb{1}_{(\|\xi\|_p \leq m)} \int_0^{t\wedge T_m} \bar{\sigma}(Z_s^{(n)};\xi^{(n)}).\, dB_s \\
&\quad + \mathbb{1}_{(\|\xi\|_p \leq m)} \int_0^{t\wedge T_m} \bar{b}(Z_s^{(n)};\xi^{(n)})\, ds \\
&= \int_0^{t\wedge T_m} \mathbb{1}_{(\|\xi\|_p \leq m)} \bar{\sigma}(Z_s^{(n)};\xi^{(n)}).\, dB_s \\
&\quad + \int_0^{t\wedge T_m} \mathbb{1}_{(\|\xi\|_p \leq m)} \bar{b}(Z_s^{(n)};\xi^{(n)})\, ds \\
&= \int_0^{t\wedge T_m} \bar{\sigma}(Z_s^{(n)} \mathbb{1}_{(\|\xi\|_p \leq m)};\xi^{(m)}).\, dB_s \\
&\quad + \int_0^{t\wedge T_m} \bar{b}(Z_s^{(n)} \mathbb{1}_{(\|\xi\|_p \leq m)};\xi^{(m)})\, ds.
\end{aligned}$$

From the above equation, we have

$$\begin{aligned}
\mathbb{1}_{(\|\xi\|_p \leq m)} Z_{t\wedge T_m}^{(n)} &= \int_0^{t\wedge T_m} \bar{\sigma}_m(Z_s^{(n)} \mathbb{1}_{(\|\xi\|_p \leq m)};\xi^{(m)}).\, dB_s \\
&\quad + \int_0^{t\wedge T_m} \bar{b}_m(Z_s^{(n)} \mathbb{1}_{(\|\xi\|_p \leq m)};\xi^{(m)})\, ds.
\end{aligned} \tag{5.21}$$

Again a.s.

$$Z_{t\wedge T_m}^{(m)} = \int_0^{t\wedge T_m} \bar{\sigma}_m(Z_s^{(m)};\xi^{(m)}).\, dB_s + \int_0^{t\wedge T_m} \bar{b}_m(Z_s^{(m)};\xi^{(m)})\, ds,\ t \geq 0.$$

Then using arguments similar to (5.21), we can show a.s. for all $t \geq 0$

$$\begin{aligned}
\mathbb{1}_{(\|\xi\|_p \leq m)} Z_{t\wedge T_m}^{(m)} &= \int_0^{t\wedge T_m} \bar{\sigma}_m(Z_s^{(m)} \mathbb{1}_{(\|\xi\|_p \leq m)};\xi^{(m)}).\, dB_s \\
&\quad + \int_0^{t\wedge T_m} \bar{b}_m(Z_s^{(m)} \mathbb{1}_{(\|\xi\|_p \leq m)};\xi^{(m)})\, ds.
\end{aligned} \tag{5.22}$$

Step 5: Now we show the uniqueness of solution of

$$dX_t = \bar{\sigma}_m(X_t;\xi^{(m)}).\, dB_t + \bar{b}_m(X_t;\xi^{(m)})\, dt,\ t \leq \theta; \quad X_0 = 0,$$

where θ is an $(\mathcal{F}_t)$ stopping time.
Let $\{X_t\}$ and $\{X'_t\}$ be two $(\mathcal{F}_t)$ adapted square integrable continuous processes satisfying the previous equation. Then under stopping by θ (Definition 2.4.8), we have a.s. for all $t \geq 0$

$$X_t^\theta - (X')_t^\theta = \int_0^{t\wedge\theta} [\bar{\sigma}_m(X_s;\xi^{(m)}) - \bar{\sigma}_m(X'_s;\xi^{(m)})]\, dB_s$$

$$+ \int_0^{t\wedge\theta} [\bar{b}_m(X_s;\xi^{(m)}) - \bar{b}_m(X'_s;\xi^{(m)})]\, ds.$$

As done in the uniqueness part of Proposition 5.2.3, we get

$$\begin{aligned}\mathbb{E}\left|X_t^\theta - (X')_t^\theta\right|^2 &\leq C(1+t)\mathbb{E}\int_0^{t\wedge\theta} \left|X_s - X'_s\right|^2 ds\\ &\leq C(1+t)\int_0^t \mathbb{E}\left|X_s^\theta - (X')_s^\theta\right|^2 ds\end{aligned}$$

for some constant $C > 0$. Consider the function

$$v(t) = \mathbb{E}\left|X_t^\theta - (X')_t^\theta\right|^2$$

on any compact time interval $[0, T]$. Then using Gronwall's inequality (Lemma 2.13.1) and the fact that $t \to v(t)$ is continuous, we get $v \equiv 0$. This proves $P(X_t^\theta = (X')_t^\theta) = 1$ for each t. Using continuity of paths of $\{X_t\}$ and $\{X'_t\}$, we conclude $P(X_t^\theta = (X')_t^\theta,\, t \geq 0) = 1$. This proves $P(X_t = X'_t,\, t \leq \theta) = 1$.
Hence from (5.21) and (5.22) we have

$$P(\mathbb{1}_{(\|\xi\|_p \leq m)} Z_t^{(m)} = \mathbb{1}_{(\|\xi\|_p \leq m)} Z_t^{(n)},\, t \leq T_m) = 1.$$

Step 6: Let $\Omega_{n,m} := \{\omega \in \Omega : \mathbb{1}_{(\|\xi\|_p \leq m)} Z_t^{(m)} = \mathbb{1}_{(\|\xi\|_p \leq m)} Z_t^{(n)},\, t \leq T_m\}$ for positive integers m, n with $m < n$. Now define

$$\widetilde{\Omega} := \left(\bigcap_{n=1}^{\infty} \bigcap_{m=1}^{n-1} \Omega_{n,m}\right).$$

Note that $\widetilde{\Omega} \in \mathcal{F}$ with $P(\widetilde{\Omega}) = 1$. Then $\widetilde{\Omega}^c$ is a null set in $\mathcal{F}_\infty^\xi$ and hence is an element of $\mathcal{F}_0^\xi$ (since the filtration satisfies the usual conditions). Therefore, $\widetilde{\Omega} \in \mathcal{F}_0^\xi$.
Now for any $\omega \in \Omega$ we have $\|\xi(\omega)\|_p < \infty$. Define $N(\omega)$ to be the least positive integer n such that $\|\xi(\omega)\|_p \leq m$ for all $m \geq n$.
Now on $\widetilde{\Omega}$ we have the following consistency relations: for each $\omega \in \widetilde{\Omega}$ and for $n > m \geq N(\omega)$

(i) $Z_t^{(m)}(\omega) = Z_t^{(n)}(\omega)$, $t \leq T_m(\omega)$.

(ii) $T_m(\omega) = \inf\{t \geq 0 : |Z_t^{(m)}(\omega)| \geq m\} = \inf\{t \geq 0 : |Z_t^{(n)}(\omega)| \geq m\}$.

(iii) From (ii),

$$\begin{aligned}T_m(\omega) &= \inf\{t \geq 0 : |Z_t^{(n)}(\omega)| \geq m\}\\ &< \inf\{t \geq 0 : |Z_t^{(n)}(\omega)| \geq n\}\\ &= T_n(\omega)\end{aligned}$$

Hence the sequence $\{T_n\}$ is eventually increasing.

Step 7: We claim that the $[0,\infty]$ valued function η defined below, is a $(\mathcal{F}_t^\xi)$ stopping time.

$$\eta(\omega) := \begin{cases} \lim_{m\to\infty} T_m(\omega),\ \forall \omega \in \widetilde{\Omega}, \\ \infty, \text{ otherwise.} \end{cases}$$

Fix $t \geq 0$. Then for any $\omega \in \widetilde{\Omega}$, the sequence $\{T_m(\omega)\}$ is eventually increasing and hence

$$\eta(\omega) = \lim_{m\to\infty} T_m(\omega) = \sup\{T_m(\omega) : m \geq N(\omega)\},\ \omega \in \widetilde{\Omega}. \tag{5.23}$$

Then for $\omega \in \widetilde{\Omega}$, $\eta(\omega) \leq t$ if and only if $T_m(\omega) \leq t, \forall m \geq N(\omega)$. By the definition of η we have

$$\begin{aligned}\{\omega \in \Omega : \eta(\omega) \leq t\} &= \{\omega \in \widetilde{\Omega} : \eta(\omega) \leq t\} \\ &= \bigcup_{n=1}^{\infty} \{\omega \in \widetilde{\Omega} : T_m(\omega) \leq t,\ \forall m \geq n\} \\ &= \bigcup_{n=1}^{\infty} \bigcap_{m \geq n} \{\omega \in \widetilde{\Omega} : T_m(\omega) \leq t\} \\ &= \widetilde{\Omega} \cap \left(\bigcup_{n=1}^{\infty} \bigcap_{m\geq n} \{\omega \in \Omega : T_m(\omega) \leq t\} \right).\end{aligned}$$

Since each T_m is an $(\mathcal{F}_t^\xi)$ stopping time, we have $(T_m \leq t) \in \mathcal{F}_t^\xi$. Hence $\{\omega \in \Omega : \eta(\omega) \leq t\} \in \mathcal{F}_t^\xi$. This proves η is an $(\mathcal{F}_t^\xi)$ stopping time.

Step 8: In this step we point out a decomposition of the space $\Omega \times [0,\infty)$. Observe that

$$\begin{aligned}\Omega \times [0,\infty) =& ((\widetilde{\Omega} \times [0,\infty)) \cap [0,\eta)) \bigsqcup ((\widetilde{\Omega} \times [0,\infty)) \cap [\eta,\infty)) \\ & \bigsqcup ((\widetilde{\Omega})^c \times [0,\infty)),\end{aligned}$$

where $[0,\eta)$ refers to the stochastic interval $\{(\omega,t) : 0 \leq t < \eta(\omega)\}$ and a similar expression holds for $[\eta,\infty)$. Again

$$(\widetilde{\Omega} \times [0,\infty)) \cap [0,\eta) = \bigcup_{k=1}^{\infty} ((\widetilde{\Omega} \cap (\|\xi\|_p \leq k)) \times [0,\infty)) \cap [0,T_k].$$

To prove the above equality, first note that the set on the right hand side is a subset of that of the left hand side.
Now let (ω,t) be an element of $(\widetilde{\Omega} \times [0,\infty)) \cap [0,\eta)$. Then there exist positive integers k_1, k_2 such that $\|\xi(\omega)\|_p \leq k_1$ and $t \leq T_{k_2}(\omega)$. Then for $k = \max\{k_1,k_2\}$ we have $(\omega,t) \in (\widetilde{\Omega} \cap (\|\xi\|_p \leq k)) \times [0,\infty) \cap [0,T_k]$. This proves the other inclusion.

Step 9: From the consistency conditions obtained in Step 6, we have

$$Z_t^{(k)}(\omega) = Z_t^{(k+1)}(\omega),\ \forall(\omega,t) \in (\widetilde{\Omega} \cap (\|\xi\|_p \le k)) \times [0,\infty) \cap [0,T_k]. \tag{5.24}$$

Define a process $\{Z_t\}$ as follows

$$Z_t(\omega) := \begin{cases} Z_t^{(k)}(\omega),\ \text{if}\, (\omega,t) \in (\widetilde{\Omega} \cap (\|\xi\|_p \le k)) \times [0,\infty) \cap [0,T_k] \\ \infty,\ \text{if}\, (\omega,t) \in ((\widetilde{\Omega} \times [0,\infty)) \cap [\eta,\infty)) \\ 0,\ \text{if}\, (\omega,t) \in (\widetilde{\Omega})^c \times [0,\infty). \end{cases}$$

Note that

$$\begin{aligned}[0,T_k] = (\{(\omega,t) : 0 \le t \le T_k(\omega)\} \cap \{(\omega,t) : T_k(\omega) < \infty\}) \\ \bigcup (\{(\omega,t) : 0 \le t < \infty\} \cap \{(\omega,t) : T_k(\omega) = \infty\}).\end{aligned}$$

From the decomposition of $\Omega \times [0,\infty)$ obtained in the previous step, it is clear that $Z_t(\omega)$ has been defined for all ω, t. Further $\{Z_t\}$ is well-defined due the consistency condition (5.24).

Step 10: We show that $\{Z_t\}$ is $(\mathcal{F}_t^\xi)$ adapted and has continuous paths on $[0,\eta)$. Observe that

$$\begin{aligned} Z_t &= (\mathbb{1}_{(\widetilde{\Omega})^c} + \mathbb{1}_{(\widetilde{\Omega})})Z_t \\ &= \mathbb{1}_{\widetilde{\Omega}} Z_t,\ (\because Z_t(\omega) = 0, \forall \omega \in (\widetilde{\Omega})^c) \\ &= \mathbb{1}_{\widetilde{\Omega}} \left(\mathbb{1}_{(t<\eta)} + \mathbb{1}_{(t\ge\eta)}\right) Z_t \\ &= \lim_k \mathbb{1}_{\widetilde{\Omega}} \mathbb{1}_{(t\le T_k)} Z_t + \infty \mathbb{1}_{\widetilde{\Omega}} \mathbb{1}_{(t\ge\eta)} \\ &= \lim_k \mathbb{1}_{\widetilde{\Omega}} \mathbb{1}_{(t\le T_k)} \mathbb{1}_{(\|\xi\|_p \le k)} Z_t + \infty \mathbb{1}_{\widetilde{\Omega}} \mathbb{1}_{(t\ge\eta)},\ (\because \mathbb{1}_{(\|\xi\|_p\le k)} \uparrow \mathbb{1}_\Omega = 1) \\ &= \lim_k \mathbb{1}_{\widetilde{\Omega}} \mathbb{1}_{(t\le T_k)} \mathbb{1}_{(\|\xi\|_p \le k)} Z_t^{(k)} + \infty \mathbb{1}_{\widetilde{\Omega}} \mathbb{1}_{(t\ge\eta)}. \end{aligned}$$

But $\widetilde{\Omega} \in \mathcal{F}_0^\xi, (\|\xi\|_p \le k) \in \mathcal{F}_0^\xi, (t \le T_k) \in \mathcal{F}_t^\xi, (t \ge \eta) \in \mathcal{F}_t^\xi$ and $\{Z^{(k)}\}$ is $(\mathcal{F}_t^\xi)$ adapted. Hence from the above equality we conclude that $\{Z_t\}$ is $(\mathcal{F}_t^\xi)$ adapted.
By definition $\eta = \infty$ on the set $(\widetilde{\Omega})^c$ and $Z_t(\omega) = 0$ whenever $(\omega,t) \in (\widetilde{\Omega})^c \times [0,\infty)$. Hence $t \mapsto Z_t(\omega)$ is continuous on $[0,\eta(\omega))$ if $\omega \in (\widetilde{\Omega})^c$.
Let $\omega \in \widetilde{\Omega}$. Then $\|\xi(\omega)\|_p \le m, \forall m \ge N(\omega)$ ($N(\omega)$ as in Step 6). By definition $Z_t(\omega) = Z_t^{(m)}(\omega)$ for all $t \in [0,T_m(\omega)], m \ge N(\omega)$. Since paths of $\{Z_t^{(m)}\}$ are continuous, we have $t \mapsto Z_t(\omega)$ is continuous on $[0,T_m(\omega)]$ for all $m \ge N(\omega)$. But $T_m(\omega)$ eventually increases to $\eta(\omega)$ and hence $t \mapsto Z_t(\omega)$ is continuous on $[0,\eta(\omega))$.

Step 11: For any positive integer k define $\theta_k := \eta \wedge \inf\{t \ge 0 : |Z_t| \ge k\}$. In this step, we show the connection between η and the θ_k's.

On $(\widetilde{\Omega})^c$, $\{t \geq 0 : |Z_t| \geq k\}$ is an empty set, since $Z_t = 0$. Hence $\inf\{t \geq 0 : |Z_t| \geq k\} = \infty$. Also by definition $\eta = \infty$ on $(\widetilde{\Omega})^c$. Therefore on $(\widetilde{\Omega})^c$, we have $\theta_k = \infty$ and $\lim_k \theta_k = \eta$.

Let $\omega \in \widetilde{\Omega}$ and $k \geq N(\omega)$. By definition $Z_t(\omega) = Z_t^{(k)}(\omega)$, $t \in [0, T_k(\omega)]$. But $T_k = \inf\{t \geq 0 : |Z_t^{(k)}| \geq k\}$. Hence $\theta_k(\omega) = T_k(\omega)$, $k \geq N(\omega)$ (note that $T_k(\omega) < \eta(\omega)$). Then $\lim_k \theta_k(\omega) = \lim_k T_k(\omega) = \eta(\omega)$.

This identification of η as a limit of θ_k's will not be used during this proof of existence. We need this in the proof of uniqueness.

Step 12: We now establish existence of some stochastic integrals which will be used in the Step 13. From Lemma 2.11.7(i), we have on $\widetilde{\Omega}$

$$|\bar{\sigma}(Z_{t\wedge T_k};\xi)| = |\left\langle \sigma \,,\, \tau_{Z_{t\wedge T_k}}\xi \right\rangle| \leq \|\sigma\|_{-p}\|\tau_{Z_{t\wedge T_k}}\xi\|_p \leq C_k\|\xi\|_p$$

for some constant $C_k > 0$. Similarly,

$$|\bar{b}(Z_{t\wedge T_k};\xi)| \leq C_k'\|\xi\|_p$$

for some constant $C_k' > 0$. Then for any $t \geq 0$

$$\mathbb{E}\int_0^{t\wedge T_k} |\bar{\sigma}(Z_s;\xi)|^2\, ds \leq C_k^2\,(\mathbb{E}\|\xi\|_p^2)\, t < \infty,$$

which shows the existence of the integral $\int_0^{t\wedge T_k} \bar{\sigma}(Z_s;\xi)\, dB_s$. Similarly we can show the existence of $\int_0^{t\wedge T_k} \bar{b}(Z_s;\xi)\, ds$. These stochastic integrals will be used in the next step. We have ignored the null set $(\widetilde{\Omega})^c$ in the above computation.

Step 13: We prove that the pair $(\{Z_t\}, \eta)$ is a solution of (5.11).

Let k be a positive integer. Let $\omega \in \widetilde{\Omega}$ with $\|\xi(\omega)\|_p \leq k$. Then

$$Z_{t\wedge T_k(\omega)}(\omega) = \begin{cases} Z_t(\omega), \text{ if } T_k(\omega) = \infty \\ Z_t(\omega), \text{ if } T_k(\omega) < \infty, t \in [0, T_k(\omega)] \\ Z_{T_k(\omega)}, \text{if } T_k(\omega) < \infty, t \in (T_k(\omega), \infty). \end{cases}$$

Now we use the definition of $\{Z_t\}$ to prove a consistency relation between $\{Z_t\}$ and $\{Z_t^{(k)}\}$.

a) If $T_k(\omega) = \infty$ and $t \in [0, T_k(\omega))$ then

$$Z_{t\wedge T_k(\omega)}(\omega) = Z_t(\omega) = Z_t^{(k)}(\omega) = Z_{t\wedge T_k(\omega)}^{(k)}(\omega)$$

b) If $T_k(\omega) < \infty$ and $t \in [0, T_k(\omega)]$ then

$$Z_{t\wedge T_k(\omega)}(\omega) = Z_t(\omega) = Z_t^{(k)}(\omega) = Z_{t\wedge T_k(\omega)}^{(k)}(\omega)$$

c) If $T_k(\omega) < \infty$ and $t \in (T_k(\omega), \infty)$ then

$$Z_{t\wedge T_k(\omega)}(\omega) = Z_{T_k(\omega)}(\omega) = Z^{(k)}_{T_k(\omega)}(\omega) = Z^{(k)}_{t\wedge T_k(\omega)}(\omega).$$

In view of the above equalities, on $\widetilde{\Omega}$

$$Z_{t\wedge T_k}\mathbb{1}_{(\|\xi\|_p\leq k)} = Z^{(k)}_{t\wedge T_k}\mathbb{1}_{(\|\xi\|_p\leq k)} \tag{5.25}$$

and hence above equality holds almost surely.
Then a.s. for all $t \geq 0$

$$\begin{aligned}
&\mathbb{1}_{(\|\xi\|_p\leq k)}Z_{t\wedge T_k}\\
&= \mathbb{1}_{(\|\xi\|_p\leq k)}Z^{(k)}_{t\wedge T_k}\\
&= \int_0^{t\wedge T_k} \bar{\sigma}_k(\mathbb{1}_{(\|\xi\|_p\leq k)}Z^{(k)}_s;\xi^{(k)}).\,dB_s\\
&+ \int_0^{t\wedge T_k} \bar{b}_k(\mathbb{1}_{(\|\xi\|_p\leq k)}Z^{(k)}_s;\xi^{(k)})\,ds,\ (\text{using}\,(5.22))\\
&= \int_0^{t\wedge T_k} \bar{\sigma}(\mathbb{1}_{(\|\xi\|_p\leq k)}Z^{(k)}_s;\xi^{(k)}).\,dB_s\\
&+ \int_0^{t\wedge T_k} \bar{b}(\mathbb{1}_{(\|\xi\|_p\leq k)}Z^{(k)}_s;\xi^{(k)})\,ds,\ (\text{using}\,(5.20))\\
&= \int_0^{t\wedge T_k} \bar{\sigma}(\mathbb{1}_{(\|\xi\|_p\leq k)}Z_s;\xi^{(k)}).\,dB_s\\
&+ \int_0^{t\wedge T_k} \bar{b}(\mathbb{1}_{(\|\xi\|_p\leq k)}Z_s;\xi^{(k)})\,ds,\ (\text{using}\,(5.25))\\
&= \int_0^{t\wedge T_k} \mathbb{1}_{(\|\xi\|_p\leq k)}\bar{\sigma}(Z_s;\xi).\,dB_s\\
&+ \int_0^{t\wedge T_k} \mathbb{1}_{(\|\xi\|_p\leq k)}\bar{b}(Z_s;\xi)\,ds,\ (\text{using}\,(5.17))\\
&= \mathbb{1}_{(\|\xi\|_p\leq k)}\left[\int_0^{t\wedge T_k} \bar{\sigma}(Z_s;\xi).\,dB_s + \int_0^{t\wedge T_k} \bar{b}(Z_s;\xi)\,ds\right],\ (\text{using}\,(5.9))
\end{aligned}$$

Let Ω' denote the set of probability 1 where the above relations hold for all positive integers k. Then on $\Omega' \cap \widetilde{\Omega}$ we have for all $t \geq 0$ and for all $k \in \mathbb{N}$,

$$\mathbb{1}_{(\|\xi\|_p\leq k)}Z_{t\wedge T_k} = \left[\int_0^{t\wedge T_k} \bar{\sigma}(Z_s;\xi).\,dB_s + \int_0^{t\wedge T_k} \bar{b}(Z_s;\xi)\,ds\right].$$

Recall that for any $\omega \in \Omega$, $\|\xi(\omega)\|_p \leq k$, $\forall k \geq N(\omega)$ ($N(\omega)$ as in Step 6) and hence for $\omega \in \Omega' \cap \widetilde{\Omega}$ we have for all $t \geq 0$ and for all $k \geq N(\omega)$,

$$Z_{t\wedge T_k(\omega)}(\omega) = \left(\int_0^{t\wedge T_k} \bar{\sigma}(Z_s;\xi).\,dB_s\right)(\omega) + \left(\int_0^{t\wedge T_k} \bar{b}(Z_s;\xi)\,ds\right)(\omega).$$

Letting k go to infinity, we have a.s.

$$Z_t = \int_0^t \bar{\sigma}(Z_s;\xi).\, dB_s + \int_0^t \bar{b}(Z_s;\xi)\, ds,\ t < \eta.$$

Hence $(\{Z_t\},\eta)$ is a solution of (5.11) and Step 13 ends here.

This concludes the proof of existence a solution of (5.11) and we now prove pathwise uniqueness. Assume that two pairs $(X^{(1)},\eta^1)$ and $(X^{(2)},\eta^2)$ solve the given equation. Define two sequences of $(\mathcal{F}_t^\xi)$ stopping times by

$$T^{1,m} := \inf\{t \geq 0 : |X_t^{(1)}| \geq m\}, \quad T^{2,m} := \inf\{t \geq 0 : |X_t^{(2)}| \geq m\}.$$

Then $\eta^i = \lim_m T^{i,m}$ for $i = 1, 2$. Hence a.s. for any $t \geq 0$, $m \geq 1$ and $i = 1, 2$

$$X^{(i)}_{t\wedge T^{1,m}\wedge T^{2,m}} = \int_0^{t\wedge T^{1,m}\wedge T^{2,m}} \bar{\sigma}(X_s^{(i)};\xi).\, dB_s + \int_0^{t\wedge T^{1,m}\wedge T^{2,m}} \bar{b}(X_s^{(i)};\xi)\, ds.$$

Note that $|X^{(i)}_{t\wedge T^{i,m}}| \leq m$, $t \geq 0, i = 1, 2$. Then using (5.9), (5.10) a.s. for all $t \geq 0$

$$\begin{aligned}
\mathbb{1}_{(\|\xi\|_p\leq m)} X^{(i)}_{t\wedge T^{1,m}\wedge T^{2,m}} &= \int_0^{t\wedge T^{1,m}\wedge T^{2,m}} \mathbb{1}_{(\|\xi\|_p\leq m)}\bar{\sigma}(X_s^{(i)};\xi).\, dB_s \\
&\quad + \int_0^{t\wedge T^{1,m}\wedge T^{2,m}} \mathbb{1}_{(\|\xi\|_p\leq m)}\bar{b}(X_s^{(i)};\xi)\, ds,\ (\text{using}\,(5.9)) \\
&= \int_0^{t\wedge T^{1,m}\wedge T^{2,m}} \bar{\sigma}(\mathbb{1}_{(\|\xi\|_p\leq m)}X_s^{(i)};\xi^{(m)}).\, dB_s \\
&\quad + \int_0^{t\wedge T^{1,m}\wedge T^{2,m}} \bar{b}(\mathbb{1}_{(\|\xi\|_p\leq m)}X_s^{(i)};\xi^{(m)})\, ds,\ (\text{using}\,(5.17)) \\
&= \int_0^{t\wedge T^{1,m}\wedge T^{2,m}} \bar{\sigma}_m(\mathbb{1}_{(\|\xi\|_p\leq m)}X_s^{(i)};\xi^{(m)}).\, dB_s \\
&\quad + \int_0^{t\wedge T^{1,m}\wedge T^{2,m}} \bar{b}_m(\mathbb{1}_{(\|\xi\|_p\leq m)}X_s^{(i)};\xi^{(m)})\, ds,\ (\text{using}\,(5.20)).
\end{aligned}$$

We can show $P(\mathbb{1}_{(\|\xi\|_p\leq m)}X_t^{(1)} = \mathbb{1}_{(\|\xi\|_p\leq m)}X_t^{(2)},\ t \leq T^{1,m}\wedge T^{2,m}) = 1$ using Step 5. Let $\bar{\Omega}$ denote the set of probability 1 where the following relation holds

$$\mathbb{1}_{(\|\xi\|_p\leq m)}X_t^{(1)} = \mathbb{1}_{(\|\xi\|_p\leq m)}X_t^{(2)},\ t \leq T^{1,m}\wedge T^{2,m}, m = 1, 2, \cdots.$$

Recall that for any $\omega \in \Omega$, $\|\xi(\omega)\|_p \leq m$, $\forall m \geq N(\omega)$ ($N(\omega)$ as in Step 6 of the proof of existence) and hence for $\omega \in \bar{\Omega}$ we have for all $t \geq 0$ and for all $m \geq N(\omega)$,

$$X_t^{(1)}(\omega) = X_t^{(2)}(\omega),\ t \leq T^{1,m}(\omega)\wedge T^{2,m}(\omega).$$

But $T^{1,m}\wedge T^{2,m} \uparrow \eta^1\wedge\eta^2$. Hence for $\omega \in \bar{\Omega}$, we have $X_t^{(1)}(\omega) = X_t^{(2)}(\omega)$, $t \leq \eta^1(\omega)\wedge\eta^2(\omega)$. This proves pathwise uniqueness. □

Remark 5.2.10. It should be possible to prove $\lim_{t\uparrow\eta}|X_t| = \infty$ on the set $(\eta < \infty)$, where (X_t, η) is the solution obtained in Theorem 5.2.9. This is a well-known property in the case of explosions in classical finite dimensional diffusions ([50, Chapter IV, Lemma 2.1]).

In Proposition 5.2.11, we obtain a criterion (which essentially is a stronger assumption on ξ) that imply a 'local Lipschitz' condition. We use this result to obtain Theorem 5.2.12, which is a version of Theorem 5.2.9.

Proposition 5.2.11. *Let $p > d + \frac{1}{2}$ and $\sigma \in \mathcal{S}_{-p}(\mathbb{R}^d)$. Then for any bounded set G in $\mathcal{S}_{p+\frac{1}{2}}(\mathbb{R}^d)$ and any positive integer n there exists a constant $C(G,n) > 0$ such that for all $x_1, x_2 \in B(0,n), y \in G$*

$$|\bar{\sigma}(x_1; y) - \bar{\sigma}(x_2; y)| \le C(G,n)|x_1 - x_2|,$$

where $B(0,n) = \{x \in \mathbb{R}^d : |x| \le n\}$.

Proof. Let $y \in \mathcal{S}_p(\mathbb{R}^d)$. Abusing notation, we denote the function $x \mapsto \langle \delta_x\, , \, y\rangle$ by y. The first order partial derivatives of function y exist and the distribution y is given by the differentiable function y (see Proposition 2.11.26). Furthermore, the first order distributional derivatives of y are given by the first order partial derivatives of y, which are continuous functions.
Let $x^1 = (x^1_1, \cdots, x^1_d), x^2 = (x^2_1, \cdots, x^2_d) \in B(0,n)$. Then for any $y \in \mathcal{S}_p(\mathbb{R}^d)$,

$$|\bar{\sigma}(x_1; y) - \bar{\sigma}(x_2; y)| \le \|\sigma\|_{-p}\|\tau_{x_1}y - \tau_{x_2}y\|_p.$$

The target of the subsequent computations is to obtain an estimate of $\|\tau_{x_1}y - \tau_{x_2}y\|_p$. Now for any $1 \le i \le d$ and $t = \lambda_i x^1_i + (1-\lambda_i)x^2_i$ with $\lambda_i \in [0,1]$

$$\begin{aligned}
&|(x^1_1, \cdots, x^1_{i-1}, t, x^2_{i+1}, \cdots, x^2_d)| \\
&= |(x^1_1, \cdots, x^1_{i-1}, \lambda_i x^1_i + (1-\lambda_i)x^2_i, x^2_{i+1}, \cdots, x^2_d)| \\
&\le |(x^1_1, \cdots, x^1_{i-1}, \lambda_i x^1_i, 0, \cdots, 0)| \\
&+ |(0, \cdots, 0, (1-\lambda_i)x^2_i, x^2_{i+1}, \cdots, x^2_d)| \\
&\le |(x^1_1, \cdots, x^1_{i-1}, x^1_i, 0, \cdots, 0)| \\
&+ |(0, \cdots, 0, x^2_i, x^2_{i+1}, \cdots, x^2_d)| \\
&\le 2n
\end{aligned}$$

Let $y \in \mathcal{S}_{p+\frac{1}{2}}(\mathbb{R}^d)$ (note that $\mathcal{S}_{p+\frac{1}{2}}(\mathbb{R}^d) \subset \mathcal{S}_p(\mathbb{R}^d)$). Then by Lemma 2.11.7, there exist constants $C_n > 0, \tilde{C}_n > 0$ independent of i such that

$$\begin{aligned}\|\tau_{(x_1^1,\cdots,x_{i-1}^1,t,x_{i+1}^2,\cdots,x_d^2)}\partial_i y\|_p &= \|\partial_i \tau_{(x_1^1,\cdots,x_{i-1}^1,t,x_{i+1}^2,\cdots,x_d^2)} y\|_p \\ &\leq C_n \, \|\tau_{(x_1^1,\cdots,x_{i-1}^1,t,x_{i+1}^2,\cdots,x_d^2)} y\|_{p+\frac{1}{2}} \\ &\leq \tilde{C}_n \, \|y\|_{p+\frac{1}{2}}\end{aligned} \tag{5.26}$$

The following is an equality of continuous functions.

$$\begin{aligned}&\tau_{(x_1^1,\cdots,x_{i-1}^1,x_i^1,x_{i+1}^2,\cdots,x_d^2)} y(\cdot) - \tau_{(x_1^1,\cdots,x_{i-1}^1,x_i^2,x_{i+1}^2,\cdots,x_d^2)} y(\cdot) \\ &= y(\cdot - (x_1^1, \cdots, x_{i-1}^1, x_i^1, x_{i+1}^2, \cdots, x_d^2)) \\ &- y(\cdot - (x_1^1, \cdots, x_{i-1}^1, x_i^2, x_{i+1}^2, \cdots, x_d^2)) \\ &= \int_{x_i^2}^{x_i^1} \partial_i y(\cdot - (x_1^1, \cdots, x_{i-1}^1, t, x_{i+1}^2, \cdots, x_d^2)) \, dt \\ &= \int_{x_i^2}^{x_i^1} \tau_{(x_1^1,\cdots,x_{i-1}^1,t,x_{i+1}^2,\cdots,x_d^2)} \partial_i y(\cdot) \, dt\end{aligned}$$

In view of (5.26), we have the equality of distributions in $\mathcal{S}_p(\mathbb{R}^d)$

$$\begin{aligned}&\tau_{(x_1^1,\cdots,x_{i-1}^1,x_i^1,x_{i+1}^2,\cdots,x_d^2)} y - \tau_{(x_1^1,\cdots,x_{i-1}^1,x_i^2,x_{i+1}^2,\cdots,x_d^2)} y \\ &= \int_{x_i^2}^{x_i^1} \tau_{(x_1^1,\cdots,x_{i-1}^1,t,x_{i+1}^2,\cdots,x_d^2)} \partial_i y \, dt.\end{aligned}$$

Then

$$\begin{aligned}&\|\tau_{(x_1^1,\cdots,x_{i-1}^1,x_i^1,x_{i+1}^2,\cdots,x_d^2)} y - \tau_{(x_1^1,\cdots,x_{i-1}^1,x_i^2,x_{i+1}^2,\cdots,x_d^2)} y\|_p \\ &= \| \int_{x_i^2}^{x_i^1} \tau_{(x_1^1,\cdots,x_{i-1}^1,t,x_{i+1}^2,\cdots,x_d^2)} \partial_i y \, dt \, \|_p \\ &\leq \left| \int_{x_i^2}^{x_i^1} \|\tau_{(x_1^1,\cdots,x_{i-1}^1,t,x_{i+1}^2,\cdots,x_d^2)} \partial_i y\|_p \, dt \right| \\ &\leq \tilde{C}_n \, \|y\|_{p+\frac{1}{2}} \, |x_i^1 - x_i^2|.\end{aligned}$$

Now

$$\begin{aligned}\tau_{x_1} y - \tau_{x_2} y = &\tau_{(x_1^1,\cdots,x_{d-1}^1,x_d^1)} y - \tau_{(x_1^1,\cdots,x_{d-1}^1,x_d^2)} y \\ &+ \tau_{(x_1^1,\cdots,x_{d-1}^1,x_d^2)} y - \tau_{(x_1^1,\cdots,x_{d-2}^1,x_{d-1}^2,x_d^2)} y \\ &+ \cdots \\ &+ \tau_{(x_1^1,x_2^2,\cdots,x_d^2)} y - \tau_{(x_1^2,\cdots,x_d^2)} y\end{aligned}$$

and hence $\|\tau_{x_1}y - \tau_{x_2}y\|_p \leq \tilde{C}_n \,\|y\|_{p+\frac{1}{2}} \,\sum_{i=1}^d |x_i^1 - x_i^2| \leq d\tilde{C}_n \,\|y\|_{p+\frac{1}{2}} \,|x_1 - x_2|$.
Then $|\bar{\sigma}(x_1;y) - \bar{\sigma}(x_2;y)| \leq \|\sigma\|_{-p}\|\tau_{x_1}y - \tau_{x_2}y\|_p \leq d\tilde{C}_n \,\|\sigma\|_{-p}\|y\|_{p+\frac{1}{2}} \,|x_1 - x_2|$. In particular if G is a bounded set in $\mathcal{S}_{p+\frac{1}{2}}(\mathbb{R}^d)$, then for any $y \in G$

$$|\bar{\sigma}(x_1;y) - \bar{\sigma}(x_2;y)| \leq d\tilde{C}_n \,\|\sigma\|_{-p} \sup_{y\in G}(\|y\|_{p+\frac{1}{2}}) \,|x_1 - x_2|,$$

i.e. the function $x \mapsto \bar{\sigma}(x;y)$ is locally Lipschitz in x for any $y \in G$ and that the Lipschitz constant can be taken uniformly in $y \in G$. □

The next result is a version of Theorem 5.2.9. We get the 'local Lipschitz' property of the coefficients from extra regularity on ξ (see Proposition 5.2.11).

Theorem 5.2.12. *Let $p > d + \frac{1}{2}$. Suppose the following are satisfied.*

(i) $\sigma, b \in \mathcal{S}_{-p}(\mathbb{R}^d)$.
(ii) ξ *is* $\mathcal{S}_{p+\frac{1}{2}}(\mathbb{R}^d)$ *valued and* $\mathbb{E}\|\xi\|^2_{p+\frac{1}{2}} < \infty$.
(iii) $\zeta = 0$.

Then there exists an $(\mathcal{F}^\xi_t)$ *stopping time* η *and an* $(\mathcal{F}^\xi_t)$ *adapted* $\widehat{\mathbb{R}^d}$ *valued process* $\{X_t\}$ *such that*

(a) $\{X_t\}$ *solves* (5.11) *up to* η *i.e. a.s.*

$$X_t = \int_0^t \bar{\sigma}(X_s;\xi).\, dB_s + \int_0^t \bar{b}(X_s;\xi)\, ds,\; 0 \leq t < \eta$$

and $X_t = \infty$ *for* $t \geq \eta$.

(b) $\{X_t\}$ *has continuous paths on the interval* $[0, \eta)$.
(c) $\eta = \lim_m \theta_m$ *where* $\{\theta_m\}$ *are* $(\mathcal{F}^\xi_t)$ *stopping times defined by* $\theta_m := \inf\{t \geq 0 : |X_t| \geq m\}$.

This is also pathwise unique in this sense: if (X'_t, η') *is another solution satisfying properties* $(a), (b), (c)$, *then* $P(X_t = X'_t, 0 \leq t < \eta \wedge \eta') = 1$.

Proof. The proof is similar to Theorem 5.2.9. We indicate the necessary changes.

(i) Fix a positive integer n. Now for each $y \in \mathcal{S}_{p+\frac{1}{2}}(\mathbb{R}^d)$, using Corollary 5.2.8 we get a globally Lipschitz function viz. $\bar{\sigma}_n(x;y)$ such that
  a) $\bar{\sigma}(x;y) = \bar{\sigma}_n(x;y)$ for all $x \in B(0,n)$.
  b) For any bounded set G in $\mathcal{S}_{p+\frac{1}{2}}(\mathbb{R}^d)$, using Proposition 5.2.11

$$|\bar{\sigma}_n(x_1;y) - \bar{\sigma}_n(x_2;y)| \leq C(G,n)\,|x_1 - x_2|,\; \forall x_1, x_2 \in \mathbb{R}^d,\; y \in G.$$

Similarly define $\bar{b}_n(x;y)$ from $\bar{b}(x;y)$. Note that $\bar{\sigma}_n(0;y) = \bar{\sigma}(0;y)$ and $\bar{b}_n(0;y) = \bar{b}(0;y)$ for any $y \in \mathcal{S}_{p+\frac{1}{2}}(\mathbb{R}^d)$. The linear growths of $x \mapsto \bar{\sigma}_n(x;y)$ and $x \mapsto \bar{b}_n(x;y)$ are established as done for $x \mapsto \bar{\sigma}(x;y)$ and $x \mapsto \bar{b}(x;y)$ in Proposition 5.2.3.

(ii) Rest of the proof is the same except that we take the variable y from $\mathcal{S}_{p+\frac{1}{2}}(\mathbb{R}^d)$.

□

We are ready to prove the main result of this section. We make two definitions extending [90, Definition 3.1 and 3.3]. Note that ξ is assumed to be independent of the Brownian motion $\{B_t\}$ and $\hat{\mathcal{S}}_p(\mathbb{R}^d) = \mathcal{S}_p(\mathbb{R}^d) \cup \{\delta\}$, where δ is an isolated point.

Definition 5.2.13. (A) We say $\{Y_t\}$ is an $\mathcal{S}_p(\mathbb{R}^d)$ valued strong solution of equation (5.1), if $\{Y_t\}$ is an $\mathcal{S}_p(\mathbb{R}^d)$ valued $(\mathcal{F}^\xi_t)$ adapted continuous process such that a.s. the following equality holds in $\mathcal{S}_{p-1}(\mathbb{R}^d)$,

$$Y_t = \xi + \int_0^t A(Y_s).\, dB_s + \int_0^t L(Y_s)\, ds;\ t \geq 0.$$

(B) By an $\hat{\mathcal{S}}_p(\mathbb{R}^d)$ valued strong local solution of equation (5.1), we mean a pair $(\{Y_t\}, \eta)$ where η is an $(\mathcal{F}^\xi_t)$ stopping time and $\{Y_t\}$ an $\hat{\mathcal{S}}_p(\mathbb{R}^d)$ valued $(\mathcal{F}^\xi_t)$ adapted continuous process such that

(1) for all $\omega \in \Omega$, the map $Y_\cdot(\omega) : [0, \eta(\omega)) \to \mathcal{S}_p(\mathbb{R}^d)$ is continuous and $Y_t(\omega) = \delta$, $t \geq \eta(\omega)$.

(2) a.s. the following equality holds in $\mathcal{S}_{p-1}(\mathbb{R}^d)$,

$$Y_t = \xi + \int_0^t A(Y_s).\, dB_s + \int_0^t L(Y_s)\, ds;\ 0 \leq t < \eta.$$

Definition 5.2.14. (A) We say strong solutions to equation (5.1) are pathwise unique, if given any two $\mathcal{S}_p(\mathbb{R}^d)$ valued strong solutions $\{Y^1_t\}$ and $\{Y^2_t\}$, we have $P(Y^1_t = Y^2_t,\ t \geq 0) = 1$.

(B) We say strong local solutions to equation (5.1) are pathwise unique, if given any two $\hat{\mathcal{S}}_p(\mathbb{R}^d)$ valued strong solutions $(\{Y^1_t\}, \eta^1)$ and $(\{Y^2_t\}, \eta^2)$, we have $P(Y^1_t = Y^2_t,\ 0 \leq t < \eta^1 \wedge \eta^2) = 1$.

Now we prove the existence and uniqueness of solutions to equation (5.1).

Theorem 5.2.15. *Suppose the following conditions are satisfied.*

(i) $\mathbb{E}\|\xi\|_p^2 < \infty$.

(ii) (Globally Lipschitz in x, locally in y) For any fixed $y \in \mathcal{S}_p(\mathbb{R}^d)$, *the functions* $x \mapsto \bar{\sigma}(x;y)$ *and* $x \mapsto \bar{b}(x;y)$ *are globally Lipschitz functions in* x *and the Lipschitz coefficient is independent of* y *when* y *varies over any bounded set* G *in* $\mathcal{S}_p(\mathbb{R}^d)$*; i.e.*

for any bounded set G in $\mathcal{S}_p(\mathbb{R}^d)$ there exists a constant $C(G) > 0$ such that for all $x_1, x_2 \in \mathbb{R}^d, y \in G$

$$|\bar{\sigma}(x_1; y) - \bar{\sigma}(x_2; y)| + |\bar{b}(x_1; y) - \bar{b}(x_2; y)| \leq C(G)|x_1 - x_2|.$$

Then equation (5.1) *has an* $(\mathcal{F}_t^\xi)$ *adapted continuous strong solution. The solutions are pathwise unique.*

First we need a characterization of the solution of equation (5.1). This is an extension of [90, Lemma 3.6] to random initial condition ξ.

Lemma 5.2.16. *Let $\xi, \bar{\sigma}, \bar{b}$ be as in Theorem 5.2.15. Let $\{Y_t\}$ be an $(\mathcal{F}_t^\xi)$ adapted $\mathcal{S}_p(\mathbb{R}^d)$ valued strong solution of equation* (5.1)*. Define a process $\{Z_t\}$ as follows:*

$$Z_t := \int_0^t \langle \sigma \,,\, Y_s\rangle \; dB_s + \int_0^t \langle b \,,\, Y_s\rangle \; ds, \; t \geq 0.$$

Then a.s. $Y_t = \tau_{Z_t}\xi$ for $t \geq 0$ and consequently Z solves equation (5.11) *with $Z_0 = 0$.*

Proof. Since $\{Y_t\}$ is a continuous $\mathcal{S}_p(\mathbb{R}^d)$ valued $(\mathcal{F}_t^\xi)$ adapted process and $\sigma \in \mathcal{S}_{-p}(\mathbb{R}^d)$, the real valued process $\{\langle \sigma \,,\, Y_t\rangle\}$ is a continuous $(\mathcal{F}_t^\xi)$ adapted process. Hence $\{\int_0^t \langle \sigma \,,\, Y_s\rangle \; dB_s\}$ is a continuous local martingale. Using similar arguments $\{\int_0^t \langle b \,,\, Y_s\rangle \; dB_s\}$ is a real valued continuous $(\mathcal{F}_t^\xi)$ adapted process.
First we define linear operator valued $(\mathcal{F}_t)$ adapted processes $\{\bar{L}(t)\}$ and $\{\bar{A}_j(t)\}$, $j = 1, \cdots, d$. For $\phi \in \mathcal{S}'(\mathbb{R}^d)$,

$$\bar{L}(t,\omega)\phi := \frac{1}{2}\sum_{i,j=1}^d (\langle \sigma \,,\, Y_t(\omega)\rangle \; \langle \sigma \,,\, Y_t(\omega)\rangle^t)_{ij} \, \partial_{ij}^2\phi - \sum_{i=1}^d (\langle\langle b \,,\, Y_t(\omega)\rangle\rangle)_i \, \partial_i\phi,$$

$$\bar{A}_j(t,\omega)\phi := -\sum_{i=1}^d (\langle\langle \sigma \,,\, Y_t(\omega)\rangle\rangle)_{ij} \, \partial_i\phi.$$

Note that $\bar{L}(t,\omega), \bar{A}_j(t,\omega)$ are linear operators from $\mathcal{S}_p(\mathbb{R}^d)$ to $\mathcal{S}_{p-1}(\mathbb{R}^d)$.
We write as $Z_t = (Z_t^1, \cdots, Z_t^d)$ and $\bar{A}(t) = (\bar{A}_1(t), \cdots, \bar{A}_d(t))$. By Theorem 5.2.2, we have the following equality in $\mathcal{S}_{p-1}(\mathbb{R}^d)$: a.s. $t \geq 0$

$$\begin{aligned}\tau_{Z_t}\xi &= \xi - \sum_{i=1}^d \int_0^t \partial_i\tau_{Z_s}\xi \, dZ_s^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial_{ij}^2\tau_{Z_s}\xi \, d[Z^i, Z^j]_s \\ &= \xi - \sum_{i=1}^d \int_0^t (\langle\langle \sigma \,,\, Y_t(\omega)\rangle\rangle)_{ij}\partial_i\tau_{Z_s}\xi \, dB_s^i - \sum_{i=1}^d \int_0^t (\langle\langle b \,,\, Y_t(\omega)\rangle\rangle)_{ij}\partial_i\tau_{Z_s}\xi \, ds \\ &+ \frac{1}{2}\sum_{i,j=1}^d \int_0^t (\langle \sigma \,,\, Y_t(\omega)\rangle \; \langle \sigma \,,\, Y_t(\omega)\rangle^t)_{ij}\partial_{ij}^2\tau_{Z_s}\xi \, ds\end{aligned}$$

$$= \xi + \int_0^t \bar{A}(s)(\tau_{Z_s}\xi).\, dB_s + \int_0^t \bar{L}(s)(\tau_{Z_s}\xi)\, ds.$$

Since $\{Y_t\}$ is a solution of equation (5.1), we also have a.s. $t \geq 0$

$$Y_t = \xi + \int_0^t \bar{A}(s)(Y_s).\, dB_s + \int_0^t \bar{L}(s)(Y_s)\, ds.$$

Define localizing sequence $\{\eta_n\}$ as

$$\eta_n := \inf\{t \geq 0 : |\langle \sigma_{ij}\, , \, Y_t\rangle| \geq n, \text{ or} \, |\langle b_i\, , \, Y_t\rangle| \geq n,\, i,j = 1,\cdots,d\},\, n \geq 1.$$

Now define $X_t^{(n)} := Y_{t\wedge\eta_n} - \tau_{Z_{t\wedge\eta_n}}\xi$. Then applying Itô formula on $\|\cdot\|_{p-1}^2$ (see Proposition 2.7.20, also see Section 2.12 Item (vi)) we get a.s.

$$\begin{aligned}\|X_t^{(n)}\|_{p-1}^2 &= \int_0^{t\wedge\eta_n} 2\sum_{i=1}^d \left\langle X_s^{(n)}\, , \, \bar{A}_i(s)X_s^{(n)}\right\rangle_{p-1} dB_s^{(i)} \\ &+ \int_0^{t\wedge\eta_n} \left[2\left\langle X_s^{(n)}\, , \, \bar{L}(s)X_s^{(n)}\right\rangle_{p-1} + \|\bar{A}(s)X_s^{(n)}\|_{HS(p-1)}^2\right] ds\end{aligned}$$

where $\{\int_0^t 2\sum_{i=1}^d \left\langle X_s^{(n)}\, , \, \bar{A}_i(s)X_s^{(n)}\right\rangle_{p-1} dB_s^{(i)}\}$ is a continuous local martingale. If for some ω, $|\langle \sigma_{ij}\, , \, Y_0(\omega)\rangle| \geq n$ or $|\langle b_i\, , \, Y_0(\omega)\rangle| \geq n$, then $\eta_n(\omega) = 0$ and such ω does not contribute to the right hand side of the above equation. Hence without loss of generality, we assume that the coefficients $\{\langle \sigma_{ij}\, , \, Y_t\rangle\}$ and $\{\langle b_i\, , \, Y_t\rangle\}$ are uniformly bounded. This in turn implies that the process $\{\int_0^{t\wedge\eta_n} 2\sum_{i=1}^d \left\langle X_s^{(n)}\, , \, \bar{A}_i(s)X_s^{(n)}\right\rangle_{p-1} dB_s^{(i)}\}$ is a continuous martingale. Since the coefficients are bounded, by the Monotonicity inequality (see [39, Theorem 2.1], Theorem 3.3.1 and Remark 3.3.2), there exists a constant $C_n > 0$ such that a.s.

$$\begin{aligned}\|X_{t\wedge\eta_n}^{(n)}\|_{p-1}^2 &\leq \int_0^{t\wedge\eta_n} 2\sum_{i=1}^d \left\langle X_s^{(n)}\, , \, \bar{A}_i(s)X_s^{(n)}\right\rangle_{p-1} dB_s^{(i)} \\ &+ C_n \int_0^{t\wedge\eta_n} \|X_{(n)}^n\|_{p-1}^2\, ds \\ &\leq \int_0^{t\wedge\eta_n} 2\sum_{i=1}^d \left\langle X_s^{(n)}\, , \, \bar{A}_i(s)X_s^{(n)}\right\rangle_{p-1} dB_s^{(i)} \\ &+ C_n \int_0^t \|X_{s\wedge\eta_n}^{(n)}\|_{p-1}^2\, ds.\end{aligned}$$

Taking expectation, we obtain $\mathbb{E}\|X_{t\wedge\eta_n}^{(n)}\|_{p-1}^2 \leq C_n \int_0^t \mathbb{E}\|X_{s\wedge\eta_n}^{(n)}\|_{p-1}^2\, ds$ for all $t \geq 0$. By the Gronwall's inequality (Lemma 2.13.1) we get $\mathbb{E}\|X_{t\wedge\eta_n}^{(n)}\|_{p-1}^2 = 0$ which implies the equality a.s. $Y_{t\wedge\eta_n} = \tau_{Z_{t\wedge\eta_n}}\xi, t \geq 0$. Since $\eta_n \uparrow \infty$, we have a.s. $Y_t = \tau_{Z_t}\xi, t \geq 0$. This implies a.s. $t \geq 0$

$$Z_t = \int_0^t \langle \sigma\, , \, Y_s\rangle\; dB_s + \int_0^t \langle b\, , \, Y_s\rangle\; ds$$

$$= \int_0^t \langle \sigma\,,\, \tau_{Z_s}\xi\rangle\; dB_s + \int_0^t \langle b\,,\, \tau_{Z_s}\xi\rangle\; ds$$
$$= \int_0^t \bar{\sigma}(Z_s;\xi).\, dB_s + \int_0^t \bar{b}(Z_s;\xi)\, ds$$

This completes the proof. □

Proof of Theorem 5.2.15. The proof is similar to that of [90, Theorem 3.4]. By Theorem 5.2.4, we have a solution $\{Z_t\}$ of (5.11) with initial condition $Z_0 = 0$. Then using the Itô formula in Theorem 5.2.2 and separating the dB and dt terms, leads to the stochastic partial differential equation (5.1), which shows $\{\tau_{Z_t}\xi\}$ is a solution.
To prove the uniqueness, let $\{Y^1_t\}, \{Y^2_t\}$ be two solutions. Then define $\{Z^1_t\}$ and $\{Z^2_t\}$ corresponding to $\{Y^1_t\}, \{Y^2_t\}$ as in Lemma 5.2.16. Then $\{Z^1_t\}, \{Z^2_t\}$ both solve (5.11) with initial condition 0. Now the uniqueness part in Theorem 5.2.4 implies a.s. $Z^1_t = Z^2_t$ for all $t \geq 0$ and hence a.s. $Y^1_t = Y^2_t$ for all $t \geq 0$. This completes the proof. □

Since $Y_t = \tau_{Z_t}\xi$ solves equation (5.1) (notations as in Theorem 5.2.15), we have $\mathbb{E}\|Y_0\|_p^2 = \mathbb{E}\|\xi\|_p^2 < \infty$. Now we prove $\mathcal{L}^2$ estimates on Y_t using two different techniques.

Proposition 5.2.17. *There exists a localizing sequence $\{\eta_n\}$ such that*

$$\mathbb{E} \sup_{t\geq 0} \|Y_t^{\eta_n}\|_p^2 \leq C_n.\mathbb{E}\|Y_0\|_p^2,$$

where the constant C_n depends only on n.

Proof. The process $\{Z_t\}$, defined in Lemma 5.2.16 is a continuous adapted process and $Z_0 = 0$. Define a localizing sequence $\{\eta_n\}$ as follows: $\eta_n := \inf\{t \geq 0 : |Z_t| \geq n\}$, $n \geq 1$. Now using Lemma 2.11.7(i) there exists a polynomial Q of degree $2([|p|] + 1)$ such that

$$\|Y_t^{\eta_n}\|_p \leq \|\xi\|_p.Q(|Z_t^{\eta_n}|) \leq \|\xi\|_p \sup_{\{x:|x|\leq n\}} Q(|x|).$$

Hence $\sup_{t\geq 0} \|Y_t^{\eta_n}\|_p^2 \leq C_n\, \|\xi\|_p^2$ with $C_n = (\sup_{\{x:|x|\leq n\}} Q(|x|))^2$. This implies the required estimate. □

using line desired

Following [38, Lemma 1], we get the next estimate.

Proposition 5.2.18. *There exists a localizing sequence $\{\eta_n\}$ such that for any positive real number T,*

$$\mathbb{E} \sup_{t\leq T} \|Y_t^{\eta_n}\|_{p-1}^2 \leq C.\mathbb{E}\|Y_0\|_{p-1}^2,$$

where the constant C depends only on n and T.

Proof. Define three localizing sequences. For any positive integer n, consider

$$\bar{\eta}_n := \inf\{t \geq 0 : \|Y_t - Y_0\|_p \geq n\},$$

and

$$\eta'_n := \inf\{t \geq 0 : |\langle \sigma , Y_t \rangle| \geq n, \text{ or } |\langle b , Y_t \rangle| \geq n\},$$

and $\eta_n := \bar{\eta}_n \wedge \eta'_n$. Note that $\|Y_t^{\eta_n}\|_p \leq \|Y_t^{\eta_n} - Y_0\|_p + \|Y_0\|_p$ and using this inequality it is easy to see that $\mathbb{E}\sup_{t\leq T} \|Y_t^{\eta_n}\|_{p-1}^2 < \infty$ for any $T > 0$. Since the following equality holds in $\mathcal{S}_{p-1}(\mathbb{R}^d)$, a.s.

$$Y_t = Y_0 + \int_0^t A(Y_s).dB_s + \int_0^t L(Y_s)\, ds,\ t \geq 0.$$

Now using Itô formula for $\|\cdot\|_{p-1}^2$ (see Proposition 2.7.20, also see Section 2.12 Item (vi)) we obtain a.s. $t \geq 0$

$$\begin{aligned}\|Y_t^{\eta_n}\|_{p-1}^2 = \|Y_0\|_{p-1}^2 &+ \int_0^{t\wedge\eta_n} 2\sum_{i=1}^d \langle Y_s^{\eta_n} , A_i Y_s^{\eta_n}\rangle_{p-1}\, dB_s^{(i)} \\ &+ \int_0^{t\wedge\eta_n} \left[2\,\langle Y_s^{\eta_n} , LY_s^{\eta_n}\rangle_{p-1} + \sum_{i=1}^d \|A_i Y_s^{\eta_n}\|_{p-1}^2\right] ds\end{aligned} \tag{5.27}$$

where $B_t^{(i)}$ denotes the i-th component of B_t. Since the coefficients $\{\langle \sigma_{ij} , Y_t\rangle\}$ and $\{\langle b_i , Y_t\rangle\}$ are uniformly bounded, $\{\int_0^{t\wedge\eta_n} 2\sum_{i=1}^d \langle Y_s^{\eta_n} , A_i Y_s^{\eta_n}\rangle_{p-1}\, dB_s^{(i)}\}$ is a continuous martingale.

If for some ω, $|\langle \sigma , Y_0\rangle|(\omega) \geq n$ or $|\langle b , Y_0\rangle|(\omega) \geq n$ then $\eta_n(\omega) = 0$. But such ω does not contribute to $\int_0^{t\wedge\eta_n} \left[2\,\langle Y_s^{\eta_n} , LY_s^{\eta_n}\rangle_{p-1} + \sum_{i=1}^d \|A_i Y_s^{\eta_n}\|_{p-1}^2\right] ds$ and hence in computing this expectation we may assume $\{|\langle \sigma , Y_t^{\eta_n}\rangle|\}$ and $\{|\langle b , Y_t^{\eta_n}\rangle|\}$ are uniformly bounded by n. Then using the Monotonicity inequality (Theorem 3.3.1 and Remark 3.3.2) and taking expectation in the previous equation yields

$$\mathbb{E}\|Y_t^{\eta_n}\|_{p-1}^2 \leq \mathbb{E}\|Y_0\|_{p-1}^2 + \gamma \int_0^t \mathbb{E}\|Y_s^{\eta_n}\|_{p-1}^2\, ds$$

where the constant γ depends only on η_n. Then Gronwall's inequality implies

$$\mathbb{E}\|Y_t^{\eta_n}\|_{p-1}^2 \leq e^{\gamma t}.\mathbb{E}\|Y_0\|_{p-1}^2,\ t \geq 0. \tag{5.28}$$

Let $\{M_t\}$ and $\{V_t\}$ respectively denote the martingale term and the finite variation term on the right hand side of (5.27). Then using the Monotonicity inequality and (5.28), we get

$$\mathbb{E}\sup_{t\leq T} V_t \leq \gamma\, \mathbb{E}\sup_{t\leq T} \int_0^t \|Y_s^{\eta_n}\|_{p-1}^2\, ds = \gamma \int_0^T \mathbb{E}\|Y_s^{\eta_n}\|_{p-1}^2\, ds \leq \widetilde{C}\, \mathbb{E}\|Y_0\|_{p-1}^2 \tag{5.29}$$

for some constant $\widetilde{C}$ depending only on η_n and T.
By Theorem 3.2.2 for each $1 \leq i \leq d$, there exists a bounded operator $\mathbb{T}_i : \mathcal{S}_{p-1}(\mathbb{R}^d) \to \mathcal{S}_{p-1}(\mathbb{R}^d)$ such that

$$\begin{aligned}
2\left\langle Y_t^{\eta_n}\,,\,A_i Y_t^{\eta_n}\right\rangle_{p-1} &= -2\sum_{j=1}^{d}\left\langle \sigma_{ji}\,,\,Y_t^{\eta_n}\right\rangle \left\langle Y_t^{\eta_n}\,,\,\partial_j Y_t^{\eta_n}\right\rangle_{p-1} \\
&= -\sum_{j=1}^{d}\left\langle \sigma_{ji}\,,\,Y_t^{\eta_n}\right\rangle \left\langle Y_t^{\eta_n}\,,\,\mathbb{T}_j Y_t^{\eta_n}\right\rangle_{p-1}.
\end{aligned}$$

Since $\{|\langle \sigma\,,\,Y_t^{\eta_n}\rangle|\}$ is uniformly bounded by n,

$$\begin{aligned}
|2\left\langle Y_t^{\eta_n}\,,\,A_i Y_t^{\eta_n}\right\rangle_{p-1}| &\leq \sum_{j=1}^{d}|\langle \sigma_{ji}\,,\,Y_t^{\eta_n}\rangle|\,|\left\langle Y_t^{\eta_n}\,,\,\mathbb{T}_j Y_t^{\eta_n}\right\rangle_{p-1}| \\
&\leq n\sum_{j=1}^{d}|\left\langle Y_t^{\eta_n}\,,\,\mathbb{T}_j Y_t^{\eta_n}\right\rangle_{p-1}| \\
&\leq \beta\|Y_t^{\eta_n}\|_{p-1}^2
\end{aligned} \tag{5.30}$$

where $\beta = nd\,\max\{\|\mathbb{T}_j\|_{\mathcal{S}_{p-1}(\mathbb{R}^d)\to\mathcal{S}_{p-1}(\mathbb{R}^d)} \mid 1 \leq j \leq d\}$.
To estimate the martingale term, we use the BDG inequalities (see Theorem 2.5.28). Note that in the following inequalities the constant C may change values from line to line, but it depends only on η_n and T.

$$\begin{aligned}
\mathbb{E}\sup_{t\leq T}|M_t| &\leq C.\mathbb{E}\,[M]_T^{\frac{1}{2}} \\
&= C.\mathbb{E}\left(\int_0^{T\wedge\eta_n} 4\sum_{i=1}^{d}\left\langle Y_s^{\eta_n}\,,\,A_i Y_s^{\eta_n}\right\rangle_{p-\frac{3}{2}}^2\,ds\right)^{\frac{1}{2}} \\
&\leq C.\mathbb{E}\left(\int_0^{T}\|Y_s^{\eta_n}\|_{p-1}^4\,ds\right)^{\frac{1}{2}},\ (\text{using } (5.30)) \\
&\leq C.\mathbb{E}\left(\sup_{t\leq T}\|Y_t^{\eta_n}\|_{p-1}^2\int_0^{T}\|Y_s^{\eta_n}\|_{p-1}^2\,ds\right)^{\frac{1}{2}}
\end{aligned}$$

using A.M - G.M inequality,

$$\begin{aligned}
&\leq \frac{C}{2}.\mathbb{E}\left(\epsilon\sup_{t\leq T}\|Y_t^{\eta_n}\|_{p-1}^2 + \frac{1}{\epsilon}\int_0^{T}\|Y_s^{\eta_n}\|_{p-1}^2\,ds\right),\ (\text{for any } \epsilon > 0) \\
&\leq C\epsilon.\mathbb{E}\sup_{t\leq T}\|Y_t^{\eta_n}\|_{p-1}^2 + C.\mathbb{E}\|Y_0\|_{p-1}^2,\ (\text{using } (5.28))
\end{aligned}$$

For the choice $\epsilon = \frac{1}{2C}$ we get

$$\mathbb{E}\sup_{t\leq T}|M_t| \leq \frac{1}{2}\mathbb{E}\sup_{t\leq T}\|Y_t^{\eta_n}\|_{p-1}^2 + C.\mathbb{E}\|Y_0\|_{p-1}^2. \tag{5.31}$$

Using (5.27), (5.29) and (5.31) we get the desired estimate. □

Remark 5.2.19. If we repeat the steps of [38, Lemma 1], then in the previous proposition we would end up with

$$\mathbb{E}\sup_{t\leq T}\|Y_t^{\eta_n}\|_{p-1}^2 \leq C.\mathbb{E}\|Y_0\|_{p-\frac{3}{2}}^2.$$

Because of Theorem 3.2.2, we are getting a better estimate.

The counterpart of Theorem 5.2.15 involving locally Lipschitz coefficients is as follows. This result is an extension of [90, Theorem 3.4].

Theorem 5.2.20. *Suppose the following conditions are satisfied.*

(i) $\mathbb{E}\|\xi\|_p^2 < \infty$.

(ii) (Locally Lipschitz in x, locally in y) for any fixed $y \in \mathcal{S}_p(\mathbb{R}^d)$ *the functions* $x \mapsto \bar{\sigma}(x;y)$ *and* $x \mapsto \bar{b}(x;y)$ *are locally Lipschitz functions in* x *and the Lipschitz coefficient is independent of* y *when* y *varies over any bounded set* G *in* $\mathcal{S}_p(\mathbb{R}^d)$*; i.e. for any bounded set* G *in* $\mathcal{S}_p(\mathbb{R}^d)$ *and any positive integer* n *there exists a constant* $C(G,n) > 0$ *such that for all* $x_1, x_2 \in B(0,n), y \in G$

$$|\bar{\sigma}(x_1;y) - \bar{\sigma}(x_2;y)| + |\bar{b}(x_1;y) - \bar{b}(x_2;y)| \leq C(G,n)|x_1 - x_2|,$$

where $B(0,n) = \{x \in \mathbb{R}^d : |x| \leq n\}$.

Then an $(\mathcal{F}_t^\xi)$ *adapted continuous strong local solution of equation* (5.1) *exists. The solutions are also pathwise unique.*

Proof. We follow the arguments in the proof of Theorem 5.2.15 and indicate the necessary changes.
First we prove the uniqueness. Let $(\{Y_t^{(1)}\}, \eta^{(1)})$ and $(\{Y_t^{(2)}\}, \eta^{(2)})$ be two $(\mathcal{F}_t^\xi)$ adapted continuous strong local solutions of equation (5.1). Now define two processes $\{Z_t^{(i)}\}, i = 1, 2$ as follows:

$$Z_t^{(i)} := \int_0^{t\wedge\eta^i} \langle \sigma\, ,\, Y_s\rangle\; dB_s + \int_0^{t\wedge\eta^i} \langle b\, ,\, Y_s\rangle\; ,\; 0 \leq t < \eta^i$$

and set $Z_t^{(i)} := \infty$, if $t \geq \eta^i$. Then as in Lemma 5.2.16 (also see [90, Lemma 3.6]) we can show a.s. $Y_t^{(i)} = \tau_{Z_t^{(i)}}\xi$, $0 \leq t < \eta^i$. Then from the definition of $\{Z_t^{(i)}\}$ we have

$$Z_t^{(i)} = \int_0^t \bar{\sigma}(Z_s^{(i)};\xi).\, dB_s + \int_0^t \bar{b}(Z_s^{(i)};\xi)\, ds,\; 0 \leq t < \eta^i.$$

From the uniqueness obtained in Theorem 5.2.9 we conclude a.s. $Z_t^{(1)} = Z_t^{(2)}$, $0 \leq t < \eta^1 \wedge \eta^2$. Hence a.s. $Y_t^{(1)} = Y_t^{(2)}$, $0 \leq t < \eta^1 \wedge \eta^2$. This proves the pathwise uniqueness.

Now we prove the existence of a strong local solution of equation (5.1). By Theorem 5.2.9 equation (5.11) has a solution $(\{Z_t\}, \eta)$. Define the $\hat{\mathcal{S}}_p(\mathbb{R}^d)$ valued process $\{Y_t\}$ as follows:

$$Y_t := \begin{cases} \tau_{Z_t}\xi, \text{ if } 0 \leq t < \eta \\ \delta, \text{ if } t \geq \eta. \end{cases}$$

Then using the Itô formula in Theorem 5.2.2, the pair $(\{\tau_{Z_t}\xi\}, \eta)$ solves (5.11). Since $\{Z_t\}$ is $(\mathcal{F}_t^\xi)$ adapted and ξ is $\mathcal{F}_0^\xi$ measurable, we have $\{Y_t\}$ is also $(\mathcal{F}_t^\xi)$ adapted. Since $\{Z_t\}$ has continuous paths on the stochastic interval $[0, \eta)$, by $\{Y_t\}$ also has continuous paths on the same stochastic interval. □

Remark 5.2.21. We describe two possible extensions of the results of this section - which is a problem for the future. The description is in terms of properties of the coefficients $\bar{\sigma}, \bar{b}$.

(i) We would like to extend the results when there is time inhomogeneity in $\bar{\sigma}, \bar{b}$.

(ii) In our case, $x \mapsto \bar{\sigma}(x; y) = \langle \sigma\, , \, \tau_x y \rangle\, , x \mapsto \bar{b}(x; y) = \langle b\, , \, \tau_x y \rangle$ are non-linear in x for all fixed $y \in \mathcal{S}_p(\mathbb{R}^d)$. We would like to extend the results to more general class of non-linear coefficients.

5.3 Stationary Solutions

We have presented some sufficient conditions under which the stochastic partial differential equation (5.1) has a unique strong solution. Now we investigate existence of stationary solutions. Our approach is to use stationary solutions, if any, of the finite dimensional stochastic differential equation (5.11). Assume that

(i) $f : \mathbb{R}^d \to \mathbb{R}^{d\times d}, g : \mathbb{R}^d \to \mathbb{R}^d$ are locally bounded, measurable functions such that the stochastic differential equation

$$dZ_t = f(Z_t)dB_t + g(Z_t)dt, \forall t \geq 0 \tag{5.32}$$

has a stationary, continuous solution and we denote the corresponding invariant measure by ν. Let $f_{ij}, g_i, 1 \leq i, j \leq d$ be the component functions of f, g.

(ii) σ_{ij}, b_i (for $i, j = 1, \cdots, d$) are tempered distributions given by functions.

Remark 5.3.1. Typically f, g will be locally Lipschitz functions such that explosions do not happen in finite time. When $f = Id$ (the $d \times d$ identity matrix), this non-explosion is guaranteed by a 'Liapunov' type criteria (see [105, 7.3.14 Corollary]).

Remark 5.3.2. Existence of invariant measures of Markov processes and finite dimensional diffusions has been studied by many authors (to cite only a few, see [11, 30, 42, 44, 45, 61], [105, Chapter VII, Section 5]).

Note that there exists a $p > 0$ such that $\sigma_{ij}, b_i \in \mathcal{S}_{-p}(\mathbb{R}^d)$ for all i, j. Fix such a $p > 0$ and consider the following subset of $\mathcal{S}_p(\mathbb{R}^d)$,

$$\begin{aligned}\mathcal{C} := \{\psi \in \mathcal{S}_p(\mathbb{R}^d) : &\int_{\mathbb{R}^d} \sigma_{ij}(y+x)\psi(y)\,dy = f_{ij}(x), \forall x \in \mathbb{R}^d;\\ &\int_{\mathbb{R}^d} b_i(y+x)\psi(y)\,dy = g_i(x), \forall x \in \mathbb{R}^d, i, j = 1, \cdots, d\}.\end{aligned} \tag{5.33}$$

Since $p > 0$, elements of $\mathcal{C}$ are given by functions. Note that $\mathcal{C}$ is a convex set.
The motivation behind above conditions requires clarification. First we want to choose a subset $\mathcal{C}$ of $\mathcal{S}_p(\mathbb{R}^d)$ such that the coefficients $\bar{\sigma}(x;\xi)$ and $\bar{b}(x;\xi)$ in the equation (5.5) remain the same, for $\xi \in \mathcal{C}$. This allows us to think of $\bar{\sigma}(x;\xi)$ and $\bar{b}(x;\xi)$ as just $\bar{\sigma}(x)$ and $\bar{b}(x)$. Second we want $\bar{\sigma} = f$ and $\bar{b} = g$ which is a choice that allows us to use the invariant measure ν of (5.32). The set $\mathcal{C}$ considered above provides exactly those conditions.

Lemma 5.3.3. *Let $\psi \in \mathcal{C}$. Then $\bar{\sigma}(x;\psi) = f(x)$ and $\bar{b}(x;\psi) = g(x)$ for all $x \in \mathbb{R}^d$.*

Proof. Observe that,

$$\langle b_i\,,\,\tau_x\psi\rangle = \langle \tau_{-x}b_i\,,\,\psi\rangle = \int_{\mathbb{R}^d} (\tau_{-x}b_i)(y)\psi(y)\,dy = \int_{\mathbb{R}^d} b_i(y+x)\psi(y)\,dy = g_i(x).$$

Proof of the other part is similar. □

We show the existence of a stationary solution of equation (5.1).

Theorem 5.3.4. *Let ξ be a $\mathcal{C}$-valued $\mathcal{F}_0$ measurable random variable with $\mathbb{E}\|\xi\|_p^2 < \infty$ and independent of $\{B_t\}$. Then $Y_t := \tau_{Z_t}\xi$ is a stationary process and solves*

$$dY_t = A(Y_t).\,dB_t + L(Y_t)\,dt; \quad Y_0 = \tau_{Z_0}\xi \tag{5.34}$$

where $\{Z_t\}$ is a stationary, continuous solution of (5.32).

Proof. We give the proof for dimension $d = 1$. The case $d > 1$ is similar.
Using the Itô formula in Theorem 5.2.2 we get a.s. for all $t \geq 0$,

$$\begin{aligned}\tau_{Z_t}\xi &= \tau_{Z_0}\xi - \int_0^t \partial(\tau_{Z_s}\xi)\,dZ_s + \frac{1}{2}\int_0^t \partial^2(\tau_{Z_s}\xi)\,d\,[Z]_s\\ &= \tau_{Z_0}\xi - \int_0^t \partial(\tau_{Z_s}\xi)f(Z_s)\,dB_s\\ &- \int_0^t \partial(\tau_{Z_s}\xi)g(Z_s)\,ds + \frac{1}{2}\int_0^t \partial^2(\tau_{Z_s}\xi)(f(Z_s))^2\,ds.\end{aligned}$$

Now we use Lemma 5.3.3. Then a.s. $t \geq 0$

$$\tau_{Z_t}\xi = \tau_{Z_0}\xi - \int_0^t \partial(\tau_{Z_s}\xi)\bar{\sigma}(Z_s;\xi)\,dB_s$$

$$- \int_0^t \partial(\tau_{Z_s}\xi)\bar{b}(Z_s;\xi)\, ds + \frac{1}{2}\int_0^t \partial^2(\tau_{Z_s}\xi)(\bar{\sigma}(Z_s;\xi))^2\, ds$$

So $Y_t = \tau_{Z_t}\xi$ solves (5.34).
Since $\{Z_t\}$ is a stationary solution to (5.32), for time points $s, t_1, t_2, \cdots, t_n \geq 0$ we have

$$(Z_{t_1}, Z_{t_2}, \cdots, Z_{t_n}) \stackrel{\mathcal{L}}{=} (Z_{s+t_1}, Z_{s+t_2}, \cdots, Z_{s+t_n}),$$

where $\stackrel{\mathcal{L}}{=}$ denotes equality in law.
Take $\psi \in \mathcal{C}$. Then the map $x \mapsto \tau_x\psi$ is continuous and hence measurable (see Lemma 2.11.7(ii)). Using this fact, for Borel sets $G_1, \cdots, G_n$ in $\mathcal{S}_p$, we have

$$\begin{aligned}&((\tau_{Z_{t_1}}\psi, \tau_{Z_{t_2}}\psi, \cdots, \tau_{Z_{t_n}}\psi) \in G_1 \times G_2 \times \cdots \times G_n)\\ &= ((Z_{t_1}, Z_{t_2} \cdots, Z_{t_n}) \in (\tau.\psi)^{-1}(G_1) \times (\tau.\psi)^{-1}(G_2) \times \cdots \times (\tau.\psi)^{-1}(G_n)).\end{aligned}$$

Now using the stationarity of $\{Z_t\}$ we have

$$\begin{aligned}&P((\tau_{Z_{t_1}}\psi, \tau_{Z_{t_2}}\psi, \cdots, \tau_{Z_{t_n}}\psi) \in G_1 \times G_2 \times \cdots \times G_n)\\ &= P((\tau_{Z_{s+t_1}}\psi, \tau_{Z_{s+t_2}}\psi, \cdots, \tau_{Z_{s+t_n}}\psi) \in G_1 \times G_2 \times \cdots \times G_n)\end{aligned} \tag{5.35}$$

Let μ_ξ denote the law of ξ on $\mathcal{S}_p$. Then using conditional probability, we have

$$\begin{aligned}&P((\tau_{Z_{t_1}}\xi, \tau_{Z_{t_2}}\xi, \cdots, \tau_{Z_{t_n}}\xi) \in G_1 \times G_2 \times \cdots \times G_n)\\ &= \int_{\mathcal{S}_p} P((\tau_{Z_{t_1}}\xi, \tau_{Z_{t_2}}\xi, \cdots, \tau_{Z_{t_n}}\xi) \in G_1 \times G_2 \times \cdots \times G_n | \xi = \psi)\, \mu_\xi(d\psi)\end{aligned}$$

since ξ is $\mathcal{C}$-valued,

$$\begin{aligned}&= \int_{\mathcal{C}} P((\tau_{Z_{t_1}}\xi, \tau_{Z_{t_2}}\xi, \cdots, \tau_{Z_{t_n}}\xi) \in G_1 \times G_2 \times \cdots \times G_n | \xi = \psi)\, \mu_\xi(d\psi)\\ &= \int_{\mathcal{C}} P((\tau_{Z_{t_1}}\psi, \tau_{Z_{t_2}}\psi, \cdots, \tau_{Z_{t_n}}\psi) \in G_1 \times G_2 \times \cdots \times G_n | \xi = \psi)\, \mu_\xi(d\psi)\end{aligned}$$

since $\{Z_t\}$ is independent of ξ,

$$= \int_{\mathcal{C}} P((\tau_{Z_{t_1}}\psi, \tau_{Z_{t_2}}\psi, \cdots, \tau_{Z_{t_n}}\psi) \in G_1 \times G_2 \times \cdots \times G_n)\, \mu_\xi(d\psi)$$

Similarly,

$$\begin{aligned}&P((\tau_{Z_{s+t_1}}\xi, \tau_{Z_{s+t_2}}\xi, \cdots, \tau_{Z_{s+t_n}}\xi) \in G_1 \times G_2 \times \cdots \times G_n)\\ &= \int_{\mathcal{C}} P((\tau_{Z_{s+t_1}}\psi, \tau_{Z_{s+t_2}}\psi, \cdots, \tau_{Z_{s+t_n}}\psi) \in G_1 \times G_2 \times \cdots \times G_n)\, \mu_\xi(d\psi).\end{aligned}$$

Using (5.35) we have

$$\begin{aligned}&P((\tau_{Z_{t_1}}\xi, \tau_{Z_{t_2}}\xi, \cdots, \tau_{Z_{t_n}}\xi) \in G_1 \times G_2 \times \cdots \times G_n)\\ &= P((\tau_{Z_{s+t_1}}\xi, \tau_{Z_{s+t_2}}\xi, \cdots, \tau_{Z_{s+t_n}}\xi) \in G_1 \times G_2 \times \cdots \times G_n)\end{aligned} \tag{5.36}$$

i.e.

$$(\tau_{Z_{t_1}}\xi, \tau_{Z_{t_2}}\xi, \cdots, \tau_{Z_{t_n}}\xi) \stackrel{\mathcal{L}}{=} (\tau_{Z_{s+t_1}}\xi, \tau_{Z_{s+t_2}}\xi, \cdots, \tau_{Z_{s+t_n}}\xi).$$

So Y_t is stationary. This completes the proof. □

Example 5.3.5. Take $d = 1, f(x) \equiv 1, g(x) = -x, \sigma = f, b = g$. Note that $\sigma \in \mathcal{S}_{-p}$ for $p > \frac{1}{4}$ and $b \in \mathcal{S}_{-p}$ for $p > \frac{3}{4}$ (see Example 2.11.17 and Example 2.11.18). Take $p > \frac{3}{4}$. First condition in the definition of set $\mathcal{C}$ (equation (5.33)) reduces to $\int_{\mathbb{R}} \psi(y)\, dy = 1$. In view of this relation, the left hand side of the second condition simplifies to

$$\int_{\mathbb{R}} -(y+x)\psi(y)\, dy = -\int_{\mathbb{R}} y\psi(y)\, dy - x\int_{\mathbb{R}} \psi(y)\, dy = -\int_{\mathbb{R}} y\psi(y)\, dy - x.$$

Hence the second condition can be written as $\int_{\mathbb{R}} y\psi(y)\, dy = 0$. Therefore

$$\mathcal{C} = \{\psi \in \mathcal{S}_p : \int_{\mathbb{R}} \psi = 1, \int_{\mathbb{R}} y\psi(y)\, dy = 0\}.$$

$\mathcal{C}$ is non empty since (centered) Gaussian densities satisfy such conditions. Consider the Ornstein-Uhlenbeck diffusion with the following initial condition:

$$dZ_t = dB_t - Z_t\, dt; \quad Z_0 \sim N\Big(0, \tfrac{1}{2}\Big), \tag{5.37}$$

where $N(0, \frac{1}{2})$ denotes the law of a Normal random variable with mean 0 and variance $\frac{1}{2}$. Recall that this gives the stationary solution (see [60, Chapter 5, 6.8 Example]). Theorem 5.3.4 asserts that $Y_t = \tau_{Z_t}\psi$ gives a stationary solution, when $\psi \in \mathcal{C}$. Note that there exist some constant $R > 0$ and a polynomial P of degree $2[|p|] + 1$ (see Lemma 2.11.7)

$$\mathbb{E}\|\tau_{Z_0}\psi\|_p^2 \le R\,\mathbb{E}\left(P(|Z_0|)\right)^2 \|\psi\|_p^2 < \infty,$$

since all absolute moments exist for Gaussian distribution.
More generally for constants $\sigma_0 > 0, \alpha_0 > 0$, we look at the stochastic differential equation

$$dZ_t = \sigma_0 dB_t - \alpha_0 Z_t\, dt; \quad Z_0 \sim N\Big(0, \tfrac{\sigma_0^2}{2\alpha_0}\Big).$$

Then for any $\psi \in \mathcal{C}$ ($\mathcal{C}$ as described above), we have the stationary solution $Y_t = \tau_{Z_t}\psi$. Note that the same subset $\mathcal{C}$ works irrespective of the constants σ_0, α_0.

The following lemma and Proposition 5.3.7 will be used in Example 5.3.8.

Lemma 5.3.6. *The tempered distribution given by the function $b(x) = x^3$, $x \in \mathbb{R}$ belongs to $\mathcal{S}_{-p}$ for $p > \frac{7}{4}$.*

Proof. We write $\mathscr{M}_x$ instead of $\mathscr{M}_1$ (see the multiplication operators defined in Example 2.11.9). Observe that $|\langle b\,,\,\phi\rangle| = |\langle 1\,,\,(\mathscr{M}_x)^3\phi\rangle|, \forall\phi\in\mathcal{S}$. Therefore

$$\begin{aligned}|\langle b\,,\,\phi\rangle| &\leq \|1\|_{-p}\|(\mathscr{M}_x)^3\phi\|_p\\ &\leq \|1\|_{-p}\|\|\mathscr{M}_x\|_{\mathcal{S}_{p+\frac{1}{2}}\to\mathcal{S}_p}\|\mathscr{M}_x\|_{\mathcal{S}_{p+1}\to\mathcal{S}_{p+\frac{1}{2}}}\|\mathscr{M}_x\|_{\mathcal{S}_{p+\frac{3}{2}}\to\mathcal{S}_{p+1}}\|\phi\|_{p+\frac{3}{2}}.\end{aligned}$$

Since $1\in\mathcal{S}_{-p}$ for $p>\frac{1}{4}$ (see Example 2.11.17), we have $b\in\mathcal{S}_{-p}$ for $p>\frac{1}{4}+\frac{3}{2}=\frac{7}{4}$. Similar computations were done in Example 2.11.18. □

We recall the set up from [105, Chapter 7 Section 3 and Section 5]. Let $g:\mathbb{R}^d\to\mathbb{R}^d$ be a continuously differentiable vector field and let $X^g:[0,\infty)\times C([0,\infty),\mathbb{R}^d)\to\mathbb{R}^d$ denote the solution of the integral equation

$$X^g(t,\omega) = \omega(t) + \int_0^t g(X^g(s,\omega))\,ds \tag{5.38}$$

up to the first time of explosion e^g (see [105, equation (7.3.4)]). Define $\Omega(g) := \{\omega\in C([0,\infty),\mathbb{R}^d) : e^g(\omega)=\infty\}$ and $L^g := \frac{1}{2}\triangle + g\cdot\nabla$. We cite a part of [105, 7.3.14 Corollary], which is of current interest.

Proposition 5.3.7 ([105, 7.3.14 Corollary])**.** *Suppose that $h\in C^2(\mathbb{R}^d,[0,\infty))$ has the properties that*

$$\lim_{|x|\to\infty} h(x) = \infty \quad \textit{and} \quad L^g h(x) \leq A + Bh(x), x\in\mathbb{R}^d,$$

for some pair $A,B\in[0,\infty)$. Then

$$\mathcal{W}_x^{(d)}(\Omega(g)) = 1$$

for all $x\in\mathbb{R}^d$, where $\mathcal{W}_x^{(d)}$ denotes the distribution of $\omega\in C([0,\infty),\mathbb{R}^d)\mapsto x+\omega\in C([0,\infty),\mathbb{R}^d)$ under $\mathcal{W}^{(d)}$ - the Wiener measure on $C([0,\infty),\mathbb{R}^d)$.

Let $U\in C^2(\mathbb{R}^d,\mathbb{R})$ be such that $\int_{\mathbb{R}^d}\exp(-2U(x))\,dx = 1$. Suppose that $g=-\nabla U$ and $\mathcal{W}_x^{(d)}(\Omega(g)) = 1, \forall x\in\mathbb{R}^d$. Let $\mathcal{B}(\mathbb{R}^d)$ denote the Borel σ-field on $\mathbb{R}^d$. Then the measure ν on $\mathbb{R}^d$ given by (see [105, equation (7.5.3)])

$$\nu(B) := \int_B \exp\left(-2U(x)\right)\,dx, \forall B\in\mathcal{B}(\mathbb{R}^d), \tag{5.39}$$

is invariant for the equation (5.38) (see [105, 7.5.18 Theorem]).

Example 5.3.8. Take $d = 1, f(x) \equiv 1, g(x) = -x^3, \sigma = f, b = g$. Note that $\sigma \in \mathcal{S}_{-p}$ for $p > \frac{1}{4}$ and $b \in \mathcal{S}_{-p}$ for $p > \frac{7}{4}$ (see Example 2.11.17 and Lemma 5.3.6). Take $p > \frac{7}{4}$. As in Example 5.3.5, the first condition in the definition of set $\mathcal{C}$ (equation (5.33)) becomes $\int_{\mathbb{R}} \psi(y)\, dy = 1$. We now simplify the second condition.

$$
\begin{aligned}
&\int_{\mathbb{R}} -(y+x)^3\psi(y)\, dy = -x^3, \ \forall x \in \mathbb{R}.\\
\iff & -\int_{\mathbb{R}} y^3\psi(y)\, dy - 3x\int_{\mathbb{R}} y^2\psi(y)\, dy - 3x^2\int_{\mathbb{R}} y\psi(y)\, dy - x^3\int_{\mathbb{R}} \psi(y)\, dy = -x^3, \ \forall x \in \mathbb{R}.\\
\iff & \int_{\mathbb{R}} y^3\psi(y)\, dy + 3x\int_{\mathbb{R}} y^2\psi(y)\, dy + 3x^2\int_{\mathbb{R}} y\psi(y)\, dy = 0, \ \forall x \in \mathbb{R}.\\
&\iff \int_{\mathbb{R}} y^3\psi(y)\, dy = \int_{\mathbb{R}} y^2\psi(y)\, dy = \int_{\mathbb{R}} y\psi(y)\, dy = 0
\end{aligned}
$$

Therefore

$$\mathcal{C} = \{\psi \in \mathcal{S}_p : \int_{\mathbb{R}} \psi = 1, \int_{\mathbb{R}} y\psi(y)\, dy = \int_{\mathbb{R}} y^2\psi(y)\, dy = \int_{\mathbb{R}} y^3\psi(y)\, dy = 0\}.$$

We show $\mathcal{C}$ is non empty since $\psi_1, \psi_2 \in \mathcal{C}$ where

$$\psi_1(y) := \exp(-y^2)\left[\frac{3}{2\sqrt{\pi}} - \frac{1}{\sqrt{\pi}}y^2\right], \psi_2(y) := \exp\left(-\frac{y^2}{2}\right)\left[\frac{3}{2\sqrt{2\pi}} - \frac{1}{2\sqrt{2\pi}}y^2\right].$$

Note that $\psi_1, \psi_2 \in \mathcal{S} \subset \mathcal{S}_p$. We need to compute certain integrals to verify other conditions. The moments of standard Normal distribution are given as follows: for positive integers n ([12, Example 21.1]),

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y^n \exp\left(-\frac{y^2}{2}\right)\, dx = \begin{cases} 0, & \text{if}\, n \,\text{is odd},\\ (n-1)!!, & \text{if}\, n \,\text{is even}, \end{cases}$$

where $(2k-1)!! = 1 \times 3 \times \cdots \times (2k-1)$ for positive integers k. More generally, for any $\sigma > 0$ we have

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} y^n \exp\left(-\frac{y^2}{2\sigma^2}\right)\, dx = \begin{cases} 0, & \text{if}\, n \,\text{is odd},\\ (n-1)!!\sigma^n, & \text{if}\, n \,\text{is even}. \end{cases}$$

Then we can compute the integrals, corresponding to $\sigma = 1$ and $\frac{1}{\sqrt{2}}$:

$$\int_{-\infty}^{\infty} y \exp\left(-\frac{y^2}{2}\right)\, dx = \int_{-\infty}^{\infty} y^3 \exp\left(-\frac{y^2}{2}\right)\, dx = 0,$$

$$\int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2}\right)\, dx = \sqrt{2\pi}, \int_{-\infty}^{\infty} y^2 \exp\left(-\frac{y^2}{2}\right)\, dx = \sqrt{2\pi}, \int_{-\infty}^{\infty} y^4 \exp\left(-\frac{y^2}{2}\right)\, dx = 3\sqrt{2\pi},$$

$$\int_{-\infty}^{\infty} \exp(-y^2)\, dx = \sqrt{\pi}, \int_{-\infty}^{\infty} y \exp(-y^2)\, dx = \int_{-\infty}^{\infty} y^3 \exp(-y^2)\, dx = 0,$$

$$\int_{-\infty}^{\infty} y^2 \exp(-y^2)\, dx = \sqrt{2\pi} \left(\frac{1}{\sqrt{2}}\right)^3 = \frac{\sqrt{\pi}}{2}, \int_{-\infty}^{\infty} y^4 \exp(-y^2)\, dx = 3\sqrt{2\pi} \left(\frac{1}{\sqrt{2}}\right)^5 = \frac{3\sqrt{\pi}}{4}.$$

Using these values, we have

$$\int_{-\infty}^{\infty} \psi_1(y)\, dy = \frac{3}{2\sqrt{\pi}}\sqrt{\pi} - \frac{1}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} = 1, \int_{-\infty}^{\infty} y^2\psi_1(y)\, dy = \frac{3}{2\sqrt{\pi}}\frac{\sqrt{\pi}}{2} - \frac{1}{\sqrt{\pi}}\frac{3\sqrt{\pi}}{4} = 0,$$

$$\int_{-\infty}^{\infty} \psi_2(y)\, dy = \frac{3\sqrt{2\pi}}{2\sqrt{2\pi}} - \frac{\sqrt{2\pi}}{2\sqrt{2\pi}} = 1, \int_{-\infty}^{\infty} y^2\psi_2(y)\, dy = \frac{3}{2\sqrt{2\pi}}\sqrt{2\pi} - \frac{1}{2\sqrt{2\pi}}3\sqrt{2\pi} = 0.$$

Other integrals, viz. $\int_{-\infty}^{\infty} y\psi_1(y)\, dy, \int_{-\infty}^{\infty} y^3\psi_1(y)\, dy, \int_{-\infty}^{\infty} y\psi_2(y)\, dy, \int_{-\infty}^{\infty} y^3\psi_2(y)\, dy$ vanish since the integrands are odd functions. This proves $\psi_1, \psi_2 \in \mathcal{C}$. Using Proposition 5.3.7, we now show that the finite dimensional diffusion (5.32) does not explode in finite time. Consider the function $h(x) = x^2, x \in \mathbb{R}$. Then $\lim_{|x|\to\infty} h(x) = \infty$ and

$$\frac{1}{2}h''(x) + g(x)h'(x) = 1 - 2x^4 \leq 1 + 0.h(x),\ \forall x \in \mathbb{R},$$

i.e. the condition is satisfied with the constants $A = 1, B = 0$. The finite dimensional diffusion (5.32) has an invariant measure ν, given by (putting $U(x) = \frac{x^4}{4}$ in equation (5.39))

$$\nu(B) := c \int_B \exp\left(-\frac{x^4}{2}\right) dx$$

for any Borel set B in $\mathbb{R}$, where c is the normalization constant. Note that

$$\begin{aligned}\int_{-\infty}^{\infty} \exp\left(-\frac{x^4}{2}\right) dx = 2\int_0^{\infty} \exp\left(-\frac{x^4}{2}\right) dx &\overset{r=\frac{x^4}{2}}{=\!=} 2\int_0^{\infty} \exp(-r)2^{-\frac{7}{4}}r^{-\frac{3}{4}}\, dr\\ = 2^{-\frac{3}{4}} \int_0^{\infty} \exp(-r)r^{\frac{1}{4}-1}\, dr = 2^{-\frac{3}{4}}\Gamma\left(\frac{1}{4}\right).\end{aligned}$$

Hence $c = 2^{\frac{3}{4}} \left(\Gamma(\frac{1}{4})\right)^{-1}$. Theorem 5.3.4 asserts that $Y_t = \tau_{Z_t}\psi$ gives a stationary solution, when $\psi \in \mathcal{C}$.

We now prove an $\mathcal{L}^1$ estimate of a stationary solution $\{Y_t\}$ in terms of Y_0.

Proposition 5.3.9. *Let $\xi, \{Z_t\}, \{Y_t\}$ be as in Theorem 5.3.4. In addition assume that ξ is norm-bounded, Z_0 has moments of orders up to $4([|p|] + 1)$ and f, g are Lipschitz continuous. Then*

(a) $\mathbb{E}\|Y_0\|_p^2 = \mathbb{E}\|\tau_{Z_0}\xi\|_p^2 < \infty.$

(b) $\mathbb{E}\sup_{t\leq T}\|Y_t\|_p \leq C\,(\mathbb{E}\|Y_0\|_p^2)^{\frac{1}{2}}$, *where* C *is a positive constant depending only on* f, g *and* T.

Proof. For any norm-bounded $\mathcal{C}$ valued random variable ξ, we have $\mathbb{E}\|\tau_{Z_0}\xi\|_p^2 \leq R\,\mathbb{E}P(|Z_0|)$ where $R > 0$ and P is a polynomial of degree $4([|p|]+1)$ (see Lemma 2.11.7). Then by our assumption, $\mathbb{E}\|\tau_{Z_0}\xi\|_p^2 < \infty$.
Observe that $Y_t = \tau_{Z_t}\xi = \tau_{Z_t - Z_0}\tau_{Z_0}\xi = \tau_{Z_t-Z_0}Y_0$. Using Lemma 2.11.7(i) we have

$$\|Y_t\|_p \leq \|Y_0\|_p\, P_k(|Z_t - Z_0|),$$

where P_k is a real polynomial of degree $k = 2([|p|]+1)$. Without loss of generality, we assume that P_k has non-negative coefficients. We use the following estimate to establish the result.

$$\mathbb{E}\sup_{t\leq T}\|Y_t\|_p \leq (\mathbb{E}\|Y_0\|_p^2)^{\frac{1}{2}}\,(\mathbb{E}\sup_{t\leq T}P_k(|Z_t - Z_0|)^2)^{\frac{1}{2}} \tag{5.40}$$

Now a.s. $Z_t - Z_0 = \int_0^t f(Z_s)\,dB_s + \int_0^t g(Z_s)\,ds$, $t \geq 0$. Hence for any positive integer m,

$$\begin{aligned}
|Z_t - Z_0|^m &= \left|\int_0^t f(Z_s)\,dB_s + \int_0^t g(Z_s)\,ds\right|^m \\
&\leq \left(\left|\int_0^t f(Z_s)\,dB_s\right| + \left|\int_0^t g(Z_s)\,ds\right|\right)^m \\
&\leq \left(\left|\int_0^t f(Z_s)\,dB_s\right| + \int_0^t |g(Z_s)|\,ds\right)^m \\
&\leq 2^{m-1}\left[\left|\int_0^t f(Z_s)\,dB_s\right|^m + \left(\int_0^t |g(Z_s)|\,ds\right)^m\right].
\end{aligned}$$

The last inequality follows from Lemma 2.13.2. Continuing from above

$$\sup_{t\leq T}|Z_t - Z_0|^m \leq 2^{m-1}\left[\sup_{t\leq T}\left|\int_0^t f(Z_s)\,dB_s\right|^m + \sup_{t\leq T}\left(\int_0^t |g(Z_s)|\,ds\right)^m\right].$$

Since f, g are Lipschitz continuous, there exist constants $\alpha, \beta > 0$ such that

$$|f(x)| \leq \alpha(1+|x|),\ |g(x)| \leq \beta(1+|x|),\ \forall x \in \mathbb{R}^d.$$

Set $C_m := \mathbb{E}((1+|Z_0|)^m)$ for integers $0 < m \leq 2k$. By the stationarity of $\{Z_t\}$, $C_m = \mathbb{E}((1+|Z_t|)^m)$ for any $t \geq 0$. Now using Jensen's inequality, for any integer $0 < m \leq 2k$

$$\begin{aligned}
\mathbb{E}\sup_{t\leq T}\left(\int_0^t |g(Z_s)|\,ds\right)^m &\leq \mathbb{E}\sup_{t\leq T} t^{m-1}\int_0^t |g(Z_s)|^m\,ds \\
&= \mathbb{E}\,T^{m-1}\int_0^T |g(Z_s)|^m\,ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \beta^m T^{m-1} \int_0^T \mathbb{E}((1+|Z_s|)^m)\, ds \\
&= \beta^m T^{m-1} \int_0^T C_m\, ds \\
&= \beta^m T^m C_m.
\end{aligned}
$$

Note that

$$
\left|\int_0^t f(Z_s)\, dB_s\right| \leq \sum_{i=1}^d \left|\sum_{j=1}^d \int_0^t f_{ij}(Z_s)\, dB_s^j\right| \leq \sum_{i,j=1}^d \left|\int_0^t f_{ij}(Z_s)\, dB_s^j\right|.
$$

Then for any integer $0 < m \leq 2k$, by Lemma 2.13.2

$$
\begin{aligned}
\left(\sup_{t\leq T}\left|\int_0^t f(Z_s)\, dB_s\right|\right)^m &\leq \left(\sum_{i,j=1}^d \sup_{t\leq T}\left|\int_0^t f_{ij}(Z_s)\, dB_s^j\right|\right)^m \\
&\leq (d^2)^{m-1} \sum_{i,j=1}^d \left(\sup_{t\leq T}\left|\int_0^t f_{ij}(Z_s)\, dB_s^j\right|\right)^m
\end{aligned}
$$

Then using BDG inequalities (see Theorem 2.5.28) there exist a suitable constant $\gamma > 0$, such that for any integer $0 < m \leq 2k$

$$
\begin{aligned}
\mathbb{E}\left(\sup_{t\leq T}\left|\int_0^t f(Z_s)\, dB_s\right|\right)^m &\leq (d^2)^{m-1} \sum_{i,j=1}^d \mathbb{E}\left(\sup_{t\leq T}\left|\int_0^t f_{ij}(Z_s)\, dB_s^j\right|\right)^m \\
&\leq (d^2)^{m-1}\gamma \sum_{i,j=1}^d \mathbb{E}\left(\int_0^T |f_{ij}(Z_s)|^2\, ds\right)^{\frac{m}{2}} \\
&\leq (d^2)^{m-1}\gamma \sum_{i,j=1}^d \mathbb{E}\left(\int_0^T |f(Z_s)|^2\, ds\right)^{\frac{m}{2}} \\
&\leq d^{2m}\gamma\, \mathbb{E}\left(\int_0^T |f(Z_s)|^2\, ds\right)^{\frac{m}{2}} \\
&\leq d^{2m}\gamma\alpha^m\, \mathbb{E}\left(\int_0^T (1+|Z_s|)^2\, ds\right)^{\frac{m}{2}} \\
&\leq d^{2m}\gamma\alpha^m T^{m-1}\, \mathbb{E}\int_0^T (1+|Z_s|)^m\, ds \\
&= d^{2m}\gamma\alpha^m T^{m-1} \int_0^T \mathbb{E}(1+|Z_0|)^m\, ds \\
&= d^{2m}\gamma\alpha^m T^m C_m
\end{aligned}
$$

From the above estimates, for any integer $0 < m \leq 2k$

$$
\mathbb{E}\sup_{t\leq T}|Z_t - Z_0|^m \leq 2^{m-1} T^m C_m (d^{2m}\gamma\alpha^m + \beta^m).
$$

Now P_k^2 has the form $(P_k(x))^2 = \sum_{m=0}^{2k} a_m x^m$ with $a_m \geq 0, \forall m$. Then

$$\begin{aligned}\mathbb{E} \sup_{t \leq T} P_k(|Z_t - Z_0|)^2 &\leq \sum_{m=0}^{2k} a_m \, \mathbb{E} \sup_{t \leq T} |Z_t - Z_0|^m \\ &\leq a_0 + \sum_{m=1}^{2k} a_m \, 2^{m-1} T^m C_m (d^{2m} \gamma \alpha^m + \beta^m).\end{aligned}$$

Hence using (5.40), we get the result. □

Remark 5.3.10. We make a few observations.

(1) If the convex set $\mathcal{C}$ (as in (5.33)) has more than one element, then we can consider probability measures on $\mathcal{C}$ which are convex combinations of Dirac measures on $\mathcal{C}$. By Theorem 5.3.4, we have the existence of infinitely many stationary solutions corresponding to each of these probability measures. To rationalize, this may be happening due to $\mathcal{C}$ being not translation invariant.

(2) We note that the set $\mathcal{C}$, as in Example 5.3.5, is not compact. To see this first take p sufficiently large so that the tempered distribution given by the function $x \mapsto x^2$ is in $\mathcal{S}_{-p}$. Then the image of $\mathcal{C}$ under this tempered distribution (a continuous linear functional on $\mathcal{S}_p$) contains $(0, \infty)$, the variances of centered Gaussian densities. So $\mathcal{C}$ is unbounded and non-compact.

CHAPTER 6

An Itô formula in $\mathcal{S}'$

6.1 Introduction

The Itô formula has been studied in various - and quite general - frameworks starting from real valued processes to processes taking values in Nuclear spaces ([22, 53, 59, 67, 74, 75, 86, 88–90, 112]). In this chapter we prove an Itô formula which generalizes a result for continuous semimartingales and is motivated by applications to stochastic partial differential equations driven by Lévy processes.
Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered complete probability space satisfying the usual conditions. Recall that $\tau_x, x \in \mathbb{R}^d$ are the translation operators on the space of tempered distributions (Example 2.11.6). Let $p \in \mathbb{R}$. Given $\phi \in \mathcal{S}_{-p}(\mathbb{R}^d)$ and an $\mathbb{R}^d$ valued $(\mathcal{F}_t)$ adapted continuous semimartingale $X_t = (X_t^1, \cdots, X_t^d)$, we have the following Itô formula (see [89, Theorem 2.3]).

Theorem 6.1.1. *$\{\tau_{X_t}\phi\}$ is an $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued continuous semimartingale and we have the equality in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$, a.s.*

$$\tau_{X_t}\phi = \tau_{X_0}\phi - \sum_{i=1}^{d}\int_0^t \partial_i\tau_{X_s}\phi\, dX_s^i + \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \partial_{ij}^2\tau_{X_s}\phi\, d[X^i, X^j]_s,\ t \geq 0.$$

This result has been used in [90] to show the existence of a solution of some stochastic differential equations in $\mathcal{S}'(\mathbb{R}^d)$. In the previous chapter, we have extended this result to the case where ϕ is an $\mathcal{F}_0$ measurable $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued random variable (see Theorem 5.2.2) and then used it in Theorem 5.2.15.

The aim of the current chapter is to prove Theorem 6.1.1 for semimartingales $\{X_t\}$ with jumps. A version of this Itô formula was also proved in [112, Theorem III.1] with equality in $\mathcal{S}'$. In [67, Theorem 3], the author has proved this formula for twice continuously (Fréchet) differentiable functions while dealing with a single Hilbert space. Note that derivatives of

$\phi \in \mathcal{S}_{-p}(\mathbb{R}^d)$ may not be in the same space. Using the technique of regularization of E' valued processes, the result [67, Theorem 3] was also proved in [77, Theorem 8] in the case of an E' valued continuous martingale, where E is a countably Hilbertian Nuclear space.

Given an $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued norm-bounded predictable process $\{G_t\}$ and an $\mathbb{R}^d$ valued semimartingale $\{X_t\}$, the stochastic integral $\int_0^t G_s\, dX_s$ can be defined (see Section 2.7 and Section 2.12). Note that for any $\phi \in \mathcal{S}(\mathbb{R}^d)$, a.s. (see Proposition 2.7.8 and Proposition 2.7.15)

$$\left\langle \int_0^t G_s\, dX_s\, ,\, \phi \right\rangle = \int_0^t \langle G_s\, ,\, \phi\rangle\; dX_s,\; t \geq 0.$$

We exploit this property to prove an Itô formula (see Theorem 6.2.3). We apply the Itô formula to a one-dimensional process X, which solves a stochastic differential equation driven by a Lévy process and show the existence of a solution of a stochastic 'partial' integro-differential equation in the Hermite-Sobolev spaces (see Theorem 6.3.1). This is similar to the solution obtained in [90] for continuous processes X. In Proposition 6.3.3 we identify the local time process of a real valued semimartingale as an $\mathcal{S}'$ valued process. Most of the results in this chapter are from [7].

6.2 An Itô formula

Given $\phi \in \mathcal{S}'(\mathbb{R}^d)$, there exists a $p > 0$ such that $\phi \in \mathcal{S}_{-p}(\mathbb{R}^d)$. Let $X_t = (X_t^1, \cdots, X_t^d)$ be an $\mathbb{R}^d$ valued $(\mathcal{F}_t)$ semimartingale with rcll paths and has the decomposition a.s.

$$X_t = X_0 + M_t + A_t,\; t \geq 0$$

where $M_t = (M_t^1, \cdots, M_t^d)$ is an $\mathbb{R}^d$ valued locally $\mathcal{L}^2$-bounded martingale and $A_t = (A_t^1, \cdots, A_t^d)$ is an $\mathbb{R}^d$ valued process of finite variation (Lemma 2.5.34). Both $\{M_t\}$ and $\{A_t\}$ have rcll paths and $M_0 = 0 = A_0$ a.s. By Lemma 2.11.7(i), $\{\tau_{X_t}\phi\}$ is an $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued process.
Recall that the process $\{X_{t-}\}$ defined by

$$X_{t-} := \begin{cases} X_0, \text{ if } t = 0. \\ \lim_{s\uparrow t} X_s, \text{ if } t > 0. \end{cases}$$

is predictable (see Proposition 2.5.4).

Lemma 6.2.1. *Let ϕ, X be as above. Then for any $1 \leq i \leq d$ and $1 \leq j \leq d$,*

(i) $\{\tau_{X_{t-}}\phi\}$ is an $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued predictable process.
(ii) $\{\partial_i \tau_{X_{t-}}\phi\}$ is an $\mathcal{S}_{-p-\frac{1}{2}}(\mathbb{R}^d)$ valued predictable process.
(iii) $\{\partial_{ij}^2 \tau_{X_{t-}}\phi\}$ is an $\mathcal{S}_{-p-1}(\mathbb{R}^d)$ valued predictable process.

Proof. By Proposition 2.5.4, $\{X_{t-}\}$ is predictable. Since $x \mapsto \tau_x \phi : \mathbb{R}^d \to \mathcal{S}_{-p}(\mathbb{R}^d)$ is continuous (see Lemma 2.11.7(ii)), the process $\{\tau_{X_{t-}}\phi\}$ is predictable.
For any $1 \leq i \leq d$, we have $\tau_x(\partial_i \phi) = \partial_i \tau_x \phi$ (see Lemma 2.11.7(iii)) and $\partial_i : \mathcal{S}_{-p}(\mathbb{R}^d) \to \mathcal{S}_{-p-\frac{1}{2}}(\mathbb{R}^d)$ (see Lemma 2.11.4). Hence $\{\partial_i \tau_{X_{t-}}\phi\}$ is an $\mathcal{S}_{-p-\frac{1}{2}}(\mathbb{R}^d)$ valued predictable process.
Similarly for $1 \leq i,j \leq d$, the processes $\{\partial_{ij}^2 \tau_{X_{t-}}\phi\}$ are $\mathcal{S}_{-p-1}(\mathbb{R}^d)$ valued predictable processes. □

Note that there exists a set $\widetilde{\Omega}$ with $P(\widetilde{\Omega}) = 1$ such that (see Corollary 2.5.41 and Lemma 2.5.42)

$$\sum_{s \leq t} | \triangle X_s|^2 < \infty, \forall t > 0, \omega \in \widetilde{\Omega}.$$

If $\omega \in \widetilde{\Omega}$, then there are at most countably many jumps of X on $[0,t]$.

Lemma 6.2.2. *Let $\phi, \{X_t\}$ be as above. Fix $\omega \in \widetilde{\Omega}$. Fix $\psi \in \mathcal{S}(\mathbb{R}^d)$. Then for all $s \leq t$*

$$\left| \left\langle \tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^d (\triangle X_s^i \, \partial_i \tau_{X_{s-}}\phi) \, , \, \psi \right\rangle \right| \leq C(t). \, |\triangle X_s|^2 \|\psi\|_{p+1},$$

and hence

$$\|\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^d (\triangle X_s^i \, \partial_i \tau_{X_{s-}}\phi)\|_{-p-1} \leq C(t).|\triangle X_s|^2, \tag{6.1}$$

where $t \mapsto C(t)$ is a positive non-decreasing function of t. In particular,

$$\tau_{X_t}\phi - \tau_{X_{t-}}\phi + \sum_{i=1}^d (\triangle X_t^i \, \partial_i \tau_{X_{t-}}\phi) = 0, \text{ if } |\triangle X_t| = 0.$$

Proof. By [89, Proposition 1.4], there exists some positive integer n such that the map $x \mapsto \tau_x \phi \in \mathcal{S}_{-n}(\mathbb{R}^d)$ is a C^2 map. For any fixed $\psi \in \mathcal{S}(\mathbb{R}^d)$ we have $x \mapsto \langle \tau_x \phi \, , \, \psi \rangle$ is a C^2 map and

$$\begin{aligned} \partial_i \langle \tau_x \phi \, , \, \psi \rangle &= \partial_i \langle \phi \, , \, \psi(\cdot + x) \rangle \\ &= \langle \phi \, , \, \partial_i \psi(\cdot + x) \rangle \\ &= \langle \phi \, , \, \tau_{-x} \partial_i \psi \rangle = - \langle \partial_i \tau_x \phi \, , \, \psi \rangle \, . \end{aligned}$$

For any $1 \leq i,j \leq d$, we have $\partial_{ij}^2 = \partial_i \partial_j = \partial_j \partial_i$ on $\mathcal{S}'(\mathbb{R}^d)$ and hence $\partial_{ij}^2 : \mathcal{S}_{-p}(\mathbb{R}^d) \to \mathcal{S}_{-p-1}(\mathbb{R}^d)$ is a bounded linear operator (see Example 2.11.3). Then there exists a constant $\alpha > 0$ such that

$$\|\partial_{ij}^2 \theta\|_{-p-1} \leq \alpha \|\theta\|_{-p}, \, , \forall \theta \in \mathcal{S}_{-p}(\mathbb{R}^d). \tag{6.2}$$

We follow the proof of [56, Theorem 23.7] and define $B(t,\omega) := \{x \in \mathbb{R}^d : |x| \leq \sup_{s\leq t}|X_s(\omega)|\}$. Then using Taylor's formula for the C^2 map $x \mapsto \langle \tau_x\phi\,,\,\psi\rangle$, we have for all $s \leq t$

$$\begin{aligned}
&\left|\left\langle \tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^{d}(\triangle X_s^i\,\partial_i\tau_{X_{s-}}\phi)\,,\,\psi\right\rangle\right| \\
=&\left|\langle \tau_{X_s}\phi\,,\,\psi\rangle - \left\langle \tau_{X_{s-}}\phi\,,\,\psi\right\rangle + \sum_{i=1}^{d}\left\langle \partial_i\tau_{X_{s-}}\phi\,,\,\psi\right\rangle\,\triangle X_s^i\right| \\
=&\left|\langle \tau_{X_s}\phi\,,\,\psi\rangle - \left\langle \tau_{X_{s-}}\phi\,,\,\psi\right\rangle - \sum_{i=1}^{d}\partial_i\left\langle \tau_{X_{s-}}\phi\,,\,\psi\right\rangle\,\triangle X_s^i\right| \\
\leq&\frac{1}{2}.|\triangle X_s|^2\left(\sum_{i,j=1}^{d}\sup_{y\in B(t,\omega)}|\left\langle \partial_{ij}^2\tau_y\phi\,,\,\psi\right\rangle|\right) \\
\leq&\frac{1}{2}.|\triangle X_s|^2\left(\sum_{i,j=1}^{d}\sup_{y\in B(t,\omega)}\|\partial_{ij}^2\tau_y\phi\|_{-p-1}\right)\|\psi\|_{p+1} \\
\leq&\frac{\alpha}{2}.|\triangle X_s|^2\left(\sup_{y\in B(t,\omega)}\|\tau_y\phi\|_{-p}\right)\|\psi\|_{p+1}\,(\text{using}\,(6.2)).
\end{aligned}$$

Define $C(t,\omega) := \frac{\alpha}{2}\left(\sup_{y\in B(t,\omega)}\|\tau_y\phi\|_{-p}\right)$. Then $C(t,\omega)$ is non-decreasing in t and for all $s \leq t$

$$\left|\left\langle \tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^{d}(\triangle X_s^i\,\partial_i\tau_{X_{s-}}\phi)\,,\,\psi\right\rangle\right| \leq C(t).\,|\triangle X_s|^2\|\psi\|_{p+1}.$$

From above estimate we have

$$\|\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^{d}(\triangle X_s^i\,\partial_i\tau_{X_{s-}}\phi)\|_{-p-1} \leq C(t).|\triangle X_s|^2.$$

In particular $\tau_{X_t}\phi - \tau_{X_{t-}}\phi + \sum_{i=1}^{d}(\triangle X_t^i\,\partial_i\tau_{X_{t-}}\phi) = 0$ if $|\triangle X_t| = 0$. □

For any $i, j = 1, \cdots, d$, let $\{[X^i, X^j]_t^c\}$ denote the continuous part of $\{[X^i, X^j]_t\}$. We now prove the main result of this chapter.

Theorem 6.2.3. *Let $p > 0$ and $\phi \in \mathcal{S}_{-p}(\mathbb{R}^d)$. Let $X = (X^1, \cdots, X^d)$ be an $\mathbb{R}^d$ valued $(\mathcal{F}_t)$ semimartingale. Let $\triangle X_s^i$ denote the jump of X_s^i. Then $\{\tau_{X_t}\phi\}$ is an $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued semimartingale and*

$$\sum_{s\leq t}\left[\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^{d}(\triangle X_s^i\,\partial_i\tau_{X_{s-}}\phi)\right]$$

is a $\mathcal{S}_{-p-1}(\mathbb{R}^d)$ valued process of finite variation and we have the following equality in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$, a.s.

$$\begin{aligned}\tau_{X_t}\phi = \tau_{X_0}\phi - \sum_{i=1}^{d}\int_0^t \partial_i\tau_{X_{s-}}\phi\, dX_s^i + \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \partial_{ij}^2\tau_{X_{s-}}\phi\, d[X^i, X^j]_s^c \\ + \sum_{s\leq t}\left[\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^{d}(\triangle X_s^i\,\partial_i\tau_{X_{s-}}\phi)\right],\ t\geq 0.\end{aligned} \tag{6.3}$$

Proof. We proceed in steps.

Step 1: Let $\widetilde{\Omega}$ be as in Lemma 6.2.2. Then $\omega\in\widetilde{\Omega}$ implies (see equation (6.1))

$$\sum_{s\leq t}\|\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^{d}(\triangle X_s^i\,\partial_i\tau_{X_{s-}}\phi)\|_{-p-1} \leq C(t)\sum_{s\leq t}|\triangle X_s|^2 < \infty. \tag{6.4}$$

Recall that if $\omega\in\widetilde{\Omega}$, then there are at most countably many jumps of X on $[0,t]$. In view of the above estimate we define for any $t\geq 0$

$$Y_t(\omega) := \sum_{s\leq t}\left[\tau_{X_s(\omega)}\phi - \tau_{X_{s-}(\omega)}\phi + \sum_{i=1}^{d}(\triangle X_s^i(\omega)\,\partial_i\tau_{X_{s-}(\omega)}\phi)\right],\ \omega\in\widetilde{\Omega}$$

and set $Y_t(\omega) := 0$, $\omega\in(\widetilde{\Omega})^c$. Then $\{Y_t\}$ is a well-defined $\mathcal{S}_{-p-1}(\mathbb{R}^d)$ valued $(\mathcal{F}_t)$ adapted process.

Step 2: Now we show $\{Y_t\}$ has rcll paths and is a process of finite variation. Fix $\omega\in\widetilde{\Omega}$. We claim

(i) $Y_{t-} = \sum_{s<t}\left[\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^{d}(\triangle X_s^i\,\partial_i\tau_{X_{s-}}\phi)\right],\ t>0.$

(ii) $Y_{t+} = \sum_{s\leq t}\left[\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^{d}(\triangle X_s^i\,\partial_i\tau_{X_{s-}}\phi)\right] = Y_t,\ t\geq 0.$

We prove (i). Proof of (ii) is similar.

Let $\{t_m\}$ be an increasing sequence converging to t. Then

$$\begin{aligned}&\left\|\sum_{s<t}\left[\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^{d}(\triangle X_s^i\,\partial_i\tau_{X_{s-}}\phi)\right] - Y_{t_m}\right\|_{-p-1}\\ &= \left\|\sum_{t_m<s<t}\left[\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^{d}(\triangle X_s^i\,\partial_i\tau_{X_{s-}}\phi)\right]\right\|_{-p-1}\\ &\leq \sum_{t_m<s<t}\left\|\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^{d}(\triangle X_s^i\,\partial_i\tau_{X_{s-}}\phi)\right\|_{-p-1}\\ &\leq C(t)\sum_{t_m<s<t}|\triangle X_s|^2\ (\text{using}\,(6.1))\\ &= C(t)\left[\sum_{s<t}|\triangle X_s|^2 - \sum_{s\leq t_m}|\triangle X_s|^2\right]\end{aligned}$$

$$\xrightarrow{m\to\infty} 0\,(\text{using Lemma}\,2.5.42(i)).$$

This proves (i). Then using $(i),(ii)$ we have on $\widetilde{\Omega}$

$$\triangle Y_t = \tau_{X_t}\phi - \tau_{X_{t-}}\phi + \sum_{i=1}^{d}(\triangle X_t^i\,\partial_i\tau_{X_{t-}}\phi),$$

and $\triangle Y_t = 0$ if $\triangle X_t = 0$. Now using (6.1), we also have

$$\sum_{s\le t}\|\triangle Y_s\|_{-p-1} \le C(t)\sum_{s\le t}|\triangle X_s|^2 < \infty,\ \omega\in\widetilde{\Omega}$$

and $Y_t = \sum_{s\le t}\triangle Y_s$. We have shown $\{Y_t\}$ has rcll paths. Now we show that $\{Y_t\}$ has paths of finite variation.
Let $\omega\in\widetilde{\Omega}$ and $t>0$. Let $\mathbb{P} = \{0 = t_0 < t_1 < \cdots < t_m = t\}$ be a partition of $[0,t]$. Then

$$\begin{aligned}
&\sum_{i=1}^{m}\|Y_{t_i} - Y_{t_{i-1}}\|_{-p-1}\\
&= \sum_{i=1}^{m}\left\|\sum_{t_{i-1}<s\le t_i}\left[\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^{d}(\triangle X_s^i\,\partial_i\tau_{X_{s-}}\phi)\right]\right\|_{-p-1}\\
&\le \sum_{i=1}^{m}\sum_{t_{i-1}<s\le t_i}\left\|\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^{d}(\triangle X_s^i\,\partial_i\tau_{X_{s-}}\phi)\right\|_{-p-1}\\
&= \sum_{s\le t}\left\|\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^{d}(\triangle X_s^i\,\partial_i\tau_{X_{s-}}\phi)\right\|_{-p-1}\\
&\le C(t)\sum_{s\le t}|\triangle X_s|^2.
\end{aligned}$$

Since the quantity $C(t)\sum_{s\le t}|\triangle X_s|^2$ is independent of the choice of the partition $\mathbb{P}$, we have $\{Y_t\}$ is of finite variation with

$$Var_{[0,t]}(Y_\cdot) \le C(t)\sum_{s\le t}|\triangle X_s|^2$$

on $\widetilde{\Omega}$.

Step 3: To complete the proof we need to verify the following equality in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$, a.s. for all $t\ge 0$

$$Y_t = \tau_{X_t}\phi - \tau_{X_0}\phi + \sum_{i=1}^{d}\int_0^t \partial_i\tau_{X_{s-}}\phi\,dX_s^i - \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \partial_{ij}^2\tau_{X_{s-}}\phi\,d[X^i,X^j]_s^c.$$

First we assume that the processes $\{X_{t-}\}, \{[X^i, X^j]^c_t\}$, $i, j = 1, \cdots, d$ are bounded. Since $\partial_i : \mathcal{S}_{-p}(\mathbb{R}^d) \to \mathcal{S}_{-p-\frac{1}{2}}(\mathbb{R}^d)$ is a bounded linear operator, by Lemma 2.11.7(i), we have for all $t \geq 0, i = 1, \cdots, d$

$$\|\partial_i \tau_{X_{t-}} \phi\|_{-p-\frac{1}{2}} \leq C.\|\tau_{X_{t-}}\phi\|_{-p} \leq C.P_k(|X_{t-}|)\|\phi\|_{-p} \leq C',$$

where $C, C' > 0$ are appropriate constants. Similarly, there exists a constant $C'' > 0$ such that

$$\|\partial_{ij}\tau_{X_{t-}}\phi\|_{-p-1} \leq C'', \forall t \geq 0, i, j = 1, \cdots, d.$$

Hence $\{\tau_{X_{t-}}\phi\}, \{\partial_i \tau_{X_{t-}}\phi\}, \{\partial^2_{ij}\tau_{X_{t-}}\phi\}$ are norm-bounded predictable processes (see Lemma 6.2.1). As per the results mentioned in the previous section, we can define stochastic integrals

$$I^1_t := \sum_{i=1}^d \int_0^t \partial_i \tau_{X_{s-}}\phi \, dX^i_s, \quad I^2_t := \sum_{i,j=1}^d \int_0^t \partial^2_{ij}\tau_{X_{s-}}\phi \, d[X^i, X^j]^c_s, \; t \geq 0$$

which are respectively $\mathcal{S}_{-p-\frac{1}{2}}(\mathbb{R}^d)$ and $\mathcal{S}_{-p-1}(\mathbb{R}^d)$ valued and have rcll paths.
For $n \in \mathbb{Z}^d_+$ applying the Itô formula (see [56, Theorem 23.7]) to the C^2 map $x \mapsto \langle \tau_x\phi\,,\, h_n\rangle$ we have, a.s. for all $t \geq 0$

$$\begin{aligned}\langle \tau_{X_t}\phi\,,\, h_n\rangle = \langle \tau_{X_0}\phi\,,\, h_n\rangle &- \underbrace{\sum_{i=1}^d \int_0^t \left\langle \partial_i\tau_{X_{s-}}\phi\,,\, h_n\right\rangle \, dX^i_s}_{=\langle I^1_t\,,\, h_n\rangle} \\ &+ \underbrace{\frac{1}{2}\sum_{i,j=1}^d \int_0^t \left\langle \partial^2_{ij}\tau_{X_{s-}}\phi\,,\, h_n\right\rangle \, d[X^i, X^j]^c_s}_{=\langle I^2_t\,,\, h_n\rangle} \\ &+ \sum_{s\leq t}\left[\langle \tau_{X_s}\phi\,,\, h_n\rangle - \left\langle \tau_{X_{s-}}\phi\,,\, h_n\right\rangle + \sum_{i=1}^d \left\langle \partial_i\tau_{X_{s-}}\phi\,,\, h_n\right\rangle \, \triangle X^i_s\right],\end{aligned} \tag{6.5}$$

where $\triangle X^i_s$ denotes the jump of X^i_s. Now varying n in the countable set $\mathbb{Z}^d_+$, we get a common null set $\widetilde{\Omega}$ such that for all $\omega \in \Omega \setminus \widetilde{\Omega}$, for all $n \in \mathbb{Z}^d_+$ and for all $t \geq 0$, we have

$$\begin{aligned}\langle\, (\tau_{X_t}\phi - \tau_{X_0}\phi + \sum_{i=1}^d \int_0^t \partial_i\tau_{X_{s-}}\phi \, dX^i_s \\ - \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial^2_{ij}\tau_{X_{s-}}\phi \, d[X^i, X^j]^c_s - Y_t), h_n\,\rangle = 0.\end{aligned}$$

Using Proposition 2.10.2, we get the required equality in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$ for semimartingales $\{X_t\}$ such that $\{X_{t-}\}, \{[X^i, X^j]^c_t\}$, $i, j = 1, \cdots, d$ are bounded.

Step 4: Now suppose at least one of $\{X_{t-}\}, \{[X^i, X^j]^c_t\}$, $i, j = 1, \cdots, d$ is not bounded. Then define

$$\bar{\sigma}_n := \inf\{t \geq 0 : |[X^i, X^j]^c_t| \geq n,\ i, j = 1, \cdots, d\}$$

and

$$\widetilde{\sigma}_n := \inf\{t \geq 0 : |X_t| \geq n\},$$

where $|\cdot|$ represents the Euclidean norms in the appropriate space $\mathbb{R}^m$ ($m = 1$ or d). Set $\sigma_n = \bar{\sigma}_n \wedge \widetilde{\sigma}_n$. Then $\{([X^i, X^j]^c)^{\sigma_n}_t\}$, $i, j = 1, \cdots, d$ are bounded.
If $|X_0(\omega)| > n$ for some w, then $\sigma_n(\omega) = 0$. Such ω does not contribute to the integral $\sum_{i=1}^d \int_0^{t\wedge\sigma_n} \|\partial_i \tau_{X_{s-}} \xi\|^2_{p-\frac{1}{2}}\, d\langle M^i\rangle_s$ etc. So we may assume the processes $\{X^{\sigma_n}_{t-}\}$ are bounded. Hence a.s. in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$ we have for all $t \geq 0$

$$\begin{aligned}\tau_{X_{t\wedge\sigma_n}}\phi =& \tau_{X_0}\phi + \sum_{i=1}^d \int_0^{t\wedge\sigma_n} \partial_i \tau_{X_{s-}}\phi\, dX^i_s \\ &- \frac{1}{2}\sum_{i,j=1}^d \int_0^{t\wedge\sigma_n} \partial^2_{ij}\tau_{X_{s-}}\phi\, d[X^i, X^j]^c_s - Y_{t\wedge\sigma_n}.\end{aligned}$$

Letting n go to infinity we get the result.

□

6.3 Two applications

In this section, we apply the Itô formula 6.2.3 firstly in Theorem 6.3.1 to obtain a solution of a certain stochastic 'partial' integro-differential equation in the Hermite-Sobolev spaces and secondly, in Remark 6.3.4 to explore some connections with the technique of 'regularization' of random linear functionals on $\mathcal{S}(\mathbb{R}^d)$. The first application is similar in spirit to the same obtained in [90, Theorem 3.4 and Lemma 3.6] for continuous processes (also see Theorem 5.2.15, Lemma 5.2.16 and Theorem 5.2.20).

Let $p \in \mathbb{R}$. Let $\phi \in \mathcal{S}_p$ and $\sigma, b \in \mathcal{S}_{-p}$. Define $\bar{\sigma}(x) := \langle \sigma\,,\, \tau_x\phi\rangle$, $\bar{b}(x) := \langle b\,,\, \tau_x\phi\rangle$, $\forall x \in \mathbb{R}$. Let $F, G : \mathcal{S}_p \times \mathbb{R} \to \mathbb{R}$ and let $\bar{F}, \bar{G} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by $\bar{F}(x, \tilde{x}) := F(\tau_x\phi, \tilde{x})$, $\bar{G}(x, \tilde{x}) := G(\tau_x\phi, \tilde{x})$. Let $\{B_t\}$ be a standard $(\mathcal{F}_t)$ Brownian motion and let N be a Poisson process driven by a Lévy measure ν. Let $\widetilde{N}$ denote the compensated measure. Assume that B and N are independent. Consider the problem of existence of a solution of the following one-dimensional equation

$$\begin{aligned}X_t =& \int_0^t \bar{b}(X_{s-})\, ds + \int_0^t \bar{\sigma}(X_{s-})\, dB_s \\ &+ \int_0^t \int_{(0<|x|<1)} \bar{F}(X_{s-}, x)\, \widetilde{N}(dsdx) + \int_0^t \int_{(|x|\geq 1)} \bar{G}(X_{s-}, x)\, N(dsdx).\end{aligned} \tag{6.6}$$

We are following the set up of [3, Chapter 6 Section 2] and we take $c = 1$. Recall that this parameter c separates the small and large jumps. One usually omits the term involving large jumps, i.e. the term involving G, since a solution of (6.6) can be obtained from the modified stochastic differential equation

$$dX_t = \bar{b}(X_{t-})\,dt + \bar{\sigma}(X_{t-})\,dB_t + \int_{0<|x|<1} \bar{F}(X_{t-}, x)\,\widetilde{N}(dtdx). \tag{6.7}$$

using interlacing ([3, Example 1.3.13]).
We assume conditions on $\bar{\sigma}, \bar{b}, \bar{F}$ which imply the existence of solutions of equation (6.7). For each $x, y \in \mathbb{R}$, define $\bar{a}(x, y) := \bar{\sigma}(x)\bar{\sigma}(y)$. Now assume the following two conditions.

(i) (Lipschitz condition) There exists $K_1 > 0$ such that

$$\begin{aligned} &|\bar{b}(y_1) - \bar{b}(y_2)|^2 + |\bar{a}(y_1, y_1) - 2\bar{a}(y_1, y_2) + \bar{a}(y_2, y_2)| \\ &+ \int_{0<|x|<1} |\bar{F}(y_1, x) - \bar{F}(y_2, x)|^2\,\nu(dx) \leq K_1\,|y_1 - y_2|^2, \forall y_1, y_2 \in \mathbb{R}. \end{aligned} \tag{6.8}$$

(ii) (Growth condition) There exists $K_2 > 0$ such that

$$|\bar{b}(y)|^2 + |\bar{a}(y, y)| + \int_{0<|x|<1} |\bar{F}(y, x)|^2\,\nu(dx) \leq K_2\,(1 + |y|^2), \forall y \in \mathbb{R}. \tag{6.9}$$

Under this conditions, a solution of the stochastic differential equation (6.7) exists (see [3, Theorem 6.2.3]) and hence that of (6.6) also exists. We denote this solution of (6.6) by $\{X_t\}$. Using the growth condition (6.9), we have

$$\int_0^t \int_{(0<|x|<1)} |\bar{F}(X_{s-}, x)|^2\,\nu(dx)ds \leq \int_0^t K_2(1 + |X_{s-}|^2)\,ds \leq K_2(1 + \sup_{s\in[0,t]} |X_{s-}|^2)t,$$

and hence the integrability condition follows: a.s.

$$\int_0^t \int_{(0<|x|<1)} |\bar{F}(X_{s-}, x)|^2\,\nu(dx)ds < \infty, \forall t \geq 0. \tag{6.10}$$

As an application of Theorem 6.2.3 we get the next result.

Theorem 6.3.1. *The $\mathcal{S}_p$ valued process Y defined by $Y_t := \tau_{X_t}\phi$ solves the following stochastic differential equation with equality in $\mathcal{S}_{p-1}$:*

$$\begin{aligned} Y_t(\phi) = \phi &+ \int_0^t A(Y_{s-}(\phi))\,dB_s + \int_0^t L(Y_{s-}(\phi))\,ds \\ &+ \int_0^t \int_{(0<|x|<1)} \Big(\tau_{F(Y_{s-}(\phi),x)} - Id + F(Y_{s-}(\phi), x)\,\partial\Big)\,Y_{s-}(\phi)\,\nu(dx)\,ds \\ &+ \int_0^t \int_{(0<|x|<1)} \Big(\tau_{F(Y_{s-}(\phi),x)} - Id\Big)\,Y_{s-}(\phi)\,\widetilde{N}(dsdx) \\ &+ \int_0^t \int_{(|x|\geq 1)} \Big(\tau_{G(Y_{s-}(\phi),x)} - Id\Big)\,Y_{s-}(\phi)\,N(dsdx), \end{aligned} \tag{6.11}$$

where the operators A, L on $\mathcal{S}_p$ are as follows:

$$A\phi := -\langle \sigma\,,\,\phi\rangle\;\partial\phi,$$

and

$$L\phi := \frac{1}{2}\langle \sigma\,,\,\phi\rangle^2\;\partial^2\phi - \langle b\,,\,\phi\rangle\;\partial\phi.$$

Proof. Observe that

$$\triangle X_t = \bar{F}(X_{t-}, \triangle X_t)\mathbb{1}_{(0<|\triangle X_t|<1)} + \bar{G}(X_{t-}, \triangle X_t)\mathbb{1}_{(|\triangle X_t|\geq 1)}. \tag{6.12}$$

From (6.12) we make two observations. Firstly, $|\bar{F}(X_{t-}, \triangle X_t)|\mathbb{1}_{(0<|\triangle X_t|<1)} \leq 1$. In particular, this implies

$$|\bar{F}(X_{t-}, \triangle X_t)|^4\mathbb{1}_{(0<|\triangle X_t|<1)} \leq |\bar{F}(X_{t-}, \triangle X_t)|^2\mathbb{1}_{(0<|\triangle X_t|<1)}.$$

Secondly, we have the following simplification.

$$\begin{aligned}
&\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \triangle X_s\,\partial\tau_{X_{s-}}\phi\\
&= (\tau_{\triangle X_s} - Id)\,\tau_{X_{s-}}\phi + \triangle X_s\,\partial\tau_{X_{s-}}\phi\\
&= \mathbb{1}_{(0<|\triangle X_s|<1)}\left(\tau_{\bar{F}(X_{s-},\triangle X_s)} - Id + \bar{F}(X_{s-}, \triangle X_s)\,\partial\right)\tau_{X_{s-}}\phi\\
&+ \mathbb{1}_{(|\triangle X_s|\geq 1)}\left(\tau_{\bar{G}(X_{s-},\triangle X_s)} - Id\right)\tau_{X_{s-}}\phi + \mathbb{1}_{(|\triangle X_s|\geq 1)}\,\bar{G}(X_{s-}, \triangle X_s)\,\partial\tau_{X_{s-}}\phi.
\end{aligned}$$

Using equation (6.1), we have

$$\begin{aligned}
&\mathbb{1}_{(0<|\triangle X_s|<1)}\left\|\left(\tau_{\bar{F}(X_{s-},\triangle X_s)} - Id + \bar{F}(X_{s-}, \triangle X_s)\,\partial\right)\tau_{X_{s-}}\phi\right\|_{-p-1}\\
&\leq C(s).\mathbb{1}_{(0<|\triangle X_s|<1)}\,|\bar{F}(X_{s-}, \triangle X_s)|^2,
\end{aligned}$$

where $t \mapsto C(t)$ is a positive non-decreasing function. Then

$$\begin{aligned}
&\int_0^t \int_{(0<|x|<1)} \left\|\left(\tau_{\bar{F}(X_{s-},x)} - Id + \bar{F}(X_{s-}, x)\,\partial\right)\tau_{X_{s-}}\phi\right\|_{-p-1}^2\;\nu(dx)ds\\
&\leq \int_0^t C(s)^2\int_{(0<|x|<1)} |\bar{F}(X_{s-}, x)|^4\,\nu(dx)ds\\
&\leq C(t)^2\int_0^t\int_{(0<|x|<1)} |\bar{F}(X_{s-}, x)|^2\,\nu(dx)ds < \infty, \text{ (by (6.10))}.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\int_0^t \int_{(0<|x|<1)} \left\|\left(\tau_{\bar{F}(X_{s-},x)} - Id\right)\tau_{X_{s-}}\phi\right\|_{-p-\frac{1}{2}}^2\;\nu(dx)ds\\
&\leq \tilde{C}(t)^2\int_0^t\int_{(0<|x|<1)} |\bar{F}(X_{s-}, x)|^2\nu(dx)ds < \infty, \text{ (by (6.10))},
\end{aligned}$$

where $t \mapsto \tilde{C}(t)$ is some non-decreasing function. Hence

$$\begin{aligned}
&\sum_{s\le t}\Big[\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \triangle X_s\,\partial\tau_{X_{s-}}\phi\Big] \\
&= \int_0^t\int_{(0<|x|<1)}\Big(\tau_{\bar{F}(X_{s-},x)} - Id + \bar{F}(X_{s-},x)\,\partial\Big)\,\tau_{X_{s-}}\phi\,N(dsdx) \\
&+ \int_0^t\int_{(|x|\ge 1)}\Big(\tau_{\bar{G}(X_{s-},x)} - Id\Big)\,\tau_{X_{s-}}\phi\,N(dsdx) \\
&+ \int_0^t\int_{(|x|\ge 1)}\bar{G}(X_{s-},x)\,\partial\tau_{X_{s-}}\phi\,N(dsdx) \\
&= \int_0^t\int_{(0<|x|<1)}\Big(\tau_{\bar{F}(X_{s-},x)} - Id + \bar{F}(X_{s-},x)\,\partial\Big)\,\tau_{X_{s-}}\phi\,\widetilde{N}(dsdx) \\
&+ \int_0^t\int_{(0<|x|<1)}\Big(\tau_{\bar{F}(X_{s-},x)} - Id + \bar{F}(X_{s-},x)\,\partial\Big)\,\tau_{X_{s-}}\phi\,\nu(dx)ds \\
&+ \int_0^t\int_{(|x|\ge 1)}\Big(\tau_{\bar{G}(X_{s-},x)} - Id\Big)\,\tau_{X_{s-}}\phi\,N(dsdx) \\
&+ \int_0^t\int_{(|x|\ge 1)}\bar{G}(X_{s-},x)\,\partial\tau_{X_{s-}}\phi\,N(dsdx).
\end{aligned}$$

Now by the Itô formula (Theorem 6.2.3)

$$\begin{aligned}
\tau_{X_t}\phi &= \tau_{X_0}\phi + \int_0^t A(\tau_{X_{s-}}\phi)\,dB_s + \int_0^t L(\tau_{X_{s-}}\phi)\,ds \\
&- \int_0^t\int_{(0<|x|<1)}\bar{F}(X_{s-},x)\,\partial\tau_{X_{s-}}\phi\,\widetilde{N}(dsdx) \\
&- \int_0^t\int_{(|x|\ge 1)}\bar{G}(X_{s-},x)\,\partial\tau_{X_{s-}}\phi\,N(dsdx) \\
&+ \sum_{s\le t}\Big[\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \triangle X_s\,\partial\tau_{X_{s-}}\phi\Big] \\
&= \phi + \int_0^t A(\tau_{X_{s-}}\phi)\,dB_s + \int_0^t L(\tau_{X_{s-}}\phi)\,ds \\
&+ \int_0^t\int_{(0<|x|<1)}\Big(\tau_{\bar{F}(X_{s-},x)} - Id + \bar{F}(X_{s-},x)\,\partial\Big)\,\tau_{X_{s-}}\phi\,\nu(dx)\,ds \\
&+ \int_0^t\int_{(0<|x|<1)}\Big(\tau_{\bar{F}(X_{s-},x)} - Id\Big)\,\tau_{X_{s-}}\phi\,\widetilde{N}(dsdx) \\
&+ \int_0^t\int_{(|x|\ge 1)}\Big(\tau_{\bar{G}(X_{s-},x)} - Id\Big)\,\tau_{X_{s-}}\phi\,N(dsdx)
\end{aligned}$$

Hence $Y_t(\phi) := \tau_{X_t}\phi$ solves the equation (6.11). □

Remark 6.3.2. We proved the existence of a solution to equation (6.11) in the previous theorem. Uniqueness of solutions of (6.11) will be taken up in future.

Given a real valued semimartingale $\{X_t\}$, consider the local time process denoted by $\{L_t(x)\}_{t\in[0,\infty),x\in\mathbb{R}}$. Note that this process is jointly measurable in (x,t,ω) and for each $x\in\mathbb{R}$, $\{L_t(x)\}$ is a continuous adapted process. Note that the occupation density formula [87, p. 216, Corollary 1] (also see [88, Proposition 4 and Theorem 3]) follows from the comparison of two versions of Itô formula in the finite dimensional case, first being the Meyer-Itô formula [87, Chapter IV, Theorem 70] (an application of which leads to the Tanaka formula) where local time appears and second the usual version for C^2 functions. By the occupation density formula, we have for any $\phi\in\mathcal{S}$, a.s.

$$\int_{-\infty}^{\infty} L_t(x)\phi(x)\,dx = \int_0^t \phi(X_{s-})d\,[X]^c_s\,, \tag{6.13}$$

where $[X]$ stands for $[X,X]$ and $[X]^c$ denotes the continuous part of $[X]$. By [87, p. 216, Corollary 2] a.s.

$$\int_{-\infty}^{\infty} L_t(x)\,dx = \int_0^t d\,[X]^c_s\,,$$

which shows a.s. for all t, the map $x\mapsto L_t(x)$ is integrable. We now identify the local time process in $\mathcal{S}'$. A version of this result was proved in [89, Lemma 2.1] for continuous semimartingales X.

Proposition 6.3.3. *The $\mathcal{S}'$ valued process $\{\int_0^t \delta_{X_{s-}} d\,[X]^c_s\}$ is $\mathcal{S}_{-p}$ valued for any $p>\frac{1}{4}$ and for each t, $\int_0^t \delta_{X_{s-}} d\,[X]^c_s$ is given by the integrable function $x\mapsto L_t(x)$.*

Proof. Note that for any fixed $x\in\mathbb{R}$, the distribution δ_x is in $\mathcal{S}_{-p}$ for any $p>\frac{1}{4}$ (see Proposition 2.11.14). Also $\tau_x\delta_0=\delta_x$ (Lemma 2.11.15). Hence by Lemma 6.2.1, $\{\delta_{X_{t-}}\}$ is an $\mathcal{S}_{-p}$ valued predictable process. By Lemma 2.11.14(ii), it is also bounded. Then we have the $\mathcal{S}_{-p}$ valued process $\{\int_0^t \delta_{X_{s-}} d\,[X]^c_s\}$, where each of the random variables $\int_0^t \delta_{X_{s-}} d\,[X]^c_s, t\geq 0$ is defined as a Bochner integral for any $p>\frac{1}{4}$.
But for any integer $n\geq 0$, by (6.13) a.s. for all $t\geq 0$

$$\begin{aligned}\left\langle \int_0^t \delta_{X_{s-}}\,d\,[X]^c_s\,,\,h_n\right\rangle &= \int_0^t \left\langle \delta_{X_{s-}}\,,\,h_n\right\rangle\,d\,[X]^c_s\\ &= \int_0^t h_n(X_{s-})\,d\,[X]^c_s\\ &= \int_{-\infty}^{\infty} L_t(x)h_n(x)\,dx\end{aligned}$$

Then there exists a P null set $\widetilde{\Omega}$ such that on $\Omega\setminus\widetilde{\Omega}$ for all integers $n\geq 0$ and all $t\geq 0$

$$\left\langle \int_0^t \delta_{X_{s-}}\,d\,[X]^c_s\,,\,h_n\right\rangle = \int_{-\infty}^{\infty} L_t(x)h_n(x)\,dx.$$

Then for each t, the $\mathcal{S}'$ valued random variable $\int_0^t \delta_{X_{s-}} d\,[X]^c_s$ is given by the function $x\mapsto L_t(x)$ (see Proposition 2.10.2). □

Remark 6.3.4. It was observed in [89, Corollary 2.5] that the Itô formula in the finite dimensional case can be written in a 'functional' form. Since $\tau_x\delta_0 = \delta_x$ (Lemma 2.11.15), for any continuous $\mathbb{R}^d$ valued $(\mathcal{F}_t)$ adapted semimartingale $\{X_t\}$, we have a.s. (see Theorem 6.1.1)

$$\delta_{X_t} = \delta_{X_0} - \sum_{i=1}^{d} \int_0^t \partial_i \delta_{X_s}\, dX_s^i + \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \partial_{ij}^2 \delta_{X_s}\, d[X^i, X^j]_s,\ t \geq 0, \tag{6.14}$$

with equality in some $\mathcal{S}_{-p}(\mathbb{R}^d)$. If $\phi \in \mathcal{S}(\mathbb{R}^d)$, then the duality $\phi(X_t) = \langle \delta_{X_t}\,,\, \phi\rangle$ together with equation (6.14) implies the Itô formula in the finite dimensional case. Using Theorem 6.2.3, a similar identification of the Itô formula in the finite dimensions can now be obtained for semimartingales $\{X_t\}$ with jumps. This identification can be stated in terms of random linear functionals on $\mathcal{S}(\mathbb{R}^d)$ (for the notion of random linear functionals on Nuclear spaces, see [117, Chapter 4]). The Itô formula for a smooth function ϕ and a continuous semimartingale $\{X_t\}$ can be written as $\phi(X_t) = \phi(X_0) + I_t^1(\phi) + I_t^2(\phi)$, $t \geq 0$ where

$$I_t^1(\phi) := \sum_{i=1}^{d}\int_0^t \partial_i\phi(X_s)\, dX_s^i, \quad I_t^2(\phi) := \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \partial_{ij}^2\phi(X_s)\, d[X^i, X^j]_s,$$

are random linear functionals on $\mathcal{S}(\mathbb{R}^d)$. In the context of Itô's regularization Theorem ([53, Theorem 2.3.2], [117, Theorem 4.1]), we can ask whether there exist $\mathcal{S}_{-q}(\mathbb{R}^d)$ (for some $q \in \mathbb{R}$) valued adapted processes $\{\widetilde{I}_t^1\}, \{\widetilde{I}_t^2\}$ such that for any $\phi \in \mathcal{S}(\mathbb{R}^d)$, a.s $t \geq 0$

$$\left\langle \widetilde{I}_t^1\,,\, \phi\right\rangle = I_t^1(\phi), \quad \left\langle \widetilde{I}_t^2\,,\, \phi\right\rangle = I_t^2(\phi).$$

The discussion at the beginning of this remark answers this question in the affirmative. This type of regularization problems have been studied in [59, Theorem 3.1.3], [77, 78] which dealt with martingales, submartingales and certain stochastic integrals and in [85] with semimartingales where '$\mathcal{S}'(\mathbb{R}^d)$ regularized' versions were obtained. This connection can be obtained in a more general setting. We say an $\mathcal{S}'(\mathbb{R}^d)$ valued $(\mathcal{F}_t)$ adapted rcll process $\{X_t\}$ is a *weak semimartingale* if for each $\phi \in \mathcal{S}(\mathbb{R}^d)$, $\{\langle X_t\,,\, \phi\rangle\}$ is a real valued semimartingale, i.e. a.s.

$$\langle X_t\,,\, \phi\rangle = X_0^\phi + M_t^\phi + A_t^\phi,\ \forall t \geq 0. \tag{6.15}$$

where X_0^ϕ is an $\mathcal{F}_0$ measurable real valued random variable, $\{M_t^\phi\}$ is a real valued local martingale with $M_0^\phi = 0$ a.s. and $\{A_t^\phi\}$ is a real valued FV process with $A_0^\phi = 0$ a.s.. Under some continuity conditions of $\phi \mapsto \mathbb{E}(M_t^\phi)^2$ and $\phi \mapsto \mathbb{E}\,Var_{[0,t]}(A^\phi)$, it is possible to obtain an $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued semimartingale $\{\widetilde{X}_t\}$ such that a.s. $X_t = \widetilde{X}_t, t \geq 0$. Similar results can be obtained for collections of random linear functionals, e.g. $\{I_t^1\}, \{I_t^2\}$ as above. A preprint about these results is under preparation.

Publications

This thesis is based on the following articles.

(1) Suprio Bhar and B. Rajeev, *Differential Operators on Hermite Sobolev Spaces*, To appear in Proc. Indian Acad. Sci. Math. Sci.

(2) Suprio Bhar, *Characterizing Gaussian flows arising from Itō's stochastic differential equations*, Communicated (2014), `arXiv:1410.4633[math.PR]`.

(3) Suprio Bhar, *An Itō formula in the space of tempered distributions*, Communicated (2014), `arXiv:1411.6145[math.PR]`.

(4) Suprio Bhar, *Stationary solutions of stochastic partial differential equations in the space of tempered distributions*, Communicated (2014), `arXiv:1412.1912[math.PR]`.

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List of symbols

$|n|$ $n_1 + \cdots + n_d$ for any multi-index $n = (n_1, \cdots, n_d) \in \mathbb{Z}_+^d$

$|x|$ Standard Euclidean norm of x, when $x \in \mathbb{R}$ or $\mathbb{R}^d$

a.s. almost surely

$\mathcal{B}(\mathbb{R}^d)$ The Borel σ-field on $\mathbb{R}^d$.

$B(0, n)$ $\{x \in \mathbb{R}^d : |x| \leq n\}$. The dimension d will be understood from the context.

$\mathbb{C}$ Set of complex numbers

$C^\infty(\mathbb{R}^d)$ The set of real valued infinitely differentiable functions on $\mathbb{R}^d$.

$\delta_x, x \in \mathbb{R}^d$ Dirac distributions.

$\partial_i, i = 1, \cdots, d$ Partial derivative operators on $\mathcal{S}'(\mathbb{R}^d)$.

$\mathcal{E}'(\mathbb{R}^d)$ The space of compactly supported distributions on $\mathbb{R}^d$.

$\stackrel{\mathcal{L}}{=}$ Equality in law.

$(2k-1)!!$ Denotes the product $1 \times 3 \times \cdots \times (2k-1)$, when k is a positive integer.

$\mathbf{H}$ Hermite operator. See Index 'Hermite operator $\mathbf{H}$' for more reference.

$h_n, n \in \mathbb{Z}_+^d$ Hermite functions on $\mathbb{R}^d$.

$\mathbb{1}_A$ indicator function of some measurable set A.

$\langle \cdot \, , \, \cdot \rangle_p$ The Hermite-Sobolev inner product, $p \in \mathbb{R}$.

$\mathcal{L}^1(\mathbb{R}^d)$ The set of real valued integrable functions on $\mathbb{R}^d$, with respect to the Lebesgue measure.

$\mathcal{L}^2(\mathbb{R}^d)$ Set of real valued square integrable functions on $\mathbb{R}^d$, with respect to the Lebesgue measure.

$\langle M, N \rangle$ Predictable quadratic variation of real valued martingales M and N

$\langle M \rangle$ Meyer process of a real valued martingale M. Shorthand for $\langle M, M \rangle$.

$\triangle X$ The jump process of a process X.

$[X, Y]$ Quadratic variation of $\mathbb{R}$ valued semimartingales X and Y.

$[X]$ Shorthand for $[X, X]$, when X is a real valued semimartingale.

$\mathscr{M}^2$ The vector space of real valued rcll $\mathcal{L}^2$ martingales.

$\mathscr{M}^2_\infty$ the vector space of real valued rcll $\mathcal{L}^2$-bounded martingales.

$\mathscr{M}^{2,c}$ the vector space of real valued continuous $\mathcal{L}^2$ martingales.

$\mathscr{M}^{2,c}_\infty$ the vector space of real valued continuous $\mathcal{L}^2$-bounded martingales.

$\mathscr{M}_i$ Operators on $\mathcal{S}'(\mathbb{R}^d)$ given by multiplication by the co-ordinate functions $x_i, i = 1, \cdots, d$. See Index 'Multiplication operators $\mathscr{M}_i$' for more reference.

X^* $\sup_{t\geq 0} |X_t|$, when $\{X_t\}$ is a real or $\mathbb{R}^d$ valued stochastic process.

X_t^* $\sup_{s\leq t} |X_s|$, when $\{X_t\}$ is a real or $\mathbb{R}^d$ valued stochastic process.

$\mathbb{N}$ Set of natural numbers

$\|\cdot\|_p$ The Hermite-Sobolev norm, $p \in \mathbb{R}$.

$\mathbb{R}$ Set of real numbers

$\mathbb{R}^d$ Cartesian Product $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$, d times. The d dimensional Euclidean space.

$\widehat{\mathbb{R}}^d$ $\mathbb{R}^d \cup \{\infty\}$, the one point compactification of $\mathbb{R}^d$

$\hat{\mathcal{S}}_p(\mathbb{R}^d)$ $\mathcal{S}_p(\mathbb{R}^d) \cup \{\delta\}$, where δ is an isolated point.

$\mathcal{S}$ abbreviated for $\mathcal{S}(\mathbb{R})$

$\mathcal{S}'$ abbreviated for $\mathcal{S}'(\mathbb{R})$

$\mathcal{S}'(\mathbb{R}^d)$ The space of tempered distributions on $\mathbb{R}^d$.

$\mathcal{S}'(\mathbb{R}^d; \mathbb{C})$ Continuous linear functionals on $\mathcal{S}(\mathbb{R}^d; \mathbb{C})$

$\mathcal{S}(\mathbb{R}^d)$ The space of real valued rapidly decreasing smooth functions on $\mathbb{R}^d$.

$\mathcal{S}(\mathbb{R}^d; \mathbb{C})$ The space of $\mathbb{C}$ valued rapidly decreasing smooth functions on $\mathbb{R}^d$.

$\mathcal{S}_p$ abbreviated for $\mathcal{S}_p(\mathbb{R})$

$\mathcal{S}_p(\mathbb{R}^d)$ Hermite-Sobolev space, Completion of $(\mathcal{S}(\mathbb{R}^d), \|\cdot\|_p)$ for $p \in \mathbb{R}$

$\mathcal{S}_p(\mathbb{R}^d; \mathbb{C})$ Hermite-Sobolev space, Completion of $(\mathcal{S}(\mathbb{R}^d; \mathbb{C}), \|\cdot\|_p)$ for $p \in \mathbb{R}$

$\tau_x, x \in \mathbb{R}^d$ Translation operators on $\mathcal{S}'(\mathbb{R}^d)$.

$Var_{[0,t]}(f)$ Total variation of a function $f : [0, t] \to \mathbb{B}$, where $\mathbb{B}$ is a Banach space.

$\mathbb{Z}$ Set of integers

$\mathbb{Z}_+^d$ Set of multi-indices $n = (n_1, \cdots, n_d)$ where n_i are non-negative integers

Index